Identification and Estimation of Treatment Effects in the Presence of Neighbourhood Interactions

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Abstract

This paper presents a counter-factual model identifying Average Treatment Effects (ATEs) by assuming conditional mean independence (CMI) when externality (or neighbourhood) effects are incorporated within the traditional Rubin's potential outcome model. As such, it tries to generalize the usual linear regression-adjustment approach, widely used in program evaluation and epidemiology, when SUTVA (i.e. Stable Unit Treatment Value Assumption) is relaxed. The paper also provides a user-written Stata routine, `ntreatreg`, suitable for an easy implementation of the model. An instructional application using simulated data shows the statistical logic of the model the computational correctness of `ntreatreg`.

Keywords: ATEs, Rubin's causal model, SUTVA, neighbourhood effects, Stata command.

JEL classification: C21, C31, C87

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1. Introduction

In observational program evaluation studies, aimed at estimating the effect of an intervention on the outcome of a set of targeted individuals, it is generally assumed that *"the treatment received by one unit does not affect other units' outcome"* (Cox, 1958). Along with other fundamental assumptions - such as, for instance, the conditional independence assumption, the exclusion restriction provided by instrumental-variables estimation, or the existence of a forcing-variable in regression discontinuity design - the no-interference assumption is required in order to obtain a consistent estimation of the (average) treatment effects (ATEs). It means that, if interference (or interaction) among units is assumed, traditional program evaluation methods such as control-function regression, selection models, matching or reweighting are bound to be biased estimations of the actual treatment effect[1].

Rubin (1978) calls this important assumption as Stable-Unit-Treatment-Value-Assumption (SUTVA), whereas Manski (2013) refers to Individualistic-Treatment-Response (ITR) to emphasize that this poses a restriction in the form of the treatment response function that the analyst considers. SUTVA (or ITR) implies that the treatment applied to a specific individual affects only the outcome of that individual, so that potential "externality effects" flowing for instance from treated to untreated subjects are sharply ruled out.

In this paper, we aim at removing this hypothesis to understand what happens to the estimation of the effect of a binary policy (treatment) in the presence of neighbourhood (externality) effects taking place among supported (treated) and non-supported (untreated) units.

Epidemiological studies have addressed this hot topic although restricting the analysis to experimental settings where treatment randomization is assumed (see, for instance: Rosenbaum, 2007; Hudgens and Halloran, 2008; Tchetgen-Tchetgen and VanderWeele, 2010; Robins et al., 2000). Differently, this paper moves along the line traced by econometric studies normally dealing with non-experimental settings where sample selection is the rule (i.e., no random draw is assumed) and an ex-post evaluation is thus envisaged (Sobel, 2006). In particular, we work within the binary potential outcome model that in many regards we aim at generalizing for taking into account neighbourhood effects. Our theoretical reference may be found in some previous works dealing with treatment effect identification in the presence of externalities and in particular in the papers by Manski (1993; 2013).

[1] The applied literature on the socio-economics of peer effect is rather vast; here we focus on that related to peer (or neighbourhood) effect within the Rubin's potential outcome model (POM). Very recently, however, Angrist (2014) has provided a comprehensive critical review of problems arising in measuring the causal effect of a peer regressor on individual performance. Such article provides also a well documented survey of the literature on the subject.

Moreover, as by-product, this work also presents a Stata routine, `ntreatreg`, for estimating Average Treatments Effects (ATEs) when neighbourhood effects are taken into account.

The paper is organized as follows: section 2 presents some related literature and positions our approach within the Manski's notion of "endogenous" neighbourhood effects; section 3 sets out the model, its assumptions and propositions; section 4 presents the model's estimation procedure; section 5 puts forward the Stata implementation of the model via the user-written routine `ntreatreg` and provides a simulated application; section 6 concludes the paper. Finally, appendix A sets out the proof of each proposition.

2. Related literature

The literature on the estimation of treatment effects under potential interference among units is a recent and challenging field of statistical and econometric study. So far, however, only few papers have dealt formally with this relevant topic (Angrist, 2014).

Rosenbaum (2007) was among the first scholars paving the way to generalize the standard randomization statistical approach for comparing different treatments to the case of units' interference. He presented a statistical model in which unit's response depends not only on the treatment individually received, but also on the treatment received by other units, thus showing how it is possible to test the null-hypothesis of no interference in a random assignment setting where randomization occurs within pre-specified groups and interference between groups is ruled out.

On the same vein, Sobel (2006) provided a definition, identification and estimation strategy for traditional average treatment effect estimators when interference between units is allowed, by taking as example the "Moving To Opportunity" (MTO) randomized social experiment. In his paper, he uses interchangeably the term interference and spillover to account for the presence of such a kind of externality. Interestingly, he shows that a potential bias can arises when no-interference is erroneously assumed, and defines a series of direct and indirect treatment effects that may be identified under reasonable assumptions. Moreover, this author shows some interesting links between the form of his estimators under interference and the Local Average Treatment Effect (LATE) estimator provided by Imbens and Angrist (1994), thus showing that – under interference – treatment effects can be identified only on specific sub-populations.

The paper by Hudgens and Halloran (2008) is probably the most relevant of this literature, as these authors develop a rather general and rigorous modelling of the statistical treatment setting under randomization when interference is potentially present. Furthermore, their approach paves the way also for extensions to observational settings. Starting from the same two-stage randomization approach of Rosenbaum (2007), these authors manage to go substantially farther by providing a

precise characterization of the causal effects with interference in randomized trials encompassing also the Sobel's approach. They define *direct*, *indirect*, *total* and *overall* causal effects showing the relation between these measures and providing an unbiased estimator of the upper bound of their variance.

Tchetgen-Tchetgen and VanderWeele (2010)'s paper follows in the footsteps traced by the approach of Hudgens and Halloran (2008) and provides a formal framework for statistical inference on population average causal effects in a finite sample setting with interference when the outcome variable is binary. Interestingly, they also present an original inferential approach for observational studies based on a generalization of the Inverse Probability Weighting (IPW) estimator when interference is present. Unfortunately, they do not provide the asymptotic variances for such estimators.

