Asset Prices and Business Cycles under Limited Commitment*

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First draft: August 8, 1997
This version: October 23, 1999

Abstract

This paper presents a business-cycle model with heterogeneous agents that have access to complete markets but face endogenous borrowing and savings constraints. These constraints are motivated by the agents' limited commitment technology. In this environment, aggregate fluctuations are close to the ones generated by Pareto Optimal (full commitment) risk-sharing arrangements. However, endogenous borrowing and savings constraints force agents to under-invest in capital and increase the volatilities of both the stochastic discount factor and the price of equity. The mechanism explains simultaneously both high average returns on equity and low average returns on bonds. This is accomplished in the economy with relatively small exogenous shocks and a high degree of patience, and a low degree of risk-aversion on the part of the agents. Previous work on limited commitment has concentrated on endowment economies and has emphasized borrowing constraints. Numerical results in this paper suggest that when capital is added to such models, savings constraints play even more central role.

Keywords: Asset Prices, Business Cycles, Heterogeneous Agents, Limited Commitment, Endogenous Borrowing and Savings Constraints, Equity Premium Puzzle.

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*This paper is preliminary and incomplete. Comments are most welcome. Revisions can be downloaded from `http://home.uchicago.edu/~jiseppal/research`.

†I am grateful to Fernando Alvarez, Marco Bassetto, Marco Cagetti, John Cochrane, Andrea Eisfeldt, Ana Fernandes, Lars Peter Hansen, Marcelo Navarro, Thomas Sargent, and Annette Vissing-Jørgensen for discussions and comments about the paper. I also received useful comments from the participants at the Hansen–Sargent Macro lunch group, Money and Banking workshop at the University of Chicago, and J.P. Morgan Emerging Markets Research group. I am especially indebted to Klaus Schittkowski for granting me a permission to use his Fortran code, to Thomas Sargent for much needed encouragement, and to Andrew Sellgren for extremely detailed comments. I would like to thank the Yrjö Jahnsson Foundation and Margaret O. Reid Fund for financial support. Parts of this paper were written while I was working at the J.P. Morgan Emerging Markets Research Department and the Bank of Finland Research Department. I would like to thank both organizations for their support. All errors are mine.

1 Introduction

Modern models of the business cycle do poorly at explaining the behavior of asset prices.[1] As the literature points out, it is particularly difficult to explain observed equity premiums (Mehra and Prescott, 1985), holding-period yields (Grossman, Melino, and Shiller, 1987), and average risk premiums in forward prices (Backus, Gregory, and Zin, 1989).

This is an important shortcoming of business-cycle models. As Cochrane and Hansen (1992) emphasize, these models center on intertemporal decisions, and asset prices provide information about intertemporal margins (marginal rates of substitution and transformation). Therefore, asset-pricing implications should be valuable in determining the relative success of different models.

The objective of my research is to refine general-equilibrium models of the business cycle, so their implications for asset prices are more in line with empirical observations. I share this goal with Boldrin, Christiano, and Fisher (1995, 1999), Christiano and Fisher (1998), Jermann (1998), and Tallarini (1999), but their approaches are different. Boldrin, Christiano, and Fisher, Christiano and Fisher, and Jermann use habit-formation preferences, whereas Tallarini has non-expected utility preferences, and I use limited-commitment constraints.

Alvarez and Jermann (1999ab) use limited commitment to explain the behavior of asset prices in endowment economies. Limited-commitment models assume that agents in the economy can default on their contracts if it is in their interests to do so. Therefore, the allocations in the economy are constrained in such a way that no agent has an incentive to default.

Alvarez and Jermann show how to decentralize an endowment economy without commitment by using endogenous solvency constraints. They have considerable success when they study the model's implications regarding the risk-free rate and equity prices. Seppälä (1999) applies the same model to the term structure of interest rates, and succeeds in matching both the sign and the magnitude of average risk premiums in forward prices.

These solvency constraints are important for three reasons. First, they introduce a wedge between marginal rates of substitution and asset prices. This wedge is the key factor in explaining asset-pricing anomalies in endowment economies. Second, they bring an endogenous justification for debt, solvency, short-selling, and other exogenous constraints that are commonly used in the literature on incomplete markets. Finally, they help explain why observed risk-sharing arrangements do not completely smooth consumption over time, space, and the states of the world.

A common approach to explaining limited risk-sharing is to assume that markets are exogenously incomplete.[2] The limited-commitment approach has three advantages over such an assumption. First, under limited commitment, allocations do not depend on a particular arbitrary set of assets that are considered to be available. Second, the markets are complete, so any security can be priced. Finally, the incompleteness of markets can be endogenenized through the above-mentioned solvency constraints.

Models with non-standard preferences and complete markets, e.g., Boldrin, Christiano, and Fisher (1995, 1999), Christiano and Fisher (1998), Jermann (1998), or Tallarini (1999), do not have anything to say about borrowing, savings, and solvency constraints. Since my second objective is to study these constraints, I decided not to follow their example.

[1]For a review of the modern business-cycle theory and its asset-pricing implications, see Cooley (1995).

[2]For a review of the literature on incomplete markets, see Magill and Quinzii (1996).

In endowment economies solvency constraints form a sequence of state-dependent borrowing constraints that may bind for at most one agent in any state. In contrast, when capital is introduced into the model, the agents can self-insure by accumulating enough capital. This has interesting implications on both solvency constraints and asset prices.

First, with capital there is a sequence of savings constraints that bind for *both agents all the time.* Rogerson (1985) obtains a somewhat similar result in very different context. He studies a repeated moral-hazard problem with a risk-neutral principal and a risk-averse agent and shows that for every outcome of every period the agent would choose to save some of his wage if he could.

In the limited commitment production economy, aggregate fluctuations are close to the ones generated by Pareto Optimal (full commitment) risk-sharing arrangements. However, the always binding savings constraints force agents to underinvest in capital and increase the volatility of investment. Together with borrowing constraints, savings contraints also increase the volatilities of both the stochastic discount factor and the price of equity. This in turn increases the equity premium considerably without increasing the volatility of either interest rates or returns on equity too much. In fact, with the standard real business cycle parameters, the model is significantly closer in matching the second moments of risk-free rates and returns on equity than habit-formation models of Boldrin, Christiano, and Fisher (1995, 1999), Christiano and Fisher (1998), and Jermann (1998).

