Measuring associations and evaluating forecasts of categorical and discrete variables

Jochem Huismans (University of Amsterdam)

Jan Willem Nijenhuis (University of Twente)

Andrei Sirchenko (Maastricht University)

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To express anything important in mere figures is so plainly impossible that there must be endless scope for well-paid advice on how to do it.

— K. A. C. Manderville, The Undoing of Lamia Gurdleneck

Input:

- (i) the values of two categorical (or discrete) variables or
- (ii) the observed values of a categorical (discrete) variable and the predicted probabilities of each category

Stata command classify

Measuring associations and evaluating forecasts

Output:

- contingency table
- general measures of overall association and correlation (and also diagnostic scores of the accuracy of probabilistic forecast)
- class-specific measures for each class as well as their simple and weighted averages

Literature on measures of association

is poorly integrated across different fields

- a wide variety of scalar statistics have been developed and used in different fields
- a similarly wide variety of nomenclature has appeared in relation to these statistics
- some of these measures have been reinvented, duplicated and renamed on multiple occasions in other fields
- confusing terminology is confounded further by different notation

Literature on measures of association

is poorly integrated across different fields

- Cohen kappa coefficient (1960)
- Heidke skill score (1926)
- Doolittle association ratio (1887)
- Galton coefficient (1892)
- Hubert–Arabie adjusted Rand index (Hubert and Arabie 1985)

Diagnostic scores for probabilistic forecasts

- Brier score (half-Brier score, quadratic score, probability score)
- Power score
- Logarithmic score (ignorance score)
- Spherical score
- Pseudospherical score
- Zero-one score
- Ranked probability score (suitable only for ordinal outcomes)

Diagnostic scores for probabilistic forecasts

- Spherical score (Winkler 1967; Winkler and Murphy 1968; Friedman 1983; $[0 \leftarrow 1]$):

$$1 - \frac{1}{n} \sum_{i=1}^{n} \frac{\sum_{k=1}^{K} \delta_{ik} \Pr(y_i = k)}{\sqrt{\sum_{k=1}^{K} [\Pr(y_i = k)]^2}}$$

- Ranked probability score (suitable only for ordinal outcomes; identical to the Brier score for binary outcomes (Epstein 1969; Murphy 1971); $[0 \leftarrow 1]$):

$$\frac{1}{n(K-1)} \sum_{i=1}^{n} \sum_{k=1}^{K-1} \left(\sum_{j=1}^{k} \Pr(y_i = j) - \sum_{j=1}^{k} \delta_{ij} \right)^2$$

	$x=1$	$x=2$	...	$x=K$	Total
$y=1$	n_{11}	n_{12}	...	n_{1K}	$n_{1+}=\sum n_{1j}$
$y=2$	n_{21}	n_{22}	...	n_{2K}	$n_{2+}=\sum n_{2j}$
...	...	...	...	...	...
$y=K$	n_{K1}	n_{K2}	...	n_{KK}	$n_{K+}=\sum n_{Kj}$
Total	$n_{+1}=\sum n_{i1}$	$n_{+2}=\sum n_{i2}$	...	$n_{+K}=\sum n_{iK}$	$n=\sum\sum n_{ij}$

Measures of association

General and class-specific measures

- General measures of the overall performance — they include explicitly all concordant (matched) pairs n_{kk}

- Class-specific measures computed for each class k — they include explicitly only one matched pair n_{kk}.

Measures of association

Symmetric measures: two types of symmetry

- A measure is transpose symmetric if it treats both variables equivalently, and so it is invariant to relabelling of them — it remains unchanged if the row variable and column variable are interchanged (if any n_{ij} and $n_{ji}, i \neq j$ are swapped).
- A measure is complement symmetric if it treats all categories equivalently, and so it is invariant to relabelling of them — it remain unchanged if any two columns and the corresponding two rows are swapped).

Measures of association

Asymmetric measures

- Asymmetric measures have typically been developed in the binary context for rare and/or extreme events.
- So the occurrences get larger weights in their definitions than the nonoccurrences.
- The true positives and true negatives are not treated equally, and the false positives and false negatives are not treated equally either: the negative matches do not mean necessarily any similarity between two objects, and type 2 errors are often more serious than type 1 errors.
- To measure association between two variables x and y, it does matter which variable is x and which is y.

