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“Third down with a yard to go”: the Dixit-Skeath conundrum on equilibria in competitive games.

Abstract

In strictly competitive games, equilibrium mixed strategies are invariant to changes in the ultimate prizes. Dixit & Skeath (1999) argue that this seems counter-intuitive. We show that this invariance is robust to dropping the independence axiom, but is removed if we drop the reduction axiom. The conditions on the resulting recursive expected-utility model to get the desired outcome are analogous to conditions used in the standard model of comparative statics under risk.

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1 Introduction

In their textbook, *Games of Strategy*, Dixit & Skeath (1999, ch. 8.5) pose a "conundrum". They consider the strictly competitive game between the offense and the defense in an American football game. It is third down and the offense has a yard to go. Suppose the offense can only choose to run or to pass. Similarly, the defense can only chose optimally to defend the run or the pass. The probability that the offense goes on to win the game depends on the actions chosen. Presumably, the probability that the offense wins is higher when its action is not matched by the defense. Run and pass, however, are not equivalent. If the offense attempts a pass against a run defense, then the offense wins the game with very high probability, but if meets a pass defense, offense loses with very high probability. On the other hand, if the offense runs against a pass defense the probability of the offense winning the game is not very much higher than if it meets a run defense. As Dixit & Skeath put it, run is the "percentage" or safe play for the offense, whereas pass is the "risky" alternative. The equilibrium of this game involves mixed strategies.

Dixit & Skeath observe:

> "...people often say that if the occasion is really important, in the sense that winning versus losing is a big difference in payoffs, then one should use the percentage play more often. Thus the offense may throw a long pass on third down with a yard to go in an ordinary season game, but in the superbowl that is too big a risk to take."

The "surprise" is that the equilibrium mixing probabilities are completely independent of the ultimate prizes.

> "The theory says that you should mix the percentage play and the risky play in exactly the same proportions on a big occasion as you would on a minor occasion. ... So which is right: theory or intuition?"

They note that "...the problem is fundamental to the use of expected utility in constructing the payoffs", and so they pose the following challenge:

> "We hope that the conundrum will spur readers of such persuasions to undertake some basic research in game theory, and to give it new foundations that do not use expected utility theory."

This paper takes up that challenge. We consider two separate departures from standard expected-utility theory. Section 2 formally sets up the problem. Section 3 shows that merely relaxing the independence axiom of expected utility does not affect the invariance of the equilibrium mixing probabilities in strictly competitive games to changes in the ultimate prizes. On the other hand, Section 4 shows that if we keep independence but instead drop the reduction of compound lotteries axiom (that is, if players have Kreps-Porteus (1978) recursive expected-utility preferences), then this "invariance property" is lost. Section 5 then gives precise conditions on the recursive expected-utility model for it to coincide with Dixit & Skeath's intuition. Under these conditions, all other things being equal, as we increase the stakes, players will be more inclined to play the safe alternative.[1] The equilibrium mixes will then have to adjust to deter this inclination.

As Dixit & Skeath point out, some people may like and some may dislike the invariance property. Our results have implications either way. If you like the invariance property, you need

[1] Reversing the inequalities in these conditions, generates the opposite effect.

not be committed to independence; but you must commit to reduction. On the other hand, if you want to get away from the invariance property, so that theory can concur with intuition, merely relaxing independence will not do the trick; but relaxing reduction can generate the desired comparative statics.

Recursive expected utility is not the only way to resolve the Dixit-Skeath conundrum. For example, one could relax probabilistic sophistication and allow players to be uncertainty averse with respect to each other's moves. Our approach, however, seems a more minimal departure from the standard model.

The recursive expected-utility model sometimes looks intimidating. The conditions that generate our comparative statics, however, are analogous to familiar conditions used in standard expected-utility comparative statics results, such as the portfolio problem. The use of this analogy illustrates a way to make recursive expected utility more tractable.