Aronow and Samii (2013) finally generalizes the approach proposed by Hudgens and Halloran (2008) going beyond the hierarchical experiment setting and providing a general variance estimation including covariates adjustment.

Previous literature assumes that the potential outcome y of unit i is a function of the treatment received by such a unit (w_i) and the treatment received by all the other units ($\mathbf{w}_{-i}$), that is:

$$y_i(w_i; \mathbf{w}_{-i}) \tag{1}$$

entailing that – with N units and a binary treatment for instance – a number of 2^N potential outcomes may arise. Nevertheless, an alternative way of modelling unit i's potential outcome may be that of assuming:

$$y_i(w_i; \mathbf{y}_{-i}) \tag{2}$$

where $\mathbf{y}_{-i}$ is the (N-1)x1 vector of other units' potential outcomes excluding unit i's potential outcome. The notion of interference entailed by expression (2) is different from that implied by expression (1). The latter, however, is well consistent with the notion of "endogenous" neighbourhood effects provided by Manski (1993, pp. 532-533). Manski, in fact, identifies three types of effects corresponding to three arguments of the individual (potential) outcome equation incorporating social effects[2]:

[2] The literature is not homogeneous in singling out a unique name of such effects: although context-dependent, authors interchangeably refer to peer, neighbourhood, social, club, interference or interaction effects.

1. *Endogenous effects*. Such effects entail that the outcome of an individual depends on the outcomes of other individuals belonging to his neighbourhood.

2. *Exogenous (or contextual) effects*. These effects concern the possibility that the outcome of an individual is affected by the exogenous idiosyncratic characteristics of the individuals belonging to his neighbourhood.

3. *Correlated effects*. They are effects due to belonging to a specific group and thus sharing some institutional/normative condition (that one can loosely define as "environment").

Contextual and correlated effects are to be assumed as exogenous, as they clearly depend on pre-determined characteristics of the individuals in the neighbourhood (case 2) or of the neighbourhood itself (case 3). Endogenous effects are on the contrary of broader interest, as they are affected by the behaviour (measured as "outcome") of other individuals involved in the same neighbourhood. This means that endogenous effects both comprise direct and indirect effects linked to a given external intervention on individuals. The model proposed in this paper incorporates the presence of endogenous neighbourhood effects as defined by Manski within a traditional binary counterfactual model and provides both an identification and an estimation procedure for the Average Treatment Effects (ATEs) in this specific case[3].

How can we position this paper within the literature? Very concisely, previous literature assumes that: (i) unit potential outcome depends on own treatment and other units' treatment; (ii) assignment is randomized or conditionally unconfounded; (iii) treatment is multiple; (iii) potential outcomes have a non-parametric form. This paper, instead, assumes that: (i) unit potential outcome depends on own treatment and other units' potential outcome; (ii) assignment is mean conditionally unconfounded; (iii) treatment is binary; (iv) potential outcomes have a parametric form.

As such, this paper suggests a simple but workable way to relax SUTVA, one that seems rather easy to implement in many socio-economic contexts of application.

[3] A combined regression model including both individual treatments and outcomes may be expressed as:

$$y_i = \mathrm{f}\,(w_i; \mathbf{y}_{-i}; \mathbf{w}_{-i})$$

Arduini, Patacchini and Rainone (2014) provides a first attempt to modelling such a regression on individuals eligible for treatment, showing that the coefficient of w_i (i.e., their measure of ATE) combines both treatments' and outcomes' direct and indirect effects on y. However, such a model is not embedded within the classical Rubin' potential outcome model (POM). Differently, the paper proposed here provides a POM-consistent approach, generalized to the case of possible interaction among units.

3. A binary treatment model with "endogenous" neighbourhood effects

This section presents a model for estimating the average treatment effects (ATEs) of a policy program (or a treatment) in a non-experimental setting in the presence of "endogenous" neighbourhood (or externality) interactions. We consider a binary treatment variable w - taking value 1 for treated and 0 for untreated units - assumed to affect an outcome (or target) variable y that can take a variety of forms.

Some notation can help in understanding the setting: N is the number of units involved in the experiment; N_1, the number of treated units; N_0 the number of untreated units; w_i the treatment variable assuming value "1" if unit i is treated and "0" if untreated; y_{1i} is the outcome of unit i when she is treated; y_{0i} is the outcome of unit i when she is untreated; $\mathbf{x}_i = (x_{1i}, x_{2i}, x_{3i}, \ldots, x_{Mi})$ is a row vector of M exogenous observable characteristics for unit $i = 1, \ldots, N$.

To begin with, as usual in this literature, we define the unit i's *Treatment Effect* (TE) as:

$$TE_i = y_{1i} - y_{0i} \tag{3}$$

TE_i is equal to the difference between the value of the target variable when the individual is treated (y_1), and the value assumed by this variable when the same individual is untreated (y_0). Since TE_i refers to the same individual at the same time, the analyst can observe just one of the two quantities feeding into (3) but never both. For instance, it might be the case that we can observe the investment behaviour of a supported company, but we cannot know what the investment of this company would have been, had it not been supported, and vice versa. The analyst faces a fundamental missing observation problem (Holland, 1986) that needs to be tackled econometrically in order to recover reliably the causal effect via some specific imputation technique (Rubin, 1974; 1977).

Both y_{1i} and y_{0i} are assumed to be independent and identically distributed (i.i.d.) random variables, generally explained by a structural part depending on observable factors and a non-structural one depending on an unobservable (error) term. Nevertheless, recovering the entire distributions of y_{1i} and y_{0i} (and, consequently, the distribution of the TE_i) may be too demanding without very strong assumptions, so that the literature has focused on estimating specific moments of these distributions and in particular the "mean", thus defining the so-called population Average Treatment Effect (hereinafter ATE), and ATE conditional on $\mathbf{x}_i$ (i.e., ATE($\mathbf{x}_i$)) of a policy intervention as:

$$ATE = E(y_{i1} - y_{i0}) \tag{4}$$

$$\mathrm{ATE}(\mathbf{x}_i) = \mathrm{E}(y_{i1} - y_{i0} \mid \mathbf{x}_i) \tag{5}$$

where $\mathrm{E}(\cdot)$ is the mean operator. ATE is equal to the difference between the average of the target variable when the individual is treated (y_1), and the average of the target variable when the same individual is untreated (y_0). Observe that, by the law of iterated expectations, $\mathrm{ATE} = \mathrm{E}_{\mathbf{x}}\{\mathrm{ATE}(\mathbf{x})\}$.

