SENSITIVITY ANALYSIS OF VALUES AT RISK

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Analyse des Sensibilités des Valeurs à Risque

Résumé

Le but de cet article est d'étudier la sensibilité des valeurs à risque vis-à-vis de la composition du portefeuille. Nous déterminons des expressions analytiques des dérivées premières et secondes de la valeur à risque et expliquons comment elles peuvent être utilisées pour simplifier l'inférence statistique et pour mener des analyses locales du risque. Une illustration empirique d'une telle analyse est donnée pour un portefeuille d'actions françaises.

Sensitivity Analysis of Values at Risk

Abstract

The aim of this paper is to analyze the sensitivity of Value at Risk (VaR) with respect to portfolio allocation. We derive analytical expressions for the first and second derivatives of the Value at Risk, and explain how they can be used to simplify statistical inference and to perform a local analysis of the Value at Risk. An empirical illustration of such an analysis is given for a portfolio of French stocks.

mots clés : Valeurs à risque, gestion des risques, portefeuille VaR efficient, isoVaR, estimation à noyau, quantile.

keywords : Value at Risk, risk management, VaR efficient portfolio, isoVaR, kernel estimators, quantile.

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1. Introduction

Value at Risk (VaR) has become a key tool for risk management of financial institutions. The regulatory environment and the need for controlling risk in the financial community have provided incentives for banks to develop proprietary risk measurement models. Among other advantages, Values at Risk provide quantitative and synthetic measures of risk, that allow to take into account various kinds of cross-dependence between asset returns, fat-tail and non normality effects, arising from the presence of financial options or default risk, for example.

There is also growing interest on the economic foundations of VaR. For a long time, economists have considered empirical behavioural models of banks or insurance companies, where these institutions maximise some utility criteria under a solvency constraint of VaR type [see Gollier, Koehl and Rochet (1996), and Santomero and Babbel (1996) and the references therein]. Similarly, other researchers have studied optimal portfolio selection under limited downside risk as an alternative to traditional mean-variance efficient frontiers [see Roy (1952), Levy and Sarnat (1972), Arzac and Bawa (1977), Jansen, Koedijk and de Vries (1998)]. Finally, internal use of VaR by financial institutions has been addressed in a delegated risk management framework in order to mitigate agency problems [Kimball (1997), Froot and Stein (1998), Stoughton and Zechner (1999)]. Indeed risk management practitioners determine VaR levels for every business unit and perform incremental VaR computations for management of risk limits within trading books. Since the number of such subportfolios is usually quite large, this involves huge calculations that preclude online risk management. One of the aims of this paper is to derive the sensitivity of VaR with respect to a modification of the portfolio allocation. Such a sensitivity has already been derived under a Gaussian and zero mean assumption by Garman (1996, 1997).

Despite of the intensive use of VaR there is a limited literature dealing with the theoretical properties of these risk measures and their consequences on risk management. Following an axiomatic approach, Artzner, Delbaen, Eber and Heath (1996, 1997) (see also Albanese (1997) for alternative axioms) have proved that VaR lacks the subadditivity property for some distributions of asset returns. This may induce an incentive to disagregate the portfolios in order to circumvent VaR constraints. Similarly VaR is not necessarily convex in the portfolio allocation, which may lead to difficulties when computing optimal portfolios under VaR constraints. Beside global properties of risk measures, it is thus also important to study their local second order behavior.

Apart from the previous economic issues, it is also interesting to discuss the estimation of the risk measure which is related to quantile estimation and tail analysis. Fully parametric approaches are widely used by practitioners (see e.g. JP Morgan Riskmetrics documentation), and most often based on the assumption of joint normality of asset (or factor) returns. These parametric approaches are rather stringent. They generally imply misspecification of the tails and VaR underestimation. Fully nonparametric approaches have also been proposed and consist in determining the empirical quantile (the historical VaR) or a smoothed version of it [Harrel and Davis (1982), Falk (1984), (1985), Jorion (1996), Ridder (1997)]. Recently semi-parametric approaches have been developped. They are based on either extreme value approximation for the tails [Bassi, Embrechts and Kafetzaki (1998), Embrechts, Resnick and Samorodnitsky (1998)] or local likelihood methods [Gouriéroux and Jasiak (1999a)].

However up to now the statistical literature has focused on the estimation of VaR levels, while in a number of cases, the knowledge of partial derivatives of VaR with respect to portfolio allocation is more useful. For instance, partial derivatives are required to check the convexity of VaR, to conduct marginal analysis of portfolios or compute optimal portfolios under VaR constraints. Such derivatives are easy to derive for multivariate Gaussian distributions, but, in most practical applications, the joint conditional p.d.f. of asset returns is not Gaussian and involves complex tail dependence [Embrechts, McNeil and Straumann (1999)]. The goal here is to derive analytical forms for these derivatives in a very general framework. These expressions can be used to ease statistical inference and to perform local risk analysis.

The paper is organized as follows. In Section 2, we consider the first and second order expansions of Value at Risk with respect to portfolio allocation. We get explicit expressions for the first and second order derivatives, which are characterized in terms of conditional moments of asset returns given the portfolio return. This allows to discuss the convexity properties of Value at Risk. In Section 3, we introduce the notion of VaR efficient portfolio. It extends the standard notion of mean-variance efficient portfolio by taking VaR as underlying risk measure. First order conditions for efficiency are derived and interpreted. Section 4 is concerned with statistical inference. We introduce kernel based approaches for estimating the Value at Risk, checking its convexity and determining VaR efficient portfolios. In Section 5 these approaches are implemented on real data, namely returns on two highly traded stocks on the Paris Bourse. Section 6 gathers some concluding remarks.

2. The sensitivity and convexity of VaR

2.1 Definition of the Value at Risk

We consider n financial assets whose prices at time t are denoted by $p_{i,t}$, $i = 1, \ldots, n$. The value at t of a portfolio with allocations a_i, $i = 1, \ldots, n$ is then : $W_t(a) = \sum_{i=1}^{n} a_i p_{i,t} = a' p_t$. If the portfolio structure is held fixed between the current date t and the future date $t+1$, the change in the market value is given by : $W_{t+1}(a) - W_t(a) = a'(p_{t+1} - p_t)$:

The purpose of VaR analysis is to provide quantitative guidelines for setting reserve amounts (or capital requirements) in phase with potential adverse changes in prices [see e.g. JP Morgan (1996), Wilson (1996), Jorion (1997), Duffie and Pan (1997), Dowd (1998), Stulz (1998) for a detailed analysis of the concept of VaR and applications in risk management]. For a loss probability level α, the Value at Risk $VaR_t(a; \alpha)$ is defined by :

$$P_t[W_{t+1}(a) - W_t(a) + VaR_t(a; \alpha) < 0] = \alpha; \qquad (2.1)$$

where P_t is the conditional distribution of future asset prices given the information available at time t. Such a definition assumes a continuous conditional distribution of returns. Typical values for the loss probability range from 1% to 5%, depending on the time horizon. Hence the VaR is the reserve amount such that the global position (portfolio plus reserve) only suffers a loss for a given small probability α over a fixed period of time, here normalized to one. The VaR can be considered as an upper quantile at level $1 - \alpha$, since :

$$P_t[-a' y_{t+1} > VaR_t(a; \alpha)] = \alpha; \qquad (2.2)$$

where $y_{t+1} = p_{t+1} - p_t$.

