Chapter 11

More on Series

11.1 Introduction: Sigma Notation

The previous chapter dealt with infinite series which contained a variable. One key question in those discussions was determining for which values of the variable the infinite series would converge; this was called the "interval of convergence." In this chapter, we study infinite series more carefully, with an eye to determining more precisely this interval of convergence. In order to do this, we will also need to define the idea of convergence more carefully.

We need to start with a review of sigma notation, because we will have to use it extensively in this chapter. The notation

$$\sum_{n=1}^{k} a_n x^n$$

is a short-hand way of writing the sum

$$a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_kx^k$$

where the numbers a_1, a_2, etc. are usually given by some formula. For example, the series

$$\sum_{n=1}^{50} \frac{x^n}{n^2} = \frac{x^1}{1} + \frac{x^2}{4} + \frac{x^3}{9} + \cdots + \frac{x^{50}}{2500}.$$

This notation also makes sense if we want to add up a series of numbers, rather than variables. We can write

$$\sum_{n=1}^{k} a_n$$

to mean

$$a_1 + a_2 + a_3 + \cdots + a_k.$$

This is now a number, not a polynomial, because there is no x in the expression. For example,

$$\sum_{n=1}^{50} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{2500}.$$

Our next job is to assign a meaning to the symbol

$$\sum_{n=1}^{\infty} a_n.$$

It's easy to write down what this means symbolically:

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n.$$

What does this mean conceptually? It means we look at the sequence of numbers

$$a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3, \ldots$$

and figure out if that sequence of numbers has a limit. If it does, then that limit is *by definition*

$$\sum_{n=1}^{\infty} a_n$$

For example, to evaluate the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

we would have to compute the numbers 1, $1 + \frac{1}{4}$, $1 + \frac{1}{4} + \frac{1}{9}$ and so on and see if they approach a limit. Here is what happens when we add up the series:

k	$1 + \frac{1}{4} + \cdots + \frac{1}{k^2}$
5	1.46361
10	1.54977
10^2	1.63498
10^3	1.64393
10^4	1.64483
10^5	1.64492
10^6	1.64493

It would be reasonable to conclude that the right-hand column of numbers seems to be converging to 1.6449 to four decimal places, and to write

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1.6449\ldots.$$

In fact, this infinite series does converge, but it is possible to find other infinite series in which the numerical evidence can be misleading. We will need to develop tests to see if a series converges or not, and these tests will be the subject of the next few sections.

Exercises

1. Write out the series

$$1 + 2 + 3 + \cdots + 100$$

using sigma notation.

2. What number does

$$\sum_{i=1}^{5} \frac{1}{i^2 + i}$$

add up to?

11.2 The Ratio Test

The *ratio test* is the simplest test to explain and use to determine if an infinite series converges or diverges.

Let a and r be constants. In Chapter 10, we saw that the infinite geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

converges if the ratio r satisfies the inequality $|r| < 1$ and diverges otherwise. If we think of this as an infinite sequence of constants $\sum a_n$, where $a_n = ar^n$, then we can compute the number $|r|$ by computing the ratio a_{n+1}/a_n, and then taking the absolute value. This is just a fancy way of saying that to go from one term (a_n) in the geometric series to the next one (a_{n+1}), we multiply by r.

What are we supposed to do if we have an infinite series that isn't a geometric series? We'd like to compute the ratio of one term to the previous one, but that won't be a constant. The idea of the ratio test is that we compute the absolute value of that ratio, and then take a limit. If that limit is less than 1, then the series converges. If the limit is greater than 1, then the series diverges.

If we put the previous paragraph into symbols, we get the following:

The Ratio Test: *Let $\sum a_n$ be an infinite series. Compute the limit*

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

If $r < 1$, then the infinite series converges. If $r > 1$, then the infinite series diverges.

Let's look at some examples to see how to apply this test. Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$$

Then $a_n = \frac{n}{2^n}$ and $a_{n+1} = \frac{n+1}{2^{n+1}}$. The ratio test tells us to look at $\frac{a_{n+1}}{a_n}$, and then take the limit as n approaches infinity. We start by looking at the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{n+1}{2n},$$

where we have used the fact that $2^{n+1}/2^n = 2$. We now are supposed to compute the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. We can ignore the absolute value signs because every number in this problem is positive, and therefore need only look at

$$\lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2} = \frac{1}{2}.$$

Because this limit is less than 1, the infinite series converges.