2 The Setting

Consider a strictly competitive contest between two players, row and column. There are two ultimate outcomes, row wins (and column loses) or row loses (and column wins). If row wins, he gets $\overline{x}$; and if he loses, he gets $\underline{x}$, where $\overline{x} > \underline{x}$. Similarly, if column wins, she gets $\overline{y}$; and if she loses, she gets $\underline{y}$, where $\overline{y} > \underline{y}$. Which outcome occurs depends not only on the players' actions but also on chance. The players' choices are simultaneous. Row chooses between S or R and column chooses between s or r. Later, we will identify S with Dixit & Skeath's "safe" (or "percentage") strategy, and identify R with their "risky" strategy. Each action-pair results in a lottery. For example, given the action-pair (R, s), let π_{Rs} be the probability that row wins $\overline{x}$ (in which case column gets $\underline{y}$); and let $1 - \pi_{Rs}$ be the probability that column wins $\overline{y}$ (in which case row gets $\underline{x}$). If we write $[\pi; \overline{x}, \underline{x}]$ for the lottery which yields $\overline{x}$ with probability π and $\underline{x}$ with probability $1 - \pi$, this contest can be summarized in the matrix below.

	s	r
S	$[\pi_{Ss}; \overline{x}, \underline{x}], [1 - \pi_{Ss}; \overline{y}, \underline{y}]$	$[\pi_{Sr}; \overline{x}, \underline{x}], [1 - \pi_{Sr}; \overline{y}, \underline{y}]$
R	$[\pi_{Rs}; \overline{x}, \underline{x}], [1 - \pi_{Rs}; \overline{y}, \underline{y}]$	$[\pi_{Rr}; \overline{x}, \underline{x}], [1 - \pi_{Rr}; \overline{y}, \underline{y}]$

We are interested in mixed-strategy equilibria so we assume: $\pi_{Rs} > \pi_{Ss}$, $\pi_{Rr} < \pi_{Sr}$, $1 - \pi_{Rs} < 1 - \pi_{Rr}$ and $1 - \pi_{Ss} > 1 - \pi_{Sr}$. We use the terminology of a mixed-strategy equilibrium being an equilibrium in beliefs.[2] Let q^* denote row's equilibrium belief that column will play s, and let p^* denote column's equilibrium belief that row chooses S.

As Dixit & Skeath note, under the standard assumption of expected utility, the equilibrium mixed-strategy profile in such 'strictly-competitive' games is invariant to the sizes of the final

[2] We adopt the "belief" rather than the "randomizing" terminology simply because we find it helps our intuitions, but nothing in our formal results depends on this. Since there are only two ultimate outcomes, there is no existence problem when we relax independence.

prizes, $\overline{x}$, $\underline{x}$, $\overline{y}$ and $\underline{y}$. In particular, row's equilibrium belief q^* is given by

$$\begin{aligned} & q^* \pi_{Ss} \left[u\left(\overline{x}\right) - u(\underline{x})\right] + \left(1 - q^*\right) \pi_{Sr} \left[u\left(\overline{x}\right) - u(\underline{x})\right] \\ = \; & q^* \pi_{Rs} \left[u\left(\overline{x}\right) - u(\underline{x})\right] + \left(1 - q^*\right) \pi_{Rr} \left[u\left(\overline{x}\right) - u(\underline{x})\right], \end{aligned} \tag{1}$$

where u denotes row's von Neumann-Morgenstern utility index. It is clear that the q^* that solves this equation does not depend on $\overline{x}$ or $\underline{x}$. Similarly, p^* does not depend on $\overline{y}$ or $\underline{y}$.[3]

Definition *We say that the invariance property of mixed equilibria in strictly competitive games is satisfied if, for all such games with $\pi_{Rs} > \pi_{Ss}$, $\pi_{Rr} < \pi_{Sr}$, $1 - \pi_{Rs} < 1 - \pi_{Rr}$ and $1 - \pi_{Ss} > 1 - \pi_{Sr}$, the equilibrium beliefs (p^*, q^*) do not depend on the ultimate prizes, $\overline{x}$, $\underline{x}$, $\overline{y}$, and $\underline{y}$.*

We are interested in whether this invariance extends beyond the expected-utility model. For our purposes, there are two key assumptions in expected-utility theory. First, is the reduction of compound lotteries axiom (hereafter, 'reduction'). Our equilibria in beliefs involve two sets of probabilities: those representing each agent's beliefs about the other's actions, and those coming from the resulting lotteries. Reduction imposes that agents treat these two the same. Agents are assumed to 'multiply through' the probabilities to form one-stage lotteries on the final prizes. More generally, we can model agents as having preferences over two-stage lotteries, the first stage representing each player's uncertainty about the other's action, and the second the 'π-lottery' over outcomes.