Given the definition of the unconditional and conditional average treatment effect in (4) and (5) respectively, it is immediate to define the same parameters in the sub-population of treated (ATET) and untreated (ATENT) units, i.e.:

$$\mathrm{ATET} = \mathrm{E}(y_{i1}-y_{i0} \mid w_i=1)$$
$$\mathrm{ATET}(\mathbf{x}_i) = \mathrm{E}(y_{i1} - y_{i0} \mid \mathbf{x}_i, w_i=1)$$

and

$$\mathrm{ATENT} = \mathrm{E}(y_{i1}-y_{i0} \mid w_i=0)$$
$$\mathrm{ATENT}(\mathbf{x}_i) = \mathrm{E}(y_{i1} - y_{i0} \mid \mathbf{x}_i, w_i=0)$$

The aim of this paper is to provide consistent parametric estimation of all previous quantities (we refer to as ATEs) in the presence of neighbourhood effects.

To that end, we start with what is observable to the analyst in such a setting, i.e. the actual status of the unit *i*, that can be obtained as:

$$y_i = y_{0i} + w_i\,(y_{1i} - y_{0i}) \tag{6}$$

Equation (6) is known as the Rubin's potential outcome model (POM), and it is the fundamental relation linking the unobservable with the observable outcome. Given Eq. (6), we first set out all the assumptions behind the next development of the proposed model.

Assumption 1. *Unconfoundedness* (or CMI). Given the set of random variables $\{y_{1i}, y_{1i}, w_i, \mathbf{x}_i\}$ as defined above, the following equalities hold:

$$\mathrm{E}(y_{ig} \mid w_i, \mathbf{x}_i) = \mathrm{E}(y_{ig} \mid \mathbf{x}_i) \quad \text{with} \quad g = \{0,1\}$$

Hence, throughout this paper, we will assume unconfoundedness, i.e. Conditional Mean Independence (CMI) to hold. As we will show, CMI is a sufficient condition for identifying ATEs also when neighbourhood effects are considered.

Once CMI has been assumed, we then need to model the potential outcomes y_{0i} and y_{1i} in a proper way so to get a representation of the ATEs (i.e., ATE, ATET and ATENT) taking into account the presence of endogenous externality effects. In this paper, we simplify further our analysis by assuming some restrictions in the form of the potential outcomes.

Assumption 2. *Restrictions on the form of the potential outcomes*. Consider the general form of the potential outcome as expressed in (2), and assume this relation to depend parametrically on a vector of real numbers $\boldsymbol{\theta} = (\boldsymbol{\theta}_0; \boldsymbol{\theta}_1)$. We assume that:

$$y_{1i}(w_i; \mathbf{x}_i; \boldsymbol{\theta}_1)$$

and

$$y_{0i}(w_i; \mathbf{x}_i; \mathbf{y}_{1,-i}; \boldsymbol{\theta}_0)$$

Assumption 2 poses two important restrictions to the form given to the potential outcomes: (i) it makes them dependent on some unknown parameters $\boldsymbol{\theta}$ (i.e., parametric form); (ii) it entails that the externality effect occurs only in one direction, from the treated individuals to the untreated, while the other way round is ruled out.

Assumption 3. *Linearity and weighting-matrix*. We assume that the potential outcomes are linear in the parameters, and that a NxN weighting-matrix $\boldsymbol{\Omega}$ of exogenous constant numbers is known.

Under Assumptions 1, 2 and 3, the model takes on this form:

$$\begin{cases} y_{1i} = \mu_1 + \mathbf{x}_i\boldsymbol{\beta}_1 + e_{1i} \\ y_{0i} = \mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma s_i + e_{0i} \\ s_i = \sum_{j=1}^{N_1} \omega_{ij} y_{1j}, \quad \text{with} \ \sum_{j=1}^{N_1} \omega_{ij} = 1 \\ y_i = y_{0i} + w(y_{1i} - y_{0i}) \\ \text{CMI holds} \end{cases} \tag{7}$$

where i = 1, ..., N and j = 1, ..., N_1, μ_1 and μ_0 are scalars, $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}_1$ are two unknown vector parameters defining the different response of unit i to the vector of covariates $\mathbf{x}$, e_0 and e_1 are two random errors with zero unconditional variance and s_i represents unit i-th neighbourhood effect due to the treatment administrated to units j (j = 1, ..., N_1). Observe that, by linearity, we have that:

$$s_i = \begin{cases} \sum_{j=1}^{N_1} \omega_{ij} y_{1j} & \text{if } i \in \{w = 0\} \\ 0 & \text{if } i \in \{w = 1\} \end{cases} \tag{8}$$

where the parameter ω_{ij} is the generic element of the weighting matrix $\boldsymbol{\Omega}$ expressing some form of *distance* between unit i and unit j. Although not strictly required for consistency, we also assume that these weights add to one, i.e. $\sum_{j=1}^{N_1} \omega_{ij} = 1$. In short, previous assumptions say that units i neighbourhood effect takes the form of a weighted-mean of the outcomes of treated units and that this "social" effect has an impact only on unit i's outcome when this unit is untreated. As a consequence, by substitution of (8) into (7), we get that:

$$y_{0i} = \mu_0 + \mathbf{x}_i \boldsymbol{\beta}_0 + \gamma \sum_{j=1}^{N_1} \omega_{ij} y_{1j} + e_{0i} \tag{9}$$

making clear that untreated unit's i outcome is a function of its own idiosyncratic characteristics ($\mathbf{x}_i$), the weighted outcomes of treated units multiplied by a sensitivity parameter γ, and a standard error term.

