MATH1007

Homework 10

Answers

1. Compute the Shapley–Shubik power of each player in the following voting systems:
(a) $[7:4,3,2,1]$.
(b) $[8:4,3,2,1]$.
(c) $[9:4,3,2,1]$.

Answer: In each case, the pivotal player is underlined:

(a)

$\langle P_1,\underline{P_2},P_3,P_4\rangle$	$\langle P_2,\underline{P_1},P_3,P_4\rangle$	$\langle P_3,P_1,\underline{P_2},P_4\rangle$	$\langle P_4,P_1,\underline{P_2},P_3\rangle$
$\langle P_1,\underline{P_2},P_4,P_3\rangle$	$\langle P_2,\underline{P_1},P_4,P_3\rangle$	$\langle P_3,P_1,\underline{P_4},P_2\rangle$	$\langle P_4,P_1,\underline{P_3},P_2\rangle$
$\langle P_1,P_3,\underline{P_2},P_4\rangle$	$\langle P_2,P_3,\underline{P_1},P_4\rangle$	$\langle P_3,P_2,\underline{P_1},P_4\rangle$	$\langle P_4,P_2,\underline{P_1},P_3\rangle$
$\langle P_1,P_3,\underline{P_4},P_2\rangle$	$\langle P_2,P_3,P_4,\underline{P_1}\rangle$	$\langle P_3,P_2,P_4,\underline{P_1}\rangle$	$\langle P_4,P_2,P_3,\underline{P_1}\rangle$
$\langle P_1,P_4,\underline{P_2},P_3\rangle$	$\langle P_2,P_4,\underline{P_1},P_3\rangle$	$\langle P_3,P_4,\underline{P_1},P_2\rangle$	$\langle P_4,P_3,\underline{P_1},P_2\rangle$
$\langle P_1,P_4,\underline{P_3},P_2\rangle$	$\langle P_2,P_4,P_3,\underline{P_1}\rangle$	$\langle P_3,P_4,P_2,\underline{P_1}\rangle$	$\langle P_4,P_3,P_2,\underline{P_1}\rangle$

We count underlines, and compute $\sigma_1 = \frac{14}{24} = \frac{7}{12} \approx 0.5833$, $\sigma_2 = \frac{6}{24} = \frac{1}{4} = 0.25$, $\sigma_3 = \frac{2}{24} = \frac{1}{12} \approx 0.0833$, and $\sigma_4 = \frac{2}{24} = \frac{1}{12} \approx 0.0833$. You do not need to reduce the fractions or convert your answers to decimals.

(b)

$\langle P_1,P_2,\underline{P_3},P_4\rangle$	$\langle P_2,P_1,\underline{P_3},P_4\rangle$	$\langle P_3,P_1,\underline{P_2},P_4\rangle$	$\langle P_4,P_1,\underline{P_2},P_3\rangle$
$\langle P_1,P_2,\underline{P_4},P_3\rangle$	$\langle P_2,P_1,\underline{P_4},P_3\rangle$	$\langle P_3,P_1,P_4,\underline{P_2}\rangle$	$\langle P_4,P_1,P_3,\underline{P_2}\rangle$
$\langle P_1,P_3,\underline{P_2},P_4\rangle$	$\langle P_2,P_3,\underline{P_1},P_4\rangle$	$\langle P_3,P_2,\underline{P_1},P_4\rangle$	$\langle P_4,P_2,\underline{P_1},P_3\rangle$
$\langle P_1,P_3,P_4,\underline{P_2}\rangle$	$\langle P_2,P_3,P_4,\underline{P_1}\rangle$	$\langle P_3,P_2,P_4,\underline{P_1}\rangle$	$\langle P_4,P_2,P_3,\underline{P_1}\rangle$
$\langle P_1,P_4,\underline{P_2},P_3\rangle$	$\langle P_2,P_4,\underline{P_1},P_3\rangle$	$\langle P_3,P_4,P_1,\underline{P_2}\rangle$	$\langle P_4,P_3,P_1,\underline{P_2}\rangle$
$\langle P_1,P_4,P_3,\underline{P_2}\rangle$	$\langle P_2,P_4,P_3,\underline{P_1}\rangle$	$\langle P_3,P_4,P_2,\underline{P_1}\rangle$	$\langle P_4,P_3,P_2,\underline{P_1}\rangle$

We count underlines, and compute $\sigma_1 = \frac{10}{24} = \frac{5}{12} \approx 0.4167$, $\sigma_2 = \frac{10}{24} = \frac{5}{12} \approx 0.4167$, $\sigma_3 = \frac{2}{24} = \frac{1}{12} \approx 0.0833$, and $\sigma_4 = \frac{2}{24} = \frac{1}{12} \approx 0.0833$.