The second assumption is the independence axiom. There is a large literature discussing reasons we might want to relax this assumption. We refer to models that relax independence but maintain reduction as atemporal nonexpected-utility models. We refer to models that relax reduction but maintain independence as recursive expected-utility models.

3 Atemporal Non-Expected Utility

Let $\mathcal{L}$ be the set of one-stage (real-valued) lotteries. Let the function $V : \mathcal{L} \to \mathbb{R}$ represent a general (complete, transitive and continuous) preference relation over $\mathcal{L}$. Under atemporal nonexpected utility, row's equilibrium belief q^* is given by

$$V\left(\left[q^* \pi_{Ss} + \left(1 - q^*\right) \pi_{Sr}; \overline{x}, \underline{x}\right]\right) = V\left(\left[q^* \pi_{Rs} + \left(1 - q^*\right) \pi_{Rr}; \overline{x}, \underline{x}\right]\right). \tag{2}$$

That is, given q^*, the (reduced) lottery over outcomes induced by S is indifferent to that induced by R. Just as with expected utility, the equilibrium belief q^* must be that which makes the total probability of 'winning' from S equal to that from R. Thus, as long as winning is strictly better than losing, this equilibrium q^* is invariant to the exact value of the prizes. Strictly speaking, this argument requires that V be strictly increasing in the probability of winning, but

[3] For these equilibrium beliefs to be well-defined, we are assuming that $u\left(\overline{x}\right) - u(\underline{x})$ is finite. We retain this or the equivalent 'boundedness' assumptions below.

most widely-used atemporal nonexpected-utility models retain such monotonicity. They assume that preferences respect first-order stochastic dominance. This discussion is summarized in the following proposition.

Proposition 1 *Provided the players' preferences respect first-order stochastic dominance, the invariance property of mixed equilibria in strictly competitive games is maintained if we relax independence but retain reduction.*

4 Recursive Expected Utility

Each action by a player results in that player facing a two-stage lottery. For example, if row believes column is choosing s with probability q, then row's choosing S yields the two-stage lottery where the second-stage lottery $[\pi_{Ss}; \overline{x}, \underline{x}]$ occurs with first-stage probability q, and the second-stage lottery $[\pi_{Sr}; \overline{x}, \underline{x}]$ with first-stage probability $1-q$. Let us write such two-stage (real-valued) lotteries in the form $X = [q; [\pi_{Ss}; \overline{x}, \underline{x}], [\pi_{Sr}; \overline{x}, \underline{x}]]$, and let L^2 be the set of such two-stage lotteries. Let the function $W : L^2 \to R$ represent a general (complete, transitive and continuous)[4] preference relation over L^2. In this context the utility function W satisfies recursive expected utility if W can be written in the following form:

$$W([q; [\pi; \overline{x}, \underline{x}], [\pi'; \overline{x}, \underline{x}]]) = qv \circ u^{-1} (U ([\pi; \overline{x}, \underline{x}])) + (1-q)\, v \circ u^{-1} (U ([\pi'; \overline{x}, \underline{x}])),$$

where U is the expected utility of lotteries that resolve in the second stage (using expected-utility index u), and v is the utility index for lotteries that resolve in the first stage. To help understand this form, notice that $u^{-1} (U ([\pi; \overline{x}, \underline{x}])) := u^{-1}(u(\overline{x}) + \pi(u(\overline{x}) - u(\underline{x})))$ is just the certainty equivalent of the second-stage lottery $[\pi; \overline{x}, \underline{x}]$ according to the expected u-utility preferences. Similarly, $u^{-1} (U ([\pi'; \overline{x}, \underline{x}]))$ is the corresponding certainly equivalent of the second-stage lottery $[\pi'; \overline{x}, \underline{x}]$. The function W is evaluated by taking the expected v-utility of the first-stage lottery that assigns probability q to the (second-stage) certainty equivalent of $[\pi; \overline{x}, \underline{x}]$ and $1-q$ to the (second-stage) certainty equivalent of $[\pi'; \overline{x}, \underline{x}]$.