We state now a series of propositions implied by previous assumptions.

Proposition 1. *Formula of* ATE *with neighbourhood interactions*. Given assumptions 2 and 3 and the implied equations established in (7), the average treatment effect (ATE) with neighbourhood interactions takes on this form:

$$\begin{aligned} &\text{ATE} = \mathrm{E}(y_{1i} - y_{0i}) = \mu + \mathrm{E}\left[\mathbf{x}_i \boldsymbol{\delta} - \left(\sum_{j=1}^{N_1} \omega_{ij} \mathbf{x}_j \right) \gamma \boldsymbol{\beta}_1 - e_i \right] = \\ &\mu + \bar{\mathbf{x}} \boldsymbol{\delta} - \bar{\mathbf{v}} \boldsymbol{\lambda} \end{aligned} \tag{10}$$

where $\boldsymbol{\lambda} = \gamma\boldsymbol{\beta}_1$, $\bar{\mathbf{x}}_i = \mathrm{E}(\mathbf{x}_i)$, $\bar{\mathbf{v}} = \mathrm{E}\left(\underbrace{\sum_{j=1}^{N_1} \omega_{ij}\mathbf{x}_j}_{\mathbf{v}_i} \right)$ is the unconditional mean of the vector $\mathbf{x}_i$, and $\mu = \mu_1 - \mu_0 - \gamma\mu_1$. The proof is in Appendix. See A1.

Indeed, by the definition of ATE as given in (4) and by (7), we can immediately show that for such a model:

$$\mathrm{ATE} = \mathrm{E}(y_{1i} - y_{0i}) = \mathrm{E}\left[\left(\mu_1 + \mathbf{x}_i\boldsymbol{\beta}_1 + e_{1i} \right) - \left(\mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma \sum_{j=1}^{N_1} \omega_{ij} y_{1j} + e_{0i} \right) \right] \tag{11}$$

where:

$$\begin{aligned} &\sum_{j=1}^{N_1} \omega_{ij} y_{1j} = \sum_{j=1}^{N_1} \omega_{ij} \left(\mu_1 + \mathbf{x}_j\boldsymbol{\beta}_1 + e_{1j} \right) = \\ &\mu_1 \sum_{j=1}^{N_1} \omega_{ij} + \sum_{j=1}^{N_1} \omega_{ij}\mathbf{x}_j\boldsymbol{\beta}_1 + \sum_{j=1}^{N_1} \omega_{ij} e_{1j} = \\ &\mu_1 + \left(\sum_{j=1}^{N_1} \omega_{ij}\mathbf{x}_j \right)\boldsymbol{\beta}_1 + \sum_{j=1}^{N_1} \omega_{ij} e_{1j} \end{aligned} \tag{12}$$

and by developing ATE further using Eq. (11), we finally get the result in (10).

Proposition 2. *Formula of* ATE($\mathbf{x}_i$) *with neighbourhood interactions*. Given assumptions 2 and 3 and the result in proposition 1, we have that:

$$\mathrm{ATE}(\mathbf{x}_i) = \mathrm{E}(y_{1i} - y_{0i} \mid \mathbf{x}_i) = \mathrm{ATE} + (\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + (\bar{\mathbf{v}} - \mathbf{v}_i)\boldsymbol{\lambda} \tag{13}$$

where it is now easy to see that ATE =$\mathrm{E}_{\mathbf{x}}\{\mathrm{ATE}(\mathbf{x})\}$. The proof is in Appendix. See A2.

Proposition 3. *Baseline random-coefficient regression*. By substitution of equations (7) into the POM of Eq. (6), we obtain the following random-coefficient regression model (Wooldridge, 1997):

$$y_i = \eta + w_i \cdot \mathrm{ATE} + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + w_i(\bar{\mathbf{v}} - \mathbf{v}_i)\boldsymbol{\lambda} + e_i \tag{14}$$

where: $\mathbf{v}_i = \sum_{j=1}^{N_1} \omega_{ij}\mathbf{x}_j$, $\bar{\mathbf{v}} = \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)$, $\boldsymbol{\lambda}=\gamma\boldsymbol{\beta}_1$, $\eta = \mu_0 + \gamma\mu_1$, and $\boldsymbol{\delta} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_0$. The proof is in Appendix. See A3.

Proposition 4. *Ordinary Least Squares (OLS) consistency*. Under assumption 1 (CMI), 2 and 3, the error tem of regression (14) has zero mean conditional on (w_i, $\mathbf{x}_i$), i.e.:

$$\mathrm{E}\left(e_i \,|w_i,\mathbf{x}_i\right) = \mathrm{E}\left(\gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} + e_{0i} + w_i(e_{1i}-e_{0i}) - w_i\gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j}\,|w_i,\mathbf{x}_i\right) = 0 \tag{15}$$

thus implying that Eq. (14) is a regression model whose parameters can be *consistently* estimated by Ordinary Least Squares (OLS). The proof is in Appendix. See A4.

Once a consistent estimation of the parameters of (14) is obtained, we can estimate ATE directly from the regression, and ATE($\mathbf{x}_i$) by plugging the estimated parameters into formula (11). This is because ATE($\mathbf{x}_i$) becomes a function of consistent estimates, and thus consistent itself:

$$\text{plim } \widehat{\text{ATE}}(\mathbf{x}_i) = \text{ATE}(\mathbf{x}_i)$$

where $\widehat{\text{ATE}}(\mathbf{x}_i)$ is the plug-in estimator of ATE($\mathbf{x}_i$). Observe, however, that the (exogenous) weighting matrix $\boldsymbol{\Omega}=[\omega_{ij}]$ needs to be provided in advance.