(c)

$\langle P_1,P_2,\underline{P_3},P_4\rangle$	$\langle P_2,P_1,\underline{P_3},P_4\rangle$	$\langle P_3,P_1,\underline{P_2},P_4\rangle$	$\langle P_4,P_1,P_2,\underline{P_3}\rangle$
$\langle P_1,P_2,P_4,\underline{P_3}\rangle$	$\langle P_2,P_1,P_4,\underline{P_3}\rangle$	$\langle P_3,P_1,P_4,\underline{P_2}\rangle$	$\langle P_4,P_1,P_3,\underline{P_2}\rangle$
$\langle P_1,P_3,\underline{P_2},P_4\rangle$	$\langle P_2,P_3,\underline{P_1},P_4\rangle$	$\langle P_3,P_2,\underline{P_1},P_4\rangle$	$\langle P_4,P_2,P_1,\underline{P_3}\rangle$
$\langle P_1,P_3,P_4,\underline{P_2}\rangle$	$\langle P_2,P_3,P_4,\underline{P_1}\rangle$	$\langle P_3,P_2,P_4,\underline{P_1}\rangle$	$\langle P_4,P_2,P_3,\underline{P_1}\rangle$
$\langle P_1,P_4,P_2,\underline{P_3}\rangle$	$\langle P_2,P_4,P_1,\underline{P_3}\rangle$	$\langle P_3,P_4,P_1,\underline{P_2}\rangle$	$\langle P_4,P_3,P_1,\underline{P_2}\rangle$
$\langle P_1,P_4,P_3,\underline{P_2}\rangle$	$\langle P_2,P_4,P_3,\underline{P_1}\rangle$	$\langle P_3,P_4,P_2,\underline{P_1}\rangle$	$\langle P_4,P_3,P_2,\underline{P_1}\rangle$

We count underlines, and get $\sigma_1 = \sigma_2 = \sigma_3 = \frac{8}{24} = \frac{1}{3}$, and $\sigma_4 = 0$.

2. Compute the Shapley–Shubik power of each player in the voting system $[6:5,1,1,1,1,1]$.

Answer: Surprisingly, this is much easier than the computation last week of Banzhaf power for the same weighted voting system. We see that P_1 is pivotal except when he/she occurs in the first position of the sequential coalition. There are $6! = 720$ sequential coalitions. Because P_1 occurs in the first position $frac16$ of the time, and that is when P_1 is *not* pivotal, we know that P_1 is pivotal in $720 - 120 = 600$ coalitions, and so $\sigma_1 = \frac{600}{720} = \frac{5}{6} \approx 0.8333$. We then know that

$\sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = \frac{1}{30}$, because the powers must sum to 1, and the powers of the remaining players must be equal.

3. Here is a preference schedule which only contains the rankings of candidate A. There are, of course, also candidates B, C, D, and E in the race.

	Number of voters		
	7	**6**	**2**
First choice	A	*	*
Second choice	*	*	*
Third choice	*	*	*
Fourth choice	*	*	*
Fifth choice	*	A	A

Is it possible for A to win using standard Borda count? If so, then fill in the preference schedule so that A wins. If not, explain why it is not possible for A to win.

Answer: It is *not* possible for A to win using standard Borda count. Here's why. We see that A has 43 points. If A were the winner, then no other candidate could have more than 42 points, and the total number of points spread among all 5 of the candidates would be at most $43 + 4 \cdot 42 = 211$. However, there are a total of 15 points on each ballot, and there are 15 ballots, so the point total must be $15 \cdot 15 = 225$.

4. Suppose that the United Nations Security Council expanded to include Germany as a sixth permanent member. In this scenario, there are 6 permanent members and 10 rotating members. A winning coalition must contain at least 12 countries, and include all 6 of the permanent members.

(*a*) Can this voting system be modelled as a weighted voting system? If so, what are the weights for the permanent and for the rotating members? If not, explain why it is not possible to find such weights.

(*b*) What is the Banzhaf power of Germany?

(*c*) What is the Shapley–Shubik power of Germany?

Answer: (*a*) This *can* be described as a weighted voting system. Suppose that we assign a weight of p to a permanent member, and a weight of r to a rotating member, and quota q, so that the system is $[q : p, \ldots, p, r, r, \ldots, r]$. We need to find values for p, q, and r so that $6p + 6r \geqslant q$ (all 6 permanent members and 6 rotating members is a winning coalition), $5p + 10r < q$ (a coalition without a permanent member loses), and $6p + 5r < q$ (a coalition with only 11 countries loses). There are infinitely many solutions to these three inequalities. One possibility is $p = 10$, $r = 1$, and $q = 66$.