Set $\varphi := v \circ u^{-1}$. Under recursive expected utility, row's equilibrium belief q^* is given by

$$\begin{aligned} & q^* \varphi (u(\underline{x}) + \pi_{Ss} (u(\overline{x}) - u(\underline{x}))) + (1-q^*) \varphi (u(\underline{x}) + \pi_{Sr} (u(\overline{x}) - u(\underline{x}))) \\ = \; & q^* \varphi (u(\underline{x}) + \pi_{Rs} (u(\overline{x}) - u(\underline{x}))) + (1-q^*) \varphi (u(\underline{x}) + \pi_{Rr} (u(\overline{x}) - u(\underline{x}))) . \qquad (3) \end{aligned}$$

That is, given q^*, the two-stage lottery induced by S is indifferent to that induced by R. In contrast to equations (1) and (2), in equation (3), the first-stage q-probabilities do not multiply the second-stage π-lotteries directly. Instead, they multiply the function φ evaluated at the U-utilities of the π-lotteries. This φ function can be non-linear. For example, if the utility index v is a concave transformation of the utility index u, then φ is concave. Concavity of φ is analogous to risk aversion in a utility index. In this case, row would prefer the U-utilities of the π-lotteries to be less spread out.

[4] In addition we assume that preferences are sufficiently smooth to allow for a twice-differentiable representation.

The following proposition makes this intuition more precise, and stands in stark contrast to the discussion of atemporal nonexpected-utility models.

Proposition 2 *The invariance property of mixed equilibria in strictly competitive games is not maintained if we relax reduction but retain independence. Indeed, for recursive expected utility, the invariance property implies reduction.*

Proof. See Appendix.

5 A Resolution of the Conundrum

It remains to show how the equilibrium beliefs change in response to changes in the prizes. Consider a setting similar to that of Dixit & Skeath, in which the action S results in almost the same lottery for row regardless of what action column chooses. The action R on the other hand results in a very good lottery for row if column chooses s and a very bad lottery for row if column chooses r. In particular, suppose that $\pi_{Rs} > \pi_{Sr} > \pi_{Ss} > \pi_{Rr}$. In this case, it is as if S is a 'safer' action for row than is R.

Following Dixit & Skeath, suppose we raise the stakes by increasing the value of winning, $\overline{x}$, or decreasing the value of losing, $\underline{x}$, or some combination of the two. For example, compare a regular season game with the superbowl. Introspection might suggest that, in the superbowl, row might be less willing to incur risk and hence more inclined to choose S. Therefore to keep row indifferent, his equilibrium belief q^* that column will play s will have to increase.

The following proposition gives conditions on the composite function $\varphi = v \circ u^{-1}$ associated with recursive expected utility that yield this comparative static.

Proposition 3 *Suppose that* $\pi_{Rs} > \pi_{Sr} > \pi_{Ss} > \pi_{Rr}$*; and that the row player's preferences over two-stage lotteries satisfy recursive expected utility. Then a sufficient condition for row's equilibrium belief* q^* *that column will play* s *to increase as* $\underline{x}$ *decreases and/or* $\overline{x}$ *increases is that the associated composite function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\varphi(0) = 0$ *is concave,* $-\varphi''(w)/\varphi'(w)$ *is decreasing and* $-w\varphi''(w)/\varphi'(w)$ *is increasing.*

Proof. See Appendix.

One way to view this result is that, if row holds the same beliefs about column's actions in the superbowl as he does in the regular season, then the safe alternative now looks more attractive. If commentators hold this naïve belief , then they would predict that row is more likely to play S. Of course, it does not follow that, in equilibrium, row will in fact play S more often. If we interpret mixed strategies as randomizations then, as usual, row's equilibrium mix is that which makes column indifferent. Under this interpretation, the result above says that column is more likely to defend the safe alternative in the superbowl than in the regular season. It is tempting to think that, for this same ordering of the π's, a similar comparative static result applies to column's beliefs (and hence to row's randomization). Since $1-\pi_{Rr} > 1-\pi_{Ss} > 1-\pi_{Sr} > 1-\pi_{Rs}$, however, it is *not* the case that s is a 'safer' action for column than is r in Dixit & Skeath's contest. In fact, it can be shown that the change in p^* for the analogous changes in $\overline{y}$ and $\underline{y}$ can not be signed for the general case.[5]

