MT903.01
Graduate Seminar: Concrete Mathematics
Final Examination
Answers

1. *(10 points)* Prove the Vandermonde identity

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$$

for integers m and n. You may not use any of the last five entries in the "Favorite Binomial Identities" table to prove this formula, because all of them are proved using Vandermonde's formula.

Answer: We prove this by induction on r. Note that both sides are polynomials in r of degree $m+n$, so if the equality holds for all positive integer r, then it must be a polynomial identity and hence true for all values of r.

We start with the case $r = 0$. In that case, the only non-zero term in the sum occurs when $m + k = 0$, so $k = -m$, and then $\binom{s}{n-k} = \binom{s}{n+m}$, which of course is the right-hand side of the equation.

Assuming that the equation is true for r, we verify it for $r + 1$. We have

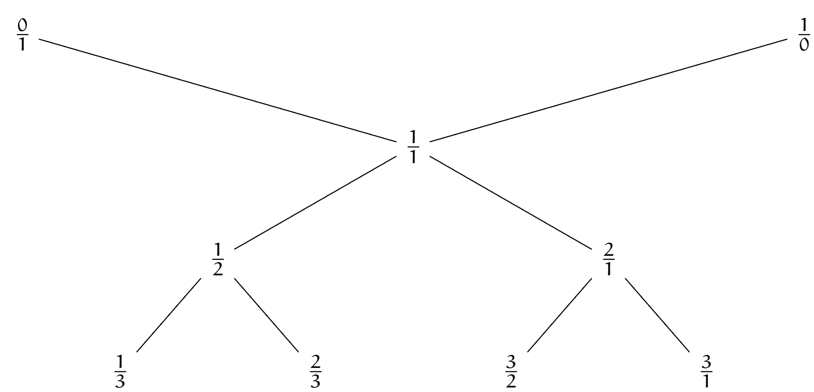
$$\begin{aligned}\sum_k \binom{r+1}{m+k}\binom{s}{n-k} &= \sum_k \left(\binom{r}{m+k} + \binom{r}{m+k-1}\right)\binom{s}{n-k} \\ &= \sum_k \binom{r}{m+k}\binom{s}{n-k} + \sum_k \binom{r}{m-1+k}\binom{s}{n-k} \\ &= \binom{r+s}{m+n} + \binom{r+s}{m-1+n} = \binom{r+s+1}{m+n}.\end{aligned}$$

2. *(5 points)* What is $\varphi(999)$? Explain how you computed your answer.

Answer: We have $\varphi(999) = \varphi(27)\varphi(37) = (27-9)(37-1) = (18)(36) = 628$.

3. *(5 points)* List the Stern-Brocot tree up to the level that includes $\frac{1}{3}$.

Answer:



4. *(10 points)* Prove

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1},$$

where n and k are positive integers.

Answer: We have

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k-1}\left[\left(\frac{n}{k+1}\right)\binom{n-1}{k}\right]\left[\left(\frac{n+1}{k}\right)\binom{n}{k-1}\right]$$
$$= \left(\frac{n}{k+1}\right)\left(\frac{n+1}{k}\right)\binom{n-1}{k-1}\binom{n-1}{k}\binom{n}{k-1}$$
$$= \binom{n+1}{k+1}\binom{n-1}{k}\binom{n}{k-1}$$

5. *(10 points)* Find a closed form for

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}}$$

for integers $n \geqslant m \geqslant 0$.

Answer: We start with a simplification:

$$\frac{\binom{m}{k}}{\binom{n}{k}} = \frac{\frac{m!}{k!(m-k)!}}{\frac{n!}{k!(n-k)!}} = \left(\frac{m!}{n!}\right)\left(\frac{(n-k)!}{(m-k)!}\right) = \left(\frac{m!(n-m)!}{n!}\right)\left(\frac{(n-k)!}{(n-m)!(m-k)!}\right) = \binom{n}{m}^{-1}\binom{n-k}{n-m}$$

Therefore,

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = \binom{n}{m}^{-1}\sum_{k=0}^{m}\binom{n-k}{n-m} = \binom{n}{m}^{-1}\sum_{k=0}^{n}\binom{n-k}{n-m}$$

In the last equality, we have used the fact that if $k > m$, then $\binom{n-k}{n-m} = 0$.

Next, let $j = n-k$, and note that if $0 \leqslant k \leqslant n$, then $0 \leqslant j \leqslant n$. We have

$$\binom{n}{m}^{-1}\sum_{k=0}^{n}\binom{n-k}{n-m} = \binom{n}{m}^{-1}\sum_{j=0}^{n}\binom{j}{n-m} = \binom{n}{m}^{-1}\binom{n+1}{n-m+1}$$
$$= \left(\frac{m!(n-m)!}{n!}\right)\left(\frac{(n+1)!}{(n-m+1)!m!}\right) = \frac{n+1}{n-m+1}.$$

6. *(10 points)* Let $f_n = 2^{2^n} + 1$. Prove that $f_m \perp f_n$ if $m < n$.