At date t the VaR is a function of past information, of the portfolio structure a and of the loss probability level α.

2.2 Gaussian case

In practice Value at Risk is often computed under the normality assumption for price changes (or returns), denoted as y_{t+1}. Let us introduce μ_t and

Ω_t, the conditional mean and covariance matrix of this Gaussian distribution. Then from (2.2) and the properties of the Gaussian distribution we deduce the expression of the Value at Risk:

$$VaR_t(a;\alpha) = -a'\mu_t + (a'\Omega_t a)^{1/2} z_{1-\alpha}; \tag{2.3}$$

where $z_{1-\alpha}$ is the quantile of level $1-\alpha$ of the standard normal distribution. This expression shows the decomposition of the VaR into two components which compensate for expected negative returns and risk, respectively.

Let us compute the first and second order derivatives of the VaR with respect to the portfolio allocation. We get:

$$\begin{aligned}
\frac{\partial VaR_t(a;\alpha)}{\partial a} &= -\mu_t + \frac{\Omega_t a}{(a'\Omega_t a)^{1/2}} z_{1-\alpha} \\
&= -\mu_t + \frac{\Omega_t a}{a'\Omega_t a}\left(VaR_t(a;\alpha) + a'\mu_t\right) \\
&= -E_t[y_{t+1} \mid a'y_{t+1} = -VaR_t(a;\alpha)]; \qquad (2.4) \\
\frac{\partial^2 VaR_t(a;\alpha)}{\partial a \partial a'} &= \frac{z_{1-\alpha}}{(a'\Omega_t a)^{1/2}}\left[\Omega_t - \frac{\Omega_t a a'\Omega_t}{a'\Omega_t a}\right] \\
&= \frac{z_{1-\alpha}}{(a'\Omega_t a)^{1/2}} V_t[y_{t+1} \mid a'y_{t+1} = -VaR_t(a;\alpha)]. \qquad (2.5)
\end{aligned}$$

In particular we note that these first and second order derivatives are affine functions of the VaR with coefficients depending on the portfolio allocation, but independent of α. In the next subsection we extend these interpretations of the first and second order derivatives of the Value at Risk in terms of first and second order conditional moments given the portfolio value.

2.3 General case

The general expressions for the first and second order derivatives of the VaR are given in the property below. They are valid as soon as y_{t+1} has a continuous conditional distribution with positive density and admits second order moments.

Property 1:

i) The first order derivative of the Value at Risk with respect to the portfolio allocation is:

$$\frac{\partial VaR_t(a;\alpha)}{\partial a} = -E_t[y_{t+1} | a'y_{t+1} = -VaR_t(a;\alpha)].$$

ii) The second order derivative of the Value at Risk with respect to the portfolio allocation is:

$$\begin{aligned}\frac{\partial^2 VaR_t(a;\alpha)}{\partial a \partial a'} &= \frac{\partial \log g_{a;t}}{\partial z}(-VaR_t(a;\alpha)) V_t[y_{t+1} | a'y_{t+1} = -VaR_t(a;\alpha)] \\ &\quad - \left\{ \frac{\partial}{\partial z} V_t[y_{t+1} | a'y_{t+1} = -z] \right\}_{z = VaR_t(a;\alpha)};\end{aligned}$$

where $g_{a;t}$ denotes the conditional p.d.f of $a'y_{t+1}$.

Proof: i) The condition defining the VaR can be written as:

$$P_t[X + a_1 Y > VaR_t(a;\alpha)] = \alpha;$$

where $X = -\sum_{i=2}^{n} a_i y_{i;t+1}$; $Y = -y_{1;t+1}$. The expression of the first order derivative directly follows from Lemma 1 in Appendix 1.

ii) The second order derivative can be deduced from the first order expansion of the first order derivative around a benchmark allocation a_o. Let us set $a = a_o + \epsilon e_j$, where ϵ is a small real number and e_j is the canonical vector, with all components equal to zero but the j^{th} equal to one. We deduce:

$$\frac{\partial VaR_t(a;\alpha)}{\partial a_i} = E_t[X | Z + \epsilon Y = 0] + o(\epsilon);$$

where:

$$X = -y_{i;t+1}; \quad Z = -a_o' y_{t+1} - VaR_t(a_o;\alpha);$$

$$Y = -y_{j;t+1} + E_t[y_{j;t+1} | Z = 0]:$$

The result follows from Lemma 3 in Appendix 2.

Q.E.D

2.4 Convexity of the VaR

It may be convenient for a risk measure to be a convex function of the portfolio allocation thus inducing incentive for portfolio diversification. From the expression of the second order derivative of the VaR, we can discuss conditions which ensure convexity. Let us consider the two terms of the decomposition given in Property 1. The first term is positive definite as soon as the p.d.f. of the portfolio price change (or return) is increasing in its left tail. This condition is satisfied if this distribution is unimodal, but can be violated in the case of several modes in the tail. The second term involves the conditional heteroscedasticity of changes in asset prices given the change in the portfolio value. It is non negative if this conditional heteroscedasticity increases with the negative level $-z$ of change in the portfolio value. This expresses the idea of increasing multivariate risk in the left tail of portfolio return. To illustrate these two components we further discuss particular examples.

1) Gaussian distribution

In the Gaussian case considered in subsection 2.2, we get:

$$\frac{\partial \log g_{a;t}(z)}{\partial z} = \frac{-z + a'\mu_t}{a'\Omega_t a}.$$

Therefore:

$$\begin{aligned} \frac{\partial \log g_{a;t}}{\partial z}(-VaR_t(a;\alpha)) &= \frac{VaR_t(a;\alpha) + a'\mu_t}{a'\Omega_t a} \\ &= \frac{z_{1-\alpha}}{(a'\Omega_t a)^{1/2}}, \text{ from equation (2.3).} \end{aligned}$$

This positive coefficient (as soon as $\alpha < 0.5$) corresponds to the multiplicative factor observed in equation (2.5). Besides the second term of the decomposition is zero due to the conditional homoscedasticity of y_t given $a'y_t$.

