Complex Reduced Rank Models for Seasonally Cointegrated Time Series

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Abstract: This paper introduces a new representation for seasonally cointegrated variables, namely the complex error correction model, which allows statistical inference to be performed by reduced rank regression. The suggested estimators and tests statistics are asymptotically equivalent to their maximum likelihood counterparts. Tables are provided for both asymptotic and finite sample critical values, and an empirical example is presented to illustrate the concepts and methods.

Keywords: Seasonal cointegration; reduced rank regression; error correction model.

JEL classification: C32

1. Introduction

Following the seminal contribution by Hylleberg et al. (1990), there has recently been a considerable interest in the seasonal cointegration analysis. The motivation for this line of research is twofold. First, a good deal of empirical evidence suggests that many macroeconomic time series are well characterised by the presence of unit roots both at the zero and seasonal frequencies (e.g. Hylleberg et al., 1993; Canova and Hansen, 1993). Secondly, estimation of the seasonal version of the error-correction model [ECM] is preliminary to other econometric analyses such as forecasting (Kunst, 1993), testing for the rational expectations hypothesis (Ermini and Chang, 1996), and the common trend-common cycle decomposition (Cubadda, 1999).

Remarkably, cointegration relations at frequencies other than zero and π are generally dynamic. This complicates the statistical analysis since polynomial cointegration vectors are entailed (see, e.g., Engle et al., 1993; Ahn and Reinsel, 1994). Limiting our discussion to maximum likelihood [ML] procedures, Lee (1992) developed inference for the particular case of synchronous cointegration at frequency $\pi/2$ whereas Johansen and Schaumburg [henceforth, JS] (1998) completed the analysis for the general case of dynamic cointegration at the complex root frequencies. Unfortunately, the JS method requires a rather involved iterative procedure to compute estimates of parameters of interest. This paper shows that an estimator and a test statistic which are asymptotically equivalent to those proposed by JS can be obtained by reduced rank regression [RRR] between complex-valued data. The basic trick is the introduction of a complex ECM, which greatly simplifies testing and estimation of polynomial cointegration vectors.

This paper is organised as follows: Section 2 introduces the complex ECM. Section 3 deals with statistical inference. In Section 4 the analysis is applied to Italian data of consumption, investment and output. Section 5 presents conclusions.

2. The complex error correction model

Let X_t be n-vector time series such that

$$\Pi(L)X_t = \Phi D_t + \varepsilon_t \tag{2.1}$$

where $\Pi(L)$ is a polynomial matrix such that $\Pi(0) = I_n$, $\Pi_j = 0$ for $j > p$, ε_t are i.i.d. $\mathrm{N}_n(0,\Omega)$, and D_t is a deterministic kernel which may contain a constant, a linear trend, and various trigonometric functions of time. We assume that the initial values are fixed and that the roots of the determinant $|\Pi(z)|$ are on or outside the unit circle.

Let $z_1,...,z_M$ be the solutions of the equation $|\Pi(z)| = 0$ such that $z_m = \exp(\iota\omega_m)$, $\iota = \sqrt{-1}$, and $\omega_m \in [0,\pi]$, $m = 1,2,...,M$.[1] Hence, we can write

$$\Pi(z_m) = -\alpha_m \beta_m^*$$

where α_m and β_m are complex-valued $(n \times r_m)$–matrices with rank equal to r_m, and β_m^* denotes the conjugate transpose of β_m.

Let us further assume that each matrix $\alpha_{m\perp}^* \dot{\Pi}(z_m)\beta_{m\perp}$ has rank equal to $(n - r_m)$, where for any complex full-rank $(n \times r)$–matrix C we denote by $C_\perp$ a complex full-rank matrix of dimension $n \times (n - r)$ such that $C^* C_\perp = 0$, and $\dot{\Pi}(z_m)$ denotes the derivative of $\Pi(z)$ at $z = z_m$.

