Technical Appendix: "Non-Walrasian Labor Market and the European Business Cycle"

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ABSTRACT

All the analytical computations of the paper quoted are provided here. These techniques were originally developed in King, Plosser, and Rebelo (1988a, b).

1 Specification of the Equilibrium

In a symmetric equilibrium the following conditions apply: $P_t(i) = P_t$, $Y_t(i) = Y_t$, $K_t(i) = K_t$, $n_t(i) = n_t$, $F_t(i) = F_t$, $W_t(i) = W_t$ and, for the government budget constraint, $\int_0^1 (1 - n(i))\, di = 1 - n_t$ for all $i \in (0,1)$ and $t = 0, 1, 2, ...$. Note that $\frac{K_t^{ss}}{K_{t-1}^{ss}} = g$, $\frac{W_t^{ss}}{W_{t-1}^{ss}} = \overline{w}$, and $\frac{P_t^{ss}}{P_{t-1}^{ss}} = \pi$ are the steady state growth rate values for capital, wage, and price respectively. The superscript "ss" means the value of the variable at the steady state.

From the structure of the model (FOCs, constraints and behavior of the exogenous components) we have the following:

The **unknown 19 variables** which need to be determined are the following:

$$[C_t, n_t, B_t, I_t, M_t, F_t, P_t, K_t, Y_t, Q_t, T_t, S_t, W_t, Z_t, \Lambda_t, \Xi_t, x_t, A_t, r_t]'.$$

The **19 equations** which determine the behavior of these variables are:

- From the household maximization problem:

$$Y_t = C_t + I_t + \frac{\phi_p}{2}\left(\frac{P_t}{P_{t-1}} - \pi\right)^2 Y_t, \tag{1}$$

$$K_{t+1} = (1-\delta)K_t + I_t - \frac{\phi_k}{2}\left(\frac{K_{t+1}}{K_t} - g\right)^2 K_t, \tag{2}$$

$$\Lambda_t = \frac{C_t^{-\frac{1}{\mu}}}{C_t^{\frac{\mu-1}{\mu}} + x_t \left(\frac{M_t}{P_t}\right)^{\frac{\mu-1}{\mu}}}, \tag{3}$$

$$\begin{aligned} 0 \;\; = \;\; & \Lambda_t \left[\phi_k \left(\frac{K_{t+1}}{K_t} - g\right) + 1\right] \\ & -\beta E_t \Lambda_{t+1} \left[(1-\delta) - \phi_k \left(\frac{K_{t+2}}{K_{t+1}} - g\right) \frac{K_{t+2}}{K_{t+1}} - \frac{\phi_k}{2} \left(\frac{K_{t+2}}{K_{t+1}} - g\right)^2 + \frac{Z_{t+1}}{P_{t+1}}\right], \end{aligned} \tag{4}$$

$$\frac{\Lambda_t}{R_t P_t} = \beta E_t \frac{\Lambda_{t+1}}{P_{t+1}}, \tag{5}$$

$$\frac{\frac{x_t}{P_t} \left(\frac{M_t}{P_t}\right)^{-\frac{1}{\mu}}}{C_t^{\frac{\mu-1}{\mu}} + x_t \left(\frac{M_t}{P_t}\right)^{\frac{\mu-1}{\mu}}} = \frac{\Lambda_t}{P_t} - \beta E_t \frac{\Lambda_{t+1}}{P_{t+1}}, \tag{6}$$

$$\ln x_t = (1-\rho_x) \ln x + \rho_x \ln x_{t-1} + \varepsilon_{xt}. \tag{7}$$

-From the representative firm optimization problem:

$$\Lambda_t \frac{W_t}{P_t} n_t = \Xi_t A_t \left\{\alpha K_t^\eta + \varphi \left[g^t n_t\right]^\eta\right\}^{\frac{1-\eta}{\eta}} \varphi \left[g^t n_t\right]^\eta, \tag{8}$$

$$\Lambda_t \frac{Z_t}{P_t} K_t = \Xi_t A_t \left\{\alpha K_t^\eta + \varphi \left[g^t n_t\right]^\eta\right\}^{\frac{1-\eta}{\eta}} \alpha K_t{}^\eta, \tag{9}$$

$$\begin{aligned} 0 \;\; = \;\; & (1-\theta) \Lambda_t - \Lambda_t \phi_p (\pi_t - \pi) \pi_t \\ & + \Xi_t \theta + \phi_p \beta E_t \Lambda_{t+1} (\pi_{t+1} - \pi) \pi_{t+1} \frac{Y_{t+1}}{Y_t}, \end{aligned} \tag{10}$$

$$Y_t = A_t \left\{\alpha K_t^\eta + \varphi \left[g^t n_t\right]^\eta\right\}^{\frac{1}{\eta}}, \tag{11}$$

$$\ln(A_t) = \rho_A \ln(A_{t-1}) + (1-\rho_A) \ln(A) + \varepsilon_{At}, \tag{12}$$

$$F_t = P_t Y_t - W_t n_t - Z_t K_t - \frac{\phi_p}{2} (\pi_t - \pi)^2 Y_t P_t. \tag{13}$$