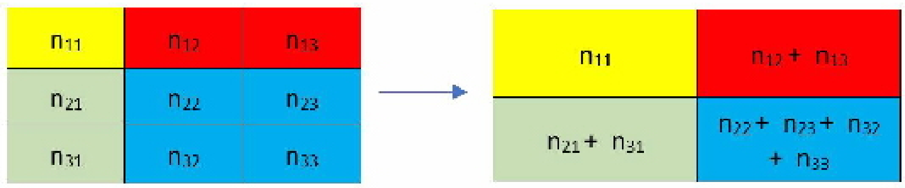
- Goodman-Kruskal λ_r coefficient (adjusted count R^2, Brennan and Prediger κ_b, Appleman (Goodman and Kruskal 1954; Brennan and Prediger 1981); $[0 \longrightarrow 1]$):

$$\frac{\sum_{k=1}^{K} n_{kk} - \max_{j=1}^{K} n_{+j}}{n - \max_{j=1}^{K} n_{+j}}$$

- Goodman-Kruskal symmetrical λ_r coefficient (Goodman and Kruskal 1954; $[-1 \longrightarrow 1]$):

$$\frac{2\sum_{k=1}^{K} n_{kk} - \max_{i=1}^{K} n_{i+} - \max_{j=1}^{K} n_{+j}}{2n - \max_{i=1}^{K} n_{i+} - \max_{j=1}^{K} n_{+j}}$$

- The class-specific measures include only one concordant pair n_{kk}, and designed for binary outcomes, mostly for a positive category.
- To compute them, any $K \times K$ contingency table can be converted (using a so-called one-vs-all binarisation strategy) to a series of K 2×2 contingency tables:



Measures of association

Class-specific contingency table

	$x = k$	$x \neq k$	Total
$y = k$	$n_{11}^{(k)} = n_{kk}$	$n_{12}^{(k)} = n_{k+} - n_{kk}$	$n_{1+}^{(k)} = n_{k+}$
$y \neq k$	$n_{21}^{(k)} = n_{+k} - n_{kk}$	$n_{22}^{(k)} = n - n_{k+} - n_{+k} + n_{kk}$	$n_{2+}^{(k)} = n - n_{k+}$
Total	$n_{+1}^{(k)} = n_{+k}$	$n_{+2}^{(k)} = n - n_{+k}$	n

- The `classify` command also computes the simple arithmetic and weighted arithmetic averages of all class-specific measures as:

$$
\begin{aligned}
Measure_{macro} &= \frac{1}{K}\sum_{k=1}^{K} Measure_k \\
Measure_{weighted} &= \sum_{k=1}^{K} Measure_k \frac{n_{+k}}{n}
\end{aligned}
$$

- The macro-averaged measures calculate unweighted (arithmetic) mean of class-specific coefficients.
- The weighted-averaged measures take a weighted mean. The weights for each class are the total number of observations of that class.

Measures of association

Class-specific measures

- Precision (positive predictive value, confidence, success ratio, post agreement, frequency of hits, correct alarm ratio (Grossmann 1898 cited in Muller 1944; Dice 1945; Wallace 1983); for negative category: negative predicted value, inverse precision, true negative accuracy; $R : [0 \longrightarrow 1]$):

$$\frac{n_{11_{(k)}}}{n_{1+_{(k)}}}$$

- F_1-score (harmonic mean of precision and recall, percent positive agreement, Gleason, Sørensen–Dice, Sørensen, Dice, Czekanowski, Nei-Li, Bray-Curtis, Upholt F, Burt, Lance–Williams, Pirlot, Tversky, Gower-Legendre T (Czekanowski 1913, 1932; Gleason 1920; Dice 1945; Sørensen 1948; Bray 1956; Bray and Curtis 1957; Lance and Williams 1966; Upholt 1977; Tversky 1977; Nei and Li 1979); $R : [0 \longrightarrow 1]$):

$$\frac{2n_{11_{(k)}}}{n_{1+_{(k)}} + n_{+1_{(k)}}}$$

Classify and be happy

" . . . there is no absolutely general measure of the degree of dependence. Every attempt to measure a conception like this by a single number must necessarily contain a certain amount of arbitrariness and suffer from certain inconveniences."

— *Cramér (1924)*