Moreover, the large equity premium is obtained without unrealistically low discount factors as in Alvarez and Jermann (1999ab) or high risk aversion parameter values as in Tallarini (1999). This result is illustrated in Figure 1 that summarizes the results in Alvarez and Jermann and relates them with the results in this paper. Alvarez and Jermann show that depending on parameter values in a limited-commitment endowment economy one can observe either full risk-sharing (as in a full commitment economy), no risk-sharing (autarky), or some risk-sharing. The third case is interesting for the purpose of asset pricing. Risk-sharing depends on parameter values as follows. When one either (i) increases risk-aversion ($\sigma \uparrow$), (ii) increases patience ($\beta \uparrow$), (iii) increases the depreciation rate of capital ($\delta \uparrow$), (iv) increases the variance of exogenous shocks ($\mathrm{var}(\epsilon_t) \uparrow$), or (v) decreases the persistence of exogenous shocks ($\rho(\epsilon_t) \downarrow$), the agents want more risk-sharing.

In order to increase the equity premium one has to reduce risk-sharing. In an endowment economy, an incentive to participate in risk-sharing is very high so that only by lowering the discount factor can autarky become tempting. In a production economy, even patient agents can consider autarky as capital allows for considerable self-insurance.

This paper also builds on earlier results by Kehoe and Levine (1993), Luttmer (1996), Kocherlakota (1996), Kehoe and Perri (1998), Hayashi (1996), and Ligon, Thomas, and Worrall (1998). To my knowledge, Kehoe and Levine is the first general-equilibrium model with endogenous solvency constraints. Like Alvarez and Jermann, Kehoe and Levine (1993) study an endowment economy. The main difference between these two papers is that Kehoe and Levine restrict agents' consumption possibility sets, whereas Alvarez and Jermann restrict agents' asset holdings.

To my knowledge, Luttmer's paper is the first to emphasize the quantitative importance of solvency constraints in explaining the behavior of asset prices. Kocherlakota studies an environment that is similar to that in Alvarez and Jermann. However, Kocherlakota considers only i.i.d. shocks in the planner's problem. Alvarez and Jermann decentralize the economy with serially correlated income shocks.

The planning problem in my paper is largely the same as that in Kehoe and Perri. The differences are as follows. First, they model an international economy. Second, they do not address

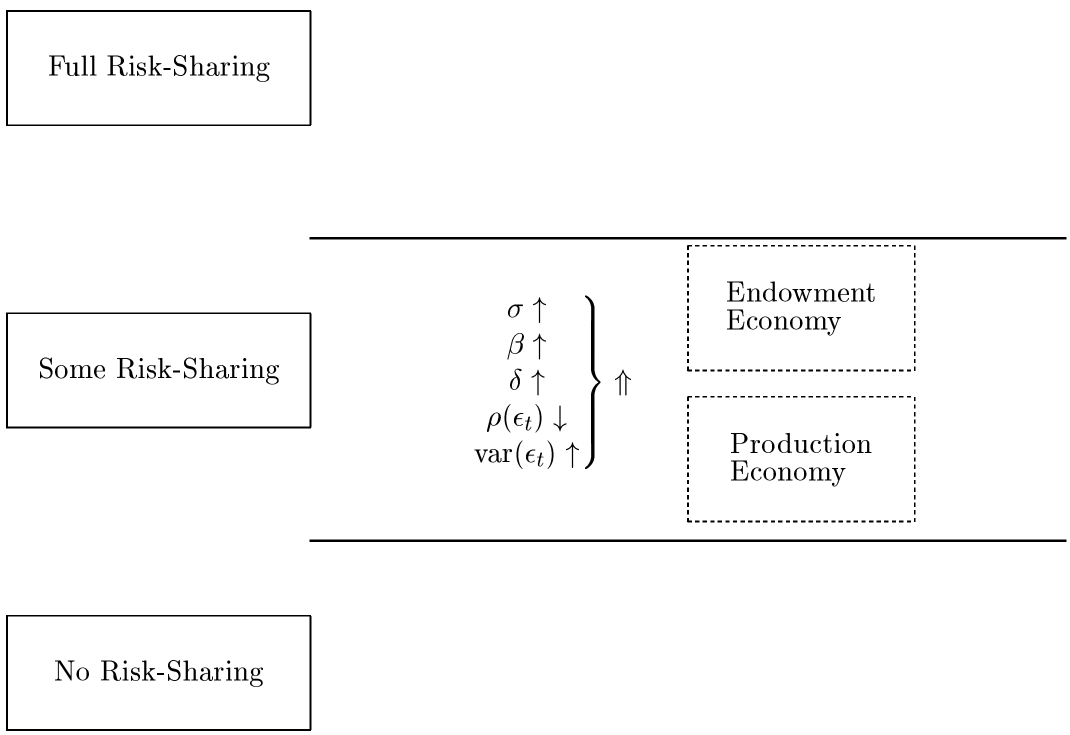

Figure 1: Parameter Values and Risk-Sharing in Production and Endowment Economies

asset prices and solvency constraints. Finally, their decentralization and solution algorithm is very different from mine.

The planning problem in this paper (in the current version) was inspired by Hayashi who only considers an endowment economy. Ligon, Thomas, and Worrall have a model of limited commitment economy with storage, but do not decentralize the economy. Some of the analytical results in this paper are similar to their results.

Cole and Kocherlakota (1997) is a related paper in the sense that they want provide microfoundations for incomplete security markets. They show that an environment with hidden income and hidden storage can be decentralized through the asset market that allows agents to trade risk-free bonds. One of the objectives of my paper is to demonstrate that the limited-commitment mechanism allows analysis of much richer environments than a mechanism that relies on adverse selection or moral hazard.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 explains how the prices and allocations can be solved numerically. Section 4 presents the parameter values used in the paper. Section 5 presents the numerical results, and Section 6 concludes. All proofs can be found in Appendix A.