(*b*) We follow the procedure from class, and count all of the winning coalitions by size. In the coalitions with 12 members, all of the members are critical. In the larger coalitions, only the permanent members are critical.

Size of coalition	Number of critical points	How many coalitions?
12	12	$\binom{10}{6} = 210$
13	6	$\binom{10}{7} = 120$
14	6	$\binom{10}{8} = 45$
15	6	$\binom{10}{9} = 10$
16	6	1

The total number of critical points is $12 \cdot 210 + 6 \cdot 120 + 6 \cdot 45 + 6 \cdot 10 + 6 \cdot 1 = 3576$. Germany (or any other permanent member) is critical $210 + 120 + 45 + 10 + 1 = 386$ times. The Banzhaf power of each of the 6 permanent members is $\frac{386}{3576} \approx 0.1079$. The power of all 10 rotating members is therefore $\frac{1260}{3576}$, and so the power of any particular rotating member is $\frac{126}{3576} \approx 0.0352$.

(c) We saw in class that it is *much* simpler to compute the Shapley–Shubik power of a rotating member. A rotating country is pivotal precisely when it occurs in position 12 and is preceded by all 6 permanent members and 5 rotating members. We count how many times this happens in 3 steps:

1. Choose the 5 rotating members. This can be done in $\binom{9}{5} = 126$ ways.
2. Jumble the 11 countries to the left of our particular rotating country. This can be done in 11! ways.
3. Jumble the 4 countries to the right of our particular rotating country. This can be done in 4! ways.

Therefore, the Shapley–Shubik power of a rotating country is given by $\frac{126 \cdot 11! \cdot 4!}{16!} = \frac{3}{520} \approx 0.0058$. The Shapley–Shubik power of all 10 rotating countries is therefor $\frac{3}{52}$. The Shapley–Shubik power of all 6 permanent countries is therefore $\frac{49}{52}$, and the Shapley–Shubik power of Germany is $\frac{49}{312} \approx 0.1571$.

5. The government of Freedonia has a king (K), a prime minister (P), and three royal advisors (A_1, A_2, and A_3). The rules of the government are:

- Every winning coalition must contain either the king or the prime minister.
- The king and the prime minister together can combine to form a winning coalition, with or without any royal advisors.
- The king and 2 or 3 royal advisors is a winning coalition.
- The prime minister and all 3 royal advisors is a winning coalition.

(a) Can this government be modelled by using a weighted voting system? If so, what are the weights for the king, the prime minister, and for a royal advisor? If not, explain why it is not possible to find such weights.

(b) Compute the Banzhaf power of the king, the prime minister, and a royal advisor.

(c) Compute the Shapley–Shubik power of the king, the prime minister, and a royal advisor. We know that there are 5! sequential coalitions to list, but if you are clever, you can solve the problem by listing fewer than half that many sequential coalitions.

Answer: (a) This can be described by a weighted voting system. Suppose that the weight of the king is k, the weight of the prime minister is p, the weight of an advisor is a, and the quota is q, so the system is $[q : k, p, a, a, a]$. We get $k + p \geqslant q$ (the king and the prime minister form a winning coalition), $k + 2a \geqslant q$ (the king and 2 advisors form a winning coalition), $p + 3a \geqslant q$ (the prime minister and 3 advisors form a winning coalition), $3a < q$ (the advisors alone are not winning), $k + a < q$ (the king and 1 advisor does not win), and $p + 2a < q$ (the king and 2 advisors does not win). Now a bit of trial and error finds $[5 : 3, 2, 1, 1, 1]$ as one of many possible solutions.

(*b*) We list the winning coalitions in increasing order of size, and underline the critical players:

$$\{\underline{K}, \underline{P}\}, \quad \{\underline{K}, \underline{P}, A_1\}, \quad \{\underline{K}, \underline{P}, A_2\}, \quad \{\underline{K}, \underline{P}, A_3\}, \quad \{\underline{K}, \underline{A_1}, \underline{A_2}\}, \quad \{\underline{K}, \underline{A_1}, \underline{A_3}\}, \quad \{\underline{K}, \underline{A_2}, \underline{A_3}\}$$
$$\{\underline{K}, A_1, A_2, A_3\}, \quad \{\underline{K}, P, A_1, A_2\}, \quad \{\underline{K}, P, A_1, A_3\}, \quad \{\underline{K}, P, A_2, A_3\}, \quad \{\underline{P}, \underline{A_1}, \underline{A_2}, \underline{A_3}\}$$
$$\{K, P, A_1, A_2, A_3\}$$

There are 25 critical points. We compute $\beta_K = \frac{11}{25}$, $\beta_P = \frac{5}{25}$, and $\beta_{A_1} = \beta_{A_2} = \beta_{A_3} = \frac{3}{25}$.