2) Gaussian model with unobserved heterogeneity

The previous example can be extended by allowing for unobserved heterogeneity. More precisely let us introduce an heterogeneity factor u and assume that the conditional distribution of asset price changes given the information held at time t has mean $\mu_t(u)$ and variance $\Omega_t(u)$. The various terms of the decomposition can easily be computed and admit explicit forms. For instance we get :

$$g_{a;t}(z) = \int g_{a;t}(z|u)\Pi(u)du,$$

where $g_{a;t}(z|u)$ is the Gaussian distribution of the portfolio price changes given the heterogeneity factor, and Π denotes the heterogeneity distribution .

We deduce that :

$$\frac{\partial \log g_{a;t}(z)}{\partial z} = \frac{\frac{\partial g_{a;t}(z)}{\partial z}}{g_{a;t}(z)} = \frac{\int \frac{\partial}{\partial z} g_{a;t}(z|u)\Pi(u)du}{\int g_{a;t}(z|u)\Pi(u)du}$$

$$= E_{\tilde{\Pi}}\left[\frac{\partial \log g_{a;t}(z|u)}{\partial z}\right];$$

where the expectation is taken with respect to the modified probability $\tilde{\Pi}$ defined by :

$$\tilde{\Pi}(u) = g_{a;t}(z|u)\Pi(u)/[\int g_{a;t}(z|u)\Pi(u)du]:$$

Due to conditional normality, we obtain :

$$\frac{\partial \log g_{a;t}}{\partial z}(-VaR_t(a;\alpha)) = E_{\tilde{\Pi}}\left[\frac{VaR_t(a;\alpha) + a'\mu_t(u)}{a'\Omega_t(u)a}\right]: \qquad (2.6)$$

Let us proceed with the second term of the decomposition. We get :

$$V_t[y_{t+1}|a'y_{t+1} = -z]$$

$$= E_\Pi V_t[y_{t+1}|a'y_{t+1} = -z; u] + V_\Pi E_t[y_{t+1}|a'y_{t+1} = -z; u]:$$

The conditional homoscedasticity given u, implies that $V_t[y_{t+1} \mid a'y_{t+1} = -z; u]$ does not depend on the level z and we deduce that:

$$
\begin{aligned}
\frac{\partial}{\partial z} V_t[y_{t+1} \mid a'y_{t+1} = -z] &= \frac{\partial}{\partial z} V_{\mid}\left[E_t[y_{t+1} \mid a'y_{t+1} = -z; u]\right] \\
&= \frac{\partial}{\partial z} V_{\mid}\left[\mu_t(u) + \frac{\Omega_t(u)a}{a'\Omega_t(u)a}(-z - a'\mu_t(u))\right]. \qquad (2.7)
\end{aligned}
$$

Let us detail formulas (2.6) and (2.7), when $\mu_t(u) = 0, \forall u$, i.e. for a conditional Gaussian random walk with stochastic volatility. From (2.6) we deduce that:

$$
\frac{\partial\, \mathrm{bgg}_{a;t}}{\partial z}(-VaR_t(a;\alpha)) = VaR_t(a;\alpha) E_{\mid}\left[\frac{1}{a'\Omega_t(a)a}\right] > 0.
$$

From (2.7), we get:

$$
\begin{aligned}
-\frac{\partial}{\partial z} V_t[y_{t+1} \mid a'y_{t+1} = -z] &= -\frac{\partial}{\partial z}\left[V_{\mid}\left[-z\frac{\Omega_t(u)a}{a'\Omega_t(u)a}\right]\right] \\
&= -\frac{\partial}{\partial z}\left[z^2 V_{\mid}\left[\frac{\Omega_t(u)a}{a'\Omega_t(u)a}\right]\right]_{z=-z} \\
&= +2zV_{\mid}\left[\frac{\Omega_t(u)a}{a'\Omega_t(u)a}\right],
\end{aligned}
$$

which is nonnegative for $z = VaR_t(a;\alpha)$. Therefore the VaR is convex when price changes follow a Gaussian random walk with stochastic volatility.

3. VaR Efficient Portfolio

Portfolio selection is based on a trade-off between expected return and risk, and requires a choice for the risk measure to be implemented. Usually the risk is evaluated by the conditional second order moment, i.e. volatility. This leads to the determination of the mean-variance efficient portfolio introduced by Markovitz (1952). It can also be based on a safety first criterion (probability of failure), initially proposed by Roy (1952) [see Levy and Sarnat (1972), Arzac and Bawa (1977), Jansen, Koedijk and de Vries (1998) for applications]. In this section we extend the theory of efficient portfolios, when Value at Risk is adopted as risk measure instead of variance.

3.1 Definition

We consider a given budget w to be allocated at time t among n risky assets and a riskfree asset. The price at t of the risky assets are p_t, whereas the price of the riskfree asset is one and the riskfree interest rate is r. The budget constraint at date t is:

$$w = a_0 + a'p_t,$$

where a_0 is the amount invested in the riskfree asset and a the allocation in the risky assets. The portfolio value at the following date is:

$$\begin{aligned} W_{t+1} &= a_0(1+r) + a'p_{t+1} \\ &= w(1+r) + a'[p_{t+1} - (1+r)p_t] \\ &= w(1+r) + a'y_{t+1} \text{ (say)}. \end{aligned}$$

The Value at Risk of this portfolio is defined by:

$$P_t[W_{t+1} < -VaR_t(a_0, a; \alpha)] = \alpha, \quad (3.1)$$

and can be written in terms of the quantile of the risky part of the portfolio.

$$VaR_t(a_0, a; \alpha) = w(1+r) + VaR_t(a; \alpha), \quad (3.2)$$

$$\text{where} \quad VaR_t(a; \alpha) \text{ satisfies: } P_t[a'y_{t+1} < -VaR_t(a; \alpha)] = \alpha. \quad (3.3)$$

We define a VaR efficient portfolio as a portfolio with allocation solving the constrained optimization problem:

$$\begin{cases} \max_a \; E_t W_{t+1} \\ \text{s.t.} \quad VaR_t(a_0, a; \alpha) \leq VaR_0, \end{cases} \quad (3.4)$$

where VaR_0 is a benchmark VaR level.