2 The Model

2.1 The Environment

The economy contains two sectors, $i = 1, 2$, each associated with a large number of infinitely-lived consumer-firm pairs with an identical production technology. Sectors can be thought as two households, villages, regions, or countries. One interpretation is that each consumer has a firm in his or her backyard, and the shock affecting the productivity of the firm is the same within each sector. That is, in each period, one sector can use only of its input resources, and the output is affected by a sector-specific technology shock.

In each period, an event s_t is realized out of a set of finitely many possible events S_t. Let $s^t = (s_0, \ldots, s_t)$ denote the history of s up to period t. The matrix Π determines the conditional probabilities for all histories $\pi(s^t|s_0)$.

The output in both sectors is produced using a technology that exhibits constant returns to scale:

$$y^i(s^t) = z^i(s^t) f(k^i(s^{t-1}), n^i(s^t)), \tag{1}$$

where the superscripts denote the sector, y is the output, z is a technology shock, $f(\cdot, \cdot)$ is a production function, and k and n denote the capital and labor input, respectively. For simplicity, I will suppress the superscripts unless needed; I will use x_t in place of $x^i(s^t)$. I use the notation that the variables dated 't' are measurable with respect to the information available at time t.

The consumers in both sectors maximize their expected lifetime utility, defined over the consumption of output c and leisure $\ell = 1 - n$:

$$\max U(c_t, \ell_t) = \sum_{j=0}^{\infty} \sum_{s^{t+j} \in S^{t+j}} \beta^j u(c_{t+j}, \ell_{t+j}) \pi(s^{t+j}|s_t), \tag{2}$$

where $\beta \in (0, 1)$ denotes the discount factor. The period utility $u(\cdot, \cdot)$ is strictly increasing, strictly concave, and C^1 in both arguments, and $u_1(0, x) = u_2(x, 0) = +\infty$ for all $0 \leq x < \infty$.

The feasibility constraint for the economy is that the combined output of both sectors can be either consumed or invested in the capital stock of the next period:

$$\sum_{i=1,2} c_t^i + k_t^i = \sum_{i=1,2} y_t^i + (1-\delta)k_{t-1}^i, \tag{3}$$

where $\delta \in [0, 1]$ denotes the depreciation rate of capital.

In addition, the agents face a participation constraint. That is, the allocations are constrained so that they make the agents better off than autarky under every possible history:

$$\sum_{j=0}^{\infty} \sum_{s^{t+j} \in S^{t+j}} \beta^j u(c_{t+j}, \ell_{t+j}) \pi(s^{t+j}|s_t) \geq V^a(k_{t-1}, s^t) \quad \forall t \geq 0, \;\; s^t \in S^t, \tag{4}$$

where $V^a(k_{t-1}, s^t)$ is the value function associated with the autarky problem:

$$V^a(k_{t-1}, s^t) = \max \sum_{j=0}^{\infty} \sum_{s^{t+j} \in S^{t+j}} \beta^j u(c_{t+j}, \ell_{t+j}) \pi(s^{t+j}|s_t) \tag{5}$$

subject to

$$c_t + k_t = z_t f(k_{t-1}, n_t) + (1-\delta)k_{t-1} \quad \forall t \geq 0. \tag{6}$$

2.2 The Planning Problem

As in Kocherlakota (1996) and Alvarez and Jermann (1999ab), I set up the planning problem for determining optimal allocations as a problem of maximizing the expected lifetime utility of agent 1 subject to feasibility and participation constraints and given some expected lifetime utility for agent 2.

The recursive formulation of the problem is given by the following functional equation:

$$TV(\omega, k_1, k_2, s) = \max u(c_1, \ell_1) + \beta \sum_{s' \in S} V(\omega_{s'}, k_1', k_2', s') \pi(s'|s) \tag{7}$$

subject to

$$c_1 + c_2 + k_1' + k_2' \leq z^1(s) f^1(k_1, 1-\ell_1) + z^2(s) f^2(k_2, 1-\ell_2) + (1-\delta)k_1 + (1-\delta)k_2 \tag{8}$$

$$u(c_2, \ell_2) + \beta \sum_{s' \in S} \omega_{s'} \pi(s'|s) \geq \omega \tag{9}$$

$$V(\omega_{s'}, k_1', k_2', s') \geq V_1^a(k_1', s') \quad \forall s' \in S \tag{10}$$

$$\omega_{s'} \geq V_2^a(k_2', s') \quad \forall s' \in S, \tag{11}$$

where primes denote next-period values, $V(\omega_{s'}, k_1', k_2', s')$ is the value function of agent 1, and ω is the utility promised to agent 2. Specifically, $\omega_{s'}$ is the promised utility of agent 2 for the next period when the state is s'. Equation (8) is the feasibility constraint, (9) is the promise-keeping constraint, and (10) and (11) are participation constraints.

Both Kocherlakota and Alvarez and Jermann present similar functional equations. As Kocherlakota considers only the i.i.d. case, his only state variable is ω. Alvarez and Jermann relax this assumption, so they have keep track of the current state s as well. When capital is introduced to the model, there are two additional state variables: k_1 and k_2.

As in Alvarez and Jermann, the problem (7)–(11) has a non-trivial domain. For each s the domain of $TV(\omega, k_1, k_2, s)$ is the set (ω, k_1, k_2) such that

$$TV(\omega, k_1, k_2, s) \geq V_1^a(k_1, s) \quad \text{and} \quad \omega \geq V_2^a(k_2, s).$$

The problem with the domain makes it difficult to obtain analytical expressions for V and considerably complicates numerical analysis. In words, if there is not much capital in the economy, the social planner cannot promise too much utility to agent 2. Since the numerical solution method used in this paper requires specification of the set (ω, k_1, k_2, s) in advance, finding the solution is not an easy matter. For this reason, in Section 3, I reformulate the problem following Hayashi (1996) in terms of promised planning weights instead of promised utilities.

