MATH1007
Homework 1
Answers

1. Consider the sequence defined by the formula $A_N = \frac{2N+3}{N+11}$.
(*a*) Find A_1.
(*b*) Find A_9.

Answer: We have $A_1 = (2 \cdot 1 + 3)/(1 + 11) = \frac{5}{12}$ and $A_9 = (2 \cdot 9 + 3)/(9 + 11) = \frac{21}{20}$.

2. Consider the sequence defined by the recursive formula

$$\begin{aligned} A_1 &= 1 \\ A_2 &= 3 \\ A_N &= A_{N-1} + 2A_{N-2}, \quad N \geqslant 3 \end{aligned}$$

List A_3, A_4, A_5, and A_8.

Answer: We have

$$\begin{aligned} A_3 &= A_2 + 2A_1 = 5 \\ A_4 &= A_3 + 2A_2 = 11 \\ A_5 &= A_4 + 2A_3 = 21 \\ A_6 &= A_5 + 2A_4 = 43 \\ A_7 &= A_6 + 2A_5 = 85 \\ A_8 &= A_7 + 2A_6 = 171 \end{aligned}$$

3. Suppose that we have a linear sequence in which $P_0 = 23$ and $P_1 = 24.1$.
(*a*) List P_2, P_3, P_4, and P_{23}.
(*b*) Use the formula for an arithmetic sum to add up $P_0 + P_1 + \cdots + P_{23}$.

Answer: (*a*) $P_2 = P_1 + 1.1 = 25.2$, $P_3 = P_2 + 1.1 = 26.3$, and $P_4 = P_3 + 1.1 = 27.4$. In general, $P_N = 23 + 1.1N$, and so $P_{23} = 48.3$.

(*b*) The formula gives $(P_0 + P_{23})24/2 = 855.6$.

4. Suppose that we have an exponential sequence in which $P_0 = 23$ and $P_1 = 24.1$.
(*a*) List P_2, P_3, P_4, and P_{23} to 4 decimal places.
(*b*) Use the formula for a geometric sum to add up $P_0 + P_1 + \cdots + P_{23}$.

Answer: (*a*) The ratio $r = 24.1/23 \approx 1.0478$. Therefore, $P_2 = rP_1 \approx 25.2526$, $P_3 = rP_2 \approx 26.4603$, and $P_4 = rP_3 \approx 27.7258$. In general, $P_n = r^n P_0$, and so $P_{23} \approx 67.3565$.

(*b*) The formula gives $23(r^{24} - 1)/(r - 1) \approx 994.8108$.

5. Suppose that we have a linear sequence in which $P_0 = 24.1$ and $P_1 = 23$.
(*a*) List P_2, P_3, P_4, and P_{14}.
(*b*) Use the formula for an arithmetic sum to add up $P_0 + P_1 + \cdots + P_{14}$.

Answer: (*a*) We have $P_2 = P_1 - 1.1 = 21.9$, $P_3 = P_2 - 1.1 = 20.8$, and $P_4 = P_3 - 1.1 = 19.7$. In general, $P_n = 24.1 - 1.1n$, and so $P_{14} = 8.7$.

(*b*) The formula is $(P_0 + P_{14})15/2 = 246$.

6. Suppose that we have an exponential sequence in which $P_0 = 24.1$ and $P_1 = 23$.

(*a*) List P_2, P_3, P_4, and P_{14} to 4 decimal places.

(*b*) Use the formula for a geometric sum to add up $P_0 + P_1 + \cdots + P_{14}$.

Answer: (*a*) Now $r = 23/24.1 \approx 0.9544$, and so $P_2 = rP_1 \approx 21.9502$, $P_3 = rP_2 \approx 20.9483$, and $P_4 = rP_3 \approx 19.9922$. In general, $P_n = 24.1r^n$, and so $P_{14} \approx 12.5305$.

(*b*) The formula gives $P_0(r^{15} - 1)/(r - 1) \approx 266.0079$.

7. The city of Sylvania currently has 401 LED streetlights. The city council has decided to install 3 additional LED streetlights at the start of each week for the next 52 weeks. Each LED streetlight costs \$0.24 to operate for a week.

(*a*) How many LED streetlights will Sylvania have at the end of 21 weeks?

(*b*) What is the cost of operating the original 401 LED streetlights for 52 weeks?

(*c*) What is the additional cost of the new LED streetlights at the end of 52 weeks?