(*c*) Let's start by figuring out when the king is pivotal. He is pivotal in coalitions of the following types, and there are 6 of each of these types (determined by the 6 different ways to replace $*$ by advisors): $\langle P, \underline{K}, *, *, *, \rangle$. He is pivotal whenever he occurs in the third position, and that happens 24 times. He is also pivotal whenever he occurs in the fourth of the five positions, and that happens 24 times. He is never pivotal in the fifth position. In all, the king is pivotal $6 + 24 + 24 = 54$ times, and $\sigma_K = \frac{54}{120}$.

The prime minister is pivotal precisely in these situations: $\langle K, \underline{P}, *, *, *\rangle$, $\langle K, *, \underline{P}, *, *\rangle$, $\langle *, K, \underline{P}, *, *\rangle$, $\langle *, *, *, \underline{P}, K\rangle$. Each of these occur 6 times. The prime minister is pivotal 24 times, and $\sigma_P = \frac{24}{120}$.

Finally, the 3 royal advisors together are pivotal $120 - (54 + 24) = 42$ times, and so each one is pivotal 14 times. We have $\sigma_{A_1} = \sigma_{A_2} = \sigma_{A_3} = \frac{14}{120}$.

6. Suppose that 5 men (α, β, γ, δ, and ε) and 5 women (A, B, C, D, and E) rank the members of the opposite gender as follows:

	A	B	C	D	E
α	1,4	3,2	2,4	4,2	5,1
β	3,2	2,3	4,2	5,1	1,5
γ	1,3	4,1	5,1	3,3	2,4
δ	5,1	1,4	2,5	4,5	3,2
ε	2,5	4,5	1,3	3,4	5,3

This chart means that α has the ranking A, C, B, D, E, and C has the ranking γ, β, ε, α, δ, for example.

(*a*) Use the Gale–Shubik matching algorithm with the men choosing to produce a stable assignment.

(*b*) Use the Gale–Shubik matching algorithm with the women choosing to produce a stable assignment.

Answer: (*a*) List the women on the left, and indicate going across how the men alter their choices:

	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7	Round 8	Round 9
A	α, γ	γ	γ	γ	γ	γ	β, γ	β	β
B	δ	δ	α, δ	α	α	α, β	α	α	α
C	ε	α, ε	ε	δ, ε	ε	ε	ε	ε	ε
D									γ
E	β	β	β	β	β, δ	δ	δ	γ, δ	δ

That is our stable match.

(b) List the men on the left, and indicate going across how the women alter their choices:

	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6
α	E	E	D, E	D	D	D
β	D	C, D	C	C	A, C	A
γ	B, C	B	B	B	B	B
δ	A	A	A	A, E	E	E
ε						C

This is the stable assignment.

7. Enormous State University has been given a total of 200 faculty positions to allocate to various departments, based on their enrollment figures. The departments and their enrollments are:

Department	A	B	C	D	E
Enrollment	1646	762	2081	1066	6945

(a) Find the standard divisor, and find each department's standard quota.
(b) Allocate the new faculty positions to the 5 departments using Hamilton's method.
(c) Allocate the new faculty positions to the 5 departments using Jefferson's method.
(d) Allocate the new faculty positions to the 5 departments using Webster's method.
(e) Allocate the new faculty positions to the 5 departments using Huntington–Hill's method.

Answer: (a) The total enrollment is 12500, and the standard divisor is $12500/200 = 62.5$. Dividing, we get:

Department	A	B	C	D	E
Enrollment	1646	762	2081	1066	6945
Standard quota	26.3360	12.1920	33.2960	17.0560	111.1200

(b) If we truncate, we get 26, 12, 33, 17, and 111, which sums to 199. The largest fractional part, 0.3360, belongs to department A, and therefore, the allocation will be 27, 12, 33, 17, and 111.

(c) If we take a modified divisor of 62, we get:

Department	A	B	C	D	E
Enrollment	1646	762	2081	1066	6945
Modifed quota	26.5484	12.2903	33.5645	17.1935	112.0161
Truncated	26	12	33	17	112

Those allocations sum to 200.

(d) If we take a modified divisor of 62.2, we get:

Department	A	B	C	D	E
Enrollment	1646	762	2081	1066	6945
Modified quota	26.4630	12.2508	33.4566	17.1383	111.6559
Rounded	26	12	33	17	112

Those allocations sum to 200.

(d) If we take a modified divisor of 62.2, we get:

Department	A	B	C	D	E
Enrollment	1646	762	2081	1066	6945
Modified quota	26.4630	12.2508	33.4566	17.1383	111.6559
H–H cut-off	26.50	12.49	33.50	17.49	111.50
H–H rounding	26	12	33	17	112

Those allocations sum to 200.