This problem is equivalent to:

$$\begin{cases} \max_a \; a' E_t y_{t+1} \\ s.t.: \quad VaR_t(a;\alpha) \leq VaR_0 - w(1+r) = \widetilde{VaR}_0, \end{cases} \tag{3.5}$$

The VaR efficient allocation depends on the loss probability α, on the bound VaR_0 limiting the authorized risk (in the context of capital adequacy requirement of the Basle Committee on Banking Supervision, usually one third or one quarter of the budget allocated to trading activities) and on the initial budget w. It is denoted by $a_t^* = a_t^*[\alpha; \widetilde{VaR}_0]$. The constraint is binding at the optimum and a_t^* solves the first order conditions:

$$\begin{cases} E_t y_{t+1} = -\lambda_t^* \dfrac{\partial VaR_t}{\partial a}(a_t^*;\alpha), \\ VaR_t(a_t^*;\alpha) = \widetilde{VaR}_0, \end{cases} \tag{3.6}$$

where λ_t^* is a Lagrange multiplier. In particular it implies proportionality at the optimum between the global and local expectations of the net gains:

$$E_t y_{t+1} = \lambda_t^* E_t \left[y_{t+1} \mid a_t^{*\prime} y_{t+1} = -\widetilde{VaR}_0 \right]. \tag{3.7}$$

4. Statistical inference

Estimation methods can be developped from stationary observations of variables of interest. Hence it is preferable to consider the sequence of returns $(p_{t+1} - p_t)/p_t$ instead of the price modifications $p_{t+1} - p_t$, and accordingly the allocations measured in values instead of shares. In this section $y_{t+1} = (p_{t+1} - p_t)/p_t$ denotes the return and a the allocation in value.

Moreover we consider the case of i.i.d. returns, which allows to avoid the dependence on past information.

4.1 Estimation of the Value at Risk

Since the portfolio value remains the same whether allocations are measured in shares or values, the VaR is still defined by:

$$P_t[-a' y_{t+1} > VaR_t(a;\alpha)] = \alpha,$$

and, since the returns are i.i.d. it does not depend on the past:

$$P[-a'y_{t+1} > VaR(a;\alpha)] = \alpha:$$

It can be consistently estimated from T observations by replacing the unknown distribution of the portfolio value by a smoothed approximation. For this purpose we introduce a Gaussian kernel and define the estimated VaR, denoted by $\widehat{VaR}$, as:

$$\frac{1}{T}\sum_{t=1}^{T}\Phi\left(\frac{-a'y_t-\widehat{VaR}}{h}\right) = \alpha; \quad (4.1)$$

where Φ is the c.d.f. of the standard normal distribution and h is the selected bandwidth. In practice equation (4.1) is solved numerically by a Gauss-Newton algorithm. If $var^{(p)}$ denotes the approximation at the p^{th} step of the algorithm, the updating is given by the recursive formula:

$$var^{(p+1)} = var^{(p)} + \frac{\frac{1}{T}\sum_{t=1}^{T}\Phi\left(\frac{-a'y_t-var^{(p)}}{h}\right)-\alpha}{\frac{1}{Th}\sum_{t=1}^{T}\varphi\left(\frac{a'y_t+var^{(p)}}{h}\right)}; \quad (4.2)$$

where φ is the p.d.f. of the standard normal distribution.

The starting values for the algorithm can be set equal to the VaR obtained under a Gaussian assumption or the historical VaR (empirical quantile).

Other choices than the Gaussian kernel may also be made without affecting the procedure substantially. The Gaussian kernel has the advantage of being easy to integrate and differentiate from an analytical point of view, and to implement from a computerized point of view.

Finally let us remark that, due to the small kernel dimension (one), we do not face the standard curse of dimensionality often encountered in kernel methods. Hence our approach is also feasible in the presence of a large number of assets.

4.2 Convexity of the VaR

From the expression of the second order derivative of the VaR provided in Property 1, we know that the Hessian $\frac{\partial^2 VaR(a;\alpha)}{\partial a \partial a'}$ is positive semidefinite if

$\frac{\partial \log g_{a,t}(z)}{\partial z} > 0$; and $\frac{\partial V[y_{t+1} | a'y_{t+1} = z]}{\partial z} \gg 0$, for negative z values. These sufficient conditions can easily be checked without having to estimate the Value at Risk. Indeed consistent estimators of the p.d.f. of the portfolio value and of the conditional variance are:

$$\hat{g}_a(z) = \frac{1}{Th} \sum_{t=1}^{T} K\left(\frac{a'y_t - z}{h}\right); \tag{4.3}$$

$$\hat{V}[y_{t+1} | a'y_{t+1} = z] = \frac{\sum_{t=1} y_t y_t' K\left(\frac{a'y_t - z}{h}\right)}{\sum_{t=1} K\left(\frac{a'y_t - z}{h}\right)} - \frac{\sum_{t=1} y_t K\left(\frac{a'y_t - z}{h}\right) \sum_{t=1} y_t' K\left(\frac{a'y_t - z}{h}\right)}{\left[\sum_{t=1} K\left(\frac{a'y_t - z}{h}\right)\right]^2}. \tag{4.4}$$

4.3 Estimation of a VaR efficient portfolio.

Due to the rather simple forms of the first and second order derivatives of the VaR, it is convenient to apply a Gauss-Newton algorithm when determining the VaR efficient portfolio. More precisely let us look for a solution to the optimization problem (3.5) in a neighbourhood of the allocation $a^{(p)}$. The optimization problem becomes equivalent to:

$$\max_a \; a'Ey_{t+1}$$

$$\text{s.t.} \quad VaR(a^{(p)};\alpha) + \frac{\partial VaR}{\partial a'}(a^{(p)};\alpha)[a - a^{(p)}]$$

$$+ \frac{1}{2}[a - a^{(p)}]' \frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)};\alpha)[a - a^{(p)}] \leq \tilde{VaR}_0.$$

This problem admits the solution:

$$a^{(p+1)} = a^{(p)} - \left[\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)};\alpha)\right]^{-1} \frac{\partial VaR}{\partial a}(a^{(p)};\alpha)$$

$$+ \left[\frac{2(\tilde{VaR}_0 - VaR(a^{(p)};\alpha)) + Q(a^{(p)};\alpha)}{Ey_{t+1}'\left[\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)};\alpha)\right]^{-1} Ey_{t+1}}\right]^{1/2}$$

$$\times\left[\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)};\alpha)\right]^{-1} E y_{t+1};$$

with : $Q(a^{(p)};\alpha) = \frac{\partial VaR}{\partial a'}(a^{(p)};\alpha)\left[\frac{\partial^2 VaR}{\partial a \partial a'}(a^{(p)};\alpha)\right]^{-1} \frac{\partial VaR}{\partial a}(a^{(p)};\alpha)$:

To get the estimate, the theoretical recursion is replaced by its empirical counterpart, in which the expectation $E y_{t+1}$ is replaced by $\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} y_t$, while the VaR and its derivatives are replaced by their corresponding kernel estimates given in the two previous subsections.