- From the monetary authority:

$$\ln(r_t) = (1-\rho_R)\ln(r) + \rho_R \ln(r_{t-1}) + \rho_\pi \ln(\frac{\pi_t}{\pi}) + \rho_y \ln(\frac{Y_t}{y}) + \varepsilon_{Rt}. \tag{14}$$

- From the union maximization problem:

$$\begin{aligned} 0 \quad = \quad & -\left[\frac{\alpha}{\left(\frac{\Lambda_t W_t}{A_t P_t \Xi_t \varphi g^{\eta t}}\right)^{\frac{\eta}{1-\eta}} - \varphi g^{\eta t}}\right]^{\frac{1}{\eta}} \frac{1}{\left(\frac{\Lambda_t W_t}{A_t P_t \Xi_t \varphi g^{\eta t}}\right)^{\frac{\eta}{1-\eta}} - \varphi g^{\eta t}} \frac{1}{1-\eta} \times \\ & \times \left(\frac{\Lambda_t}{A_t P_t \Xi_t \varphi g^{\eta t}}\right)^{\frac{\eta}{1-\eta}} W_t^{\frac{2\eta - 1}{1-\eta}} K_t \left[W_t - S_t\right] + \left[\frac{\alpha}{\left(\frac{\Lambda_t W_t}{A_t P_t \Xi_t \varphi g^{\eta t}}\right)^{\frac{\eta}{1-\eta}} - \varphi g^{\eta t}}\right]^{\frac{1}{\eta}} K_t + \\ & -\phi_w \left(\frac{W_t}{W_{t-1}} - \overline{w}\right) \frac{W_t}{W_{t-1}} - \frac{\phi_w}{2}\left(\frac{W_t}{W_{t-1}} - \overline{w}\right)^2 + \beta \phi_w E_t \left(\frac{W_{t+1}}{W_t} - \overline{w}\right)\left(\frac{W_{t+1}}{W_t}\right)^2 \end{aligned} \tag{15}$$

- From the government constraint:

$$Q_t + \frac{B_t}{R_t} - B_{t-1} + M_t - M_{t-1} = (1-n_t) S_t + T_t. \tag{16}$$

- The market clearing conditions:

$$B_t = 0, \tag{17}$$ [1]

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$$S_t = S, \tag{18}$$

$$Q_t = Q. \tag{19}$$

2 Transformed System

To remove the trend from the variables which grow over time we define:

$y_t = \frac{Y_t}{g^t}$, $c_t = \frac{C_t}{g^t}$, $i_t = \frac{I_t}{g^t}$, $k_t = \frac{K_t}{g^t}$, $\lambda_t = \Lambda_t g^t$, $\xi_t = \Xi_t g^t$, $w_t = \frac{W_t}{P_t g^t}$, $f_t = \frac{F_t}{P_t g^t}$, $m_t = \frac{M_t}{P_t g^t}$, $z_t = \frac{Z_t}{P_t}$, $b_t = \frac{B_t}{M_t}$, $s_t = \frac{S_t}{M_t}$, $q_t = \frac{Q_t}{M_t}$, $t_t = \frac{T_t}{M_t}$, and $\mu_{t+1} = \frac{M_t}{M_{t+1}}$. As a result, note that: $\frac{k_t}{k_{t-1}} = 1$, $\frac{w_t}{w_{t-1}} = \frac{\overline{w}}{g\pi}$. Also, define the gross inflation rate as $\pi_t = \frac{P_t}{P_{t-1}}$.

[1]This equality is imposed because the no Ponzi game condition says that the present value of assets is asymptotically non negative and debt, B_t, cannot grow as fast as R_t.

Using these definitions we can write our original non linear system of equilibrium equations in terms of stationary variables.

$$y_t = c_t + i_t + \frac{\phi_p}{2}\left(\pi_t - \pi\right)^2 y_t \tag{1}$$

$$g k_{t+1} = (1-\delta)k_t + i_t - \frac{\phi_k}{2}\left(\frac{k_{t+1}}{k_t}g - g\right)^2 k_t \tag{2}$$

$$\lambda_t = \frac{c_t^{-\frac{1}{\mu}}}{c_t^{\frac{\mu-1}{\mu}} + x_t m_t^{\frac{\mu-1}{\mu}}} \tag{3}$$

$$\begin{aligned} 0 \quad = \quad & g\lambda_t \left[\phi_k\left(\frac{k_{t+1}}{k_t}g - g\right) + 1\right] + \\ & -\beta E_t \lambda_{t+1}\left[(1-\delta) - \phi_k\left(\frac{k_{t+2}}{k_{t+1}}g - g\right)\frac{k_{t+2}}{k_{t+1}}g - \frac{\phi_k}{2}\left(\frac{k_{t+2}}{k_{t+1}}g - g\right)^2 + z_{t+1}\right] \end{aligned} \tag{4}$$

$$\frac{\lambda_t g}{R_t} = \frac{\beta E_t \lambda_{t+1}}{\pi_{t+1}} \tag{5}$$

$$\frac{x_t m_t^{-\frac{1}{\mu}}}{c_t^{\frac{\mu-1}{\mu}} + x_t m_t^{\frac{\mu-1}{\mu}}} = \lambda_t - \beta E_t \frac{\lambda_{t+1}}{\pi_{t+1} g} \tag{6}$$