Proposition 3 resembles standard results in comparative statics for expected-utility theory originally due to Pratt (1964).[6] To require $-\varphi''(w)/\varphi'(w)$ to be decreasing is analogous to assuming decreasing absolute risk aversion. Similarly, to require $-w\varphi''(w)/\varphi'(w)$ to be increasing is analogous to assuming increasing relative risk aversion. This analogy should not, however, be taken literally. The function φ is not a utility index. Loosely speaking, rather than mapping outcomes to utils, it maps utils to utils. Grant, Kajii & Polak (1998) show that curvature of φ does not capture risk aversion, but rather captures the agent's intrinsic attitude toward information. The following example satisfies the conditions of Proposition 3, but does not require either of the underlying utility indices, u or v, to be risk averse.

Example *Let $u(x) = (x+1)^a - 1$ and $v(x) = (x+1)^b - 1$, where $a > b > 0$. Then $\varphi := v \circ u^{-1}$ is given by $\varphi(w) = (w+1)^{b/a} - 1$, which is concave and has $\varphi(0) = 0$. Moreover,*

$$\begin{aligned} \frac{-\varphi''(w)}{\varphi'(w)} &= \left(\frac{a-b}{a}\right)\frac{1}{w+1} \quad \text{and} \\ \frac{-w\varphi''(w)}{\varphi'(w)} &= \left(\frac{a-b}{a}\right)\frac{w}{w+1}. \end{aligned}$$

So the former is decreasing in w, and the latter is increasing in w, as required. The underlying utility indices are only risk averse, however, if their respective coefficients, a and b, are less than one.

Although the interpretation of φ is not the same as that of a utility index, the analogy allows us to borrow Pratt's machinery of standard comparative statics under risk. Proposition 3 illustrates just one example of this method, but one could similarly translate other results.[7]

Appendix.

Proof of Proposition 2. Let $\underline{u} := u(\underline{x})$, and let $\Delta u := u(\overline{x}) - u(\underline{x})$. It is enough to show that q^* cannot be invariant both to changes in $\underline{u}$ (holding Δu fixed) and to changes in Δu (holding $\underline{u}$ fixed) unless the preferences satisfy reduction. For q^* to be invariant to changes in $\underline{u}$ (holding Δu fixed), differentiating equation (3) with respect to $\underline{u}$, we require

$$\begin{aligned} & q^*\varphi'(\underline{u} + \pi_{Ss}\Delta u) + (1-q^*)\varphi'(\underline{u} + \pi_{Sr}\Delta u) \\ = \; & q^*\varphi'(\underline{u} + \pi_{Rs}\Delta u) + (1-q^*)\varphi'(\underline{u} + \pi_{Rr}\Delta u). \end{aligned}$$

[5] The problem is that the difference in the relevant 'mean values', $\overline{\pi}(\pi_{Rr}, \pi_{Rs}) - \overline{\pi}(\pi_{SS}, \pi_{Sr})$, can not be uniquely signed.

[6] For other examples of comparative statics under expected utility, see Gollier (1995).

[7] For example, sufficient conditions for q^* to decrease as the value of both prizes, $\underline{x}$ and $\overline{x}$ are increased by the same amount are those of Proposition 3 plus risk aversion with respect to lotteries that are degenerate in the first stage.

Evaluating this expression at $\underline{u} = 0$ and $\Delta u = 1$, rearranging and substituting for q^* from expression (3), we get

$$\begin{aligned} & \operatorname{sign}\left[\frac{dq^*}{d\underline{u}}\right] \\ = & -\operatorname{sign}\left[q^*\left(\varphi'\left(\pi_{Rs}\right)-\varphi'\left(\pi_{Ss}\right)\right)-\left(1-q^*\right)\left(\varphi'\left(\pi_{Sr}\right)-\varphi'\left(\pi_{Rr}\right)\right)\right] \\ = & \operatorname{sign}\left[\frac{\left(\varphi'\left(\pi_{Sr}\right)-\varphi'\left(\pi_{Rr}\right)\right)}{\left(\varphi\left(\pi_{Sr}\right)-\varphi\left(\pi_{Rr}\right)\right)}-\frac{\left(\varphi'\left(\pi_{Rs}\right)-\varphi'\left(\pi_{Ss}\right)\right)}{\left(\varphi\left(\pi_{Rs}\right)-\varphi\left(\pi_{Ss}\right)\right)}\right]. \end{aligned} \tag{4}$$