Based on Cubadda (1995) and JS, we know that the processes $\nabla_m(L)X_t^{(m)}$ and $\beta_m(L)' X_t^{(m)}$ have no unit roots, where

$$\nabla_m(L) = \begin{cases} (1 - z_m^{-1}L), \text{ if } \omega_m = 0 \text{ or } \omega_m = \pi \\ (1 - 2\cos(\omega_m)L + L^2), \text{ if } \omega_m \in (0,\pi) \end{cases},$$

$$X_t^{(m)} = \left(\prod_{j \neq m}^{M} \nabla_j(L) \right) X_t, \text{ and}$$

[1] Note that, since $\Pi(L)$ has real coefficient matrices, also $z_1^{-1},...,z_m^{-1}$ must be roots of $|\Pi(z)|$.

$$\beta_m(L) = \begin{cases} \beta_m, \text{ if } \omega_m = 0 \text{ or } \omega_m = \pi \\ \mathrm{Re}\{\beta_m\} - \mathrm{Im}\{\beta_m\}\left(1/\tan(\omega_m) - L/\sin(\omega_m)\right), \text{ if } \omega_m \in (0,\pi) \end{cases} \tag{2.2}$$

If we expand the polynomial matrix $\Pi(L)$ around 0 and all the unit roots of $|\Pi(z)|$, we get the following representation of series X_t:

$$X_t^{(0)} = \Phi D_t + \sum_{m=1}^{M} A_m(L)\beta_m(L)' X_{t-1}^{(m)} + \Psi(L) X_{t-1}^{(0)} + \varepsilon_t \tag{2.3}$$

where $X_t^{(0)} = \left(\prod_{j=1}^{M} \nabla_j(L)\right) X_t$, $A_m(L)$ is related to $\alpha_m \left(z_m \prod_{j \neq m}^{M} \nabla_j(z_m) \right)^{-1}$ as $\beta_m(L)$ to β_m in equation (2.2), and $\Psi(L)$ is a polynomial matrix.

Let us now consider the expansion of $\Pi(L)$ around 0 and $z_1,\ldots,z_M$. In this case, the VAR model (2.1) can be rewritten as follows:

$$Y_t^{(0)} = \Phi D_t + \sum_{m=1}^{M} \alpha_m \beta_m^* Y_{t-1}^{(m)} + \Gamma(L) Y_{t-1}^{(0)} + \varepsilon_t \tag{2.4}$$

where $Y_t^{(0)} = \left(\prod_{j=1}^{M} \Delta_j(L)\right) X_t$, $\Delta_m(L) = \left(1 - z_m^{-1} L\right)$, $Y_t^{(m)} = \left(z_m \prod_{j \neq m}^{M} \Delta_j(z_m) \right)^{-1} \left(\prod_{j \neq m}^{M} \Delta_j(L) \right) X_t$, and $\Gamma(L)$ is a polynomial matrix.

Equation (2.3) resembles the usual ECM where the stationary variables $X_t^{(0)}$ react to the equilibrium errors $\beta_m(L)' X_{t-1}^{(m)}$ through the (polynomial) adjustment matrices $A_m(L)$. Interpretation of equation (2.4) is less neat. However, we note from JS that complex processes $Y_t^{(0)}$ and $\beta_m^* Y_t^{(m)}$ do not possess the unit roots $z_1,\ldots,z_M$. Henceforth, equation (2.4) will be called the complex ECM of series X_t.

3. Statistical inference

The statistical analysis of the complex ECM is based on partial canonical correlations between $Y_t^{(0)}$ and $Y_{t-1}^{(m)}$. In fact, when focusing on cointegration at a given frequency we can safely ignore reduced rank restrictions at other frequencies since processes with different unit roots are asymptotically uncorrelated. Moreover, the same argument implies that the tests statistics and estimators given later are asymptotically equivalent to their maximum likelihood counterparts, see Lee (1992) and JS.