Answer: Note that $2^{2^m} \equiv -1 \pmod{2^{2^m}+1}$. Raise both sides of this congruence to the power $2^{2^n - 2^m}$ (which is an even number), and we get $2^{2^n} \equiv 1 \pmod{f_m}$. Therefore, $f_n \equiv 2 \pmod{f_m}$, meaning that $f_n = qf_m + 2$ for some integer q. Therefore, if p is any prime which divides both f_n and f_m, p must divide 2, meaning that p would be 2. But obviously, f_n and f_m are both odd, and hence not multiples of 2, showing that there is no prime dividing both f_m and f_n. This shows that $f_m \perp f_n$.

7. *(10 points)* Suppose that $f(n)$ and $g(m)$ are both multiplicative functions. Define

$$h(m) = \sum_{d|m} f(d)g(\frac{m}{d}).$$

Show that $h(m)$ is a multiplicative function.

Answer: Suppose that $m = m_1m_2$, with $m_1 \perp m_2$. We must show that $h(m_1m_2) = h(m_1)h(m_2)$. The critical point is that if $d|m_1m_2$, then d can be written *uniquely* in the form $d = d_1d_2$, with $d_1|m_1$, $d_2|m_2$, and $d_1 \perp d_2$. Conversely, if $d_1|m_1$ and $d_2|m_2$, then $d_1 \perp d_2$ and if $d = d_1d_2$, then $d|m$.

We therefore have

$$h(m_1m_2) = \sum_{d|m_1m_2} f(d)g(\frac{m_1m_2}{d}) = \sum_{d_1d_2|m_1m_2} f(d_1d_2)g(\frac{m_1m_2}{d_1d_2}) = \sum_{d_1|m_1,d_2|m_2} f(d_1d_2)g(\frac{m_1}{d_1}\cdot\frac{m_2}{d_2})$$
$$= \sum_{d_1|m_1,d_2|m_2} f(d_1)f(d_2)g(\frac{m_1}{d_1})g(\frac{m_2}{d_2}) = \sum_{d_1|m_1} f(d_1)g(\frac{m_1}{d_1})\sum_{d_2|m_2} f(d_2)g(\frac{m_2}{d_2}) = h(m_1)h(m_2).$$

8. *(10 points)* Show that

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. How can you know when each case will occur?

Answer: Notice that $\lceil \frac{2x+1}{4} \rceil - \lfloor \frac{2x+1}{4} \rfloor$ will be 1 unless $\frac{2x+1}{4}$ is an integer. Suppose $\frac{2x+1}{4} = n$. Then $2x + 1 = 4n$, so $x = \frac{4n-1}{2} = 2n - \frac{1}{2}$. This means that we have

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \left\lceil \frac{2x+1}{2} \right\rceil - 1 + [x = 2n - \tfrac{1}{2}].$$

Now, write $x = m + \alpha$, where $m = \lfloor x \rfloor$ and $0 \leqslant \alpha < 1$. We have

$$\begin{aligned} \left\lceil \frac{2x+1}{2} \right\rceil - 1 + [x = 2n - \tfrac{1}{2}] &= \left\lceil \frac{2(m+\alpha)+1}{2} \right\rceil - 1 + [m + \alpha = 2n - \tfrac{1}{2}] \\ &= \left\lceil \frac{2m+2\alpha+1}{2} \right\rceil - 1 + [m = 2n-1][\alpha = \tfrac{1}{2}] \\ &= m + \left\lceil \alpha + \frac{1}{2} \right\rceil - 1 + [m = 2n-1][\alpha = \tfrac{1}{2}] \end{aligned}$$

If $0 \leqslant \alpha < \frac{1}{2}$, this expression simplifies to m, which is $\lfloor x \rfloor$. If $\alpha > \frac{1}{2}$, this simplifies to $m+1 = \lceil x \rceil$. If $\alpha = \frac{1}{2}$ and m is odd, this simplifies to $m + 1 = \lceil x \rceil$, while if $\alpha = \frac{1}{2}$ and m is even, this simplifies to $m = \lfloor x \rfloor$.

9. *(10 points)* Prove or disprove:

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leqslant \lfloor 2x \rfloor + \lfloor 2y \rfloor .$$

Answer: Write $x = m + \alpha$, where $m = \lfloor x \rfloor$ and $0 \leqslant \alpha < 1$, and $y = n + \beta$, where $n = \lfloor y \rfloor$ and $0 \leqslant \beta < 1$. We have

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor = m + n + \lfloor m + n + \alpha + \beta \rfloor = 2m + 2n + \lfloor \alpha + \beta \rfloor$$

while

$$\lfloor 2x \rfloor + \lfloor 2y \rfloor = \lfloor 2m + 2\alpha \rfloor + \lfloor 2n + 2\beta \rfloor = 2m + 2n + \lfloor 2\alpha \rfloor + \lfloor 2\beta \rfloor .$$

So the problem reduces to seeing if $\lfloor \alpha + \beta \rfloor$ is always less than or equal to $\lfloor 2\alpha \rfloor + \lfloor 2\beta \rfloor$.