For another example, consider the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots .$$

In this infinite series, $a_n = \frac{1}{n!}$ and $a_{n+1} = \frac{1}{(n+1)!}$. The ratio a_{n+1}/a_n is the ratio $\frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!}$, and this ratio is tricky enough so that it should be explained in detail. The number $(n+1)!$ is the product $(n+1)\cdot n\cdot(n-1)\cdots 3\cdot 2\cdot 1$, while the number $n!$ is the product $n\cdot(n-1)\cdots 3\cdot 2\cdot 1$. Therefore, the ratio

$$\frac{n!}{(n+1)!} = \frac{n\cdot(n-1)\cdots 3\cdot 2\cdot 1}{(n+1)\cdot n\cdot(n-1)\cdots 3\cdot 2\cdot 1}$$

and now we can do some cancellation and get

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

We can now compute the limit that we need:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Since this limit is less than 1, the ratio test tells us that this series converges.

Unfortunately, all too often you will compute the ratio r using the ratio test and find that $r = 1$. In that case, you can conclude *nothing*. The series might converge or it might diverge.

For example, consider the series

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots ,$$

which is called the harmonic series. If we try to use the ratio test, we have $a_n = \frac{1}{n}$ and $a_{n+1} = \frac{1}{n+1}$, and our ratio is $a_{n+1}/a_n = \frac{n}{n+1}$, and then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1.$$

We will see in a few pages that this series diverges.

We can also consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots .$$

Here we have $a_n = \frac{1}{n^2}$ and $a_{n+1} = \frac{1}{(n+1)^2}$, and our ratio is $a_{n+1}/a_n = \frac{n^2}{(n+1)^2}$. The ratio test now tells us to consider the limit

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1.$$

Again, the ratio test is inconclusive. We saw earlier that the numerical evidence strongly suggests that this series converges, and we will see in a few pages that this is indeed the case.

When is the ratio test likely to lead to an inconclusive answer? Typically, if the numerator and denominator of a_n are both polynomials, then the ratio test probably will not be useful. If either the numerator or denominator are exponential functions or factorials, then the ratio test will probably give a ratio other than 1, and therefore will lead to a conclusion about whether the series converges or diverges.

Exercises

Attempt to use the ratio test on each of the following series to determine which converge and which diverge. *Warning:* The ratio test will not be able to tell whether some of the following converge or diverge.

1. $\sum_{n=1}^{\infty} e^n$
2. $\sum_{n=1}^{\infty} \ln n$
3. $\sum_{n=1}^{\infty} \frac{1}{2n}$
4. $\sum_{n=1}^{\infty} \frac{1}{2^n}$
5. $\sum_{n=1}^{\infty} (-1)^n$
6. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$
7. $\sum_{n=1}^{\infty} \frac{1}{n^n}$
8. $\sum_{n=1}^{\infty} \frac{5^n}{n^2}$
9. $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$
10. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

11.3 The Ratio Test and Power Series

The ratio test is particularly well-suited to computing the interval of convergence of a power series. Suppose that we need to see for which values of x the series

$$\sum_{n=1}^{\infty} c_n x^n$$

or even the series

$$\sum_{n=1}^{\infty} c_n (x-a)^n$$

converges. We can let $a_n = c_n x^n$ or $a_n = c_n(x-a)^n$ and compute the ratio r using the ratio test. After that, we can write an inequality $r < 1$ and solve this inequality for x. Typically, this gives a set of x-values for which the series converges.

Take for example the infinite series

$$\sum_{n=1}^{\infty} \frac{x^n}{n4^n}.$$

We let $a_n = \frac{x^n}{n4^n}$ and $a_{n+1} = \frac{x^{n+1}}{(n+1)4^{n+1}}$. Then the ratio we have to compute is

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}/(n+1)4^{n+1}}{x^n/n4^n} = x\frac{n}{4(n+1)}$$

We now take absolute values and limits:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|x\frac{n}{4(n+1)}\right| = \lim_{n\to\infty} |x|\frac{n}{4(n+1)} = \left|\frac{x}{4}\right|.$$

In other words, the ratio test in this case gives us a ratio of $\left|\frac{x}{4}\right|$. So if $\left|\frac{x}{4}\right| < 1$, which means that $|x| < 4$, then the series converges; if $|x| > 4$, the series diverges. The ratio test also doesn't let us conclude anything if $|x| = 4$.

