MATH4410
Homework 10
Due April 30, 2021
Your Name Here

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EDT on April 30. Please name your file hw10-lastname-firstname.pdf. For example, my solution file is hw10-gross-robert.pdf. Please use hyphens and not underlines, please use all lower-case letters, and please do not leave any spaces in the name of your file. You will have 5 points added to your homework score if you follow these instructions.

I will try to acknowledge receipt of each e-mail.

1. Let
$$h(t) = \begin{cases} 4 - 2t & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$
Write $h(t)$ using the unit step-function $u_c(t)$, and compute $\mathscr{L}(h)$.

2. Solve
$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \sec t \\ 0 \end{bmatrix} \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3. Solve
$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ u_5(t) \end{bmatrix} \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

4. Let $\mathbf{A}$ be a 3×3 matrix, and suppose that the characteristic polynomial for $\mathbf{A}$ factors as $(\lambda - c)^3$, where c is a fixed real number. If the matrix $\mathbf{A}$ has 3 linearly independent eigenvectors, we know how to solve the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. We now consider what to do when this is not the case.

   (a) Suppose that we can find only one linearly independent eigenvector $\mathbf{v}$, so that $\mathbf{A}\mathbf{v} = c\mathbf{v}$. We know that one solution of the differential equation is $\mathbf{v}e^{ct}$. We need to find two other linearly independent solutions.

   (i) Suppose that a second linearly independent solution has the form $\mathbf{u}_1 e^{ct} + \mathbf{u}_2 t e^{ct}$. Write down the equations that the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ must satisfy.

   (ii) Suppose that a third linearly independent solution has the form $\mathbf{w}_1 e^{ct} + \mathbf{w}_2 t e^{ct} + \mathbf{w}_3 t^2 e^{ct}$. Write down the equations that the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ must satisfy.

   It turns out that those equations can always be solved nontrivially.

   (b) Suppose that we can find two linearly independent eigenvectors $\mathbf{u}$ and $\mathbf{v}$, so that $\mathbf{A}\mathbf{u} = c\mathbf{u}$ and $\mathbf{A}\mathbf{v} = c\mathbf{v}$. We know that two linearly independent solutions of the differential equation are $\mathbf{u}e^{ct}$ and $\mathbf{v}e^{ct}$. We need to find one more linearly independent solution. Suppose that a solution has the form $\mathbf{w}_1 e^{ct} + \mathbf{w}_2 t e^{ct}$. Write down the equations that the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ must satisfy. Again, it turns out that this equation can always be solved nontrivially.

5. Suppose that we start with a second-order linear homogeneous differential equation
$$y'' + p(t)y' + q(t)y = 0.$$

   (a) Show that it is always possible to multiply this equation by a function $f(t)$ so that the resulting equation
$$f(t)y'' + f(t)p(t)y' + f(t)q(t)y = 0$$

can be rewritten as

$$[a(t)y']' + b(t)y = 0.$$

(b) Start with the differential equation

$$y'' - 2ty' + \lambda y = 0$$

(where λ is any real number) and find the functions $f(t)$, $a(t)$, and $b(t)$.

6. Suppose that we try to solve the differential equation from the previous problem,

$$y'' - 2xy' + \lambda y = 0$$

We have switched the independent variable to x, because we would like to solve using a power series expansion. Show that if λ is a nonnegative even integer $2m$, then one solution is a polynomial of degree m.