One additional problem relative to Kocherlakota and Alvarez and Jermann in analyzing the planner's problem is that the constraints need not be convex, due to constraints (10) and (11). Nevertheless, it is possible to show the following:

Proposition 1. *If autarky is not the only allocation satisfying the participation constraints then the participation constraint of at most one agent is binding in each period.*

Non-convexity of constraints means that it is not trivial to show that the value function is concave or that the decision rules are single valued. Therefore, I make the following two assumptions:

Assumption 1. *The value function is once differentiable.*

Assumption 2. *The decision rules are single valued.*

I hope to be able to show that these assumptions hold and hence the following proposition and its corollary hold more generally.[3]

Proposition 2. *Under Assumptions 1 and 2, if both agents are unconstrained for s', then their marginal rates of substitution are equalized. If one agent is constrained then his marginal rate of substitution will be strictly smaller than that of the other agent.*

In Section 3, I reformulate the problem in terms of promised planning weights instead of promised utilities. When one replicates the analysis in the proof of Proposition 2 in this new environment, the following corollary is immeadiate.

Corollary 1. *Under Assumptions 1 and 2, if both agents are unconstrained for s', then their planning weights stay constant. If one agent is constrained then his planning weight will be increased.*

[3]In the numerical example considered in Sections 4 and 5, the proposition held.

Notice that binding participation constraint implies higher planning weight for the constrained agent. The planner has to make sure that the agent who would like to default wants to stay in the risk-sharing arrangement. This can obtained two ways: either by increasing the current utility by increasing current consumption and leisure or by increasing the promised about future utilities. However, since the agents are risk-averse they will prefer the latter scheme as the first would increase the volatility of allocations too much. Increasing the promised utilities or planning weights allows the planner to smooth increase over several time periods.

In order to analyze asset prices, this economy has to be decentralized. The above propositions are useful in tightening the link between the solution to the planning problem and its decentralization.

2.3 Competitive Equilibrium and Asset Pricing

The decentralization I have in mind is the following. There are two sectors both having separate labor and capital markets. In each period in both sectors the firms rent capital and labor services from the households in their respective sector. Households can consume and save the income from these rental services and also (partially) hedge against the shocks by trading state-contingent securities with each other. However, the fact that the households cannot commit to their contracts rules out large negative positions in the state-contingent securities. The households with large debt payments would like to default instead of continuing the risk-sharing arrangement. Also, the households that have accumulated enough wealth (capital) have no need to participate in risk-sharing. In the decentralization both these possibilities are ruled out by endogenously generated solvency constraints—borrowing and savings constraints—that restrict how much the households can borrow from each other and how much they can save in terms of capital.

The environment differs from the usual Arrow-Debreu world only through the above-mentioned solvency constraints. I assume that in addition to the households and firms the economy contains financial intermediaries that take care of risk-sharing arrangements. Financial intermediaries operate in a competitive manner and are the only agents in the economy with an access to commitment technology.

Firms

Firms solve a static profit maximization problem

$$\max z_t f(k_{t-1}, n_t) - w_t n_t - r_t k_{t-1},$$

where w and r denote the wage rate and the rental rate of capital, respectively.

Households

The households maximize (2) subject to their budget constraint:

$$c_t + k_t + \sum_{s^{t+1} \in S^{t+1}} q(s^{t+1}, s^t) a(s^{t+1}) \leq r_t k_{t-1} + w_t n_t + (1-\delta) k_{t-1} + a_t \tag{12}$$

and the borrowing constraints:

$$a(s^{t+1}) \geq B(s^{t+1}) \quad \forall s^{t+1} \in S^{t+1},$$

and the savings constraint:

$$k_t \leq C_t,$$

where $q(s^{t+1}, s^t)$ determines the price of a security $a(s^{t+1})$ that pays one unit of consumption good at the beginning of the next period when the next period's state is s^{t+1} and the current period's state is s^t. $B(s^{t+1})$ is the minimum asset position the agent can take in an asset that pays when the state is s^{t+1}, and C_t is the maximum capital stock position the agent is allowed to hold. From now on, I will call $B(s^{t+1})$ a *borrowing constraint* and C_t a *savings constraint.*[4]

Notice that only when the horizon is infinite the autarky is not the only equilibrium for this economy. If there exists the final period T, the households with $a_T < 0$ would default. Since everybody knows that, nobody would participate in the risk-sharing arrangement in the period $T-1$. Therefore, the households with $a_{T-1} < 0$ would default. Repeating the argument, we obtain that nobody will participate in the risk-sharing arrangement in any period unless the horizon is infinite.

Financial Intermediaries

There is a large number of financial intermediaries that arrange the risk-sharing arrangement between the sectors. Financial intermediaries operate in a competitive manner and are the only agents in the economy with an access to commitment technology. Each household is matched with a financial intermediary who observes the household's wealth (capital stock and state-contingent debt) and makes the household an offer. The household is allowed to save in capital only up to the amount C_t and issue state-contingent debt up to the amount $B(s^{t+1})$. The household has three options. First, the household can accept the offer and issue new state-contingent debt. Second, the household can go to see other financial intermediary. Third, the household can revert to autarky. In the equilibrium, the household will always accept the offer. Finally, the financial intermediaries sell the state-contingent debt in competitive markets.

Hence, the agents are kept within the risk-sharing arrangement by setting the savings and borrowing constraints in the right way. The introduction of the savings constraint may seem counter-intuitive. The following example from Obstfeld and Rogoff (1996) may help to understand why they are needed. Suppose that the world is deterministic and both sectors have the production function $y = f(k)$. Let $R^\star$ be the inverse of Arrow pricing function q in the steady state. When the sector reaches the steady state capital stock $k^\star$ at which $f'(k^\star) = R^\star$ the sector has no need for financial markets. The purpose of the savings constraint is to prevent this from happening.