5. An empirical illustration

This section illustrates the implementation of the estimation procedures described in Section 4.[1] We analyze two companies listed on the Paris Bourse: Thomson-CSF (electronic devices) and L'Oréal (cosmetics). Both stocks belong to the French stock index CAC 40. The data are daily returns recorded from 04/01/1997 to 05/04/1999, i.e. 546 observations. The return mean and standard deviation are 0.0049% and 1.262% for the first stock, 0.0586% and 1.330% for the second stock. Minimum returns are -4.524% and -4.341%, while maximum values are 3.985% and 4.013%, respectively. We have for skewness -0.2387 and 0.0610, and for kurtosis 4.099 and 4.295. This indicates that the data cannot be considered as normally distributed (it is confirmed by the values 387.5 and 420.0 taken by the Jarque-Bera (1980) test statistic). The correlation is 0.003%. We have checked the absence of dynamics by examining the autocorrelograms, partial autocorrelograms and Ljung-Box statistics.

Figure 1 shows the estimated VaR of a portfolio including these two stocks with different allocations. The allocations range from 0 (1) to 1 (0) in Thomson-CSF (L'Oréal) stock. The loss probability level is 1%. The dashed line provides the estimated VaR based on the kernel estimator (4.1). We have selected the bandwidth according to the classical proportionality rule: $h = (4/3)^{1/5}\sigma_a T^{-1/5}$, where σ_a is the standard deviation of the portfolio return with allocation a. We also provide the estimates given by (2.3) based on the normality assumption (solid line) and the estimates using the empirical first percentile (dashed line). The standard VaR based on the normality assumption are far below the other estimated values as it could have been

[1] Gauss programs developped for this section are available on request.

expected from the skewness and kurtosis exhibited by the individual stock returns. This standard VaR leads to an underestimation of the reserve amount aimed to cover potential losses. We note that the kernel based estimator and percentile based estimator lead to similar results with a smoother pattern for the first one.

INSERT Figure 1 : Estimated VaR

Let us now examine the sensitivities. Estimated first partial derivatives of the portfolio VaR are given in Figure 2. The solid line provides the estimate of the partial derivative for the first stock Thomson-CSF based on a kernel approach. The dotted line conveys its Gaussian counterpart and does not reflect the non monotonicity of the first derivative. The two other dashed lines give the analogous curves for the second stock L'Oréal. At the portfolio corresponding to the minimum VaR in Figure 1, the first derivatives w.r.t. each portfolio allocation are equal as seen on Figure 2, and coincide with the Lagrange mutiplier associated with the constraint $a_1 + a_2 = 1$.

INSERT Figure 2 : Estimated sensitivity

What could be said about VaR convexity when a particular allocation a is adopted ? Both conditions $\frac{\partial \log g_{a,t}(z)}{\partial z} > 0$ and $\frac{\partial V[y_{t+1} | a' y_{t+1} = z]}{\partial z} \gg 0$ for negative z values can be verified in order to check VaR convexity. We can use the estimators based on (4.3) and (4.4) for such a verification. Let us take a diversified portfolio with allocation $a = (0.5, 0.5)'$. Figure 3 gives the estimated log derivative of the p.d.f. of the portfolio returns (see (4.3)) and shows that the first condition is not empirically satisfied.

INSERT Figure 3 : First condition for convexity

Moreover the second condition is also not empirically met. Indeed we can observe in Figure 4 that the solid and dashed lines representing the two eigenvalues of the estimated conditional variance (see (4.4)) are not strictly positive for negative z values. Hence we conclude to the local non convexity of the VaR for a portfolio evenly invested in Thomson-CSF and L'Oréal. Such a finding is not necessarily valid for other allocation structures.

INSERT Figure 4 : Second condition for convexity

We end this section by discussing the shapes of the estimated VaR. We compare the Gaussian and kernel approaches in Figures 5 and 6. The asset allocations range from -1 to 1 in both assets. The contour plot corresponds to increments in the estimated VaR by 0.5%. Hence the contour lines correspond to successive isoVaR curves with levels 0.5%, 1%, 1.5%, ..., starting from 0 (allocation $a = (0;0)'$). Under a Gaussian assumption, the isoVaR corresponds to an elliptical surface (see Figure 5). The isoVaR obtained by the kernel approach are provided in Figure 6. We observe that the corresponding VaR are always higher than the Gaussian ones, and that symmetry with respect to the origin is lost. In particular, without the Gaussian assumption, the directions of steepest (resp. flattest) ascent are no more straight lines. However under both computations of isoVaR the portfolios with steepest (resp. flattest) ascent are obtained for allocation of the same (resp. opposite) signs.

Finally the isoVaR curves can be used to characterize the VaR efficient portfolios. The estimated efficient portfolio for a given authorized level $\widehat{VaR}_0$ is given by the tangency point between the isoVaR curve of level $\widehat{VaR}_0$ and the set of lines with equation : $a_1\hat{\mu}_1 + a_2\hat{\mu}_2 = constant$, where $\hat{\mu}_1$, $\hat{\mu}_2$ denote the estimated means. Since the isoVaR curves do not differ substantially on our empirical example the efficient portfolios are not very much affected by the use of the Gaussian or the kernel approach. This finding would be challenged if assets with nonlinear payoffs, such as options, were introduced in the portfolio.

INSERT Figure 5 : IsoVaR Curves by Gaussian approach

INSERT Figure 6: IsoVaR Curves by kernel approach

6. Concluding remarks

We have considered the local properties of the Value at Risk. In particular we have derived explicit expressions for the sensitivities of the risk measures with respect to the portfolio allocation and applied the results to the determination of VaR efficient portfolios. The empirical application points out the difference between a VaR analysis based on a Gaussian assumption for asset returns and a direct nonparametric approach.

This analysis has been performed under two restrictive conditions, namely i.i.d. returns and constant portfolio allocations. These conditions can be weakened. For instance we can introduce nonparametric Markov models for returns, allowing for nonlinear dynamics, and compute the corresponding conditional VaR together with their derivatives. Such an extension is under current development. The assumption of constant holdings until the benchmark horizon can also be questioned. Indeed in practice the portfolio can be frequently updated and a major part of the risk can be due to unappropriate updating. The effect of a dynamic strategy on the VaR can only be valuated by Monte-Carlo methods [see for instance the impulse response analysis in Gouriéroux and Jasiak (1999b)]. It has also to be taken into account when determining a dynamic VaR efficient hedging strategy [see Foellmer and Leukert (1998)]. Finally let us remark that our kernel based approach can be used to analyse the sensitivity of the expected shortfall, i.e. the expected loss knowing that the loss is larger that a given loss quantile. This is also under current development.