The suggested inferential procedure goes as follows. Regress $Y_t^{(0)}$ and $Y_{t-1}^{(m)}$ on other regressors in equation (2.4) to form residuals $R_t^{(0)}$ and $R_t^{(m)}$ respectively. Then, perform canonical correlation analysis between $R_t^{(0)}$ and $R_t^{(m)}$ by computing the product moment matrices

$$S_{i,j} = T^{-1}\sum_{t=1}^{T} R_t^{(i)} R_t^{(j)*}, \text{ for } i, j = 0, M$$

and solving the eigenvalue problem

$$\left|\lambda S_{m,m} - S_{m,0} S_{0,0}^{-1} S_{0,m}\right| = 0$$

for eigenvalues $\hat{\lambda}_1 > \ldots > \hat{\lambda}_n$ and eigenvectors $\hat{V} = (\hat{v}_1, \ldots, \hat{v}_n)$ normalised such that $\hat{V}^* S_{m,m} \hat{V} = I_n$.[2] Hence, the RRR estimator is $\hat{\beta}_m = (\hat{v}_1, \ldots, \hat{v}_{r_m})$ and a test statistic for the hypothesis $\Pi(z_m) = -\alpha_m \beta_m^*$ is

$$TR = -2T \sum_{i=r_m+1}^{n} \ln(1 - \hat{\lambda}_i)$$

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[2] See Brillinger (1981) for details on regression and canonical correlation analysis between complex variables.

Finally, the estimate of the polynomial vector $\hat{\beta}_m(L)$ is found by inserting $\hat{\beta}_m$ in equation (2.2).

The limit distribution of the test statistic TR for the complex root case is given in the following theorem.

Theorem 2.1. In the model (2.4) and for the case $\omega_m \in (0,\pi)$ we have

$$TR \Rightarrow \operatorname{tr}\left\{\int_0^1 (\mathrm{d}\,B_c)F_c^*\left[\int_0^1 F_c F_c^*\,\mathrm{d}\,u\right]^{-1}\int_0^1 F_c(\mathrm{d}B_c)^*\right\} \tag{3.1}$$

where $\Rightarrow$ denotes weak convergence in distribution, $\operatorname{tr}\{\cdot\}$ denotes the trace of the matrix in argument, $B_c(u) = B_r(u) + \imath B_i(u)$, $B_r(u)$ and $B_i(u)$ are independent standard Brownian motions of dimension $(n - r_m)$, and $F_c(u) = B_c(u)$ if D_t does not include the trigonometric functions $[\cos(\omega_m t), \sin(\omega_m t)]$, $F_c(u) = B_c(u) - \int_0^1 B_c(s)\,\mathrm{d}\,s$ otherwise.

Proof. The proof is only sketched since it involves simple applications of earlier results in this area of research. For the moment, let us assume that X_t is a complex-valued process, $\Delta_m(L)X_t$ is a real-valued stationary process, $\beta_m^* X_t$ is stationary, and there are no deterministic terms in the model, i.e. $\Phi = 0$. We can easily deduce from Corollary 7 in JS that

$$T^{-1}S_{m,m} \Rightarrow \frac{1}{2}C_m\left(\int_0^1 W_c W_c^*\,\mathrm{d}\,u\right)C_m^*$$

$$S_{m,0}\alpha_{m\perp} \Rightarrow \frac{1}{2}C_m\int_0^1 W_c(\mathrm{d}\,W_c)^*\alpha_{m\perp}$$

$$S_{0,0} \to \Sigma_{0,0}, \quad \beta_m^* S_{m,m}\beta_m \to \Sigma_{\beta,\beta}, \quad \beta_m^* S_{m,0} \to \Sigma_{\beta,0}$$

where $C_m = -z_m^{-1}\beta_{m\perp}\left(\alpha_{m\perp}^{*}\,\dot{\Pi}(z_m)\beta_{m\perp}\right)^{-1}\alpha_{m\perp}^{*}$, $W_c(u) = W_r(u) + \imath W_i(u)$, $W_r(u)$ and $W_i(u)$ are independent n-dimensional Brownian motions with variance matrix Ω, $\to$ denotes almost sure convergence, and

$$\operatorname{Var}\left\{\begin{matrix} Y_t^{(0)} \\ \beta_m^{*} Y_{t-1}^{(m)} \end{matrix}\left|(Y_{t-1}^{(0)},\dots,Y_{t-p+M}^{(0)}, Y_{t-1}^{(j)},\ j \neq m = 1,\dots,M)\right.\right\} = \begin{bmatrix} \Sigma_{0,0} & \Sigma_{0,\beta} \\ \Sigma_{\beta,0} & \Sigma_{\beta,\beta} \end{bmatrix}.$$

