7 January, 2000

DYNAMIC FACTOR ANALYSIS WITH ARMA FACTORS

by

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Abstract

In this paper we present a new approach to the specification of dynamic factor models. Our model has three advantages over existing work. Firstly, it is based on a minimal-dimension state-space representation giving some gain in computational efficiency over existing methods. Secondly, it easily accommodates hypothesis tests about the order of the factor-filter. Thirdly, by allowing the factor-filter to have a common polynomial factor, ARMA-factor models may be estimated with little extra computational expense over the AR-factor case. We illustrate the use of our model with an application to business cycle analysis.

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Introduction

Since the work of Jöreskog and Lawley and Maxwell in the late 1960s, factor analysis has enjoyed popularity in a wide range of empirical disciplines. In economics however, its use has been infrequent to the extent that it is rarely seen in the literature and is not featured in many econometrics software packages. Given the prevalence of latent variables in economic theory (aggregate demand, willingness-to-pay, expected inflation, etc.) and the fact that most work in economics is multivariate, this may seem peculiar, but it is undoubtedly due to the inapplicability of classical factor analysis techniques to models of serially correlated time series data. Some work has been done to remedy this. Geweke (1977) exploits the asymptotic independence of periodogram ordinates at harmonic frequencies to fit factor models to frequency bands of labor force data using standard techniques modified for complex variables. This work is expanded in Geweke and Singleton (1981). However, the factors in this model are not constrained to be causal to the measured variables. Watson and Engle (1983) specify a model in which a vector of industry wages is the sum of an AR(2) factor and an AR(1) vector of industry-specific errors. They estimate the model using both scoring and the EM algorithm. More recently, Stock and Watson (1998) have extended the work of Connor and Korajczyk (1986) to estimate factor models with large cross-sections using an approach based on principle components. There also exists some work on identifiability by Heij, Scherrer and Deistler (1997), but this issue is far from being resolved.

In this paper, we propose an approach to dynamic factor analysis that represents an improvement over existing work for three reasons. Firstly, it utilizes a minimum-dimension state-space form, resulting in improved computational efficiency over the matrix-stacking approaches commonly used for autoregressive factor models. Secondly, hypothesis tests of the order of the factor-filter are easily constructed. Thirdly, by imposing a common factor restriction on the factor filter matrix, ARMA-factor models may be estimated with little increase in computational complexity over AR-factor models.

A Dynamic Factor Model

An observed px1 vector of variables y_t is said to have a factor structure[1] if

$$y_t = B(L)f_t + \varepsilon_t,$$

f_t is a kx1 vector of unobserved factors with $k<p$, ε_t a px1 vector of mutually orthogonal errors, and B(L) a vector of lag polynomials of order q. If q=0 and ε_t and f_t are not serially correlated, then the model is the classical static factor model of Jöreskog (1967) and Lawley and Maxwell (1971). In what follows we

[1] Chamberlain and Rothchild (1983) and Stock and Watson (1998) would refer to this as an exact factor structure.

initially treat the case in which k=1 and ε_t is white noise. Thus, the dynamic structure of y_t is determined by the dynamic structure and filtering of a single unobserved factor. We then discuss the case in which the factor is a k-dimensional block-identifiable vector autoregression. A more general k-dimensional factor model is under development. The extension to colored ε_t is straightforward.

We begin by considering the case of a scalar autoregressive factor

$$f_t = \frac{\eta_t}{\phi(L)}$$

where $\phi(L)$ is a lag polynomial of order m. As yet little is known about identification in dynamic factor models, but we note that if p=m=q=1 then y_t is an ARMA(1,1) process plus white noise which is not identified. Accordingly, it seems reasonable to impose the restriction that q<m.

The minimum-dimension state-space representation of the factor is

$$\begin{aligned} f_t &= e_1' Z_t \\ Z_t &= AZ_t + \eta_t e_1 \end{aligned}$$

where $Z_t = \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-m+1} \end{pmatrix}$ is the mx1 state vector, $A = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_m \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$ is a so-called companion matrix (Kailath, 1980) and e_1 is an mx1 vector with 1 in the first element and zeros elsewhere.

Now note that with q<m, $B(L)f_t = BZ_t$ where $B = \begin{pmatrix} \beta_0 & \cdots & \beta_q & 0 & \cdots & 0 \end{pmatrix}$ is a pxm matrix. Thus, the dynamic AR-factor model may be written as follows:

$$\begin{aligned} y_t &= BZ_t + \varepsilon_t \\ Z_t &= AZ_{t-1} + \eta_t e_1 \end{aligned}$$

A number of comments may be made at this point. Firstly, as is done in the existing literature on AR-factor models, this model may easily be estimated using the EM algorithm. Secondly, the state vector is of dimension m. This is less than the state dimension of pm attained by the matrix stacking procedure used in Stock and Watson (1998) offering some gain in computational efficiency. Thirdly, likelihood ratio tests of the order of the lag polynomial m can easily be constructed. Fourthly, if we model the factor as a k-dimensional block-identifiable vector autoregression then the compact state-space representation just described extends easily with A becoming a block matrix with ϕ_i as kxk matrices and the 1s in A replaced by kxk identity matrices. A more general theory is under development.

We now consider the extension to the ARMA-factor case. Write the factor-filter vector as

$$B(L) = \begin{pmatrix} b_1(L) \\ \vdots \\ b_p(L) \end{pmatrix}$$

and assume that the $b_r(L)$ have a common lag-polynomial factor b(L) so that

$$b_r(L) = b^*{}_r(L)b(L)$$

The model may then be written as

$$y_t = B^*(L)f_t + \varepsilon_t$$

where $f_t = \dfrac{b(L)}{\phi(L)}\eta_t$ and $B^*(L)$ is a vector of lag polynomials. It is natural to call f_t the factor rather than $\dfrac{\eta_t}{\phi(L)}$. The dynamic ARMA-factor model may then be written as

$$\begin{aligned} y_t &= BZ_t + \varepsilon_t \\ Z_t &= AZ_{t-1} + \eta_t e_1 \\ B &= \beta b \end{aligned}$$

where β is a lower triangular Toeplitz matrix with entries made up from the coefficients in $b^*{}_r(L)$ and b is a vector whose entries are the coefficients in b. This model is easily estimated using the EM algorithm. In the M-step, cyclic ascent may be used to estimate β and b.

An ARMA-Factor Model of the Business Cycle

The preceding ideas will be illustrated with an application of the ARMA-factor model to an analysis of business cycles.

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