Some kind of savings constraint is always needed in models with commitment problems and capital. The reason is that with full commitment the capital is chosen using a marginal condition

[4] Notice that since the production technology exhibits constant returns to scale, one could decentralize the economy without firms. Households in both sectors would have a backyard production technology:

$$c_t + k_t + \sum_{s^{t+1} \in S^{t+1}} q(s^{t+1}, s^t) a(s^{t+1}) \leq z_t f(k_{t-1}, n_t) + (1-\delta) k_{t-1} + a_t.$$

that equates expected marginal rates of substitution and transformation. As participation constraints will bind either agent with small probability, they bring a wedge between marginal rates of substitution and transformation. Savings constraints are needed in order to account for this wedge. Kehoe and Perri (1998) take care of this wedge by introducing a government that taxes the capital exactly the right amount.[5]

In my setup, in order to stay within risk-sharing arrangement, the sector has to precommit to its investment strategy. The financial intermediary is not only interested in how much each sector wants to borrow. He also wants to know what the sector is going to do with its resources. Since this is a world of perfect information, it is not possible to overinvest and participate in the financial markets at the same time. The punishment from overinvestment would be the exclusion from the financial markets forever. One way to think that about this is to observe that in this economy capital is an illiquid asset. Financial intermediaries do not want agents to accumulate assets that cannot be confiscated.

Competitive Equilibrium

Definition 1. The equilibrium *given the solvency constraints* $\{B_t^i, C_t^i\}$ *is a set of prices* $\{w_t^i, r_t^i, q_t\}$ *and allocations* $\{c_t^i, \ell_t^i, k_t^i, a_{t+1}^i\}$ *such that*

1. *Taking prices as given the allocations solve both the firms' and the households' optimization problems.*

2. *Markets clear:*

$$a^1(s^{t+1}) + a^2(s^{t+1}) = 0 \quad \forall t \geq 0, \forall s^{t+1} \in S^{t+1}.$$

3. *Feasibility is satisfied:*

$$\sum_{i=1,2} c_t^i + k_t^i = \sum_{i=1,2} y_t^i + (1-\delta)k_{t-1}^i. \tag{13}$$

The optimality conditions for the firms are:

$$r_t = z_t f_1(k_{t-1}, 1-\ell_t) \quad \text{and} \tag{14}$$
$$w_t = z_t f_2(k_{t-1}, 1-\ell_t). \tag{15}$$

[5] Kehoe and Perri consider an international economy with two countries. The government of each country can tax payments made to foreignors and capital income and then rebate the proceeds in a lump sum fashion to private agents. The governments in the two countries sequentially choose policy in an optimal fashion to maximize the welfare of their residents. The household's budget constraint in their economy is

$$c_t + k_t + \sum_{s^{t+1} \in S^{t+1}} q(s^{t+1}, s^t) a(s^{t+1}) \leq (1-\tau_t^k)(1 + r_t - \delta)k_{t-1} + (1-\tau_t^a)a_t + w_t n_t + T_t,$$

where τ_t^k and τ_t^a and tax rates on capital and asset holdings, respectively, and T_t are lump sump transfers of the government. These are chosen in such a way that the participation constraints for both countries are satisfied.

Households have first-order conditions:

$$u_2(c_t, \ell_t) = w_t u_1(c_t, \ell_t) \tag{16}$$

$$u_1(c_t, \ell_t) \begin{cases} = \beta E_t[u_1(c_{t+1}, \ell_{t+1})(1 + r_{t+1} - \delta)] \text{ if } k_t < C_t), \\ \leq \beta E_t[u_1(c_{t+1}, \ell_{t+1})(1 + r_{t+1} - \delta)] \text{ if } k_t = C_t. \end{cases} \tag{17}$$

$$q(s^{t+1}, s^t) u_1(c_t, \ell_t) \begin{cases} \geq \beta \pi(s^{t+1}|s_t) u_1(c_{t+1}, \ell_{t+1}) \text{ if } a(s^{t+1}) = B(s^{t+1}), \\ = \beta \pi(s^{t+1}|s_t) u_1(c_{t+1}, \ell_{t+1}) \text{ if } a(s^{t+1}) > B(s^{t+1}). \end{cases} \tag{18}$$

and transversality conditions:

$$\lim_{t \to \infty} E_0 \beta^t u_1(c_t, \ell_t)[a_t - B_t] = 0$$
$$\lim_{t \to \infty} E_0 \beta^t u_1(c_t, \ell_t)[C_t - k_t] = 0.$$

Asset Pricing

As in Luttmer (1996), Cochrane and Hansen (1992), and Alvarez and Jermann (1999ab), the prices of Arrow securities are given by the maximum of the marginal rates of substitution of agent 1 and 2. That is,

$$q(s^{t+1}, s^t) = \max_{i=1,2} \beta \frac{u_1(c^i_{t+1}, \ell^i_{t+1})}{u_1(c^i_t, \ell^i_t)} \pi(s^{t+1}|s_t). \tag{22}$$

The economic intuition is that the unconstrained agent in the economy does the pricing. As $B(s^{t+1})$ gives the minimum amount of an asset one can buy, the constrained agent would like sell that asset and hence his marginal valuation of the asset is lower. To make the same point in other words, the constrained agent has an internal interest rate that is higher than the market rate. Therefore, he would like to borrow more than is feasible to keep the autarky constraints satisfied.

These $q(s^{t+1}, s^t)$ determine the *pricing kernel* or the *stochastic discount factor* m_{t+1} as follows:

$$m_{t+1} \equiv \max_{i=1,2} \beta \frac{u_1(c^i_{t+1}, \ell^i_{t+1})}{u_1(c^i_t, \ell^i_t)}.$$

The price of a one-period bond p^b_t is equal to the price of a portfolio containing equal weights of one-period Arrow securities, each priced according to (22):

$$p^b_t = \sum_{s^{t+1} \in S^{t+1}} \max_{i=1,2} \beta \frac{u_1(c^i_{t+1}, \ell^i_{t+1})}{u_1(c^i_t, \ell^i_t)} \pi(s^{t+1}|s_t) = E_t[m_{t+1}].$$

As in Bulow and Rogoff (1989), I define the equity to be a claim to an infinite stream of future net income. Hence, the price of equity p^e_t is as follows:

$$p^e_t = \sum_{j=1}^{\infty} \sum_{s^{t+j} \in S^{t+j}} q(s^{t+j}|s_t) \sum_{i=1,2} y^i_{t+j} - (k^i_{t+j} - (1 - \delta) k^i_{t+j-1}) \tag{23}$$

$$= E_t \left[\sum_{j=1}^{\infty} m_{t+j} \sum_{i=1,2} y^i_{t+j} - (k^i_{t+j} - (1 - \delta) k^i_{t+j-1}) \right], \tag{24}$$

where $q(s^{t+j}|s_t)$ is the price of an Arrow security from the state s_t to the state s^{t+j}, which is given by

$$q(s^{t+j}|s_t) = \prod_{k=t}^{t+j-1} q(s^{k+1}, s_k), \tag{25}$$

where $q(s^{k+1}, s_k)$ is given by (22).