Rewriting this without the absolute value signs, we have worked out that the series $\sum \frac{x^n}{n4^n}$ converges if $-4 < x < 4$ and diverges if $x < -4$ or $x > 4$. We can't yet conclude anything at all about $x = 4$ or $x = -4$, because those two cases give a ratio of 1.

Some strange things can happen using the ratio test to compute intervals of convergence. For example, consider the series $\sum n!x^n$. We have $a_n = n!x^n$ and $a_{n+1} = (n+1)!x^{n+1}$. The ratio that we need to consider is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!x^{n+1}}{n!x^n} = x(n+1).$$

Then we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x(n+1)| = \lim_{n\to\infty} |x|\,(n+1).$$

If $x = 0$, then this limit is 0. If $x \neq 0$, then this limit is ∞. Therefore, the only x value for which the series converges is $x = 0$.

The ratio test can also be used for intervals of convergence for Taylor series for values of a other than 0. Suppose we consider the power series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}(x-3)^n.$$

Here we have $a_n = \frac{n}{2^n}(x-3)^n$ and $a_{n+1} = \frac{n+1}{2^{n+1}}(x-3)^{n+1}$. The limit we need to consider is

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)(x-3)^{n+1}/2^{n+1}}{n(x-3)^n/2^n}\right| \\ &= \lim_{n\to\infty} \left|\frac{(n+1)(x-3)}{2n}\right| = \lim_{n\to\infty} |x-3|\,\frac{n+1}{2n} = |x-3|\frac{1}{2}.\end{aligned}$$

Therefore, this series converges if $|x-3|\frac{1}{2} < 1$, which is that same as $|x-3| < 2$, and it diverges if $|x-3| > 2$. We are unable to say anything if $|x-3| = 2$, because that is when the ratio test gives a ratio of 1. Re-arranging the inequalities to remove the absolute value sign, we see that the series converges if $1 < x < 5$ and diverges if $x < 1$ or $x > 5$. We can't conclude anything at all if $x = 1$ or $x = 5$.

Exercises

1. The Taylor series for e^x is
$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
Use the ratio test to show that this series converges for any value of x.
2. The Taylor series for $\sin x$ is
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$
Use the ratio test to show that this series converges for any value of x.
3. The Taylor series for $\cos x$ is
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$
Use the ratio test to show that this series converges for any value of x.
4. The Taylor series for $\ln(1+x)$ is
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$
Use the ratio test to show that this series converges if $|x| < 1$ and diverges if $|x| > 1$.
5. The binomial series expansion for $(1+x)^a$ is
$$1 + \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{1 \cdot 2 \cdots n} x^n.$$
Use the ratio test to show that this series converges if $|x| < 1$ and diverges if $|x| > 1$.
6. Write out the binomial series expansion of $\sqrt{1+x}$ up to and including the x^{10} term. The previous problem told you that this series converges if $|x| < 1$ and diverges if $|x| > 1$. Now try substituting $x = 1$ and $x = -1$ into your expansion and guess if the series converges for those two values of x. If you have access to a computer or programmable calculator, try adding up the first 100 terms of the binomial series with $x = \pm 1$.

11.4 The Integral Test

The *integral test* is a test that is particularly well-suited to determining the convergence or divergence of series which are made up of polynomials. The basic idea is to compare the infinite series with an improper integral, and then use the methods of Chapter 7 to decide if the integral converges or diverges.