Answer: (*a*) The formula for the sequence is $P_n = 401 + 3n$. At the end of 21 weeks, we have 464 lights. (*b*) The cost of the original 401 lights is $401 \cdot 52 \cdot 0.24 = \$5004.48$. (*c*) We need to compute $0.24(3 + 6 + 9 + \cdots + 156) = 0.24(3 + 156)(52)/2 = \$992.16$.

8. This problem asks you to experiment with the logistic growth model

$$p_n = rp_{n-1}(1 - p_{n-1})$$

for various values of p_0 and r. Do all of your work to at least 4 decimal places.

(*a*) Suppose that $r = 0.5$ and $p_0 = 0.3$. Compute p_1 up to p_{10}.

(*b*) Suppose that $r = 1.5$ and $p_0 = 0.3$. Compute enough terms of the sequence for you to observe a pattern.

(*c*) Suppose that $r = 2.5$ and $p_0 = 0.3$. Compute enough terms of the sequence for you to observe a pattern.

(*d*) Suppose that $r = 3.2$ and $p_0 = 0.3$. Compute enough terms of the sequence for you to observe a pattern.

(*e*) Suppose that $r = 3.5$ and $p_0 = 0.3$. Compute enough terms of the sequence for you to observe a pattern.

Answer: (*a*) For this part of the problem, we show all of the work:

$$\begin{aligned}
p_1 &= 0.5(0.3000)(1 - 0.3000) &= 0.1050\\
p_2 &= 0.5(0.1050)(1 - 0.1050) &= 0.0470\\
p_3 &= 0.5(0.0470)(1 - 0.0470) &= 0.0224\\
p_4 &= 0.5(0.0224)(1 - 0.0224) &= 0.0109\\
p_5 &= 0.5(0.0109)(1 - 0.0109) &= 0.0054\\
p_6 &= 0.5(0.0054)(1 - 0.0054) &= 0.0027\\
p_7 &= 0.5(0.0027)(1 - 0.0027) &= 0.0013\\
p_8 &= 0.5(0.0013)(1 - 0.0013) &= 0.0007\\
p_9 &= 0.5(0.0007)(1 - 0.0007) &= 0.0003\\
p_{10} &= 0.5(0.0003)(1 - 0.0003) &= 0.0002
\end{aligned}$$

(*b*) The resulting sequence (starting with p_1) is:

0.3150	0.3237	0.3284	0.3308	0.3321	0.3327
0.3330	0.3332	0.3333	0.3333	0.3333	0.3333
0.3333	0.3333	0.3333	0.3333	0.3333	0.3333

The sequence settles down to a limit of $\frac{1}{3}$.

(c) The resulting sequence (starting with p_1) is:

0.5250	0.6234	0.5869	0.6061	0.5968	0.6016	0.5992
0.6004	0.5998	0.6001	0.6000	0.6000	0.6000	0.6000

The sequence settles down to a limit of $\frac{3}{5}$.

(d) The resulting sequence (starting with p_1) is:

0.6720	0.7053	0.6651	0.7128	0.6551	0.7230
0.6408	0.7365	0.6210	0.7532	0.5949	0.7712
0.5647	0.7866	0.5371	0.7956	0.5204	0.7987
0.5146	0.7993	0.5133	0.7994	0.5131	0.7995
0.5131	0.7995	0.5130	0.7995	0.5130	0.7995

The sequence eventually oscillates between 0.5130 and 0.7995.

(e) The resulting sequence (starting with p_1) is:

0.7350	0.6817	0.7594	0.6394	0.8070	0.5452	0.8678	0.4014
0.8410	0.4680	0.8714	0.3922	0.8343	0.4839	0.8741	0.3852
0.8289	0.4964	0.8750	0.3829	0.8270	0.5007	0.8750	0.3828
0.8269	0.5009	0.8750	0.3828	0.8269	0.5009	0.8750	0.3828

The sequence eventually cycles between 4 numbers.

9. Suppose that we consider the logistic equation with $r = 3.4$. What value of p_0 (other than 0) will produce a constant sequence in which $p_0 = p_1 = p_2$?

Answer: To get $p_1 = p_0$, we solve the equation $p_0 = 3.4p_0(1 - p_0)$. Divide by p_0 to get $1 = 3.4(1 - p_0)$. Divide by 3.4 to get $0.2941 = 1 - p_0$ and so $p_0 = 1 - 0.2941 = 0.7059$. Once we have $p_1 = p_0$, it is then automatic that $p_2 = p_1$.