So, for $dq^*/d\underline{u}$ to be zero, we require the difference in the last square bracket to be zero for all π's such that a mixed-strategy equilibrium applies. This in turn implies that

$$\frac{\left(\varphi'\left(\pi_1\right)-\varphi'\left(\pi_2\right)\right)}{\left(\varphi\left(\pi_1\right)-\varphi\left(\pi_2\right)\right)}=\alpha$$

for all $\pi_1 > \pi_2$, where α is a constant. Allowing $\pi_2 = \pi_1 + \Delta\pi$, and taking $\Delta\pi$ arbitrarily small, by l'Hôpital's rule, we obtain

$$\frac{\varphi''(\pi_1)}{\varphi'(\pi_1)}=\alpha$$

for all π_1 (where $\varphi'(\pi_1) > 0$). It is as if the composite function φ must exhibit "constant absolute risk-aversion" (although, we do not require the function to be concave).

Similarly, for q^* to be invariant to changes in Δu (holding $\underline{u}$ fixed) we require

$$\begin{aligned} & q^*\pi_{Ss}\varphi'\left(\underline{u}+\pi_{Ss}\Delta u\right)+(1-q^*)\pi_{Sr}\varphi'\left(\underline{u}+\pi_{Sr}\Delta u\right) \\ = \; & q^*\pi_{Rs}\varphi'\left(\underline{u}+\pi_{Rs}\Delta u\right)+(1-q^*)\pi_{Rr}\varphi'\left(\underline{u}+\pi_{Rr}\Delta u\right). \end{aligned}$$

Again, evaluating this expression at $\underline{u} = 0$ and $\Delta u = 1$, rearranging and substituting for q^* from expression (3), we get

$$\operatorname{sign}\left[\frac{dq^*}{d\left(\Delta u\right)}\right]=\operatorname{sign}\left[\frac{\left(\pi_{Sr}\varphi'\left(\pi_{Sr}\right)-\pi_{Rr}\varphi'\left(\pi_{Rr}\right)\right)}{\left(\varphi\left(\pi_{Sr}\right)-\varphi\left(\pi_{Rr}\right)\right)}-\frac{\left(\pi_{Rs}\varphi'\left(\pi_{Rs}\right)-\pi_{Ss}\varphi'\left(\pi_{Ss}\right)\right)}{\left(\varphi\left(\pi_{Rs}\right)-\varphi\left(\pi_{Ss}\right)\right)}\right]. \tag{5}$$

So, for $dq^*/d\left(\Delta u\right)$ to be zero, we require the difference in the last square bracket to be zero for all π's such that a mixed-strategy equilibrium applies. This in turn implies that

$$\frac{\left(\pi_1\varphi'\left(\pi_1\right)-\pi_2\varphi'\left(\pi_2\right)\right)}{\left(\varphi\left(\pi_1\right)-\varphi\left(\pi_2\right)\right)}=\rho$$

for all $\pi_1 > \pi_2$, where ρ is a constant. Again, allowing $\pi_2 = \pi_1 + \Delta\pi$, and applying l'Hôpital's rule, we obtain

$$\frac{\pi_1\varphi''(\pi_1)}{\varphi'(\pi_1)}=\rho-1$$

for all π_1. It is as if the composite function φ must also exhibit "constant relative risk-aversion".