Suppose that $f(x)$ is a decreasing positive function. This means that $f(1) > f(2) > f(3) > \cdots$ and $f(n) > 0$ for any value of n. We want to study the convergence or divergence of the sum
$$\sum_{n=1}^{\infty} f(n).$$

Remember that this really means studying the limit

$$\lim_{k\to\infty}\sum_{n=1}^{k} f(n) = \lim_{k\to\infty} f(1)+f(2)+\cdots+f(k).$$

Think instead for a moment about the integral $\int_1^k f(x)\,dx$. Because $f(x)$ is a decreasing function, we know that any left-hand sum is bigger than the integral, and any right-hand sum is smaller than the integral. In particular, let's take sums with $\Delta x = 1$, and then we get

$$f(1)+f(2)+\cdots+f(k-1) > \int_1^k f(x)\,dx > f(2)+f(3)+\cdots+f(k).$$

Suppose that $\int_1^\infty f(x)\,dx$ converges to some number T. Then we would know that

$$T > f(2)+f(3)+\cdots+f(k)$$

for any value of k. If we add $f(1)$ to each side of the inequality, we get

$$T+f(1) > f(1)+f(2)+f(3)+\cdots+f(k)$$

and this means that the infinite series

$$f(1)+f(2)+\cdots$$

must converge.

On the other hand, suppose that the integral $\int_1^\infty f(x)\,dx$ diverges. The inequality

$$f(1)+f(2)+\cdots+f(k-1) > \int_1^k f(x)\,dx$$

means that we can make the sum $f(1)+f(2)+\cdots+f(k-1)$ as large as we want by taking a large value of k. In other words, the infinite series

$$f(1)+f(2)+\cdots$$

diverges.

If we summarize the above briefly, we get:

The Integral Test: *Suppose that the function $f(x)$ is positive and decreasing. Then the infinite series*

$$\sum_{n=1}^{\infty} f(n)$$

and the improper integral

$$\int_1^\infty f(x)\,dx$$

both converge or both diverge.

Consider for example the infinite sum that we started the chapter with: $\sum n^{-2}$. We can take $f(x) = x^{-2}$, and a very small amount of checking shows that $f(x)$ is positive and decreasing. (You can check that by looking at the graph, if you aren't very particular. If you want to be a bit more precise, then you can compute $f'(x)$ and see that it is negative; therefore, $f(x)$ is decreasing.) Therefore, in order to see if the infinite series $\sum n^{-2}$ converges or diverges, we need only look at the improper integral

$$\int_1^\infty \frac{dx}{x^2}$$

and we saw in Chapter 7 that this improper integral converges. Therefore, the infinite series also converges.

Similarly, we can consider the infinite series

$$\sum_{n=2}^\infty \frac{1}{\ln n} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \cdots$$

(We start the infinite series with $n = 2$ because $\ln 1 = 0$, and that would involve a fraction with 0 in the denominator.) We set $f(x) = \frac{1}{\ln x}$, and check (again, either graphically or by studying $f'(x)$) that $f(x)$ is positive and decreasing. Therefore, in order to see if

$$\sum_{n=2}^\infty \frac{1}{\ln n} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \cdots$$

converges or diverges, we need only see if

$$\int_2^\infty \frac{dx}{\ln x}$$

converges or diverges. We already saw in Chapter 7 that this integral diverges, and therefore the series also diverges.

Warning!

The integral test has two major problems.

1. The functions used in writing down the series might not be functions of a real number x. What does this mean? Suppose that we are trying to study the infinite series $\sum \frac{1}{n!}$. We can't do this by setting $f(x) = \frac{1}{x!}$, because $x!$ makes no sense if x isn't an integer. This problem usually occurs in series with factorials in them, and in that case usually the ratio test is the way to decide about convergence or divergence.

2. The integral that we get might be hard or even impossible to handle. Suppose that we are looking at $\sum \frac{1}{\sqrt{n^3+1}}$. We'd like to decide if this converges by looking at

   $$\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$$

   but that integral doesn't show up in the tables.

   We'll see how to handle this problem when we talk about the limit comparison test.

Exercises

1. Let a be a positive number. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^a}$$

converges if $a > 1$ and diverges otherwise.
2. Use the integral test to see if these series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2-2} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+2}}$$

3. Use the integral test to determine if

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

converges or diverges.
4. Use the integral test to determine if

$$\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$$

converges or diverges.
5. Add up some terms in each of the following series to see if you can tell numerically whether they converge or diverge.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)} \qquad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \qquad \sum_{n=2}^{\infty} \frac{1}{n^{1.001}}$$

11.5 Comparison Tests

If you look back at Chapter 7, you see that one effective way to decide if an improper integral converges or diverges is to compare it to another one. If an improper integral is larger than one which diverges, it too must diverge. If an improper integral is smaller than one which converges, it too must converge.