Based on the above results, we can proceed analogously to Johansen (1988) for the real root case in order to prove that $T(\hat{\lambda}_{r_m+1},\dots,\hat{\lambda}_n)$ converge in distribution to the ordered solutions of the equation

$$\left|\rho\int_0^1 B_c B_c^{*}\,\mathrm{d}u - \frac{1}{2}\int_0^1 B_c(\mathrm{d}B_c)^{*}\int_0^1(\mathrm{d}B_c)B_c^{*}\right| = 0 \tag{3.2}$$

Notice that when the trigonometric functions $[\cos(\omega_m t), \sin(\omega_m t)]$ are included in the model, the complex-valued Brownian motion $B_c(u)$ in equation (3.2) must be replaced with its demeaned counterpart, i.e. $B_c(u) - \int_0^1 B_c(s)\,\mathrm{d}s$. Moreover, the limit distribution (3.2) is unaffected by the inclusion of deterministic terms having no spectral mass at frequency ω_m, see Lee and Siklos (1995) and JS. Finally, by writing $TR = 2T\sum_{i=r_m+1}^{n}\hat{\lambda}_i + \mathrm{o}_p(1)$, we complete the proof of the theorem for the particular case under consideration.

Regarding the general case of a real-valued process X_t with various unit roots, we know that the asymptotic distribution (3.1) is invariant to the presence of unit roots other than z_m (including z_m^{-1}) since processes being I(1) at different frequencies are asymptotically independent, see again Lee (1992) and JS. Notice that the distribution (3.1) is an equivalent formulation of the limit distribution of the LR statistic for the model with no deterministic term given in JS. However, these distributions do not coincide for the model with seasonal dummies due to the different treatment of the periodic term, compare Lee and Siklos (1995) with Franses and Kunst (1999). ■

Quantiles of the limit distribution (3.1) with $F_c(u) = B_c(u) - \int_0^1 B_c(s)\,\mathrm{d}\,s$ are reported in Table 1. This distribution is simulated by approximating the process B_c with a 400-step random walk where the increments are replications of a $(n - r_m)$-complex i.i.d. variable η_t such that

$$\left(\mathrm{Re}\{\eta_t\}', \mathrm{Im}\{\eta_t\}'\right)' \sim \mathrm{N}_{2(n-r_m)}\left(0, I_{2(n-r_m)}\right) \quad (n - r_m = 1, 2, 3),$$

and the statistic is computed $100\,000$ times.

Tables 2-5 report finite sample critical values of the LR statistic for cointegration at frequency $\pi/2$ with quarterly data. In particular, the following data-generating process is considered

$$\Delta_4 X_t = \varepsilon_t \quad (t = 1,2,\ldots,T)$$

for $T = 50,100,150,200,$ where ε_t is i.i.d. $N(0, I_n)$ for $n = 1,2,3,$ and initial values are set to zero. The finite sample quantiles are obtained by $30\,000$ replications using the complex RRR model (2.4) where the deterministic kernel D_t may include a constant, seasonal dummies and a time-trend.

Finally, notice that the asymptotic equivalence of the RRR estimator to the ML estimator implies that $\hat{\beta}_m$ is consistent and its limit distribution is mixed Gaussian, see Johansen (1996). Hence, linear hypothesis on polynomial cointegration vectors can be investigated with asymptotic χ^2 tests.

4. Empirical example: seasonal cointegration in a small macroeconomic system

In order to illustrate the practical value of the concepts and methods previously discussed, we consider Italian quarterly time series on household consumption (c_t), fixed investment (i_t)

and gross domestic production (y_t) in log per-capita form for the period 1973.2 through 1997.1 (data from 1970.1 are taken as starting values). These series are graphed in Figure 1.

