Mathematics 805
Homework 11
Due Friday, April 24, 1 PM

1. As before, let $B_n(x)$ be the Bernoulli polynomial of degree n. Let p be a positive integer larger than 1, and suppose $0 \le x \le 1$. Show that

$$B_k(px) = p^{k-1} \sum_{b=0}^{p-1} B_k\left(x + \frac{b}{p}\right).$$

2. We saw in a previous homework that if $P_k(x) = \frac{B_k(x+1) - B_k(1)}{k}$, then $P_k(n) = 1^{k-1} + 2^{k-1} + \cdots + n^{k-1}$. A quick computation gives

$$\begin{aligned}
P_2(x) &= \frac{x(x+1)}{2} \\
P_3(x) &= \frac{x(x+1)(2x+1)}{6} \\
P_4(x) &= \frac{x^2(x+1)^2}{4} \\
P_5(x) &= \frac{x(x+1)(2x+1)(3x^2+3x-1)}{30} \\
P_6(x) &= \frac{x^2(x+1)^2(2x^2+2x-1)}{12} \\
P_7(x) &= \frac{x(x+1)(2x+1)(3x^4+6x^3-3x+1)}{42} \\
P_8(x) &= \frac{x^2(x+1)^2(3x^4+6x^3-x^2-4x+2)}{24}
\end{aligned}$$

Some obvious patterns suggest themselves.

(a) Prove that $P_k(0) = P_k(-1) = 0$ if $k \ge 2$. (In other words, $x(x+1)|P_k(x)$.)

(b) If $k \ge 3$ is odd, prove that $P_k(-\frac{1}{2}) = 0$. (In other words, $(2x+1)|P_k(x)$ if $k \ge 3$ is odd.)

(c) If $k \ge 4$ is even, prove that $P_k'(0) = P_k'(1) = 0$. (In other words, $x^2(x+1)^2|P_k(x)$ if $k \ge 4$ is even.)

3. (a) Show that $\frac{t}{e^t - 1} + \frac{t}{2} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} t^{2n}$.

(b) Let $t = iy$ in the previous equation, and after some algebraic manipulation of the left-hand side of the equation, find a power series expansion for $y \cot y$.

(c) Prove that $\cot z - 2\cot 2z = \tan z$.

(d) Combine these results to show that

$$\tan z = \sum_{k=1}^{\infty} \frac{(2^{2k}-1)2^{2k}(-1)^{k-1}}{(2k)!} B_{2k} z^{2k-1}.$$