MT903.01
Graduate Seminar: Concrete Mathematics
Final Examination
May 11, 2009, 9 AM, Gasson 201

Please do all of your work in the blue books, and label your answers clearly. You must be explicit in discussing how you arrived at your solutions. Little or no credit will be given for solutions without explanation.

You should read the entire examination before starting. The problems are not arranged in order of increasing difficulty.

Cheating will be severely punished.

1. *(10 points)* Prove the Vandermonde identity

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$$

for integers m and n. You may not use any of the last five entries in the "Favorite Binomial Identities" table to prove this formula, because all of them are proved using Vandermonde's formula.

2. *(5 points)* What is $\varphi(999)$? Explain how you computed your answer.

3. *(5 points)* List the Stern-Brocot tree up to the level that includes $\frac{1}{3}$.

4. *(10 points)* Prove

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1},$$

where n and k are positive integers.

5. *(10 points)* Find a closed form for

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}}$$

for integers $n \geqslant m \geqslant 0$.

6. *(10 points)* Let $f_n = 2^{2^n} + 1$. Prove that $f_m \perp f_n$ if $m < n$.

7. *(10 points)* Suppose that $f(n)$ and $g(m)$ are both multiplicative functions. Define

$$h(m) = \sum_{d \mid m} f(d) g(\frac{m}{d}).$$

Show that $h(m)$ is a multiplicative function.

8. *(10 points)* Show that

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. How can you know when each case will occur?

9. *(10 points)* Prove or disprove:

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \leqslant \lfloor 2x \rfloor + \lfloor 2y \rfloor .$$

10. *(20 points)* Sometimes induction arguments work in unusual ways. Consider the statement

$$P(n): \qquad x_1 \cdots x_n \leqslant \left(\frac{x_1 + \cdots + x_n}{n} \right)^n, \qquad \text{if } x_1, \ldots, x_n \geqslant 0$$

(a) Prove that $P(2)$ is true.

(b) By setting $x_n = (x_1 + \cdots + x_{n-1})/(n-1)$, prove that $P(n)$ implies $P(n-1)$ whenever $n > 1$.

(c) Show that $P(n)$ and $P(2)$ imply $P(2n)$.

(d) Explain why these three steps imply that $P(n)$ is true for all positive integers n.

Favorite Binomial Identities

$$
\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} && \text{integers } n \geqslant k \geqslant 0 \text{ (\textit{factorial expansion})} \\
\binom{n}{k} &= \binom{n}{n-k} && \text{integer } n \geqslant 0, \text{ integer } k \text{ (\textit{symmetry})} \\
\binom{r}{k} &= \frac{r}{k}\binom{r-1}{k-1} && \text{integer } k \neq 0 \text{ (\textit{absorption/extraction})} \\
\binom{r}{k} &= \binom{r-1}{k} + \binom{r-1}{k-1} && \text{integer } k \text{ (\textit{addition/induction})} \\
\binom{r}{k} &= (-1)^k \binom{k-r-1}{k} && \text{integer } k \text{ (\textit{upper negation})} \\
\binom{r}{m}\binom{m}{k} &= \binom{r}{k}\binom{r-k}{m-k} && \text{integers } m, k \text{ (\textit{trinomial revision})} \\
\sum_k \binom{r}{k} x^k y^{r-k} &= (x+y)^r && \text{integer } r \geqslant 0 \text{ or } |x/y| < 1 \text{ (\textit{binomial theorem})} \\
\sum_{k \leqslant n} \binom{r+k}{k} &= \binom{r+n+1}{n} && \text{integer } n \text{ (\textit{parallel summation})} \\
\sum_{0 \leqslant k \leqslant n} \binom{k}{m} &= \binom{n+1}{m+1} && \text{integers } m, n \geqslant 0 \text{ (\textit{upper summation})} \\
\sum_k \binom{r}{m+k}\binom{s}{n-k} &= \binom{r+s}{m+n} && \text{integers } m, n \text{ (\textit{Vandermonde convolution})} \\
\sum_k \binom{l}{m+k}\binom{s}{n+k} &= \binom{l+s}{l-m+n} && \text{integer } l \geqslant 0, \text{ integers } m, n \\
\sum_k \binom{l}{m+k}\binom{s+k}{n}(-1)^k &= (-1)^{l+m}\binom{s-m}{n-l} && \text{integer } l \geqslant 0, \text{ integers } m, n \\
\sum_{k \leqslant l} \binom{l-k}{m}\binom{s}{k-n}(-1)^k &= (-1)^{l+m}\binom{s-m-1}{l-m-n} && \text{integers } l, m, n \geqslant 0 \\
\sum_{0 \leqslant k \leqslant l} \binom{l-k}{m}\binom{q+k}{n} &= \binom{l+q+1}{m+n+1} && \text{integers } l, m \geqslant 0, \text{ integers } n \geqslant q \geqslant 0
\end{aligned}
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