The theoretical background is represented by the neoclassical model of seasonal fluctuations proposed by Chatterjee and Ravikumar (1992). A relevant implication of this model is that deviations of c_t, i_t, and y_t from a common deterministic trend can be decomposed in two parts: a deterministic seasonal component that reflects periodic shifts in preferences and technology, and a stochastic transitory component that captures the effects of non-seasonal shocks to the economy. Following King et al. (1988), we know that the common trend becomes stochastic when labour augmenting technology is assumed to follow a random walk with drift rather than a deterministic function of time. Similarly, stochastic seasonality can arise from persistent seasonal variations in productivity and tastes, see Wells (1997). In this case, seasonal cointegration analysis may reveal the number of independent shocks that drive the seasonal fluctuations in the economy.

As a first step of the empirical analysis, a VAR(13) model is selected according to the longest significant lag rule and usual diagnostic tests give no sign of misspecification for this model. Note that seasonal dummies and linear trends are included in the regressions thus rendering preliminary pre-testing for univariate unit-roots unnecessary.

The results of the LR cointegration tests at the zero frequency, reported in Table 6, suggest the existence of a single cointegration vector such that $(1, 0.06, -1.64)'$.[3] Hence, there is no evidence of balanced growth for the Italian economy. From Table 6 we also see that there is evidence of one cointegration relationship at frequency π. The associated eigenvector is $(1, -0.13, -0.15)'$.

Regarding cointegration at the annual frequency, the results of the trace test for polynomial cointegration and the Lee's test for synchronous cointegration are both reported at Table 7. We see that the test based on the complex ECM provides strong evidence in favour of a non contemporaneous cointegration relationship which is not detected by the Lee's test. The RRR estimate of the polynomial cointegration vector is the following

$$(1, -0.12 + 0.05L, -0.27 + 0.58L)'.$$

[3] Notice that an LR test for restricting to zero the coefficient of i_t produces an $\chi^2(1)$ equal to 1.82, which is insignificant at a 10% level.

Further insights on the cointegration properties of variables can be understood by checking the significance of the various error correction terms in the seasonal ECM. From Table 8 we see that the first lag of the annual error correction terms is insignificant. Hence, we can omit this redundant variable and test on the remaining error correction terms. The results, reported in Table 8, confirm the relevance of the cointegration relationships at the different frequencies.

Finally, we can compare the selected specification of the seasonal ECM with a model where the annual cointegration vector is estimated by the Lee's procedure. The test for the former encompassing the latter gives raise to a $F(3,58)$ statistic equal to 1.38, which is insignificant at the 20% level. The test statistic for the reverse encompassing comparison is equal 9.88, which is overwhelmingly significant.

5. Conclusions

This paper considers the complex ECM for seasonally cointegrated time series. It offers a reduced rank estimator of polynomial cointegration vectors and a trace test for determining the cointegration rank at frequencies different from zero and π. The asymptotic distribution theory is discussed and the relevant critical values are computed. The methods are applied to Italian macroeconomic data, and evidence is provided for an annual cointegration relationship which is not detected by the usual Lee's test for synchronous cointegration.

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Table 1

Asymptotic critical values of the TR statistic

Model (2.4) where $[\cos(\omega_m t), \sin(\omega_m t)] \subset D_t$

$(n-r_m)$	50%	75%	80%	85%	90%	95%	97.5%	99%
1	5.4	8.2	9.0	9.9	11.2	13.2	15.1	17.5
2	19.2	23.6	24.7	26.1	28.0	30.9	33.5	36.8
3	40.9	46.8	48.4	50.2	52.7	56.4	59.7	63.6