Notice that $\lfloor \alpha + \beta \rfloor$ is either 0 or 1. If $\lfloor \alpha + \beta \rfloor = 1$, then $\alpha + \beta \geqslant 1$, which implies that either $\alpha \geqslant \frac{1}{2}$ or $\beta \geqslant \frac{1}{2}$. In the first case, we must have $\lfloor 2\alpha \rfloor = 1$, while in the second, we have $\lfloor 2\beta \rfloor = 1$. In either event, the inequality is true.

10. *(20 points)* Sometimes induction arguments work in unusual ways. Consider the statement

$$P(n): \qquad x_1 \cdots x_n \leqslant \left(\frac{x_1 + \cdots + x_n}{n} \right)^n, \qquad \text{if } x_1, \ldots, x_n \geqslant 0$$

(a) Prove that $P(2)$ is true.

(b) By setting $x_n = (x_1 + \cdots + x_{n-1})/(n-1)$, prove that $P(n)$ implies $P(n-1)$ whenever $n > 1$.

(c) Show that $P(n)$ and $P(2)$ imply $P(2n)$.

(d) Explain why these three steps imply that $P(n)$ is true for all positive integers n.

Answer: (a) $P(2)$ is the statement that $ab \leqslant \left(\frac{a+b}{2}\right)^2$. To prove this inequality, start with $\left(\frac{a-b}{2}\right)^2 \geqslant 0$ (which is true because a square is always non-negative), and then add ab to both sides of the inequality.

(b) Suppose that $P(n)$ is true, and set $x_n = (x_1 + \cdots + x_{n-1})/(n-1)$. We have

$$\begin{aligned}
x_1 \cdots x_{n-1} x_n &\leqslant \left(\frac{x_1 + \cdots + x_{n-1} + x_n}{n}\right)^n \\
x_1 \cdots x_{n-1} \left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right) &\leqslant \left(\frac{x_1 + \cdots + x_{n-1}}{n} + \frac{x_n}{n}\right)^n = \left(\frac{x_1 + \cdots + x_{n-1}}{n} + \frac{x_1 + \cdots + x_{n-1}}{n(n-1)}\right)^n \\
&= \left(\frac{(x_1 + \cdots + x_{n-1})(n-1)}{n(n-1)} + \frac{x_1 + \cdots + x_{n-1}}{n(n-1)}\right)^n \\
&= \left(\frac{(x_1 + \cdots + x_{n-1})(n)}{n(n-1)}\right)^n = \left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right)^n \\
x_1 \cdots x_{n-1} &\leqslant \left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right)^{n-1}
\end{aligned}$$

This is $P(n-1)$, so we have shown that $P(n)$ implies $P(n-1)$.

(c) Suppose that both $P(2)$ and $P(n)$ are true. We will prove $P(2n)$ by applying first $P(n)$ and then $P(2)$. We have

$$\begin{aligned}
x_1 \cdots x_n x_{n+1} \cdots x_{2n} = (x_1 \cdots x_n)(x_{n+1} \cdots x_{2n}) &\leqslant \left(\frac{x_1 + \cdots + x_n}{n}\right)^n \left(\frac{x_{n+1} + \cdots + x_{2n}}{n}\right)^n \\
(x_1 \cdots x_n x_{n+1} \cdots x_{2n})^{\frac{1}{n}} &\leqslant \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{x_{n+1} + \cdots + x_{2n}}{n}\right) \\
&\leqslant \left(\frac{\left(\frac{x_1 + \cdots + x_n}{n}\right) + \left(\frac{x_{n+1} + \cdots + x_{2n}}{n}\right)}{2}\right)^2 = \left(\frac{x_1 + \cdots + x_{2n}}{2n}\right)^2 \\
x_1 \cdots x_n x_{n+1} \cdots x_{2n} &\leqslant \left(\frac{x_1 + \cdots + x_{2n}}{2n}\right)^{2n}
\end{aligned}$$

This last inequality is $P(2n)$.

(d) We have now shown that $P(2) \& P(n) \Rightarrow P(2n)$ and $P(n) \Rightarrow P(n-1)$. The first implication shows that $P(2^k)$ is true for all k, and then the second, applied repeatedly, shows that $P(n)$ is true for all n.

Grade	Number of people
84	1
80	1
68	1
65	1
61	1
59	1
52	1
29	1

Mean: 62.25
Standard deviation: 16.01