It is well-known, however, that the only functions that satisfy constant relative and constant absolute risk aversion are linear. By the definition of φ, this implies that the utility index v is an affine transformation of the utility index u. In this case, we can write $v(x) = a + bu(x)$ (where

$b > 0$) and

$$\begin{aligned}
& W([q;[\pi;\overline{x},\underline{x}],[\pi';\overline{x},\underline{x}]]) \\
= \ & q\left[a + bu\circ u^{-1}\left(U\left([\pi;\overline{x},\underline{x}]\right)\right)\right] + (1-q)\left[a + bu\circ u^{-1}\left(U\left([\pi';\overline{x},\underline{x}]\right)\right)\right] \\
= \ & a + b\left(qU\left([\pi;\overline{x},\underline{x}]\right) + (1-q)U\left([\pi';\overline{x},\underline{x}]\right)\right) \\
= \ & a + bU\left([q\pi + (1-q)\pi';\overline{x},\underline{x}]\right).
\end{aligned}$$

But $[q\pi + (1-q)\pi';\overline{x},\underline{x}]$ is simply the one-stage lottery that is the reduction of the two-stage lottery $[q;[\pi;\overline{x},\underline{x}],[\pi';\overline{x},\underline{x}]]$. That is, if the invariance property of mixed equilibria in strictly competitive games applies under the recursive expected-utility preferences W then the preferences also satisfy reduction. ■

Proof of Proposition 3. Set $\underline{u} := u(\underline{x})$ and $\Delta u := u(\overline{x}) - u(\underline{x})$. As $\underline{x}$ decreases, $\underline{u}$ decreases. As $\underline{x}$ decreases and/or $\overline{x}$ increases, Δu increases. So, it is enough to show that q^* is decreasing in $\underline{u}$ (that is, expression (4) is negative), and increasing in Δu (that is, expression (5) is positive).

We deal first with changes in $\underline{u}$. Set $\pi_1 > \pi_2$. Let $w_i := \underline{u} + \pi_i \Delta u$ and let $\zeta_i := \varphi(w_i)$ for $i = 1, 2$. Then, by the mean value theorem,

$$\begin{aligned}
& \varphi' \circ \varphi^{-1}\left(\varphi\left(w_1\right)\right) - \varphi' \circ \varphi^{-1}\left(\varphi\left(w_2\right)\right) \\
= \ & \varphi' \circ \varphi^{-1}\left(\zeta_1\right) - \varphi' \circ \varphi^{-1}\left(\zeta_2\right) \\
= \ & \left(\zeta_1 - \zeta_2\right)\frac{d}{d\zeta}\varphi' \circ \varphi^{-1}\left(\bar{\zeta}\left(\zeta_1,\zeta_2\right)\right)
\end{aligned}$$

for some $\bar{\zeta}\left(\zeta_1,\zeta_2\right)$ in the interval $\left(\zeta_2,\zeta_1\right)$. Applying the inverse function rule, we get

$$\varphi'\left(w_1\right) - \varphi'\left(w_2\right) = \left(\varphi\left(w_1\right) - \varphi\left(w_2\right)\right)\frac{\varphi''\left[\bar{w}\left(w_1,w_2\right)\right]}{\varphi'\left[\bar{w}\left(w_1,w_2\right)\right]}$$

for some $\bar{w}\left(w_1,w_2\right)$ in the interval $\left(w_2,w_1\right)$ (since $\varphi'(w) > 0$).

Let $A(w) := -\varphi''(w)/\varphi'(w)$. We next show that the mean value is increasing in its arguments. By the implicit function theorem,

$$\frac{\partial}{\partial w_1}\bar{w}\left(w_1,w_2\right) = \frac{\left[A(w_1) - A\left(\bar{w}\left(w_1,w_2\right)\right)\right]\varphi'\left(w_1\right)}{A'\left(\bar{w}\left(w_1,w_2\right)\right)\left[\varphi\left(w_1\right) - \varphi\left(w_2\right)\right]},$$

which is greater than zero, since (by our assumptions) A is strictly monotone. Similarly, $\bar{w}\left(w_1,w_2\right)$ is also increasing in w_2.

Putting w_{Sr} for $\underline{u} + \pi_{Sr}\Delta u$ etc., and using this 'mean-value' method, we can rewrite the bracketed term in expression (4) as

$$\frac{\varphi''\left(\bar{w}\left(w_{Sr},w_{Rr}\right)\right)}{\varphi'\left(\bar{w}\left(w_{Sr},w_{Rr}\right)\right)} - \frac{\varphi''\left(\bar{w}\left(w_{Rs},w_{Ss}\right)\right)}{\varphi'\left(\bar{w}\left(w_{Rs},w_{Ss}\right)\right)}.$$

Since $\pi_{Rs} > \pi_{Sr} > \pi_{Ss} > \pi_{Rr}$, we have $\bar{w}\left(w_{Sr},w_{Rr}\right) < \bar{w}\left(w_{Rs},w_{Ss}\right)$. Hence the expression is negative. That is, q^* is decreasing in $\underline{u}$.