Appendix 1
Expansion of a quantile

Lemma 1: Let us consider a bivariate continuous vector $(X;Y)$ and the quantile $Q(\varepsilon;\alpha)$ defined by:

$$P[X + \varepsilon Y > Q(\varepsilon;\alpha)] = \alpha.$$

Then:

$$\frac{\partial}{\partial \varepsilon} Q(\varepsilon;\alpha) = E[Y \mid X + \varepsilon Y = Q(\varepsilon;\alpha)].$$

Proof: Let us denote by $f(x;y)$ the joint p.d.f. of the pair $(X;Y)$. We get:

$$P[X + \varepsilon Y > Q(\varepsilon;\alpha)] = \alpha$$

$$\iff \int \left[\int_{Q(\varepsilon;\alpha) - \varepsilon y}^{\infty} f(x;y)\,dx \right] dy = \alpha.$$

The differentiation with respect to ε provides.

$$\int \left[\frac{\partial Q(\varepsilon;\alpha)}{\partial \varepsilon} - y \right] f(Q(\varepsilon;\alpha) - \varepsilon y; y)\,dy = 0,$$

which leads to:

$$\frac{\partial Q(\varepsilon;\alpha)}{\partial \varepsilon} = \frac{\int y f(Q(\varepsilon;\alpha) - \varepsilon y; y)\,dy}{\int f(Q(\varepsilon;\alpha) - \varepsilon y; y)\,dy}$$

$$= E[Y \mid X + \varepsilon Y = Q(\varepsilon;\alpha)].$$

Q.E.D

Appendix 2
Expansion of the conditional expectation

Lemma 2: Let us consider a continuous three dimensional vector (X, Y, Z); then:

$$E[X \mid Z + \varepsilon Y = 0]$$

$$= E[X \mid Z = 0] - \varepsilon \left[\frac{\partial \log g(z)}{\partial z}\right]_{z=0} \operatorname{Cov}[X, Y \mid Z = 0]$$

$$- \varepsilon \frac{\partial}{\partial z} \operatorname{Cov}[X, Y \mid Z = z]|_{z=0} + \varepsilon E[Y \mid Z = 0] \frac{\partial}{\partial z} E[X \mid Z = z]|_{z=0} + o(\varepsilon);$$

where g is the marginal p.d.f. of Z.

Proof: Let us denote by $f(x, y, z)$ the joint p.d.f. of the triple (X, Y, Z) and by $f(x, y \mid z) = \dfrac{f(x, y, z)}{g(z)}$ the conditional p.d.f. of X, Y given $Z = z$. The conditional expectation is given by:

$$E[X \mid Z + \varepsilon Y = 0] = \frac{\int\int x f(x, y, -\varepsilon y)\, dx\, dy}{\int\int f(x, y, -\varepsilon y)\, dx\, dy}$$

$$= \frac{\int\int x f(x, y, 0)\, dx\, dy - \varepsilon \int\int xy \frac{\partial}{\partial z} f(x, y, 0)\, dx\, dy}{\int\int f(x, y, 0)\, dx\, dy - \varepsilon \int\int y \frac{\partial}{\partial z} f(x, y, 0)\, dx\, dy} + o(\varepsilon)$$

$$= E[X \mid Z = 0] - \varepsilon E\left[XY \frac{\partial \log f}{\partial z}(X, Y, 0) \mid Z = 0\right]$$

$$+ \varepsilon E[X \mid Z = 0] E\left[Y \frac{\partial \log f}{\partial z}(X, Y, 0) \mid Z = 0\right] + o(\varepsilon)$$

$$
\begin{aligned}
&= E[X|Z=0] - \epsilon \operatorname{Cov}[X, Y \frac{\partial \log f}{\partial z}(X, Y, 0)|Z=0] + o(\epsilon) \\
&= E[X|Z=0] - \epsilon \frac{\partial \log g(z)}{\partial z}|_{z=0} \operatorname{Cov}[X, Y|Z=0] \\
&\quad - \epsilon \operatorname{Cov}[X, Y \frac{\partial \log f}{\partial z}(X, Y|0)|Z=0] + o(\epsilon). \qquad (A1)
\end{aligned}
$$

Let us now consider the derivative of the conditional covariance. We get:

$$
\begin{aligned}
&\frac{\partial}{\partial z} \operatorname{Cov}[X, Y|Z=z] \\
&= \frac{\partial}{\partial z}[E[XY|Z=z] - E[X|Z=z]E[Y|Z=z]] \\
&= E\left[XY \frac{\partial \log f}{\partial z}(X, Y|z)|Z=z\right] - E[X|Z=z]E\left[Y \frac{\partial \log f}{\partial z}(X, Y|z)|Z=z\right] \\
&\quad - \frac{\partial}{\partial z} E[X|Z=z]E[Y|Z=z] \\
&= \operatorname{Cov}\left[X, Y \frac{\partial \log f}{\partial z}(X, Y|z)|Z=z\right] - \frac{\partial}{\partial z} E[X|Z=z]E[Y|Z=z].
\end{aligned}
$$

The expansion of Lemma 2 directly follows after substitution in equation (A1).

Q.E.D

Lemma 3: When $E[Y|Z=0]=0$, the expansion reduces to:

$$
\begin{aligned}
&E[X|Z+\epsilon Y=0] \\
&= E[X|Z=0] - \epsilon \frac{\partial \log g(z)}{\partial z}|_{z=0} \operatorname{Cov}[X, Y|Z=0] \\
&\quad - \epsilon \frac{\partial}{\partial z} \operatorname{Cov}[X, Y|Z=z]|_{z=0} + o(\epsilon).
\end{aligned}
$$

Lemma 4: When $E[Y|Z=0]=0$, and $Cov[X,Y|Z=z]$ is independent of z (conditional homoscedasticity) the expansion reduces to:

$$E[X|Z+\varepsilon Y=0]$$

$$= E[X|Z=0] - \varepsilon \frac{\partial \log g(z)}{\partial z}\Big|_{z=0} Cov[X,Y|Z=0] + o(\varepsilon).$$