Once the formulas for ATE and ATE($\mathbf{x}_i$) are available, it is also possible to recover the Average Treatment Effect on Treated (ATET) and on non-Treated (ATENT) as:

$$\text{ATET} = \text{ATE} + \frac{1}{\sum_{i=1}^{N} w_i}\sum_{i=1}^{N} w_i\left[(\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + (\bar{\mathbf{v}} - \mathbf{v}_i)\boldsymbol{\lambda}\right] \tag{16}$$

and:

$$\text{ATENT} = \text{ATE} + \frac{1}{\sum_{i=1}^{N}(1-w_i)}\sum_{i=1}^{N}(1-w_i)\left[(\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + (\bar{\mathbf{v}} - \mathbf{v}_i)\boldsymbol{\lambda}\right] \tag{17}$$

These quantities are functions of observable components and parameters consistently estimated by OLS (see next section). Once these estimates are available, standard errors for ATET and ATENT can be correctly obtained via bootstrapping (see Wooldridge, 2010, pp. 911-919).

4. Estimation

Starting from previous section's results, a simple protocol for estimating ATEs can be suggested. Given an i.i.d. sample of observed variables for each individual *i*:

$$\{y_i, w_i, \mathbf{x}_i\} \text{ with } i = 1, \ldots, N$$

1. provide a weighting matrix $\Omega=[\omega_{ij}]$ measuring some type of distance between the generic unit *i* (untreated) and unit *j* (treated);

2. estimate by an OLS a regression model of:

$$y_i \quad \text{on} \quad \{1,\ w_i,\ \mathbf{x}_i,\ w_i(\mathbf{x}_i - \bar{\mathbf{x}}),\ w_i(\bar{\mathbf{v}} - \mathbf{v}_i)\}$$

3. obtain $\left\{\hat{\boldsymbol{\beta}}_0,\ \hat{\boldsymbol{\delta}},\ \hat{\gamma},\ \hat{\boldsymbol{\beta}}_1\right\}$ and put them into the formulas of ATEs.

By comparing for instance the formula of ATE *with* ($\gamma \neq 0$) and *without* ($\gamma = 0$) neighbourhood effect, we get the *neighbourhood-bias* defined as:

$$\begin{aligned} \text{Bias} &= \left|\text{ATE}_{\text{no-neigh}} - \text{ATE}_{\text{with-neigh}}\right| = \left|(\bar{\mathbf{v}} - \mathbf{v}_i)\boldsymbol{\lambda}\right| = \\ &\left| \left[\frac{1}{N} \sum_{i=1}^{N} \left(\sum_{j=1}^{N_1} \omega_{ij} \mathbf{x}_j \right) - \sum_{j=1}^{N_1} \omega_{ij} \mathbf{x}_j \right] \boldsymbol{\lambda} \right| \end{aligned} \tag{18}$$

This can also be seen as the *externality effect* produced by the evaluated policy: it depends on the weights employed, on the average of the observable confounders considered into **x**, and on the magnitude of the coefficients γ and $\boldsymbol{\beta}_1$. Observe that such bias may be positive as well as negative. Furthermore, by defining:

$$\gamma \boldsymbol{\beta}_1 = \boldsymbol{\lambda} \tag{19}$$

it is also possible to test whether this bias is or is not statistically significant by simply testing the following null-hypothesis:

$$H_0: \ \lambda_1 = \lambda_2 = ... = \lambda_M = 0$$

If this hypothesis is rejected, we cannot exclude that neighbourhood effects are pervasive, thus affecting significantly the estimation of the causal parameters ATEs. Finally, in a similar way, we can also get an estimation of the neighbourhood-bias for ATET and ATENT.

5. Stata implementation via `ntreatreg`

The previous model can be easily estimated by using the author-written Stata routine `ntreatreg`. The syntax of `ntreatreg` is a very common one for a Stata command and takes on this form:

ntreatreg *outcome treatment varlist* **, hetero(***varlist_h***)**
spill(*matrix***) graphic**

where:

outcome: is the *y* of the previous model, i.e. the target variable of the policy considered.

treatment: is the *w* of the previous model, i.e. the binary policy (treatment) indicator.

varlist: is the **x** of the previous model, i.e. the vector of observable unit characteristics.

hetero(*varlist_h***)**: is an optional subset of **x** to allow for observable heterogeneity.

spill(*matrix***)**: is the weighting-matrix **Ω,** to be provided by the user.

graphic: returns a graph of the distribution of ATE(**x**), ATET(**x**) and ATENT(**x**).

In the next two sub-sections we provide two instructional applications of the model presented in this paper and of its Stata implementation: the first one on the effect of housing location on crime; the second one on the effect of education on fertility. Results are also compared with a no-interaction setting.

5.1 A simulation exercise

In order to check the reliability of `ntreatreg` and better understand the statistical setting of the model, we perform a simulation exercise providing the data generating process (DGP) underlying

the model fitted by `ntreatreg`. The Stata code is reported below where, for illustrative purposes, we consider a random treatment:

```
************************************
* 1. Generate the matrix "omega"
************************************
* Generate the matrix "omega"
. clear
. set matsize 1000 , permanently
. set obs 200
. set seed 10101
. gen w=rbinomial(1,0.5)
. gsort - w
. count if w==1
. global N1=r(N)
. global N0=_N-$N1
. mat def M=J(_N,_N,0)
. global N=_N

* Generate a matrix M from a Uniform distribution
forvalues i=1/$N{
forvalues j=1/$N1{
mat M[`i',`j']=runiform()
}
}

* Generate a vector SUM containing the column sum of M
mat def SUM=J(_N,1,0)
forvalues i=1/$N{
forvalues j=1/$N1{
mat SUM[`i',1] = SUM[`i',1] + M[`i',`j']
}
}

* Generate the matrix omega as defined in figure #
forvalues i=1/$N{
forvalues j=1/$N1{
mat M[`i',`j']=M[`i',`j']/SUM[`i',1]
}
}
mat omega=M

*************************************************
* 2. Define the model's data generating process (DGP)
*************************************************
* Declare a series of parameters
scalar mu1=2
scalar b11=5
scalar b12=3
scalar e1=rnormal()
scalar mu0=5
scalar b01=7
scalar b02=1
scalar e0=rnormal()
gen x1=rnormal()
gen x2=rnormal()
scalar gamma=0.8

* Sort the treatment so to have the "ones" first
gsort - w
```

```
* Generate "y1"
gen y1 = mu1 + x1*b11 + x2*b12 + e1
gen y1_obs=w*y1
mkmat y1_obs , mat(y1_obs)

* Generate "s"
mat s = omega*y1_obs
mat list s
svmat s

* Generate "y0" and finally "y"
gen y0 = mu0 + x1*b01 + x2*b02 + gamma*s1 + e0
gen y = y0 + w*(y1-y0)

* Generate the treatment effect "te"
gen te=y1-y0
sum te

* Put the ATE into a scalar
scalar ATE=r(mean)
di ATE

*************************************
* 3. Estimate the model using ntreatreg
*************************************
* y: dependent variable
* w: treatment
* x: [x1; x2] are the covariates
* Matrix of spillovers: OMEGA

* Estimate the model using "NTREATREG" ///
set more off
xi: ntreatreg y w x1 x2  , ///
hetero(x1 x2) spill(omega) graphic
scalar ate_neigh = _b[w]  // put ATE into a scalar
di ate_neigh

* END OF THE SIMULATION
```