Table 2

Finite sample $(T = 50)$ critical values of the TR statistic

$(n-r_m)$	(I, SD, Tr)	50%	75%	80%	85%	90%	95%	97.5%	99%
1	(0,0,0)	1.5	3.0	3.5	4.1	4.9	6.4	7.8	9.8
	(1,0,0)	1.5	3.0	3.5	4.1	4.9	6.4	7.8	9.8
	(1,0,1)	1.5	3.0	3.5	4.1	4.9	6.4	7.9	9.8
	(1,1,0)	5.6	8.4	9.2	10.2	11.6	13.8	15.8	18.4
	(1,1,1)	5.6	8.5	9.3	10.3	11.7	13.9	16.1	18.6
2	(0,0,0)	11.8	15.3	16.3	17.5	19.1	21.8	24.2	27.4
	(1,0,0)	11.9	15.5	16.4	17.7	19.3	22.0	24.6	27.8
	(1,0,1)	12.1	15.6	16.6	17.8	19.4	22.2	24.9	28.2
	(1,1,0)	20.6	25.4	26.7	28.4	30.5	33.7	36.9	41.0
	(1,1,1)	20.9	25.8	27.1	28.8	31.0	34.4	37.7	41.6
3	(0,0,0)	31.8	37.5	39.1	41.0	43.4	47.2	50.9	55.8
	(1,0,0)	32.2	38.1	39.7	41.7	44.1	48.2	51.8	56.8
	(1,0,1)	32.7	38.7	40.3	42.3	44.9	48.9	52.5	57.4
	(1,1,0)	46.3	53.6	55.6	58.1	61.3	66.5	71.1	77.5
	(1,1,1)	47.2	54.7	56.6	59.1	62.6	67.9	72.6	78.6

Note: I = constant, SD = seasonal dummies, Tr = deterministic linear trend.

Table 3

Finite sample $(T=100)$ critical values of the TR statistic

$(n-r_m)$	(I,SD,Tr)	50%	75%	80%	85%	90%	95%	97.5%	99%
1	(0,0,0)	1.5	3.0	3.4	4.0	4.8	6.3	7.7	9.5
	(1,0,0)	1.5	3.0	3.4	4.0	4.8	6.3	7.7	9.5
	(1,0,1)	1.5	3.0	3.4	4.0	4.8	6.3	7.7	9.5
	(1,1,0)	5.5	8.3	9.1	10.0	11.2	13.2	15.2	17.6
	(1,1,1)	5.5	8.4	9.1	10.1	11.3	13.3	15.2	17.7
2	(0,0,0)	11.5	14.9	15.8	16.9	18.5	20.9	23.2	26.0
	(1,0,0)	11.6	14.9	15.8	16.9	18.5	21.0	23.1	26.0
	(1,0,1)	11.6	14.9	15.8	17.0	18.6	21.0	23.2	26.1
	(1,1,0)	19.7	24.2	25.4	26.8	28.7	31.7	34.6	38.0
	(1,1,1)	19.9	24.3	25.5	26.9	28.9	31.9	34.8	38.2
3	(0,0,0)	30.1	35.3	36.6	38.3	40.4	43.7	46.7	51.0
	(1,0,0)	30.3	35.4	36.8	38.4	40.5	43.9	47.0	50.9
	(1,0,1)	30.4	35.5	36.9	38.5	40.6	44.0	47.1	51.2
	(1,1,0)	42.7	49.0	50.6	52.6	55.2	59.2	62.7	67.5
	(1,1,1)	42.9	49.2	50.9	52.8	55.4	59.4	62.9	67.9

See note to Table 2 for details.

Table 4

Finite sample $(T = 150)$ critical values of the TR statistic

$(n-r_m)$	(I, SD, Tr)	50%	75%	80%	85%	90%	95%	97.5%	99%
	(0,0,0)	1.5	3.0	3.4	4.0	4.8	6.2	7.6	9.2
	(1,0,0)	1.5	3.0	3.4	4.0	4.8	6.2	7.6	9.3
1	(1,0,1)	1.5	2.9	3.4	4.0	4.8	6.2	7.6	9.3
	(1,1,0)	5.4	8.3	9.1	10.0	11.2	13.2	15.2	17.6
	(1,1,1)	5.5	8.3	9.1	10.1	11.3	13.3	15.2	17.7
	(0,0,0)	11.5	14.8	15.7	16.9	18.4	20.9	23.1	25.9
	(1,0,0)	11.5	14.8	15.7	16.9	18.4	20.9	23.1	26.0
2	(1,0,1)	11.5	14.8	15.7	16.9	18.4	20.9	23.1	26.0
	(1,1,0)	19.6	24.0	25.2	26.6	28.5	31.4	34.3	37.8
	(1,1,1)	19.6	24.1	25.3	26.7	28.6	31.5	34.3	37.8
	(0,0,0)	29.8	35.0	36.3	37.9	40.0	43.2	46.2	50.0
	(1,0,0)	29.9	35.0	36.4	38.0	40.0	43.3	46.2	50.1
3	(1,0,1)	29.9	35.1	36.4	38.0	40.1	43.4	46.3	50.1
	(1,1,0)	42.2	48.1	49.7	51.6	54.2	58.3	61.6	66.1
	(1,1,1)	42.3	48.2	49.9	51.8	54.3	58.4	61.8	66.1

See note to Table 2 for details.