The argument for changes in Δu is similar but more involved. Using the same notation as before, by the mean value theorem,

$$\begin{aligned} & \pi_1\varphi'(w_1) - \pi_2\varphi'(w_2) \\ = & \left(\frac{\varphi^{-1}(\zeta_1) - \underline{u}}{\Delta u}\right)\varphi'\left(\varphi^{-1}(\zeta_1)\right) - \left(\frac{\varphi^{-1}(\zeta_2) - \underline{u}}{\Delta u}\right)\varphi'\left(\varphi^{-1}(\zeta_2)\right) \\ = & (\zeta_1 - \zeta_2)\frac{d}{d\zeta}\left[\left(\frac{\varphi^{-1}(\bar{\zeta}) - \underline{u}}{\Delta u}\right)\varphi'\left(\varphi^{-1}(\bar{\zeta})\right)\right] \end{aligned}$$

where $\bar{\zeta}$ is in the interval (ζ_2, ζ_1). Applying the inverse function rule, the last expression becomes:

$$\frac{(\zeta_1 - \zeta_2)}{\Delta u}\left[\frac{\varphi'(\bar{w})}{\varphi'(\bar{w})} + \bar{\pi}\Delta u\frac{\varphi''(\bar{w})}{\varphi'(\bar{w})}\right],$$

where $\bar{\pi}$ is in the interval (π_2, π_1), and $\bar{w} = \underline{u} + \bar{\pi}\Delta u$. That is,

$$\pi_1\varphi'(w_1) - \pi_2\varphi'(w_2) = \frac{(\varphi(w_1) - \varphi(w_2))}{\Delta u}\left[1 - \frac{\bar{\pi}\Delta u}{\bar{w}}R(\bar{w})\right]$$

where $R(w) = -w\varphi''(w)/\varphi'(w)$.

We next show that the mean value is increasing in π_1 and π_2. By the implicit function theorem

$$-\frac{\partial\bar{\pi}}{\partial\pi_1}\frac{1}{\bar{w}^2}\left[\underline{u}R(\bar{w}) + \bar{\pi}\Delta u\bar{w}R'(\bar{w})\right] = \left[\frac{\bar{\pi}\Delta u}{\bar{w}}R(\bar{w}) - \frac{\pi_1\Delta u}{w_1}R(w_1)\right].$$

Given our assumptions, R is positive and increasing, so $\bar{\pi}$ is increasing in π_1. Similarly, it is increasing in π_2.

Let $\bar{\pi}(\pi_{Sr}, \pi_{Rr})$ be the relevant mean-value for π_{Sr} and π_{Rr}, and let $\bar{\pi}(\pi_{Rs}, \pi_{Ss})$ be the relevant mean-value for π_{Rs} and π_{Ss}. Since $\pi_{Rs} > \pi_{Sr} > \pi > \pi_{Rr}$, we have $\bar{\pi}(\pi_{Rs}, \pi_{Ss}) > \bar{\pi}(\pi_{Sr}, \pi_{Rr})$. Applying the mean-value method, we can rewrite the bracketed term in expression (5) as:

$$\frac{\bar{\pi}(\pi_{Rs}, \pi_{Ss})\Delta u}{\underline{u} + \bar{\pi}(\pi_{Rs}, \pi_{Ss})\Delta u}R\left(\underline{u} + \bar{\pi}(\pi_{Rs}, \pi_{Ss})\Delta u\right) - \frac{\bar{\pi}(\pi_{Sr}, \pi_{Rr})\Delta u}{\underline{u} + \bar{\pi}(\pi_{Sr}, \pi_{Rr})\Delta u}R\left(\underline{u} + \bar{\pi}(\pi_{Sr}, \pi_{Rr})\Delta u\right)$$

which is positive since R is positive and increasing. Hence q^* is increasing in Δu. ■

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