Previous Stata code: (i) starts by providing the matrix **Ω**; (ii) form the model DGP as defined in (7); (iii) estimate the model by `ntreatreg` using the DGP simulated data.

By running this code, we get a value of ATE as predicted by `ntreatreg` equal to -1.553163, which is very close to the DGP value of the ATE, that is -1.5643281. We can run many simulations getting a similar result. This implies that `ntreatreg` correctly estimates the model as defined in (7).

6. Conclusion

This paper has presented a possible solution to incorporate *externality* (or *neighbourhood*) *effects* within the traditional Rubin's potential outcome model under conditional mean independence. As such, it generalizes the traditional parametric models of program evaluation when SUTVA is relaxed. As by-product, this work has also put forward `ntreatreg`, a Stata routine for estimating

Average Treatments Effects (ATEs) when social interactions are present. In order to check the reliability of `ntreatreg`, we perform a simulation experiment providing the data generating process (DGP) underlying the model fitted by `ntreatreg`. We show that `ntreatreg` correctly estimates the model as defined in (7).

Of course, this approach presents also some limitations, and in what follows we list some of its potential developments. Indeed, the model might be improved by:

- allowing also for treated units to be affected by other treated units' outcome;
- extending the model to "multiple" or "continuous" treatment, when treatment may be multi-valued or fractional for instance, by still holding CMI;
- identifying ATEs with neighbourhood interactions when *w* may be endogenous (i.e., relaxing CMI) by implementing GMM-IV estimation;
- trying to go beyond the potential outcomes' parametric form, by relying on a semi-parametric specification;

Finally, an interesting issue deserving further inquiry regards the assumption of exogeneity concerning the weighting matrix Ω. Indeed, a challenging question might be: what happens if individuals strategically modify their weighting weights to better profit of others' treatment outcome? It is clear that weights do become endogenous, thus yielding severe identification problems for previous causal effects. Future studies should tackle situations in which this possibility may occur.

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Appendix A

In this appendix, we show how to obtain the formulas of ATE and ATE(**x**) set out in (12) and (13). Then, we show how regression (14) can be obtained and, finally, we prove that Assumption 1 is sufficient for consistently estimating the parameters of regression (14) by OLS.

A1. *Formula of* ATE *with neighbourhood interactions.*

Given assumptions 2 and 3, and the implied equations in (7), we get that:

$$y_{1i} = \mu_1 + \mathbf{x}_i\boldsymbol{\beta}_1 + e_{1i}$$
$$y_{0i} = \mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma s_i + e_{0i}$$
$$s_i = \sum_{j=1}^{N_1}\omega_{ij}y_{1j}$$

$$\text{ATE} = \text{E}(y_{1i} - y_{0i}) = \text{E}\left[\left(\mu_1 + \mathbf{x}_i\boldsymbol{\beta}_1 + e_{1i}\right) - \left(\mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma\left[\mu_1 + \left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\boldsymbol{\beta}_1 + \sum_{j=1}^{N_1}\omega_{ij}e_{1j}\right] + e_{0i}\right)\right] =$$

$$= \text{E}\left[\mu_1 + \mathbf{x}_i\boldsymbol{\beta}_1 + e_{1i} - \left(\mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma\mu_1 + \gamma\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\boldsymbol{\beta}_1 + \gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} + e_{0i}\right)\right] =$$

$$= \text{E}\left[\mu_1 + \mathbf{x}_i\boldsymbol{\beta}_1 + e_{1i} - \mu_0 - \mathbf{x}_i\boldsymbol{\beta}_0 - \gamma\mu_1 - \gamma\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\boldsymbol{\beta}_1 - \gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} - e_{0i}\right] =$$

$$= \text{E}\left[\mu_1 - \gamma\mu_1 - \mu_0 + \mathbf{x}_i\boldsymbol{\beta}_1 - \mathbf{x}_i\boldsymbol{\beta}_0 - \gamma\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\boldsymbol{\beta}_1 - \gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} + e_{1i} - e_{0i}\right] =$$

$$= \text{E}\left[\mu_1(1-\gamma) - \mu_0 + \mathbf{x}_i(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) - \left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\gamma\boldsymbol{\beta}_1 - \gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} + e_{1i} - e_{0i}\right] =$$

$$= \text{E}\left[\mu_1(1-\gamma) - \mu_0 + \mathbf{x}_i\boldsymbol{\delta} - \left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\gamma\boldsymbol{\beta}_1 - \gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} + e_{1i} - e_{0i}\right] =$$

$$= \mu + \text{E}\left[\mathbf{x}_i\boldsymbol{\delta} - \left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\gamma\boldsymbol{\beta}_1 - e_i\right] = \mu + \text{E}(\mathbf{x}_i)\boldsymbol{\delta} - \gamma\text{E}\left(\underbrace{\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j}_{\mathbf{v}_i}\right)\boldsymbol{\beta}_1$$