The same is true of infinite series, but most students find a different type of comparison test to be much more useful. The key idea is that if two infinite series are more or less "the same," then they both converge or both diverge. What is meant by "the same?" One method of expressing the answer precisely is:

The Limit Comparison Test: *Suppose that $\sum a_n$ and $\sum b_n$ are both series of positive terms. Suppose that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Then the two series behave similarly: they either both converge or both diverge.

In practice, how can we use this test? The idea is to find the "most important" terms in the numerator and denominator of a_n, ignore all the others, and use that for the series b_n. "Most important," as we shall see in some examples, usually just means biggest.

Consider the series that ended the last chapter:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$$

Let $a_n = \frac{1}{\sqrt{n^3+1}}$. The most important term in the numerator is the only term in the numerator: 1. The most important term in the denominator is $\sqrt{n^3}$. Therefore, we try to let $b_n = \frac{1}{\sqrt{n^3}}$, and see if this is good enough. We need to have two things happen. First, we need to have $\lim \frac{a_n}{b_n} = 1$. Second, we need to be able to figure out if $\sum b_n$ converges or diverges.

The first question is simple enough:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n^3+1}}{1/\sqrt{n^3}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n^3}}} = 1.$$

But how do we decide if $\sum b_n$ converges or diverges? We use the integral test. Let $f(x) = \frac{1}{\sqrt{x^3}}$, and then check that $f(x) > 0$ and $f(x)$ is decreasing. All we need to do is look at the improper integral

$$\int_1^{\infty} \frac{dx}{\sqrt{x^3}}$$

and decide if it converges or diverges. If we write this as

$$\int_1^{\infty} x^{-3/2}\,dx$$

then we can either check back to Chapter 7 and see that this converges, or else just do the integral:

$$\int_1^{\infty} x^{-3/2}\,dx = \lim_{a\to\infty} -2x^{-1/2}\Big|_1^a = \lim_{a\to\infty} \frac{-2}{\sqrt{a}} + 2 = 2.$$

To summarize, we have the following chain of reasoning: The improper integral $\int_1^{\infty} \frac{dx}{\sqrt{x^3}}$ converges. Therefore, the infinite series $\sum \frac{1}{\sqrt{n^3}}$ converges. And therefore (whew!) the infinite series $\sum \frac{1}{\sqrt{n^3+1}}$ also converges.

Let's look at another example. Suppose that we need to see if

$$\sum_{n=1}^{\infty} \frac{n^2+3n+2}{n^3+5}$$

converges or diverges. We let $a_n = \frac{n^2+3n+2}{n^3+5}$. The most important term in the numerator is n^2, and the most important term in the denominator is n^3. Therefore, we set $b_n = \frac{n^2}{n^3} = \frac{1}{n}$. We first need to check if $\lim a_n/b_n = 1$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+3n+2/n^3+5}{1/n} = \lim_{n\to\infty} \frac{n^3+3n^2+2n}{n^3+5} = \lim_{n\to\infty} \frac{1+\frac{3}{n}+\frac{2}{n^2}}{1+\frac{5}{n^3}} = 1$$

This lets us know that $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

And how do we decide if $\sum b_n$ converges or diverges? This sum is the famous *harmonic series*, and we'll talk about it in the next section.

The Other Comparison Test

The version of the comparison test given above in fact is the most useful one for most purposes. However, just to be complete, here is the more traditional form of the comparison test:

The Comparison Test: *Suppose that $\sum a_n$ and $\sum b_n$ are both infinite sums which contain only positive numbers. Suppose that for each n, $a_n \geq b_n$. Then*

- *If $\sum a_n$ converges, then $\sum b_n$ converges.*
- *If $\sum b_n$ diverges, then $\sum a_n$ diverges.*

The problem with using this form of the comparison test is that you need to decide whether the series that you are studying converges or diverges. If you think that it converges, then you need to search for a larger series which also converges. If you think that it diverges, then you need to look for a smaller series that also diverges. But the test doesn't give you any guidance about whether you should be looking for a larger series or a smaller series for comparison purposes.