Existence of Equilibrium

Let $J(a, k, s)$ denote the value function in the household's problem:

$$J(a, k, s) = \max u(c, \ell) + \beta \sum_{s' \in S} J(a(s'), k', s')\pi(s'|s)$$

subject to

$$c + k' + \sum_{s' \in S'} q(s', s)a(s') \leq rk + wn + (1 - \delta)k + a$$
$$a(s') \geq B(s') \quad \forall s' \in S',$$
$$k' \leq C,$$

Let $J^a(k, s)$ denote the household's value function in autarchy and k^a optimal amount of capital the household would choose in autarky.

Definition 2. Borrowing constraints are not too tight *if they satisfy*

$$J(B(s'), C, s') = J^a(k^a, s') \quad \forall s' \in S'.$$

This condition guarantees that *given the limit in capital accumulation* the borrowing constraints prevent default by not letting the agents accumulate more debt than they are willing to pay back.

Definition 3. The prices of Arrow securities are not too high *if the infinite sums of the form*

$$\sum_{j=1}^{\infty} \sum_{s^{t+j} \in S^{t+j}} q(s^{t+j}|s_t)x_{t+j}$$

are finite for all equilibrium objects x_{t+j}.

Given that the borrowing constraints are not too tight and that the prices of Arrow securities are not too high, the transversality conditions are satisfied and the household's problem has a concave objective and convex constraints. By standard arguments the equilibrium exists. In addition, the equilibrium is uniquely determined by the first-order conditions and the transversality conditions.

2.4 Second Welfare Theorem

Proposition 3. *Given an allocation* $\{c_t^i, \ell_t^i, k_t^i\}$ *that satisfies*

1. *the feasibility condition (13) at any period and state,*
2. *the participation constraints (4) at any period and state,*
3. *intratemporal optimality condition*

$$u_2(c_t, \ell_t) = z_t f_2(k_{t-1}, 1 - \ell_t) u_1(c_t, \ell_t)$$

at any period,

4. *that the implied prices of Arrow securities are not too high, and*
5. *that the marginal utility of consumption stays finite:*

$$\lim_{t \to \infty} E_0 \beta^t u_1(c_t, \ell_t) < \infty,$$

then there exist processes $\{a_t^i, B_t^i, C_t^i, r_t^i, w_t^i, q_t\}$ *such that a sequence* $\{c_t^i, \ell_t^i, k_t^i, a_{t+1}^i\}$ *is a competitive equilibrium given the solvency constraints* $\{B_t^i, C_t^i\}$ *and the prices* $\{w_t^i, r_t^i, q_t\}$. *In addition, the borrowing constraints are not too tight.*

The link between participation constraints in the planner's problem and the solvency constraints in the household's problem is the following. When a participation constraint in the planner's problem binds, the corresponding solvency constraint in that state will bind for one agent. However, since the constraints on capital are for the expectation over tomorrow's states, the bounds on capital holdings (savings) may bind for both agents in every period.

2.5 Solvency Constraints and Risk-Sharing

To characterize the effects of solvency constraints, I apply the Recursive Saddle Point (RSP) formulation developed in Marcet and Marimon (1992, 1998). Unlike in the planner's problem, RSP analysis is valid in the household's problem, since it has a concave objective and convex constraints.

Let $R_t \equiv 1 + r_t - \delta$ and $x_t \equiv c_t + k_t - R_t k_{t-1} - w_t n_t$. Solving the household's budget constraint (12) forward gives

$$a_t \geq \sum_{j=0}^{\infty} \sum_{s^{t+j} \in S^{t+j}} q(s^{t+j}|s_t) x_{t+j} = E_t \left[\sum_{j=0}^{\infty} m_{t,t+j} x_{t+j} \right], \tag{26}$$

where $m_{t,t+j} \pi(s^{t+j}|s_t) \equiv q(s^{t+j}|s_t)$ and $q(s^t|s_t) = m_{t,t} = 1$. In addition, at $t = 0$ the household must honor its initial state-contingent claim a_0 inherited from the past:

$$a_0 \geq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} q(s^t|s_0) x_t = E_0 \left[\sum_{t=0}^{\infty} m_{0,t} x_t \right]. \tag{27}$$

Using (26) and (27) at equality, the household's problem is associated with the Lagrangian

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ u(c_t, \ell_t) + \theta_t \Bigg[E_t \sum_{j=0}^{\infty} m_{t,t+j} x_{t+j} - B_t \Bigg] + \Delta \Bigg[a_0 - E_0 \sum_{j=0}^{\infty} m_{0,j} x_j \Bigg] + \nu_t (C_t - k_t) \Bigg\}.$$

Using the law of iterated expectations, noticing that $m_{t,t+2} = m_{t,t+1} m_{t+1,t+2}$, and applying summation by parts, the Lagrangian can rewritten as

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \big\{ u(c_t, \ell_t) + \mu_t x_t - \theta_t B_t + \nu_t (C_t - k_t) \big\} + \Delta a_0,$$

or

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \big\{ u(c_t, 1 - n_t) + \mu_t (c_t + k_t - R_t k_{t-1} - w_t n_t) - \theta_t B_t + \nu_t (C_t - k_t) \big\} + \Delta a_0,$$

where

$$\mu_t = \beta^{-1} m_{t-1,t} \mu_{t-1} + \theta_t,$$

with initial condition $\mu_0 = \theta_0 - \Delta$. First-order conditions with respect to c_t, n_t, k_t are as follows:

$$\begin{aligned} u_1(c_t, \ell_t) + \mu_t &= 0 \\ u_2(c_t, \ell_t) + \mu_t w_t &= 0 \\ \mu_t - \nu_t &= E_t[\beta R_{t+1} \mu_{t+1}]. \end{aligned}$$