$$\ln x_t = (1-\rho_x)\ln x + \rho_x \ln x_{t-1} + \varepsilon_{xt} \tag{7}$$

$$\lambda_t w_t n_t = \xi_t A_t \left[\alpha k_t^\eta + \varphi n_t^\eta\right]^{\frac{1-\eta}{\eta}} \varphi n_t^\eta \tag{8}$$

$$\lambda_t z_t k_t = \xi_t A_t \left[\alpha k_t^\eta + \varphi n_t^\eta\right]^{\frac{1-\eta}{\eta}} \alpha k_t^\eta \tag{9}$$

$$\lambda_t(1-\vartheta) - \phi_p\left(\pi_t - \pi\right)\lambda_t \pi_t + \beta E_t \lambda_{t+1}\phi_p\left(\pi_{t+1} - \pi\right)\pi_{t+1}\frac{y_{t+1}}{y_t} + \vartheta\xi_t = 0 \tag{10}$$

$$y_t = A_t \left[\alpha k_t^\eta + \varphi n_t^\eta\right]^{\frac{1}{\eta}} \tag{11}$$

$$\ln(A_t) = \rho_A \ln(A_{t-1}) + (1-\rho_A)\ln(A) + \varepsilon_{At} \tag{12}$$

$$f_t = y_t - w_t n_t - z_t k_t - \frac{\phi_p}{2}\left(\pi_t - \pi\right)^2 y_t \tag{13}$$

$$\ln(r_t) = (1-\rho_R)\ln(r) + \rho_R \ln(r_{t-1}) + \rho_\pi \ln(\frac{\pi_t}{\pi}) + \rho_y \ln(\frac{y_t}{y}) + \varepsilon_{Rt} \tag{14}$$

$$\begin{aligned} 0 \quad = \quad & -\left[\frac{\alpha}{\left(\frac{\lambda_t w_t}{\varphi \xi_t A_t}\right)^{\frac{\eta}{1-\eta}} - \varphi}\right]^{\frac{1}{\eta}} \frac{1}{\left[\left(\frac{\lambda_t w_t}{\varphi \xi_t A_t}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \frac{1}{1-\eta}\left(\frac{\lambda_t}{\varphi \xi_t A_t}\right)^{\frac{\eta}{1-\eta}} w_t^{\frac{2\eta-1}{1-\eta}} k_t (w_t - s_t m_t) + \\ & + \left[\frac{\alpha}{\left(\frac{\lambda_t w_t}{\varphi \xi_t A_t}\right)^{\frac{\eta}{1-\eta}} - \varphi}\right]^{\frac{1}{\eta}} k_t - \frac{\phi_w}{2}\left(\frac{w_t}{w_{t-1}}\pi g - \overline{w}\right)^2 - \phi_w \left(\frac{w_t}{w_{t-1}}\pi g - \overline{w}\right)\frac{w_t}{w_{t-1}}\pi g + \\ & + \beta \phi_w \left(\frac{w_{t+1}}{w_t}\pi g - \overline{w}\right)\left(\frac{w_{t+1}}{w_t}\right)^2 \pi g \end{aligned} \tag{15}$$

$$q_t + \frac{b_t}{R_t} - b_{t-1} + 1 - \frac{1}{\mu_t} = (1-n_t)s_t + t_t \tag{16}$$

$$b_t = 0 \tag{17}$$

$$s_t = s \tag{18}$$

$$q_t = q \tag{19}$$

With these new notations, the vector of unknown 19 variables becomes:

$$[c_t, n_t, b_t, i_t, m_t, f_t, \pi_t, k_t, y_t, q_t, t_t, s_t, w_t, z_t, \lambda_t, \xi_t, x_t, A_t, r_t]'.$$

3 Steady State

In absence of shocks, the economy converges to a steady state where each of the transformed variables does not change over time. Let the money growth, μ, be chosen by policy. The shock equations (7) and (12) define x and A.

The procedure we use to find the steady state for the 19 variables divides in two parts: $i)$ we rewrite the system of the previous section at its steady state, and $ii)$ we compute the value of all the variables at the steady state. In particular, the first part is carried out explicitly because it helps to simplify equations when the log-linearization is performed.

3.1 The System at the Steady State

The transformed system becomes at the steady state:

$$y = c + i \tag{1}$$

$$gk = (1-\delta)k + i \tag{2}$$

$$\lambda = \frac{c^{-\frac{1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}} \tag{3}$$

$$g = \beta\left[(1-\delta) + z\right] \tag{4}$$

$$\frac{g}{R} = \frac{\beta}{\pi} \tag{5}$$

$$\frac{xm^{-\frac{1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}} = \lambda - \beta\frac{\lambda}{\pi g} \tag{6}$$

$$x = x \tag{7}$$

$$\lambda wn = \xi A\left[\alpha k^{\eta} + \varphi n^{\eta}\right]^{\frac{1-\eta}{\eta}} \varphi n^{\eta} \tag{8}$$