Let us remark that Lemma 4 is in particular valid for a Gaussian vector (X,Y,Z). In this specific case, we get:

$$\log g(z) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log VZ - \frac{1}{2}\frac{(z-EZ)^2}{VZ};$$

$$\frac{\partial \log g(z)}{\partial z}\Big|_{z=0} = \frac{EZ}{VZ}.$$

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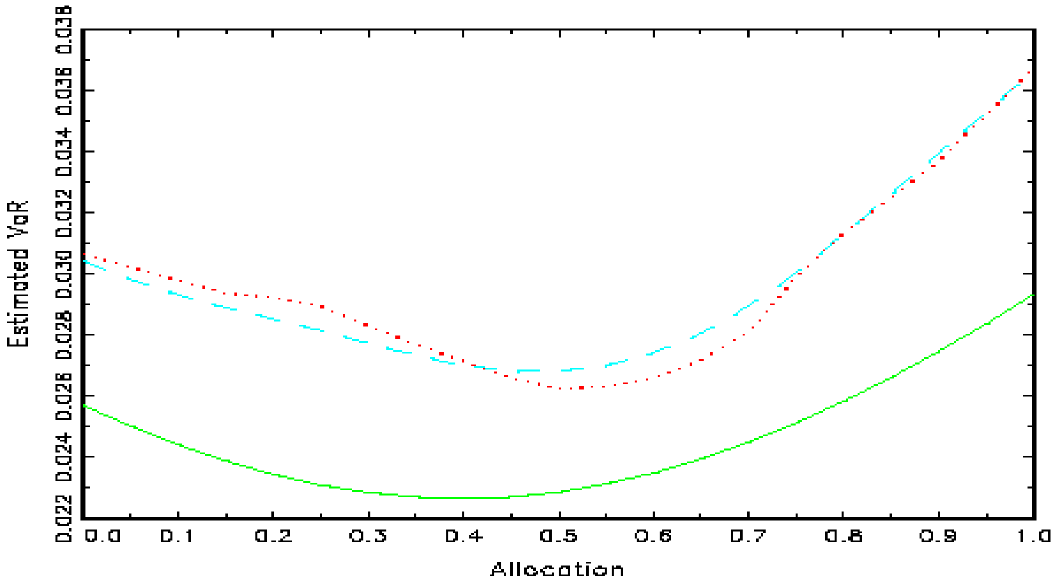

