MATH4410
Homework 5
Due March 19, 2021
Your Name Here

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EDT on March 19. Please name your file hw05-lastname-firstname.pdf. For example, my solution file is hw05-gross-robert.pdf. Please use hyphens and not underlines, please use all lower-case letters, and please do not leave any spaces in the name of your file. You will have 5 points added to your homework score if you follow these instructions.

I will try to acknowledge receipt of each e-mail.

1. Consider the differential equation $y' = y(a-y)$, with $y(0) = y_0$. The behavior of solutions changes dramatically as a changes from negative to zero to positive.

(a) Suppose that $a < 0$, and write $a = -b^2$. Solve the differential equation. Show that there are 2 equilibrium solutions. Decide if each solution is asymptotically stable, semistable, or unstable.

(b) Suppose that $a = 0$. Solve the differential equation $y' = -y^2$, and analyze the behavior of the equilibrium solution(s) by studying $\lim_{t\to\infty} y(t)$.

(c) Suppose that $a > 0$, and write $a = b^2$. Solve the differential equation. Show that there are 2 equilibrium solutions. Decide if each solution is asymptotically stable, semistable, or unstable.

This is yet another example of a bifurcation at $a = 0$. This type of bifurcation is called a *transcritical* bifurcation; we say that there has been an *exchange of stability* as a passes through the bifurcation point $a = 0$.

2. Let ν be a non-negative real constant. The differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0$$

has solutions called *Bessel functions*. Without solving the differential equation, compute the Wronskian of two Bessel functions by using Abel's Theorem and an undetermined constant.

3. In class, we saw that the method of reduction of order gives one way to find the second solution of a second-order homogeneous differential equation when one solution is already known. This problem and the next explore other methods of finding a second solution.

Suppose that y_1 is a known non-trivial solution of the differential equation $y'' + p(t)y' + q(t)y = 0$. Suppose that y_2 is an unknown second solution.

(a) Abel's theorem tells us that $W(y_1, y_2) = C\exp[\int -p\,dt]$. Suppose that $C = 1$. Show that we can find a first-order differential equation for y_2 in terms of p and y_1.

(b) In particular, consider the differential equation $y'' + 2ay' + a^2y = 0$. We know that $y_1 = e^{-at}$ is one solution of this equation. Use Abel's Theorem to find a second independent solution y_2. This second solution will of course be the same one that we found in class using reduction of order.

4. Suppose that r_1 and r_2 are the unequal roots of $ar^2 + br + c = 0$. We know that $\lambda_1 e^{r_1 t} + \lambda_2 e^{r_2 t}$ solves $ay'' + by' + cy = 0$ for any values of λ_1 and λ_2. In particular, we know that $\phi(t, r_1, r_2) = (e^{r_1 t} - e^{r_2 t})/(r_1 - r_2)$ is a solution.

Now think of r_1 as fixed, and compute $\lim_{r_2\to r_1} \phi(t, r_1, r_2)$, using l'Hôpital's rule. Verify that this limit is the second solution we found when $ar^2 + br + c$ has two equal roots.

5. We learned earlier this semester that the solution of

$$y' + a(t)y = 0$$

is

$$y(t) = \exp\left(\int_c^t a(s)\,ds\right)$$

for any constant c.

Suppose that we try to solve

$$y'' + a_1(t)y' + a_2(t)y = 0$$

with a solution of the form

$$y(t) = \exp\left(\int_c^t p(s)\,ds\right)$$

for some function $p(s)$. Show that p must satisfy

$$p'(t) = -p(t)^2 - a_1(t)p(t) - a_2(t).$$