$$\lambda zk = \xi A\left[\alpha k^{\eta} + \varphi n^{\eta}\right]^{\frac{1-\eta}{\eta}} \alpha k^{\eta} \tag{9}$$

$$\lambda(1-\vartheta) + \vartheta\xi = 0 \tag{10}$$

$$y = A\left[\alpha k^{\eta} + \varphi n^{\eta}\right]^{\frac{1}{\eta}} \tag{11}$$

$$A = A \tag{12}$$

$$f = y - wn - zk \tag{13}$$

$$r = r \tag{14}$$

$$\frac{1}{1-\eta}\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}-\frac{1}{1-\eta}\left(\frac{\lambda}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}w^{\frac{2\eta-1}{1-\eta}}sm=\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}-\varphi \tag{15}$$

$$q-(1-n)s+1-\frac{1}{\mu}=t \tag{16}$$

$$b_t=0 \tag{17}$$

$$s_t=s \tag{18}$$

$$q_t=q \tag{19}$$

3.2 Values of the Variables at the Steady State

In this section the 19 variables $\{c, n, b, i, m, f, \pi, k, y, q, t, s, w, z, \lambda, \xi, x, A, r\}$ are expressed in function of the "structural" parameters. Values of inflation π_t, unemployment subsidies s, and lump-sum taxation q are chosen by policy.

$$b=0,$$

from equations (16), (5), (4), and (10) we obtain:

$$t=1-\frac{1}{\mu}+q-(1+n)s,$$

$$\pi=\frac{\mu}{g},$$

$$z=\frac{g}{\beta}-(1-\delta),$$

$$\xi=\frac{\lambda(\vartheta-1)}{\vartheta}.$$

Equations (3) and (6) determine:

$$m=\left(\frac{x\pi g}{\pi g-\beta\lambda}\right)\frac{1}{\lambda}\left[\left(\frac{\pi g-\beta\lambda}{x\pi g}\right)^{\mu-1}+x\right]^{-1},$$

$$c=\left[x\left(\frac{\pi g}{\pi g-\beta\lambda}\right)\right]^{-\mu}m.$$

Equation (15) yields:

$$1-\frac{sm}{w}=(1-\eta)-\varphi\left(\frac{\lambda w}{\varphi\xi A}\right)^{-\frac{\eta}{1-\eta}}$$

Since this equation is highly non linear, it needs to be solved numerically.[2]

Equations (2), (9), (1), (8), and (13) give:

$$i=k\left(g-1+\delta\right),$$

$$k=\frac{c}{\frac{\alpha\lambda z}{\xi}+\frac{\varphi\lambda z}{\xi}\left[\frac{\alpha}{\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}-\varphi}\right]-(g-1+\delta)},$$

$$n=\left\{\frac{\alpha}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}-\varphi\right]}\right\}^{\frac{1}{\eta}}k,$$

$$y=\left\{\frac{\alpha\lambda z}{\xi}+\frac{\varphi\lambda z}{\xi}\left[\frac{\alpha}{\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}-\varphi}\right]\right\}k,$$

and

$$f=y-wn-zk.$$

Finally, equation (11) allows to compute the value for the Lagrange multiplier, λ:

$$\lambda=\left[\alpha+\frac{\varphi\alpha}{\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}-\varphi}\right]\left\{\frac{\alpha z}{\xi}+\frac{\varphi z}{\xi}\left[\frac{\alpha}{\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}-\varphi}\right]\right\}^{-1}.$$

[2] Appendix 1 treats the case when $s=\mathbf{0}$, which simplifies a bit the computations.

4 Log-Linearization

To study the dynamic of the system in response to shocks, we log-linearize it around its steady state. In this way we can examine how the 19 variables fluctuate about their steady state in response to shocks. For notational purposes define $\widehat{c}_t = \ln\left(\frac{c_t}{c}\right)$, $\widehat{n}_t = \ln\left(\frac{n_t}{n}\right)$, $\widehat{b}_t = \ln\left(\frac{b_t}{b}\right)$, $\widehat{i}_t = \ln\left(\frac{i_t}{i}\right)$, $\widehat{m}_t = \ln\left(\frac{m_t}{m}\right)$, $\widehat{f}_t = \ln\left(\frac{f_t}{f}\right)$, $\widehat{\pi}_t = \ln\left(\frac{\pi_t}{\pi}\right)$, $\widehat{k}_t = \ln\left(\frac{k_t}{k}\right)$, $\widehat{y}_t = \ln\left(\frac{y_t}{y}\right)$, $\widehat{q}_t = \ln\left(\frac{q_t}{q}\right)$, $\widehat{t}_t = \ln\left(\frac{t_t}{t}\right)$, $\widehat{s}_t = \ln\left(\frac{s_t}{s}\right)$, $\widehat{w}_t = \ln\left(\frac{w_t}{w}\right)$, $\widehat{z}_t = \ln\left(\frac{z_t}{z}\right)$, $\widehat{\lambda}_t = \ln\left(\frac{\lambda_t}{\lambda}\right)$, $\widehat{\xi}_t = \ln\left(\frac{\xi_t}{\xi}\right)$, $\widehat{x}_t = \ln\left(\frac{x_t}{x}\right)$, $\widehat{A}_t = \ln\left(\frac{A_t}{A}\right)$, and $\widehat{r}_t = \ln\left(\frac{r_t}{r}\right)$. Then, the transformed system can be written as:

$$0 = y\widehat{y}_t + c\widehat{c}_t + i\widehat{i}_t \tag{1}$$

$$0 = -kg\widehat{k}_{t+1} + (1-\delta)k\widehat{k}_t + i\widehat{i}_t \tag{2}$$

$$\begin{aligned} 0 \;=\; & -\widehat{\lambda}_t + \frac{1}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}}\left[\left(-c^{\frac{\mu-1}{\mu}} - \frac{1}{\mu}xm^{\frac{\mu-1}{\mu}}\right)\widehat{c}_t + \frac{1-\mu}{\mu}\left(\frac{xm^{\frac{\mu-1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}}\right)\widehat{m}_t\right] + \\ & -\frac{1}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}}\left[\left(\frac{xm^{\frac{\mu-1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}}\right)\widehat{x}_t\right] \end{aligned} \tag{3}$$

$$\begin{aligned} 0 \;=\; & \widehat{\lambda}_t + \phi_k\left(g\right)E_t\widehat{k}_{t+1} - \phi_k\left(g\right)\widehat{k}_t - E_t\widehat{\lambda}_{t+1} + \\ & -\beta E_t\left[-\phi_k\widehat{k}_{t+2} + \phi_k g\widehat{k}_{t+1} + \frac{1}{g}z\widehat{z}_{t+1}\right] \end{aligned} \tag{4}$$

$$0 = -\widehat{\lambda}_t + \widehat{R}_t + E_t\widehat{\lambda}_{t+1} - E_t\widehat{\pi}_{t+1} \tag{5}$$

$$\begin{aligned} 0 \;=\; & \widehat{x}_t + \frac{-\frac{1}{\mu}\left(c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}\right) + \left(\frac{1-\mu}{\mu}\right)xm^{\frac{\mu-1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}}\widehat{m}_t + \frac{\left(\frac{\mu-1}{\mu}\right)c^{\frac{\mu-1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}}\widehat{c}_t + \\ & -\left(1 - \frac{\beta\lambda}{\pi g}\right)\left[\widehat{\lambda}_t - \frac{\beta\lambda}{\pi g}\left(E_t\widehat{\lambda}_{t+1} - \widehat{\pi}_t\right)\right] \end{aligned} \tag{6}$$

$$\widehat{x}_t = \rho_x\widehat{x}_{t-1} + \varepsilon_{xt} \tag{7}$$

$$\begin{aligned} 0 \;=\; & -\widehat{\lambda}_t - \widehat{w}_t + \widehat{\zeta}_t + \widehat{A}_t + \left[\left(\alpha k^{\eta} + \varphi n^{\eta}\right)^{-1}\left(1-\eta\right)\alpha k^{\eta}\right]\widehat{k}_t + \\ & + \left[\left(1-\eta\right)\left(\alpha k^{\eta} + \varphi n^{\eta}\right)^{-1}\varphi n^{\eta} + \eta - 1\right]\widehat{n}_t \end{aligned} \tag{8}$$

$$
\begin{aligned}
0 \quad = \quad & -\widehat{\lambda}_t - \widehat{z}_t + \widehat{\zeta}_t + \widehat{A}_t + \left[\left(\alpha k^{\eta} + \varphi n^{\eta} \right)^{-1} (1-\eta) \varphi n^{\eta} \right] \widehat{n}_t + \\
& + \left[(1-\eta) \left(\alpha k^{\eta} + \varphi n^{\eta} \right)^{-1} \alpha k^{\eta} + \eta - 1 \right] \widehat{k}_t
\end{aligned} \tag{9}
$$

$$
0 = (1-\vartheta) \lambda \widehat{\lambda}_t - \phi_p \lambda \pi^2 \widehat{\pi}_t + \beta \lambda \phi_p \pi^2 E_t \widehat{\pi}_{t+1} + \vartheta \xi \widehat{\xi}_t \tag{10}
$$

$$
0 = -\widehat{y}_t + \widehat{A}_t + \left(\alpha k^{\eta} + \varphi n^{\eta} \right)^{-1} \alpha k^{\eta} \widehat{k}_t + \left(\alpha k^{\eta} + \varphi n^{\eta} \right)^{-1} \varphi n^{\eta} \widehat{n}_t \tag{11}
$$

$$
\widehat{A}_t = \rho_A \widehat{A}_{t-1} + \varepsilon_{At} \tag{12}
$$

$$
0 = -f \widehat{f}_t + y \widehat{y}_t - wn \widehat{w}_t - wn \widehat{n}_t - zk \widehat{z}_t - zk \widehat{k}_t \tag{13}
$$

$$
0 = -\widehat{r}_t + \rho_r \widehat{r}_{t-1} + \rho_y \widehat{y}_t + \rho_\pi \widehat{\pi}_t + \varepsilon_{Rt} \tag{14}
$$