Two special cases are worth considering. First, if the solvency constraint never binds then $\theta_t = \nu_t = 0 \ \forall t \geq 0$. In this case, the first-order conditions for both agents satisfy the familiar full risk-sharing condition:

$$\frac{u_1(c_t^1, \ell_t^1)}{u_1(c_t^2, \ell_t^2)} = \frac{\Delta^1}{\Delta^2}.$$

Second, suppose that there is some $t^\star$ such that, $\forall t < t^\star$, $\theta_t = \nu_t = 0$, but $\theta_{t^\star}^1 > 0$ and $\theta_{t^\star}^2 = 0$. In addition, suppose that $\theta_{t^\star+1} = 0$. In this case:

$$\begin{aligned} \frac{u_1(c_{t^\star-1}^1, \ell_{t^\star-1}^1)}{u_1(c_{t^\star-1}^2, \ell_{t^\star-1}^2)} &= \frac{\Delta^1}{\Delta^2} \\ \frac{u_1(c_{t^\star}^1, \ell_{t^\star}^1)}{u_1(c_{t^\star}^2, \ell_{t^\star}^2)} &= \frac{\Delta^1 - \frac{\beta \theta_{t^\star}^1}{m_{0,t^\star}}}{\Delta^2} \\ \frac{u_1(c_{t^\star+1}^1, \ell_{t^\star+1}^1)}{u_1(c_{t^\star+1}^2, \ell_{t^\star+1}^2)} &= \frac{\Delta^1 - \frac{\beta \theta_{t^\star}^1}{m_{0,t^\star}}}{\Delta^2} = \frac{u_1(c_{t^\star}^1, \ell_{t^\star}^1)}{u_1(c_{t^\star}^2, \ell_{t^\star}^2)}. \end{aligned}$$

That is, the one-period impulse from the solvency constraint makes the solution in those subsequent periods in which the constraint is not binding equivalent to a version of the full risk-sharing problem with higher weight assigned to agent 1. Recall Corollary 1 from Section 2.2: If one agent is constrained then his planning weight will be increased. The analysis in this section shows that binding participation constraint in the planner's problem corresponds to a binding solvency constraint in the household's problem.

In this section I only analyzed the effects of a one-period shock on the borrowing constraint for one agent. The analysis of the effects of savings constraints is more complicated because they may bind for both agents at the same time. For this reason and to better understand the effects of solvency constraints at the aggregate level, I now resort to numerical simulations.

3 Algorithm

3.1 New Formulation

I mentioned in Section 2.2 that the problem with the domain makes it difficult to obtain analytical expressions for V and considerably complicates numerical analysis. That is, if there is not much capital in the economy, the social planner cannot promise too much utility to agent 2. Since the numerical solution method used in this paper requires specification of the set (ω, k_1, k_2, s) in advance, finding the solution is not an easy matter. I was not able find the solution using the original formulation for realistic values of the depreciation rate ($\delta < 0.8$), and hence reformulated the problem following Hayashi (1996) in terms of promised planning weights instead of promised utilities. The advantage of using planning weights instead of promised utilities is that the sampling region for the former does not change with the preference and technology parameters.

The planning problem is now

$$\begin{aligned} TW(\lambda, k_1, k_2, s) = \max \lambda \Big\{ u(c_1, \ell_1) + \beta \sum_{s' \in S} V_1(\lambda_{s'}, k_1', k_2', s')\pi(s'|s) \Big\} + \\ (1-\lambda)\Big\{ u(c_2, \ell_2) + \beta \sum_{s' \in S} V_2(1-\lambda_{s'}, k_2', k_1', s')\pi(s'|s) \Big\} \end{aligned} \tag{28}$$

subject to

$$c_1 + c_2 + k_1' + k_2' \leq z^1(s) f^1(k_1, 1-\ell_1) + z^2(s) f^2(k_2, 1-\ell_2) + (1-\delta)k_1 + (1-\delta)k_2 \tag{29}$$

$$V_1(\lambda_{s'}, k_1', k_2', s') \geq V_1^a(k_1', s') \quad \forall s' \in S \tag{30}$$

$$V_2(1-\lambda_{s'}, k_2', k_1', s') \geq V_2^a(k_2', s') \quad \forall s' \in S, \tag{31}$$

where λ denotes the planning weight, (λ, k_1, k_2, s) is the social welfare function, $V_1(\lambda_{s'}, k_1', k_2', s')$ is the value function of agent 1, and $V_2(1-\lambda_{s'}, k_2', k_1', s')$ is the value function of agent 2. Notice that W depends on both V_1 and V_2. For three unknown functions, one has to specify a system of functional equations. The obvious additional functional equations are

$$TV_1(\lambda, k_1, k_2, s) = u(c_1, \ell_1) + \beta \sum_{s' \in S} V_1(\lambda_{s'}, k_1', k_2', s')\pi(s'|s)$$

$$TV_2(1-\lambda, k_2, k_1, s) = u(c_2, \ell_2) + \beta \sum_{s' \in S} V_2(1-\lambda_{s'}, k_2', k_1', s')\pi(s'|s),$$

where the variables on right-hand side come from the planning problem above. A fixed point of operator T gives the value to the problem of maximizing a weighted sum of utilities. The Pareto Optimum (full commitment) problem can be solved from the same system equations, but without imposing (30) and (31). Implication of Corollary 1 is that if the participations constraints do not bind then λ's stay constant. Hence, in the full commitment problem, $\lambda_{t+1} = \lambda_t = \ldots = \lambda_0 \ \forall t$.

3.2 Solution Strategy

The algorithm I use for solving the model numerically is based on ideas presented in Judd (1998), den Haan (1996, 1997), and Christiano and Fisher (1997). The idea is to parameterize the unknown value functions, $V_i(\lambda, k_1, k_2, s)$ and $V_i^a(k, s)$, $i = 1, 2$, using Chebyshev approximation. The details are presented in Appendix B.