However, this form of the comparison test is sometimes helpful in allowing you to simplify the series under consideration. For example, to determine if

$$\sum_{n=1}^{\infty} \frac{1}{n^n n!}$$

converges, you can notice that

$$\sum_{n=1}^{\infty} \frac{1}{n^n n!} < \sum_{n=1}^{\infty} \frac{1}{n!}.$$

Since this latter series converges (use the ratio test), we know that the smaller series also converges.

Exercises

Determine if the following series converge or diverge

1. $\sum_{n=1}^{\infty} \frac{n}{n^2+5}$
2. $\sum_{n=1}^{\infty} \frac{n}{n^3+5}$
3. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+5}$
4. $\sum_{n=1}^{\infty} \frac{n}{n!}$
5. $\sum_{n=1}^{\infty} \frac{n^2+2n+1}{n^3+3}$
6. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)\ln n}$
7. $\sum_{n=1}^{\infty} \frac{\ln n}{2^n}$
8. $\sum_{n=1}^{\infty} \frac{n^2}{n!}$
9. $\sum_{n=1}^{\infty} \frac{2^n}{\ln n}$
10. $\sum_{n=1}^{\infty} \frac{n^3+3}{n^3+4}$

11.6 The Harmonic Series

The divergence of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is a question that mathematicians have studied for over five hundred years. The series does indeed diverge, and in this section we'll look at three different proofs that it diverges.

Why give three different proofs? Because the fact that this series diverges is so confusing that many students keep insisting that it converges. Perhaps if you read the following three (very different) proofs, you'll become convinced that it really does diverge.

The first demonstration is the simplest: we can use the integral test. Let $f(x) = x^{-1}$, and then $f(x)$ is easily seen to be positive and decreasing. Therefore, the harmonic series converges or diverges depending on whether

$$\int_1^{\infty} \frac{dx}{x}$$

converges or diverges. But we already saw in Chapter 7 that this improper integral diverges:

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \ln x \Big|_1^b = \lim_{b\to\infty} \ln b - \ln 1 = \infty.$$

The second proof is the oldest, going back to a French mathematician named Oresme, well before Newton. He thought about grouping the series in sets of increasing length:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \left(\frac{1}{17} + \cdots + \frac{1}{32}\right) + \cdots$$

His next observation was that we only make the series smaller if we replace each term in a group with the last term in the group. In other words, replace $\frac{1}{3}$ with $\frac{1}{4}$, and replace $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$ with $\frac{1}{8}$, and so on. We get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \left(\frac{1}{17} + \cdots + \frac{1}{32}\right) + \cdots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) + \left(\frac{1}{32} + \cdots + \frac{1}{32}\right) + \cdots\end{aligned}$$

Now we look at the sum of the terms in each group:

$$\begin{aligned}\frac{1}{4} + \frac{1}{4} &= \frac{1}{2} \\ \frac{1}{8} + \cdots + \frac{1}{8} &= \frac{1}{2} \\ \frac{1}{16} + \cdots + \frac{1}{16} &= \frac{1}{2}\end{aligned}$$

In other words, the harmonic series is larger than $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$, and clearly this sum can be made as large as like by adding in more terms. So the harmonic series also can be made as big as we like by adding in more terms, and so it diverges.

The third proof depends on an inequality:

$$\frac{1}{n-1} + \frac{1}{n+1} > \frac{2}{n}.$$

To prove this, we put the left-hand side over the common denominator of $n^2 - 1$, and then we get

$$\begin{aligned} \frac{n+1}{n^2-1} + \frac{n-1}{n^2-1} &> \frac{2}{n} \\ \frac{2n}{n^2-1} &> \frac{2}{n} \end{aligned}$$

If we cross-multiply, we get $2n^2 > 2n^2 - 2$, which is indeed correct.

If we take our inequality, and add $\frac{1}{n}$ to each side, we get

$$\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} > \frac{3}{n}$$

Think of this in words as saying that the sum of the reciprocals of three consecutive numbers is larger than 3 times the reciprocal of the middle one.

