Mathematics 805
Homework 11
Due Friday, April 24, 1 PM

1. As before, let $B_n(x)$ be the Bernoulli polynomial of degree n. Let p be a positive integer larger than 1, and suppose $0 \le x \le 1$. Show that

$$B_k(px) = p^{k-1} \sum_{b=0}^{p-1} B_k\left(x + \frac{b}{p}\right).$$

Answer: We have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\sum_{b=0}^{p-1} B_n\left(x+\frac{b}{p}\right)}{n!} t^n &= \sum_{b=0}^{p-1} \frac{te^{\left(x+\frac{b}{p}\right)t}}{e^t-1} = \frac{t}{e^t-1} \sum_{b=0}^{p-1} e^{\left(x+\frac{b}{p}\right)t} \\
&= \frac{t}{e^t-1}\left(\frac{e^{xt}-e^{(x+1)t}}{1-e^{t/p}}\right) = \frac{t}{e^t-1}\left(e^{xt}\frac{1-e^t}{1-e^{t/p}}\right) \\
&= \frac{te^{xt}}{e^{t/p}-1} = p\frac{(t/p)e^{px(t/p)}}{e^{t/p}-1} = p\sum_{n=0}^{\infty} \frac{B_n(px)}{n!}\left(\frac{t}{p}\right)^n
\end{aligned}$$

Equating the coefficient of t^n yields

$$\sum_{b=0}^{p-1} B_n\left(x+\frac{b}{p}\right) = p^{1-n} B_n(px),$$

which is the desired result.

2. We saw in a previous homework that if $P_k(x) = \frac{B_k(x+1)-B_k(1)}{k}$, then $P_k(n) = 1^{k-1} + 2^{k-1} + \cdots + n^{k-1}$. A quick computation gives

$$\begin{aligned}
P_2(x) &= \frac{x(x+1)}{2} \\
P_3(x) &= \frac{x(x+1)(2x+1)}{6} \\
P_4(x) &= \frac{x^2(x+1)^2}{4} \\
P_5(x) &= \frac{x(x+1)(2x+1)(3x^2+3x-1)}{30} \\
P_6(x) &= \frac{x^2(x+1)^2(2x^2+2x-1)}{12} \\
P_7(x) &= \frac{x(x+1)(2x+1)(3x^4+6x^3-3x+1)}{42} \\
P_8(x) &= \frac{x^2(x+1)^2(3x^4+6x^3-x^2-4x+2)}{24}
\end{aligned}$$

Some obvious patterns suggest themselves.
(a) Prove that $P_k(0) = P_k(-1) = 0$ if $k \ge 2$. (In other words, $x(x+1)|P_k(x)$.)
(b) If $k \ge 3$ is odd, prove that $P_k(-\frac{1}{2}) = 0$. (In other words, $(2x+1)|P_k(x)$ if $k \ge 3$ is odd.)
(c) If $k \ge 4$ is even, prove that $P_k'(0) = P_k'(-1) = 0$. (In other words, $x^2(x+1)^2|P_k(x)$ if $k \ge 4$ is even.)

Answer: (a) We have $P_k(0) = (B_k(1) - B_k(1))/k = 0$ and $P_k(-1) = (B_k(0) - B_k(1))/k = 0$, because $B_k(0) = B_k(1)$.

(b) If k is odd, we have $P_k(-\frac{1}{2}) = (B_k(\frac{1}{2}) - B_k(1))/k = 0$, because we saw on the last homework assignment that $B_k(\frac{1}{2}) = 0$ if k is odd, and we showed earlier that $B_k(1) = 0$ if k is odd.

(c) We have $P_k'(0) = B_k'(1)/k = B_{k-1}(1) = 0$, because $k-1$ is odd, and $P_k'(-1) = B_k'(0)/k = B_{k-1}(0) = 0$, again because $k-1$ is odd.

3. (a) Show that $\frac{t}{e^t-1} + \frac{t}{2} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} t^{2n}$.

(b) Let $t = iy$ in the previous equation, and after some algebraic manipulation of the left-hand side of the equation, find a power series expansion for $y \cot y$.

(c) Prove that $\cot z - 2\cot 2z = \tan z$.

(d) Combine these results to show that

$$\tan z = \sum_{k=1}^{\infty} \frac{(2^{2k}-1)2^{2k}(-1)^{k-1}}{(2k)!} B_{2k} z^{2k-1}.$$

Answer: (a) We know that $\frac{t}{e^t-1} = 1 - \frac{t}{2} + \sum_{n=2}^{\infty} \frac{B_n}{n!} t^n = 1 - \frac{t}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} t^{2n}$, because $B_{2k+1} = 0$ for $k > 0$. Transferring $\frac{t}{2}$ to the other side of the equation gives the desired result.