To simplify the specification of the log-linearization of the wage equation we use the following notation:

$$
\overline{U} = - \left[\frac{\alpha}{\left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \varphi} \right]^{\frac{1}{\eta}} \frac{1}{\left[\left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \varphi \right]} \frac{1}{1-\eta} \left(\frac{\lambda}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} w^{\frac{2\eta - 1}{1-\eta}} k (w - sm)
$$

$$
\overline{V} = \left[\frac{\alpha}{\left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \varphi} \right]^{\frac{1}{\eta}} k
$$

With this notation the wage equation can be written as:

$$
\begin{aligned}
& \overline{U} \times \left\{ - \frac{1}{\left[\left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \varphi \right]} \left(\frac{\lambda}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} w^{\frac{2\eta-1}{1-\eta}} - \frac{2}{\left[\left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \varphi \right]} \left(\frac{\eta}{1-\eta} \right) \left(\frac{\lambda}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} w^{\frac{2\eta-1}{1-\eta}} + \left(\frac{2\eta-1}{1-\eta} \right) w_t^{-1} + \frac{1}{(w-sm)} \right\} w \widehat{w}_t + \\
& + \overline{U} \times \left\{ - \frac{1}{\left[\left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \varphi \right]} \left(\frac{w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} \lambda^{\frac{2\eta-1}{1-\eta}} - \frac{2}{\left[\left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \varphi \right]} \left(\frac{\eta}{1-\eta} \right) \left(\frac{w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} \lambda^{\frac{2\eta-1}{1-\eta}} + \frac{\eta}{1-\eta} \lambda^{-1} \right\} \lambda \widehat{\lambda}_t + \\
& + \overline{U} \times \left\{ \frac{1}{\left[\left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \varphi \right]} \left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} + \frac{2}{\left[\left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \varphi \right]} \left(\frac{\eta}{1-\eta} \right) \left(\frac{\lambda w}{\varphi \xi A} \right)^{\frac{\eta}{1-\eta}} - \frac{\eta}{1-\eta} \right\} \left(\widehat{\xi}_t + \widehat{A}_t \right) +
\end{aligned} \tag{15}
$$

$+\overline{U}\times\widehat{k}_t-\overline{U}\times\frac{sm}{(w-sm)}\widehat{s}_t-\overline{U}\times\frac{sm}{(w-sm)}\widehat{m}_t+$

$+\overline{V}\times\left\{-\frac{1}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}-\varphi\right]}\frac{1}{1-\eta}\left(\frac{\lambda}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}w^{\frac{2\eta-1}{1-\eta}}\right\}w\widehat{w}_t+$

$+\left\{\overline{V}\times\frac{1}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}-\varphi\right]}\frac{1}{1-\eta}\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}\right\}\left(\widehat{\xi}_t+\widehat{A}_t\right)+$

$-\phi_w\overline{w}\widehat{w}_t+\phi_w\overline{w}\widehat{w}_{t-1}+\beta\phi_w\overline{w}^2\widehat{w}_{t+1}\frac{1}{\pi g}-\beta\phi_w\overline{w}^2\widehat{w}_t\frac{1}{\pi g}=0$

$$0=-q\widehat{q}_t-\frac{b}{R}\widehat{b}_t+\frac{b}{R}\widehat{R}_t+b\widehat{b}_{t-1}-\frac{1}{\mu}\widehat{\mu}_t-sn\widehat{n}_t+(1-n)s\widehat{s}_t+t\widehat{t}_t \tag{16}$$

For the solution of the model is convenient to rewrite equations (4) and (15) so that $\widehat{k}_{t+2}$ and $\widehat{w}_{t+1}$vanish. Equation (4) can be rewritten using equation (2):

$$\begin{aligned} 0 &= \widehat{\lambda}_t-\phi_k(g)\widehat{k}_t-E_t\widehat{\lambda}_{t+1}+\frac{\beta\phi_k i}{kg}E_t\widehat{i}_{t+1}-\frac{\beta}{g}zE_t\widehat{z}_{t+1}+ \\ &\quad +E_t\widehat{k}_{t+1}\left[\beta\phi_k\frac{(1-\delta)}{g}+\phi_k g-\beta\phi_k g\right]. \end{aligned} \tag{4}$$

In equation (15) we substitute $\widehat{w}_{t+1}$with:

$$\widehat{w}_{t+1}=-\frac{f}{wn}\widehat{f}_{t+1}+\frac{y}{wn}\widehat{y}_{t+1}-\widehat{n}_{t+1}-\frac{zk}{wn}\widehat{z}_{t+1}-\frac{zk}{wn}\widehat{k}_{t+1},$$

which is given by equation (13).