Figure 1 : Estimated VaR

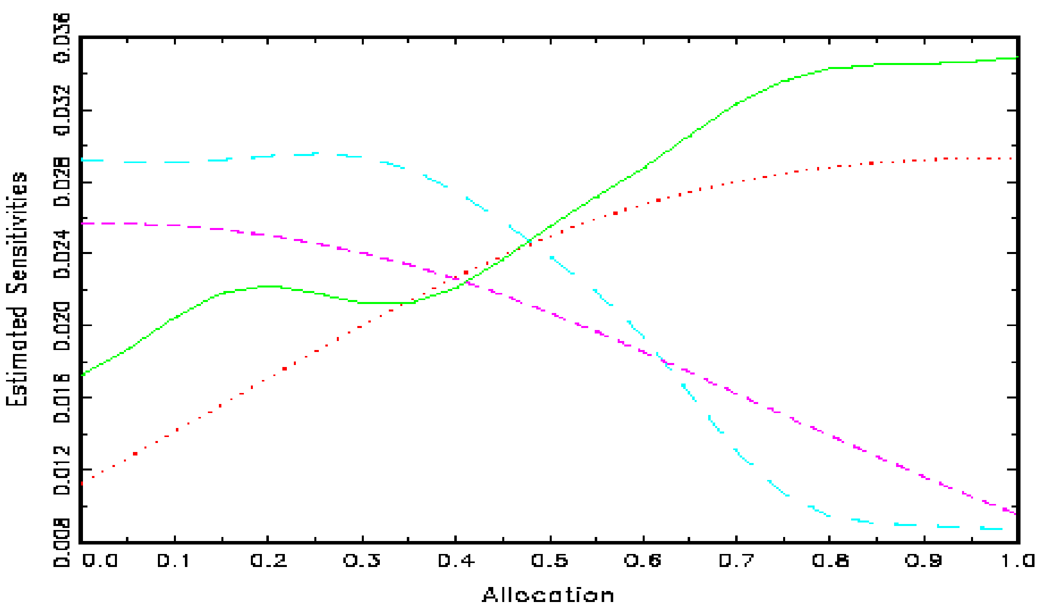

Figure 2 : Estimated sensitivity

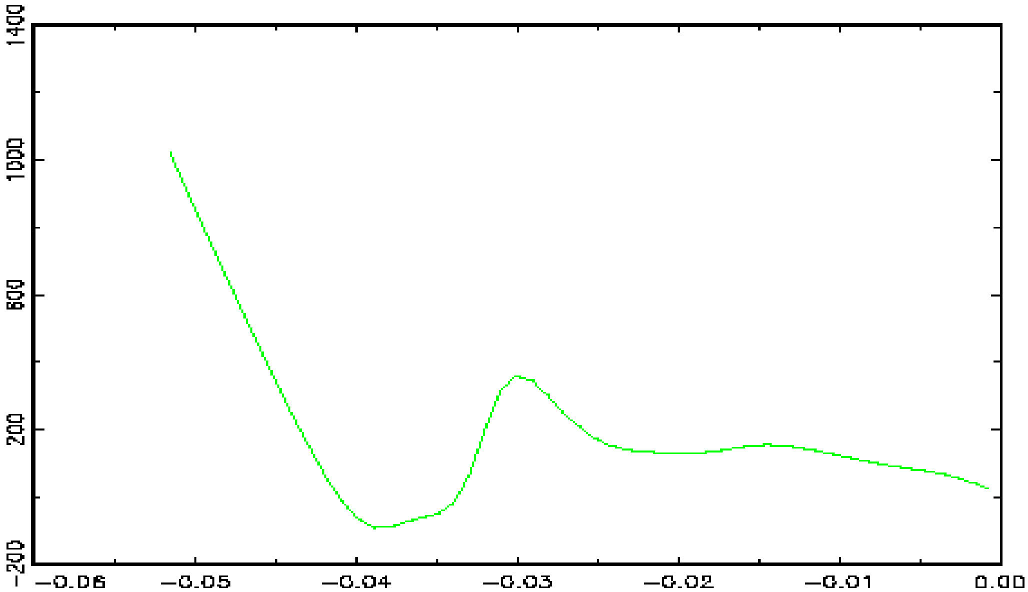

Figure 3 : First condition for convexity

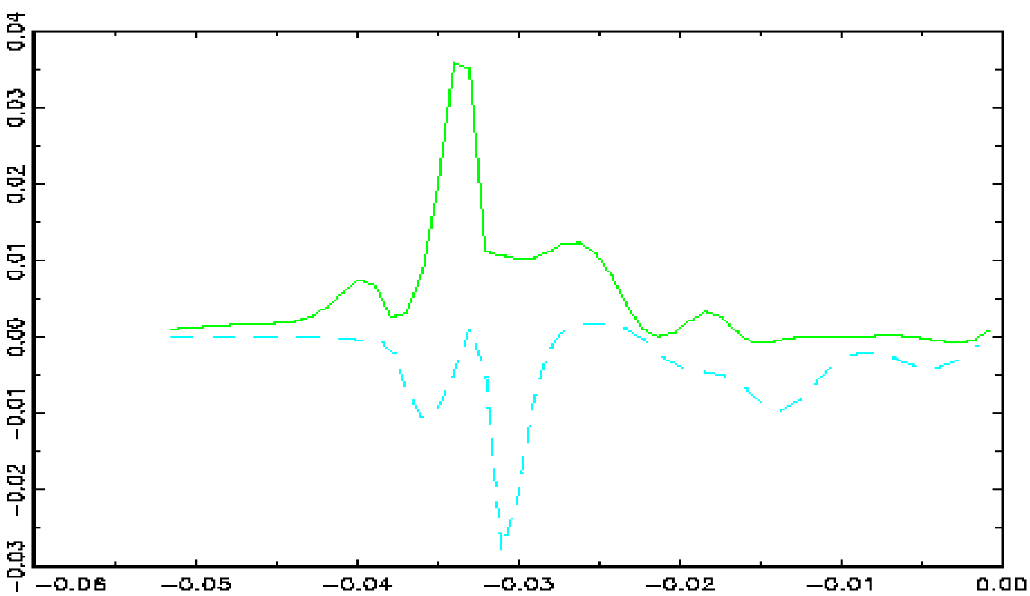

Figure 4 : Second condition for convexity

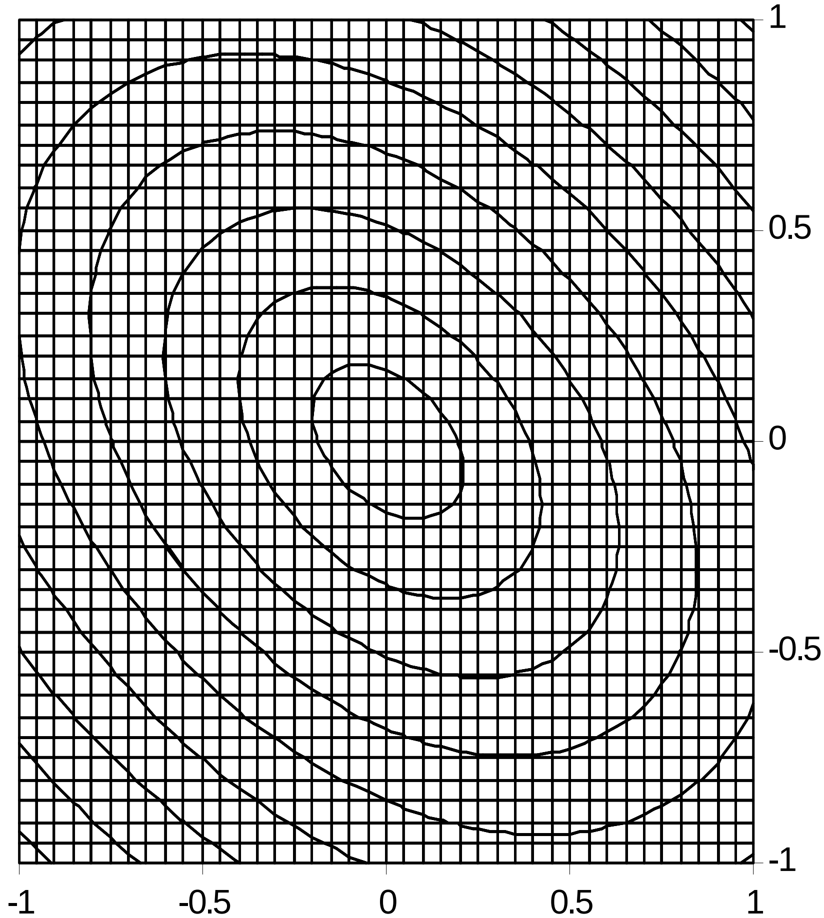

Figure 5 : IsoVaR curves by Gaussian approach

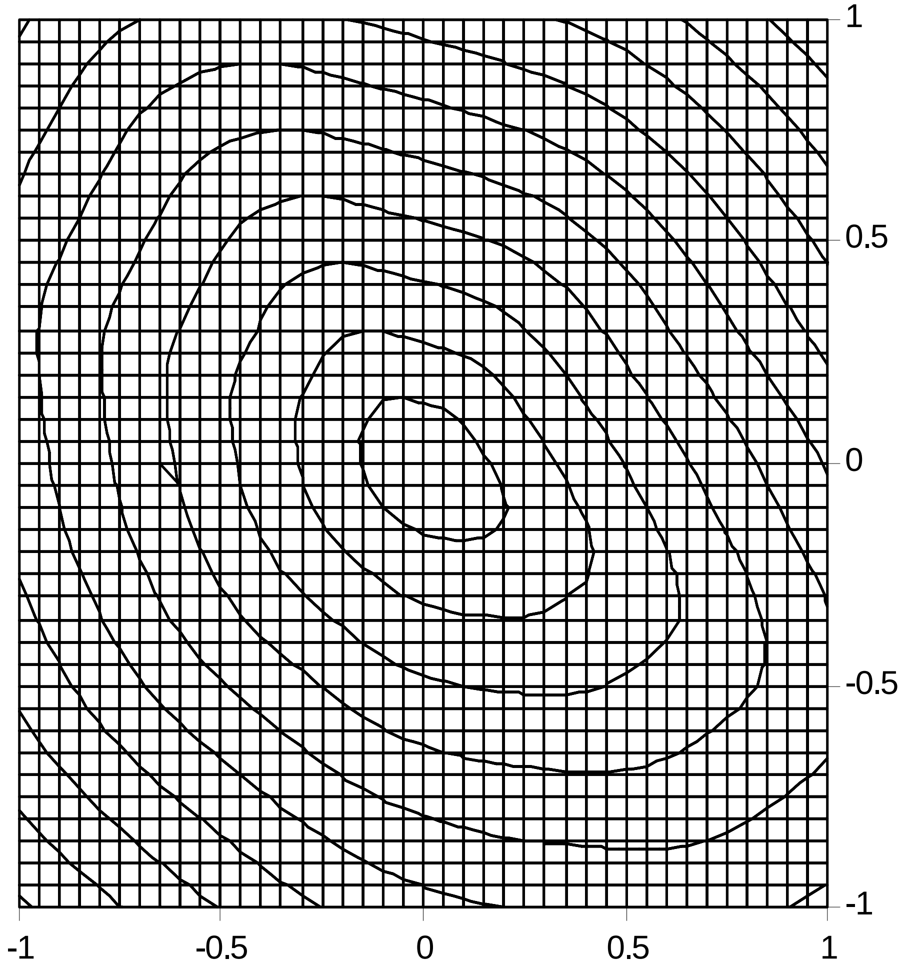

Figure 6 : IsoVaR curves by kernel approach