What is the relationship of this to the harmonic series? Suppose for a moment that the harmonic series indeed converged, to some number D. We would then have

$$D = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

After the initial term, parenthesize the reciprocals in groups of 3, and then we get:

$$\begin{aligned} D =& 1 \\ &+ \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \\ &+ \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \\ &+ \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) + \\ &+ \left(\frac{1}{11} + \frac{1}{12} + \frac{1}{13}\right) + \cdots \end{aligned}$$

Now we apply our inequality and simplify:

$$
\begin{aligned}
D =& 1 \\
&+ \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \\
&+ \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) \\
&+ \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) \\
&+ \left(\frac{1}{11} + \frac{1}{12} + \frac{1}{13}\right) + \cdots \\
>& 1 \\
&+ \frac{3}{3} \\
&+ \frac{3}{6} \\
&+ \frac{3}{9} \\
&+ \frac{3}{12} + \cdots \\
=& 1 \\
&+ 1 \\
&+ \frac{1}{2} \\
&+ \frac{1}{3} \\
&+ \frac{1}{4} + \cdots
\end{aligned}
$$

But this is nearly the harmonic series again! More precisely, we have the very strange inequality $D > 1 + D$.

This of course is not true for any number D. The contradiction shows that the harmonic series can't add up to any number D, so it can't converge.

11.7 Alternating Series

So far the results we have discussed have mainly been useful to study the convergence of series which contained only positive numbers. There is one other type of series which is very easy to study (perhaps easier than anything that we have discussed so far): a series in which positive and negative numbers alternate. In order to deal only with positive numbers, we write such a series as

$$a_1 - a_2 + a_3 - a_4 + \cdots .$$

If we need to use $\sum$-notation, then we write this as

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n.$$

(We need the exponent to be $n+1$ if we want to get a positive result when we plug in $n=1$ and a negative one when we substitute in $n=2$.) The result here is a very simple one:

The Alternating Series Test: *Suppose that*

$$a_1 > a_2 > a_3 > \cdots,$$

and

$$\lim_{n\to\infty} a_n = 0.$$

Then the alternating infinite series

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

converges. On the other hand, if

$$\lim_{n\to\infty} a_n \neq 0,$$

then the alternating series diverges.

The simplest example to consider is the "alternating harmonic series":

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Because $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} \cdots$ and $\lim \frac{1}{n} = 0$, this series converges. Notice that this is different from the situation with positive series; we saw in the last chapter that this same series with all positive signs diverges.

Because this test is so simple to apply, it makes more sense to discuss why it rather than to work any more examples.

Remember that to see if a series converges, we need to study the numbers

$$a_1,\ a_1 - a_2,\ a_1 - a_2 + a_3, \ldots$$

and see if those numbers tend to a limit. Let's write

$$\begin{aligned}
s_1 &= a_1\\
s_2 &= a_1 - a_2\\
s_3 &= a_1 - a_2 + a_3\\
s_4 &= a_1 - a_2 + a_3 - a_4\\
s_5 &= a_1 - a_2 + a_3 - a_4 + a_5\\
s_6 &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6
\end{aligned}$$

and so on. We need to see if $\lim s_n$ exists.

Let's start with an easy observation. Remember that $a_1 > a_2 > a_3 > \cdots$. If we notice that $s_4 = s_2 + (a_3 - a_4)$, then we know that $s_4 > s_2$ because the number in parentheses must be positive. Similarly, $s_6 = s_4 + (a_5 - a_6)$, and because $a_5 > a_6$, the number in parentheses is positive and $s_6 > s_4$. If we turn around the direction of the inequalities, we have

$$s_2 < s_4 < s_6 < s_8 \cdots$$

We can do something similar with the odd subscripts. Write $s_3 = s_1 - (a_2 - a_3)$, and again the term in parentheses is positive. That says that $s_3 < s_1$. Similarly, we can write $s_5 = s_3 - (a_4 - a_5)$, and then $s_5 < s_3$. We get a sequence of inequalities

$$\cdots s_7 < s_5 < s_3 < s_1.$$

And how can we relate the even and the odd subscripts? This is particularly easy. For example, $s_8 = s_7 - a_8$, and since a_8 is positive, we know right away that $s_8 < s_7$. This final inequality means that we can chain together our two lines above and get

$$s_2 < s_4 < s_6 < s_8 < \cdots < s_7 < s_5 < s_3 < s_1.$$

In words, half of the numbers are getting bigger, and half are getting smaller. We only need to decide if somewhere in the middle, there is some limit L that both sides are getting close to.