Table 5

Finite sample $(T = 200)$ critical values of the TR statistic

$(n-r_m)$	(I, SD, Tr)	50%	75%	80%	85%	90%	95%	97.5%	99%
1	(0,0,0)	1.5	3.0	3.4	4.0	4.8	6.2	7.6	9.2
	(1,0,0)	1.5	3.0	3.4	4.0	4.9	6.2	7.6	9.3
	(1,0,1)	1.5	2.9	3.4	4.0	4.8	6.2	7.6	9.3
	(1,1,0)	5.4	8.3	9.0	10.0	11.2	13.2	15.2	17.6
	(1,1,1)	5.5	8.3	9.1	10.0	11.3	13.3	15.2	17.6
2	(0,0,0)	11.5	14.8	15.6	16.8	18.3	20.7	22.9	25.6
	(1,0,0)	11.5	14.8	15.6	16.8	18.3	20.7	22.8	25.7
	(1,0,1)	11.5	14.8	15.6	16.8	18.3	20.7	22.9	25.7
	(1,1,0)	19.6	23.9	25.0	26.5	28.4	31.4	34.0	37.2
	(1,1,1)	19.6	23.9	25.1	26.6	28.5	31.5	34.1	37.2
3	(0,0,0)	29.7	34.7	36.1	37.7	39.8	42.9	45.9	49.8
	(1,0,0)	29.7	34.8	36.1	37.8	39.8	42.9	46.0	49.9
	(1,0,1)	29.7	34.7	36.1	37.8	39.8	42.9	46.0	49.8
	(1,1,0)	41.8	47.9	49.5	51.3	53.9	57.6	61.2	65.5
	(1,1,1)	41.9	47.9	49.6	51.4	54.0	57.7	61.1	65.7

See note to Table 2 for details.

Table 6

Trace tests for cointegration at frequencies 0 and π

Frequency					
0			π		
Null hypothesis	Test statistic	5% critical value	Null hypothesis	Test statistic	5% critical value
$r_1 = 0$	46.5	43.1	$r_2 = 0$	42.4	34.6
$r_1 \leq 1$	12.4	25.5	$r_2 \leq 1$	13.1	19.4
$r_1 \leq 2$	4.5	12.3	$r_2 \leq 2$	2.9	8.7

Note: r_1 = cointegration rank at frequency zero, r_2 = cointegration rank at frequency π.

Table 7

Trace tests for cointegration at frequency $\pi/2$

	Polynomial cointegration		Synchronous cointegration	
Null hypothesis	Test statistic	5% critical value	Test statistic	5% critical value
$r_3 = 0$	93.0	59.4	31.9	40.9
$r_3 \leq 1$	28.3	31.9	9.3	24.5
$r_3 \leq 2$	6.7	13.3	0.0	12.0

Note: r_3 = cointegration rank at frequency $\pi/2$.

Table 8

Significance tests on the error correction terms

	Unrestricted model		Restricted model	
Variable	$F(3,58)$ test statistic	p-value	$F(3,58)$ test statistic	p-value
$ecm1_{t-1}$	8.53	0.00	8.68	0.00
$ecm2_{t-1}$	8.78	0.00	8.87	0.00
$ecm3_{t-1}$	0.95	0.42	–	–
$ecm3_{t-2}$	15.13	0.00	15.48	0.00

Note: $ecm1_t$ = zero-frequency error correction term, $ecm2_t$ = biannual frequency error correction term, $ecm3_t$ = annual frequency error correction term.

Figure 1

Consumption, investment and output in log per-capita form