This implies that $\text{ATE} = \text{E}(y_{1i} - y_{0i}) = \mu + \text{E}(\mathbf{x}_i)\boldsymbol{\delta} - \gamma\text{E}(\mathbf{v}_i)\boldsymbol{\beta}_1$ whose sample equivalent is:

$$\text{ATE} = \hat{\mu} + \frac{1}{N}\left(\sum_{i=1}^{N}\mathbf{x}_i\right)\hat{\boldsymbol{\delta}} - \hat{\gamma}\frac{1}{N}\left(\sum_{i=1}^{N}\mathbf{v}_i\right)\hat{\boldsymbol{\beta}}_1 = \mu + \frac{1}{N}\left(\sum_{i=1}^{N}\mathbf{x}_i\right)\hat{\boldsymbol{\delta}} - \hat{\gamma}\frac{1}{N}\left(\sum_{i=1}^{N}\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\right)\hat{\boldsymbol{\beta}}_1$$

where $\mu = \mu_1(1-\gamma) - \mu_0$, and $\boldsymbol{\delta} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_0$. ■

As an example, consider the case in which N=4, and N_1=N_0=2. Suppose that the matrix $\boldsymbol{\Omega}$ is organized as follows:

	T	C
T	$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}$	$\begin{bmatrix} \omega_{13} & \omega_{14} \\ \omega_{23} & \omega_{24} \end{bmatrix}$
C	$\begin{bmatrix} \omega_{31} & \omega_{32} \\ \omega_{41} & \omega_{42} \end{bmatrix}$	$\begin{bmatrix} \omega_{33} & \omega_{34} \\ \omega_{34} & \omega_{44} \end{bmatrix}$

Suppose to have just one confounder x. In this case, we have:

$$\text{ATE} = \hat{\mu} + \frac{1}{4}\left(\sum_{i=1}^{4} x_i\right)\hat{\delta} - \hat{\gamma}\cdot\hat{\beta}_1\frac{1}{4}\left(\sum_{i=1}^{4}\left(\sum_{j=1}^{2}\omega_{ij}x_j\right)\right) = \hat{\mu} + \frac{1}{4}\left(\sum_{i=1}^{4} x_i\right)\hat{\delta} - \hat{\gamma}\cdot\hat{\beta}_1\frac{1}{4}\left(\sum_{i=1}^{4}\left[\omega_{i1}x_1 + \omega_{i2}x_2\right]\right) =$$

$$\hat{\mu} + \bar{x}\hat{\delta} - \hat{\gamma}\cdot\hat{\beta}_1\frac{1}{4}\left(\sum_{i=1}^{4}\underbrace{\left[\omega_{i1}x_1 + \omega_{i2}x_2\right]}_{v_i}\right) = \hat{\mu} + \bar{x}\hat{\delta} - \hat{\gamma}\cdot\hat{\beta}_1\bar{v}$$

Observe that:

$$\begin{aligned} v_1 &= \omega_{11}x_1 + \omega_{12}x_2 \\ v_1 &= \omega_{21}x_1 + \omega_{22}x_2 \\ v_3 &= \omega_{31}x_1 + \omega_{32}x_2 \\ v_4 &= \omega_{41}x_1 + \omega_{42}x_2 \end{aligned}$$

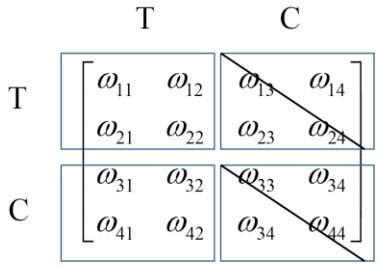
implying:

$$\text{ATE} = \hat{\mu} + \bar{x}\hat{\delta} - \hat{\gamma}\cdot\hat{\beta}_1\frac{1}{4}\left(\sum_{i=1}^{4}\underbrace{\left[\omega_{i1}x_1 + \omega_{i2}x_2\right]}_{v_i}\right) = \hat{\mu} + \bar{x}\hat{\delta} - \hat{\gamma}\cdot\hat{\beta}_1\left[\bar{\omega}_{\cdot 1}x_1 + \bar{\omega}_{\cdot 2}x_2\right]$$

where:

$$\bar{\omega}_{\cdot 1} = \frac{1}{4}\sum_{i=1}^{4}\omega_{i1} \quad \text{and} \quad \bar{\omega}_{\cdot 2} = \frac{1}{4}\sum_{i=1}^{4}\omega_{i2}$$

This means that, by assuming that the externality effect only comes from treated to untreated units thus excluding other types of feedbacks, is equivalent to consider *only* the first two columns of **Ω** in the calculation of the externality component, those refereeing to the treated units, i.e.:



where no use of the two columns referring to the control group occurs.

A2. *Formula of* ATE($\mathbf{x}_i$) *with neighbourhood interactions.*

Given assumptions 2 and 3, and the result in A1, we get:

$$\mathrm{ATE}(\mathbf{x}_i) = \mathrm{E}(y_{1i} - y_{0i} \mid \mathbf{x}_i) = \mu + \mathrm{E}\left[\mathbf{x}_i\boldsymbol{\delta} - \left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\gamma\boldsymbol{\beta}_1 - e_i \mid \mathbf{x}_i\right] = \mu + \mathbf{x}_i\boldsymbol{\delta} - \left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\gamma\boldsymbol{\beta}_1 +$$

$$+[\bar{\mathbf{x}}\boldsymbol{\delta} - \bar{\mathbf{x}}\boldsymbol{\delta}] + \left[\mathrm{E}\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\gamma\boldsymbol{\beta}_1 - \mathrm{E}\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\gamma\boldsymbol{\beta}_1\right] =$$

$$\left(\mu + \bar{\mathbf{x}}\boldsymbol{\delta} - \mathrm{E}\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\gamma\boldsymbol{\beta}_1\right) + (\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + \left[\mathrm{E}\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right) - \left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\right]\gamma\boldsymbol{\beta}_1 =$$

$$\mathrm{ATE} + (\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + (\bar{\mathbf{v}} - \mathbf{v}_i)\boldsymbol{\lambda}$$

. where $\boldsymbol{\lambda}=\gamma\boldsymbol{\beta}_1$.