But this last question is also easy to answer. We know that $\lim a_n = 0$. Furthermore, to go from s_{n-1} to s_n, you either add or subtract a_n. So $\lim s_{n-1} - s_n = \lim \pm a_n = 0$. This says that the difference between even and odd subscripts gets smaller and smaller, so somewhere in the middle there is indeed the limit L. The final picture looks like this:

$$s_2 < s_4 < s_6 < s_8 < \cdots < L < \cdots < s_7 < s_5 < s_3 < s_1.$$

Error analysis

If you check the above discussion a bit more carefully, you find that in fact

$$|L - s_n| < a_{n+1}$$

In words, the error that you make by chopping the series off after some term and using that to approximate the limit L is less than the next term.

Here's a particular example. We saw in Chapter 10 that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

If we wanted to use this series to approximate π, we would need to stop adding at some point. Suppose that we compute

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{99}$$

as an approximation to π. We would then know that our answer is closer than $\frac{1}{101}$ to π. In this case, that's not very close at all, which is why this series isn't used very often to compute π.

Conditional and Absolute Convergence

Mathematicians find it convenient in many situations to distinguish between two different types of convergent alternating series. If

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

converges, but

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

diverges, then we say that the infinite series $a_1 - a_2 + a_3 - a_4 + \cdots$ is *conditionally convergent*. If

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

and

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

both converge, then we say that the infinite series $a_1 - a_2 + a_3 - a_4 + \cdots$ is *absolutely convergent*.

For example, the *alternating harmonic series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent, because the harmonic series diverges. On the other hand,

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

is absolutely convergent, because the infinite series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

converges.

Exercises

1. Does the binomial series

$$\sqrt{1+x}$$

converge or diverge when $x = 1$?

11.8 Putting Everything Together

Let's work an example to see how all the pieces fit together. Suppose that we wanted to determine the set of all x-values for which the Taylor series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n3^n}$$

converges.

We start by using the ratio test to figure out the interval of convergence. Let $a_n = \frac{(x+1)^n}{n3^n}$, and then $a_{n+1} = \frac{(x+1)^{n+1}}{(n+1)3^{n+1}}$. We compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(x+1)^{n+1}/(n+1)3^{n+1}}{(x+1)^n/n3^n} = \lim_{n\to\infty}(x+1)\frac{n}{3(n+1)} = \lim_{n\to\infty}(x+1)\frac{1}{3+\frac{3}{n}} = \frac{x+1}{3}.$$

We now need to set the absolute value of this ratio to be less than 1, and that gives the inequality

$$\left|\frac{x+1}{3}\right| < 1$$
$$|x+1| < 3$$

Solving this gives the inequalities $-4 < x < 2$.

This means the our power series converges for $-4 < x < 2$, and diverges if $x < -4$ or $x > 2$. We now need to discuss what happens when $x = -4$ and $x = 2$.

When $x = -4$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

and we saw in the section about alternating series that this series converges.

When $x = 2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

and we are by now well aware that this series diverges.

To summarize: the infinite series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n3^n}$$

converges for $-4 \le x < 2$, and diverges for other values of x.

Exercises

Determine precisely for which x values the following series converge, and for which they diverge.

1. $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$
2. $\sum_{n=1}^{\infty} 3^n x^n$
3. $\sum_{n=1}^{\infty} \frac{n!x^n}{2^n}$
4. $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$
5. $\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{2^n}$
6. $\sum_{n=1}^{\infty} \frac{3^n(x+5)^n}{4^n}$
7. $\sum_{n=1}^{\infty} \frac{(x+1)^{2n+1}}{n^2+4}$
8. $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{(2n)!}$
9. $\sum_{n=1}^{\infty} \frac{(2x-3)^n}{4^{2n}}$
10. $\sum_{n=1}^{\infty} \frac{(x-4)^n}{(n+1)^2}$