(b) Notice that $\frac{t}{e^t-1} + \frac{t}{2} = \frac{te^t+t}{2(e^t-1)} = \frac{t}{2} \cdot \frac{e^t+1}{e^t-1} = \frac{t}{2} \cdot \frac{e^t+1}{e^t-1} \cdot \frac{e^{-t/2}}{e^{-t/2}} = \frac{t}{2} \cdot \frac{e^{t/2}+e^{-t/2}}{e^{t/2}-e^{-t/2}}$. We therefore have

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^n y^{2n} = \left(\frac{iy}{2}\right) \frac{e^{iy/2}+e^{-iy/2}}{e^{iy/2}-e^{-iy/2}} = \left(\frac{y}{2}\right) \frac{e^{iy/2}+e^{-iy/2}}{2} \cdot \frac{2i}{e^{iy/2}-e^{-iy/2}} = \frac{y\cos(y/2)}{2\sin(y/2)} = \frac{y}{2}\cot(y/2).$$

Let $y = 2z$, and we have

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^n 2^{2n} z^{2n} = z \cot z$$

(c) We have $\cot z - \tan z = \frac{\cos z}{\sin z} - \frac{\sin z}{\cos z} = \frac{\cos^2 z - \sin^2 z}{\sin z \cos z} = 2\frac{\cos 2z}{\sin 2z} = 2\cot 2z$. Therefore, $\cot z - 2\cot 2z = \tan z$.

(d) We have

$$z\cot z = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^n 2^{2n} z^{2n}$$

$$2z\cot 2z = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^n 2^{2n} (2z)^{2n} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^n 2^{4n} z^{2n}$$

$$z(\cot z - 2\cot 2z) = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^n (2^{2n} - 2^{4n}) z^{2n} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^{n-1} (2^{4n} - 2^{2n}) z^{2n}$$

Notice that the $n = 0$ term in this sum is 0, so we have

$$z\tan z = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^{n-1} (2^{4n} - 2^{2n}) z^{2n}$$

$$\tan z = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^{n-1} (2^{4n} - 2^{2n}) z^{2n-1} = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(-1)^{n-1} 2^{2n}(2^{2n}-1) z^{2n-1}$$

4. **10.7.25.** For a function $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, define the partial functions $f_x(t) := f(x,t)$ and $f_t(x) := f(x,t)$. Suppose that $f_t \in L^1$ for each $t \in \mathbf{R}$ and that f_x is differentiable for each $x \in \mathbf{R}$ and such that, with $\frac{\partial f}{\partial t} := \frac{df_x}{dt}$ and some $g \in L^1$, we have

$$\left| \frac{\partial}{\partial t} f(x,t) \right| \le g(x) \qquad (\forall x, t \in \mathbf{R}).$$

Then the function $x \to \frac{\partial}{\partial t} f(x,t)$ is integrable and we have

$$\frac{d}{dt}\int f(x,t)\,dx = \int \frac{\partial}{\partial t} f(x,t)\,dx.$$

Answer: Note that the hypotheses tell us that $\displaystyle\int f(x,t)\,dx$ is defined for each $t \in \mathbf{R}$. Let h_n be any sequence of positive real numbers with $\lim h_n = 0$. The Mean Value Theorem says that $f(x,t+h_n) - f(x,t) = h_n \frac{\partial f}{\partial t}(x, t+k_n)$ where $0 < k_n < h_n$. Define $f_n(x,t) = h_n^{-1}(f(x,t+h_n) - f(x,t))$. Notice that $f_n \in L^1$, and that $|f_n(x,t)| \le g(x)$.

$$\frac{d}{dt}\int f(x,t)\,dx = \lim_{n\to\infty}\int \frac{f(x,t+h_n) - f(x,t)}{h_n}\,dx = \lim_{n\to\infty}\int f_n(x,t)\,dx$$

Now, we can apply Lebesgue's Dominated Convergence Theorem to tell us that

$$\lim_{n\to\infty}\int f_n(x,t)\,dx = \int \lim_{n\to\infty} f_n(x,t)\,dx = \int \frac{\partial f}{\partial t}(x,t)\,dx.$$

We can then repeat the argument, picking h_n to be any sequence of negative real numbers.

5. **10.7.27.** Consider the function

$$F(t) = \int_0^\infty e^{-tx}\frac{\sin x}{x}\,dx \qquad (\forall t > 0),$$

where $\dfrac{\sin x}{x} = 1$ if $x = 0$.

(*a*) Differentiating under the integral sign, show that $F'(t) = -\frac{1}{t^2+1}$ and deduce that $F(t) = C - \arctan t$ for all $t > 0$ and some constant $C \in \mathbf{R}$.