■

A3. *Obtaining regression (14).*

By substitution of the potential outcome as in (7) into the potential outcome model, we get that:

$$y_i = \left(\mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma\sum_{j=1}^{N_1}\omega_{ij}y_{1j} + e_{0i}\right) + w\left[\left(\mu_1 + \mathbf{x}_i\boldsymbol{\beta}_1 + e_{1i}\right) - \left(\mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma\sum_{j=1}^{N_1}\omega_{ij}y_{1j} + e_{0i}\right)\right] =$$

$$= \left(\mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma\sum_{j=1}^{N_1}\omega_{ij}y_{1j} + e_{0i} \right) + w_i(\mu_1 - \mu_0) + w_i\mathbf{x}_i(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) + w_i(e_{1i} - e_{0i}) - w_i\gamma\sum_{j=1}^{N_1}\omega_{ij}y_{1j} =$$

$$= \mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma\mu_1 + \left(\gamma\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j \right)\boldsymbol{\beta}_1 + \gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} + e_{0i} + w_i(\mu_1 - \mu_0) + w_i\mathbf{x}_i(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) + w_i(e_{1i} - e_{0i}) -$$

$$-w_i\gamma\mu_1 - w_i\gamma\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\boldsymbol{\beta}_1 - w_i\gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} =$$

$$= \mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma\mu_1 + \underbrace{[\gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} + e_{0i} + w_i(e_{1i} - e_{0i}) - w_i\gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j}]}_{e_i} + w_i(\mu_1 - \mu_0) + w_i\mathbf{x}_i(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) -$$

$$-w_i\gamma\mu_1 - w_i\gamma\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j \right)\boldsymbol{\beta}_1 =$$

$$= \mu_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + \gamma\mu_1 + w_i(\mu_1 - \mu_0) + w_i\mathbf{x}_i(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) - w_i\gamma\mu_1 - w_i\gamma\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j \right)\boldsymbol{\beta}_1 + e_i =$$

$$= (\mu_0 + \gamma\mu_1) + w_i(\mu_1 - \mu_0 - \gamma\mu_1) + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i\mathbf{x}_i(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) - w_i\gamma\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j \right)\boldsymbol{\beta}_1 + e_i =$$

$$= (\mu_0 + \gamma\mu_1) + w_i(\mu_1 - \mu_0 - \gamma\mu_1) + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i\mathbf{x}_i\boldsymbol{\delta} - w_i\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j \right)\gamma\boldsymbol{\beta}_1 + e_i =$$

$$= (\mu_0 + \gamma\mu_1) + w_i(\mu_1 - \mu_0 - \gamma\mu_1) + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i\mathbf{x}_i\boldsymbol{\delta} - w_i\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j \right)\gamma\boldsymbol{\beta}_1 + e_i +$$

$$+\left[w_i\bar{\mathbf{x}}_i\boldsymbol{\delta} - w_i\bar{\mathbf{x}}_i\boldsymbol{\delta} \right] + \left[w_i\mathrm{E}\left(\underbrace{\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j}_{\mathbf{v}_i} \right)\gamma\boldsymbol{\beta}_1 - w_i\mathrm{E}\left(\underbrace{\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j}_{\mathbf{v}_i} \right)\gamma\boldsymbol{\beta}_1 \right] =$$

$$= (\mu_0 + \gamma\mu_1) + w_i(\mu + \bar{\mathbf{x}}_i\boldsymbol{\delta} - \mathbf{v}_i\boldsymbol{\lambda}) + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + w_i(\bar{\mathbf{v}} - \mathbf{v}_i)\boldsymbol{\lambda} + e_i$$

Therefore, we can conclude that:

$$y_i = \eta + w_i \cdot \mathrm{ATE} + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + w_i(\bar{\mathbf{v}} - \mathbf{v}_i)\boldsymbol{\lambda} + e_i$$

where: $\mathbf{v}_i = \sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j$, $\bar{\mathbf{v}} = \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j \right)$, $\boldsymbol{\lambda}=\gamma\boldsymbol{\beta}_1$, $\eta = \mu_0 + \gamma\mu_1$, and $\boldsymbol{\delta} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_0$

■

A4. *Ordinary Least Squares (OLS) consistency*.

Under Assumption 1 (CMI), the parameters of regression (14) can be consistently estimated by OLS. Indeed, it is immediate to see that the mean of e_i conditional on (w_i; $\mathbf{x}_i$) is equal to zero:

$$\mathrm{E}\left[\gamma \sum_{j=1}^{N_1} \omega_{ij} e_{1j} + e_{0i} + w_i (e_{1i} - e_{0i}) - w_i \gamma \sum_{j=1}^{N_1} \omega_{ij} e_{1j} \,\middle|\, w_i, \mathbf{x}_i \right] =$$

$$\mathrm{E}\left[\gamma \sum_{j=1}^{N_1} \omega_{ij} e_{1j} \,\middle|\, w_i, \mathbf{x}_i \right] + \mathrm{E}\left[e_{0i} \,\middle|\, w_i, \mathbf{x}_i \right] + \mathrm{E}\left[w_i (e_{1i} - e_{0i}) \,\middle|\, w_i, \mathbf{x}_i \right] - \mathrm{E}\left[w_i \gamma \sum_{j=1}^{N_1} \omega_{ij} e_{1j} \,\middle|\, w_i, \mathbf{x}_i \right] =$$

$$\gamma \sum_{j=1}^{N_1} \omega_{ij} \mathrm{E}\left[e_{1j} \,\middle|\, \mathbf{x}_i \right] + \mathrm{E}\left[e_{0i} \,\middle|\, \mathbf{x}_i \right] + w_i \mathrm{E}\left[(e_{1i} - e_{0i}) \,\middle|\, \mathbf{x}_i \right] - w_i \gamma \sum_{j=1}^{N_1} \omega_{ij} \mathrm{E}\left[e_{1j} \,\middle|\, \mathbf{x}_i \right] = 0$$

where $\eta = \mu_0 + \gamma \mu_1$. ■