MATH1007
Homework 8
Answers

1. Baseball's Cy Young Award uses a variant of Borda count in which the point are awarded using a 7-4-3-2-1 scheme. In other words, first place is worth 7 points, second place is worth 4 points, third place is worth 3 points, fourth place is worth 2 points, and fifth place is worth 1 point.

Construct a preference schedule with 5 candidates so that one candidate wins when using the standard Borda count, with 5-4-3-2-1 awards, and a different candidate wins when using the modified 7-4-3-2-1 scheme.

Answer: Consider this preference schedule:

Number of voters	**1**	**1**
First choice	A	E
Second choice	B	B
Third choice	C	D
Fourth choice	D	A
Fifth choice	E	C

Using the standard 5-4-3-2-1 count, A has 7 points, B has 8 points, C has 4 points, D has 5 points, and E has 6 points, and B wins the election. Using the modified 7-4-3-2-1 count, A has 9 points, B has 8 points, C has 4 points, D has 5 points, and E has 8 points, and A wins the election.

2. Another modification of the Borda count awards 0 points for last place, 1 point for next-to-last, and so on. In an election with 3 candidates, this modification gives 2 points for first-place, 1 point for second place, and 0 points for last place. Show that in an election with 3 candidates, the modified 2-1-0 Borda count will always rank the candidates in the same order as the standard 3-2-1 Borda count.

Answer: On each ballot, each candidate will receive precisely one fewer point, regardless of his/her place on the ballot. Suppose that there are N voters. At the end of the process, the result of the 2-1-0 count will give each candidate N fewer points than using the 3-2-1 count. Subtracting N from each candidate's point total will not change the ranking of the candidates.

3. Yet another modification of the Borda count awards 1 point for *first* place, 2 points for *second* place, and so on. In this version, the winner is the candidate with the *fewest* points.

Show that in an election with 3 candidates, awarding the points using the 1-2-3 modification, and ranking the candidates from fewest to most points, yields the same election result as the standard Borda count (ranking the candidates in the usual way, from most to fewest points).

Answer: Suppose that a candidate has x first-place votes, y second-place votes, and z third place votes. This means that the number of voters is $x + y + z = N$.

Using the standard Borda count, he/she would get $3x + 2y + z = k$ points. Using the modified Borda count, he/she would get $x + 2y + 3z$ points. The sum of the two is $4(x + y + z) = 4N$. In other words, if a candidate gets k points using the standard Borda count, then he/she gets $4N - k$ points using the altered Borda count.

The person in first place has the largest value for k, which gives the smallest value for $4N - k$. That means that this candidate is still in first place.

4. Suppose that in an election among 3 candidates, we award 5 points for first place, 3 points for second place, and 1 point for third place. Does this method produce the same result as the standard Borda count method? If this method yields the same ranking of candidates as the standard Borda count, explain why. If it is not equivalent, give a preference schedule which demonstrates that this method can give a different result.

Answer: The results of using 5-3-1 are exactly the same as using 3-2-1. We can see this in a sequence of steps:

- The result of using the 3-2-1 count is the same as the result using a modified 2-1-0 count.
- The result using a modified 2-1-0 count is the same as the result using a modified 4-2-0 count.
- The result using a modified 4-2-0 count is the same as the result using a modified 5-3-1 count.

5. In class, we saw that Borda count can violate the majority criterion.

(a) As a warm up, recapitulate the example given in class with 3 candidates, in which one candidate has a majority of first-place votes, and a different candidate wins the Borda count.

(b) When you look at your answer to part (a), it seems as if the problem is that first-place is not worth enough points. Let's try a variation of Borda count in which first place is worth 20 points, second place is worth 2 points, and third place is worth 1 point. Is it now possible for a candidate to have a majority of first-place votes and not win the Borda count? If you think that it is possible, give a preference table that shows a candidate with the majority of first-place votes does not win the Borda count. If you think that it is not possible, give a convincing argument in support of your position.

Answer: (a) Consider this preference schedule:

	Number of voters	
	4	**3**
First choice	A	B
Second choice	B	C
Third choice	C	A

A has a majority of first-place votes. In the Borda count, A has 15 points, B has 17 points, and C has 10 points.