(*b*) Using the sequence $(F(n))_{n\in\mathbf{N}}$, show that $C = \dfrac{\pi}{2}$, and deduce that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Answer: (*a*) We need to verify the hypotheses of the previous problem in order to differentiate under the integral sign. Fix $\delta > 0$, and suppose that $t \ge \delta$. We know that $|\frac{\sin x}{x}| < 1$, so $\left|e^{-tx}\frac{\sin x}{x}\right| < e^{-\delta x} \in L^1(0,\infty)$. Moreover, $\left|\frac{\partial}{\partial t}\left(e^{-tx}\frac{\sin x}{x}\right)\right| = |-e^{-tx}\sin x| \le e^{-\delta x} \in L^1(0,\infty)$. Thus, *provided that* $t \ge \delta$, we can differentiate under the integral sign.

We have

$$\begin{aligned}
F'(t) &= \int_0^\infty \frac{\partial}{\partial t}(e^{-tx})\frac{\sin x}{x}\,dx = \int_0^\infty -e^{-tx}\sin x\,dx = \frac{1}{2i}\int_0^\infty -e^{-tx}(e^{ix} - e^{-ix})\,dx \\
&= \frac{1}{2i}\int_0^\infty \left(e^{x(-i-t)} - e^{x(i-t)}\right)dx = \frac{1}{2i}\left(\frac{e^{x(-i-t)}}{-i-t} - \frac{e^{x(i-t)}}{i-t}\right]_0^\infty\Bigg) = -\frac{1}{2i}\left(\frac{1}{-i-t} - \frac{1}{i-t}\right) \\
&= -\frac{1}{2i}\left(\frac{(i-t) - (-i-t)}{t^2+1}\right) = -\frac{1}{2i}\left(\frac{2i}{t^2+1}\right) = -\frac{1}{t^2+1}.
\end{aligned}$$

Therefore, $F'(t) = -\frac{1}{t^2+1}$, so $F(t) = C - \arctan t$.

(*b*) Returning to the original definition of the function, we see that $\displaystyle\lim_{n\to\infty} F(n) = \lim_{n\to\infty}\int_0^\infty e^{-nx}\frac{\sin x}{x}\,dx = \int_0^\infty \lim_n e^{-nx}\frac{\sin x}{x}\,dx = 0$. The interchange of limit and integral is justified by Lebesgue's Dominated Convergence Theorem, because $|e^{-nx}\frac{\sin x}{x}| < e^{-x} \in L^1(0,\infty)$. Therefore, $0 = \displaystyle\lim_{n\to\infty}(C - \arctan n) = C - \frac{\pi}{2}$, so $C = \dfrac{\pi}{2}$ and $F(t) = \dfrac{\pi}{2} - \arctan t$.

Note that $F(0) = \frac{\pi}{2} - \arctan 0 = \frac{\pi}{2}$. We'd like to write $F(0) = \int_0^\infty \frac{\sin x}{x}\, dx$, but more work is needed to interchange the limit and integral here, because the function $\frac{\sin x}{x} \notin L^1(0, \infty)$.

We know from a previous homework problem that $\lim_{b\to\infty} \int_0^b \frac{\sin x}{x} dx$ converges. Given $\epsilon > 0$, choose B so that if $b > B$, then $\left| \int_b^\infty \frac{\sin x}{x}\, dx \right| < \epsilon$.

Remember that if $x \geq 0$ and $\delta \geq 0$, then $e^{-\delta x} < 1$, so $\left| \int_b^\infty e^{-\delta x} \frac{\sin x}{x}\, dx \right| < \epsilon$. So

$$\begin{aligned}
\left| F(\delta) - \int_0^\infty \frac{\sin x}{x}\, dx \right| \leq \left| F(\delta) - \int_0^b \frac{\sin x}{x}\, dx \right| + \left| \int_b^\infty \frac{\sin x}{x}\, dx \right| &= \left| F(\delta) - \int_0^b \frac{\sin x}{x}\, dx \right| + \epsilon \\
&= \left| \int_0^\infty e^{-\delta x} \frac{\sin x}{x}\, dx - \int_0^b \frac{\sin x}{x}\, dx \right| + \epsilon \\
&\leq \left| \int_0^b e^{-\delta x} \frac{\sin x}{x}\, dx - \int_0^b \frac{\sin x}{x}\, dx \right| + \left| \int_b^\infty e^{-\delta x} \frac{\sin x}{x} \right| + \epsilon \\
&\leq \left| \int_0^b e^{-\delta x} \frac{\sin x}{x}\, dx - \int_0^b \frac{\sin x}{x}\, dx \right| + 2\epsilon \\
&= \left| \int_0^b \left(e^{-\delta x} - 1\right) \frac{\sin x}{x}\, dx \right| + 2\epsilon
\end{aligned}$$

Now, we can apply the Lebesgue Dominated Convergence Theorem and conclude that

$$\lim_{\delta\to 0} \left| F(\delta) - \int_0^\infty \frac{\sin x}{x}\, dx \right| \leq \left| \int_0^b \lim_{\delta\to 0} \left(e^{-\delta x} - 1\right) \frac{\sin x}{x}\, dx \right| + 2\epsilon = 2\epsilon.$$

Because ϵ is arbitrary, we can conclude that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$