5 The Linearized System in Matrix Form

The linearized system can be written in matrix form. Let define the vectors of flow and state variables respectively as

$$\mathbf{f}_t^0=\left[\widehat{c}_t,\widehat{n}_t,\widehat{i}_t,\widehat{f}_t,\widehat{\pi}_t,\widehat{y}_t,\widehat{q}_t,\widehat{t}_t,\widehat{s}_t,\widehat{z}_t,\widehat{\lambda}_t,\widehat{\xi}_t,\widehat{\mu}_t\right]',$$

and

$$\mathbf{s}_t^0=\left[\widehat{m}_{t-1},\widehat{k}_t,\widehat{w}_{t-1},\widehat{r}_{t-1}\right]',$$

with the vector of the shocks given by

$$\mathbf{z}_t^0=\left[\widehat{x}_t,\widehat{A}_t,\varepsilon_{Rt}\right]'.$$

Using this notation, equations (1)-(6), (8)-(11), and (13)-(16) can be written as

$$AE_t\mathbf{x}_{t+1}^0 + B\mathbf{x}_t^0 + CE_t\mathbf{z}_{t+1}^0 + D\mathbf{z}_t^0 = 0 \tag{1}$$

where

$$\mathbf{x}_t^0 = \begin{pmatrix} \mathbf{f}_t^0 \\ \mathbf{s}_t^0 \end{pmatrix},$$

with A and B are 14×17 and C and D are 14×3. Instead, equations (7) and (12) can be written as

$$\mathbf{z}_{t+1}^0 = P\mathbf{z}_t^0 + \varepsilon_{t+1} \tag{2}$$

where

$$P = \begin{bmatrix} \rho_x & 0 & 0 \\ 0 & \rho_A & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \varepsilon_t = [\varepsilon_{xt}, \varepsilon_{At}, \varepsilon_{Rt}]'.$$

The vector ε_t is assumed to be normally distributed with zero mean and uncorrelated innovations, which satisfy

$$E_t\varepsilon_t\varepsilon_t' = \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_A^2 & 0 \\ 0 & 0 & \sigma_R^2 \end{bmatrix}.$$

In the case of our linearized system matrixes A, B, C, and D are defined as follows.

Equation (1) gives:

$b_{1,1} = c, \quad b_{1,3} = i, \quad b_{1,6} = -y.$

Equation (2) gives:

$a_{2,15} = -kg$, $b_{2,3} = i$, $b_{2,15} = (1-\delta)k$.

Equation (3) gives:

$a_{3,14} = \frac{1}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}} \frac{1-\mu}{\mu} \left(\frac{xm^{\frac{\mu-1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}} \right)$, $b_{3,1} = \frac{1}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}} \left(-c^{\frac{\mu-1}{\mu}} - \frac{1}{\mu} xm^{\frac{\mu-1}{\mu}} \right)$, $b_{3,11} = -1$, $d_{3,1} = -\frac{1}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}} \left(\frac{xm^{\frac{\mu-1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}} \right)$.

Equation (4) gives:

$a_{4,3} = \frac{\beta i \phi_k}{kg}$, $a_{4,10} = -\frac{\beta}{g} z$, $a_{4,11} = -1$, $a_{4,15} = \beta\phi_k \frac{(1-\delta)}{g} + \phi_k g - \beta\phi_k g$, $b_{4,11} = 1$, $b_{4,15} = -\phi_k g$.

Equation (5) gives:

$a_{5,5} = -1$, $a_{5,11} = 1$, $a_{5,17} = 1$, $b_{5,11} = -1$.

Equation (6) gives:

$a_{6,11} = \left(1 - \frac{\beta\lambda}{\pi g}\right) \frac{\beta\lambda}{\pi g}$, $a_{6,14} = \frac{-\frac{1}{\mu}\left(c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}\right) + \left(\frac{1-\mu}{\mu}\right) xm^{\frac{\mu-1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}}$, $b_{6,1} = \frac{\left(\frac{\mu-1}{\mu}\right) c^{\frac{\mu-1}{\mu}}}{c^{\frac{\mu-1}{\mu}} + xm^{\frac{\mu-1}{\mu}}}$, $b_{6,5} = -\left(1 - \frac{\beta\lambda}{\pi g}\right) \frac{\beta\lambda}{\pi g}$, $b_{6,11} = -\left(1 - \frac{\beta\lambda}{\pi g}\right)$.

Equation (8) gives:

$a_{8,11} = -1$, $a_{8,16} = -1$, $b_{8,2} = (1-\eta)\left(\alpha k^{\eta} + \varphi n^{\eta}\right)^{-1} \varphi n^{\eta} + \eta - 1$, $b_{8,12} = 1$, $b_{8,15} = \left(\alpha k^{\eta} + \varphi n^{\eta}\right)^{-1} (1-\eta)\, \alpha k^{\eta}$, $d_{8,2} = 1$.

Equation (9) gives:

$b_{9,2} = \left(\alpha k^{\eta} + \varphi n^{\eta}\right)^{-1} (1-\eta)\, \varphi n^{\eta}$, $b_{9,10} = -1$, $b_{9,11} = -1$, $b_{9,12} = 1$, $b_{9,15} = (1-\eta)\left(\alpha k^{\eta} + \varphi n^{\eta}\right)^{-1} \alpha k^{\eta} + \eta - 1$, $d_{9,2} = 1$.

Equation (10) gives:

$a_{10,5} = \beta\lambda\phi_p\pi^2$, $b_{10,5} = -\phi_p\lambda\pi^2$, $b_{10,11} = (1-\vartheta)\,\lambda$, $b_{10,12} = \vartheta\xi$.

Equation (11) gives:

$b_{11,2} = \left(\alpha k^{\eta} + \varphi n^{\eta}\right)^{-1} \varphi n^{\eta}$, $b_{11,6} = -1$, $b_{11,15} = \left(\alpha k^{\eta} + \varphi n^{\eta}\right)^{-1} \alpha k^{\eta}$, $d_{11,2} = 1$.