(b) It is still possible for A to have a majority of first-place votes and lose the election: Consider this preference schedule:

	Number of voters	
	20	**19**
First choice	A	B
Second choice	B	C
Third choice	C	A

A has a majority of first place votes. Using the modified 20-2-1 count, A has 419 points, B has 420 points, and C has 58 points. The winner is B.

6. For many years, elections in Louisiana have been conducted using what is called an "open" primary system, also sometimes called "plurality with runoff" or "jungle primary." If someone gets a majority, that candidate wins. Otherwise, the top two candidates compete in a runoff election. We will assume that no voters change their preferences between the two elections.

(a) Give an example to show that plurality with runoff can produce a different outcome from plurality with elimination.

(b) Give an example to show that plurality with runoff violates the monotonicity condition.

(c) Give an example to show that a Condorcet winner does not necessarily win using plurality with runoff.

Answer: Plurality with runoff is only different from plurality with elimination if there are 4 or more candidates.

(a) Consider this schedule:

	Number of voters			
	8	**7**	**6**	**3**
First choice	A	B	C	D
Second choice	B	C	B	C
Third choice	C	A	A	A
Fourth choice	D	D	D	C

Plurality with runoff has only two rounds:

Round 1:	
A	8
B	7
C	6
D	3

Round 2:	
A	11
B	13

B wins. On the other hand, plurality with elimination (IRV) has 3 rounds:

Round 1:	
A	8
B	7
C	6
D	3

Round 2:	
A	8
B	7
C	9

Round 3:	
A	8
C	16

C is the winner.

(b) Here's an example with 5 candidates in the preference schedule:

	Number of voters				
	10	**9**	**8**	**7**	**6**
First choice	A	B	C	D	E
Second choice	B	A	A	C	C
Third choice	C	C	B	A	B
Fourth choice	D	D	E	B	A
Fifth choice	E	E	D	D	D

Plurality with runoff has only two rounds:

Round 1:	
A	10
B	9
C	8
D	7
E	6

Round 2:	
A	25
B	15

Take the same preference schedule, and move A to first place and B to second place in the second column, to produce:

	Number of voters				
	10	**9**	**8**	**7**	**6**
First choice	A	A	C	D	E
Second choice	B	B	A	C	C
Third choice	C	C	B	A	B
Fourth choice	D	D	E	B	A
Fifth choice	E	E	D	D	D

The outcome:

Round 1:	
A	19
B	0
C	8
D	7
E	6

Round 2:	
A	19
C	21

C is the winner.

(*c*) Consider this schedule:

	Number of voters			
	8	**7**	**6**	**3**
First choice	A	B	C	D
Second choice	D	D	D	C
Third choice	B	C	A	B
Fourth choice	C	A	B	A

In head-to-head pairings, D beats A 16 to 8, D beats B 17 to 7, and D beats C 18 to 6. None of those contests are close. Yet D is eliminated in the first round of a jungle primary:

Round 1:	
A	8
B	7
C	6
D	3

Round 2:	
A	14
B	10

A wins.

7. Here is a preference schedule:

	Number of voters			
	6	**5**	**4**	**2**
First choice	G	M	D	S
Second choice	M	G	S	D
Third choice	D	D	M	G
Fourth choice	S	S	G	M

(*a*) Compute the winner of an election using instant runoff voting.
(*b*) Now suppose that in the last column, G switches place with D, giving this new preference schedule:

	Number of voters			
	6	**5**	**4**	**2**
First choice	G	M	D	S
Second choice	M	G	S	$\boxed{\text{G}}$
Third choice	D	D	M	$\boxed{\text{D}}$
Fourth choice	S	S	G	M

Who wins using instant runoff voting?
(*c*) Comment on this situation.

Answer: (*a*) We have

Round 1:	
G	6
M	5
D	4
S	2

Round 2:	
G	6
M	5
D	6

Round 3:	
G	11
D	6

G is the winner.

(*b*) Now we have

Round 1:	
G	6
M	5
D	4
S	2

Round 2:	
G	8
M	5
D	4

Round 3:	
G	8
M	9

M is the winner.

(*c*) This is a violation of monotonicity.