

MT90301  
 Graduate Seminar: Concrete Mathematics  
 Examination 1  
 Answers

1. (20 points) Let  $m$  and  $n$  be positive integers. Show that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor.$$

*Answer:* Suppose  $\left\lceil \frac{n}{m} \right\rceil = k$ . Then  $k-1 < \frac{n}{m} \leq k$ , and multiplying by  $m$  yields  $mk - m < n \leq mk$ . Now all of the terms are integers, so we can shift by 1 and get  $mk - m + 1 \leq n < mk + 1$ . Subtract 1, and we have  $mk - m \leq n - 1 < mk$ . Now add  $m$ , and the result is  $mk \leq n + m - 1 < mk + m$ . Finally, divide by  $m$ , and we get  $k \leq \frac{n+m-1}{m} < k+1$ . But this precisely says that  $k = \left\lfloor \frac{n+m-1}{m} \right\rfloor$ .

2. (20 points) Prove using induction that

$$\sum_{0 \leq k < n} H_k = nH_n - n.$$

*Answer:* Let  $S_n = \sum_{0 \leq k < n} H_k$ . Then  $S_1 = H_0 = 0 = 1H_1 - 1$ , so we can start the induction. Just to be sure, we can check for  $n = 2$  also:  $S_2 = H_1 = 1 = 2H_2 - 2$ , because  $H_2 = \frac{3}{2}$ .

Now,

$$\begin{aligned} S_{n+1} &= S_n + H_n = nH_n - n + H_n = (n+1)H_n - n \\ &= (n+1)H_n + 1 - (n+1) = (n+1)H_n + (n+1)\frac{1}{n+1} - (n+1) \\ &= (n+1)\left(H_n + \frac{1}{n+1}\right) - (n+1) = (n+1)H_{n+1} - (n+1), \end{aligned}$$

completing the inductive step.

3. (20 points) Try to sum the series

$$\sum_{1 \leq k \leq 2n} (-1)^k k^2$$

by using perturbation. What sum do you compute instead?

*Answer:* Let  $S_n = \sum_{1 \leq k \leq 2n} (-1)^k k^2$ . Then

$$\begin{aligned} S_n - (2n+1)^2 + (2n+2)^2 &= S_{n+1} = -1 + 4 + \sum_{3 \leq k \leq 2n+2} (-1)^k k^2 = 3 + \sum_{1 \leq k \leq 2n} (-1)^{k+2} (k+2)^2 \\ &= 3 + \sum_{1 \leq k \leq 2n} (-1)^k (k^2 + 4k + 4) = 3 + S_n + \sum_{1 \leq k \leq 2n} (-1)^k (4k + 4) \\ -(4n^2 + 4n + 1) + (4n^2 + 8n + 4) &= 3 + \sum_{1 \leq k \leq 2n} (-1)^k (4k + 4) \\ 4n + 3 &= 3 + \sum_{1 \leq k \leq 2n} (-1)^k (4k + 4) \\ n &= \sum_{1 \leq k \leq 2n} (-1)^k (k + 1) \end{aligned}$$

Because  $\sum_{1 \leq k \leq 2n} (-1)^k = 0$ , we can actually conclude that  $\sum_{1 \leq k \leq 2n} (-1)^k k = n$ .

4. (20 points) Let  $m$  and  $n$  be positive integers. Prove that

$$\frac{x^m}{(x-n)^m} = \frac{x^n}{(x-m)^n},$$

provided that neither denominator is 0.

*Answer:* We know that  $x^{n+m} = x^n(x-n)^m$ , and also  $x^{n+m} = x^m(x-m)^n$ . This says that  $x^n(x-n)^m = x^m(x-m)^n$ . Now divide by  $(x-n)^m(x-m)^n$ , and the result follows.

5. (20 points) Prove by rearranging terms that

$$\sum_{1 \leq j < k \leq n} (a_j b_k - a_k b_j)^2 = \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right) - \left( \sum_{k=1}^n a_k b_k \right)^2.$$

*Answer:* Let  $S = \sum_{1 \leq j < k \leq n} (a_j b_k - a_k b_j)^2$ . Reverse  $j$  and  $k$ , and  $S = \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2$ . But  $(c-d)^2 = (d-c)^2$ , so we have  $S = \sum_{1 \leq k < j \leq n} (a_j b_k - a_k b_j)^2$ . Finally, note that  $\sum_{1 \leq k = j \leq n} (a_j b_k - a_k b_j)^2 = 0$ . Adding, we have  $2S = \sum_{1 \leq j, k \leq n} (a_j b_k - a_k b_j)^2$ . Now we have something which is much simpler to expand:

$$\begin{aligned} 2S &= \sum_{1 \leq j, k \leq n} (a_j b_k - a_k b_j)^2 = \sum_{1 \leq j, k \leq n} a_j^2 b_k^2 - 2a_j a_k b_j b_k + a_k^2 b_j^2 \\ &= \sum_{1 \leq j, k \leq n} (a_j^2 b_k^2 + a_k^2 b_j^2) - 2 \sum_{1 \leq j, k \leq n} a_j a_k b_j b_k \\ &= 2 \sum_{1 \leq j, k \leq n} a_j^2 b_k^2 - 2 \sum_{1 \leq j, k \leq n} a_j a_k b_j b_k \\ &= 2 \left( \sum_{1 \leq j \leq n} a_j^2 \right) \left( \sum_{1 \leq k \leq n} b_k^2 \right) - 2 \left( \sum_{1 \leq j \leq n} a_j b_j \right) \left( \sum_{1 \leq k \leq n} a_k b_k \right) \\ &= 2 \left( \sum_{1 \leq j \leq n} a_j^2 \right) \left( \sum_{1 \leq j \leq n} b_j^2 \right) - 2 \left( \sum_{1 \leq j \leq n} a_j b_j \right) \left( \sum_{1 \leq j \leq n} a_j b_j \right) \\ &= 2 \left( \sum_{1 \leq j \leq n} a_j^2 \right) \left( \sum_{1 \leq j \leq n} b_j^2 \right) - 2 \left( \sum_{1 \leq j \leq n} a_j b_j \right)^2 \end{aligned}$$

Divide by 2, and we have the result, which proves the Cauchy-Schwarz-Bouniakowski inequality

$$\left( \sum_{1 \leq j \leq n} a_j b_j \right)^2 \leq \left( \sum_{1 \leq j \leq n} a_j^2 \right) \left( \sum_{1 \leq j \leq n} b_j^2 \right).$$

This might be more familiar from linear algebra as  $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \cdot \|\vec{y}\|$  for  $\vec{x}, \vec{y} \in \mathbf{R}^n$ .