Equation (13) gives:

$a_{13,16} = -wn$, $b_{13,2} = -wn$, $b_{13,4} = -f$, $b_{13,6} = y$, $b_{13,10} = -zk$, $b_{13,15} = -zk$.

Equation (14) gives:

$a_{14,17} = -1$, $b_{14,5} = \rho_{\pi}$, $b_{14,6} = \rho_{y}$, $b_{14,17} = \rho_{R}$, $d_{14,3} = 1$.

Equation (15) gives:

$a_{15,2} = -\beta\phi_w \frac{\overline{w}^2}{\pi g}$, $a_{15,4} = -\beta\phi_w \frac{\overline{w}^2}{\pi g} \frac{f}{wn}$, $a_{15,6} = \beta\phi_w \frac{\overline{w}^2}{\pi g} \frac{y}{wn}$, $a_{15,10} = -\beta\phi_w \frac{\overline{w}^2}{\pi g} \frac{zk}{wn}$, $a_{15,15} = -\beta\phi_w \frac{\overline{w}^2}{\pi g} \frac{zk}{wn}$,
$a_{15,16} = \overline{U} \times \left\{ -\frac{1}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \left(\frac{\lambda}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} w^{\frac{2\eta-1}{1-\eta}} - \frac{2}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \left(\frac{\eta}{1-\eta}\right) \left(\frac{\lambda}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} w^{\frac{2\eta-1}{1-\eta}} + \left(\frac{2\eta-1}{1-\eta}\right) w_t^{-1} + \frac{1}{(w-sm)} \right\} w$
$\phi_w\overline{w} - \beta\phi_w \frac{\overline{w}^2}{\pi g}$, $b_{15,9} = -\overline{U} \times \frac{sm}{(w-sm)}$, $b_{15,11} = \overline{U} \times \left\{ -\frac{1}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \left(\frac{w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} \lambda^{\frac{2\eta-1}{1-\eta}} - \frac{2}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \left(\frac{\eta}{1-\eta}\right) \left(\frac{w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}}\right.$
$b_{15,12} = \overline{U} \times \left\{ \frac{1}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} + \frac{2}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \left(\frac{\eta}{1-\eta}\right) \left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \frac{\eta}{1-\eta} \right\} + \left\{ \overline{V} \times \frac{1}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \frac{1}{1-\eta} \left(\frac{\lambda w}{\varphi\xi A}\right)^{\overline{1}}\right.$
$b_{15,15} = \overline{U}$, $b_{15,16} = \phi_w\overline{w}$, $d_{15,2} = \overline{U} \times \left\{ \frac{1}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} + \frac{2}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \left(\frac{\eta}{1-\eta}\right) \left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \frac{\eta}{1-\eta} \right\} +$
$\left\{ \overline{V} \times \frac{1}{\left[\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} - \varphi\right]} \frac{1}{1-\eta} \left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} \right\}$.

Equation (16) gives:

$a_{16,17} = \frac{b}{R}$, $b_{16,2} = -sn$, $b_{16,7} = q$, $b_{16,8} = t$, $b_{16,9} = (1-n)s$, $b_{16,13} = -\frac{1}{\mu}$.

6 Solving the Model

Klein (2000) shows that the system written in the form of the previous section as equations (1) and (2) has as a solution the state space representation

$$\mathbf{s}_t = \boldsymbol{\Psi}\mathbf{s}_{t-1} + \boldsymbol{\Omega}\boldsymbol{\varepsilon}_t,$$

and

$$\mathbf{f}_t = \mathbf{U}\mathbf{s}_t,$$

where

$$\mathbf{f}_t = \left[\widehat{c}_t, \widehat{n}_t, \widehat{i}_t, \widehat{f}_t, \widehat{\pi}_t, \widehat{y}_t, \widehat{q}_t, \widehat{t}_t, \widehat{s}_t, \widehat{z}_t, \widehat{\lambda}_t, \widehat{\xi}_t, \widehat{\mu}_t\right]',$$

$$\mathbf{s}_t = \left[\widehat{m}_{t-1}, \widehat{k}_t, \widehat{w}_{t-1}, \widehat{r}_{t-1}, \widehat{x}_t, \widehat{A}_t, \varepsilon_{Rt}\right]',$$

and

$$\boldsymbol{\Omega} = \begin{pmatrix} \underset{4\times 3}{0} \\ \underset{3\times 3}{I} \end{pmatrix}.$$

7 Appendix 1

Steady state values in case of $s = 0$.

The steady state value of wage becomes:

$$w = \left[\frac{\varphi(\eta-1)}{\eta}\right]^{\frac{1-\eta}{\eta}} \frac{\varphi\xi A}{\lambda}$$

Using this value we can simplify the original notations in the extent that:

$$\left(\frac{\lambda w}{\varphi\xi A}\right)^{\frac{\eta}{1-\eta}} = g\left(\frac{1-\eta}{\eta}\right).$$

Where this last expression does not appear, the system is the same as the original one.

8 REFERENCES

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