Kehoe and Perri (1998) solve a similar problem using the Recursive Saddle Point formulation, but that approach will not work here because this problem need not be convex, due to constraints (10) and (11), so a solution strategy that uses Euler equations may not find the correct solution. Below, I present the solution scheme for $V_1(\lambda, k_1, k_2, s)$.[6] The algorithm for finding $V_1^a(k, s)$ is similar, but simpler.

Recall, M is the number of elements in S. Let $M = 4$; fix $b_{ijk}(s)$; and use (33) to approximate the value function. I use Chebyshev polynomials of degree $N = 5$, and I set $K = 7$ as the number of grid points for each endogenous state variable. For each grid point the maximization problem has a nonlinear objective function over 10 control variables $(c_1, c_2, k_1', k_2', \ell_1, \ell_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The maximization is subject to nine nonlinear inequality constraints. In addition, some of the inequality constraints need not be convex.

This is a very difficult nonlinear programming problem. For example, neither NAG nor IMSL, two popular Fortran libraries for solving mathematical problems, contains a routine that could find a solution to this problem *even when the problem is initialized using the correct* $b_{ijk}(s)$'s. I solve it using a method of sequential quadratic programming developed by Schittkowski (1985).

Solving the maximization problem for each grid point leads to the following linear system:

$$\mathbf{X}\mathbf{b} = \mathbf{Y},$$

[6] Notice that one has to solve for only V_1 as due to symmetricity of preferences and shocks, $V_1(\lambda, k_1, k_2, s) = V_2(\lambda, k_1, k_2, \hat{s})$, where

$$\hat{s} = \begin{cases} 1 & \text{if } s = 1, \\ 3 & \text{if } s = 2, \\ 2 & \text{if } s = 3, \\ 4 & \text{if } s = 4. \end{cases}$$

where

$$\underset{343\times 35}{\mathbf{X}} = [P_i(\varphi_\lambda(\lambda))P_j(\varphi_{k1}(k_1))P_k(\varphi_{k2}(k_2))]$$
$$\underset{35\times 4}{\mathbf{b}} = [b_{ijk}(s)]$$
$$\underset{343\times 4}{\mathbf{Y}} = [TV(\lambda, k_1, k_2; s)].$$

Solving the system by OLS gives a new solution:

$$S(\mathbf{b}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

and the coefficients can be updated by using a relaxation parameter, $0 \le \rho \le 1$:

$$\mathbf{b}_{i+1} = \rho S(\mathbf{b}_i) + (1-\rho)\mathbf{b}_i.$$

From the discrete orthogonality property of Chebyshev polynomials it follows that $(\mathbf{X}'\mathbf{X})^{-1}$ is a diagonal matrix. I iterate the procedure until $\| S(\mathbf{b}_i) - \mathbf{b}_i \|_\infty$ is less than 10^{-9}.

3.3 Asset Prices and Holdings

After finding the value function and decision rules, I solve for the price of equity and asset holdings. Using (22), (24), and (25) notice that

$$p_t^e = E_t\left[m_{t+1}\left(p_{t+1}^e + \sum_{i=1,2} y_{t+1}^i - (k_{t+1}^i - (1-\delta)k_t^i)\right)\right]$$

or

$$p_t^e = E_t\left[m_{t+1}\left(p_{t+1}^e + \sum_{i=1,2} d_t^i)\right)\right],$$

where $d_t^i = y_{t+1}^i - (k_{t+1}^i - (1-\delta)k_t^i)$, $i = 1, 2$, is the dividend from equity. Notice that $\sum_{i=1,2} d_t^i = \sum_{i=1,2} c_t^i$. That is, equity is a claim to aggregate consumption.

The problem is solving for the price of equity is a fixed-point equation in an unknown function $p^e(\lambda, k_1, k_2; s)$. Asset holdings are determined by a functional equation (26)

$$a_t = E_t[m_{t+1}(c_t + k_t - R_t k_{t-1} - w_t n_t + a_{t+1})].$$

Again, I apply Chebyshev approximation to the unknown functions. The difference is that the above equations are linear in the unknown coefficients so that they can solved exactly in one step.

Table 1: Parameter values

Discount factor	$\beta = 0.96$
Curvature parameter	$\sigma = 1.0$
Consumption share	$\gamma = 0.34$
Capital share	$\alpha = 0.36$
Depreciation rate	$\delta = 0.1$

4 Numerical Examples

In order to relate the results of this paper to existing literature, I followed functional forms and parameter values for an annual model given in Prescott (1986). I considered two different cases to illustrate how the variance of exogenous shocks affects the equity premium.

In both cases, the agents' one-period utility function was

$$u(c,\ell) = \begin{cases} \frac{(c^{\gamma}\ell^{1-\gamma})^{1-\sigma}}{1-\sigma} & \text{if } \sigma \neq 1, \\ \gamma \log(c) + (1-\gamma)\log(\ell) & \text{if } \sigma = 1, \end{cases}$$

and the production function was Cobb-Douglas: $f(k,n) = k^{\alpha}n^{1-\alpha}$. Table 1 presents the parameter values I used.

Let the technology shock take two different values in both sectors. Therefore, the total number of exogenous states is four. In the first case, different states are as follows:

$$z_1 = \begin{bmatrix} 1.1 \\ 1.1 \\ 0.9 \\ 0.9 \end{bmatrix}, \quad \text{and} \quad z_2 = \begin{bmatrix} 1.1 \\ 0.9 \\ 1.1 \\ 0.9 \end{bmatrix},$$

and in the second case:

$$z_1 = \begin{bmatrix} 1.2 \\ 1.2 \\ 0.8 \\ 0.8 \end{bmatrix}, \quad \text{and} \quad z_2 = \begin{bmatrix} 1.2 \\ 0.8 \\ 1.2 \\ 0.8 \end{bmatrix},$$

In both cases,

$$\Pi = \begin{bmatrix} 0.9 & 0.025 & 0.025 & 0.05 \\ 0.025 & 0.5 & 0.45 & 0.025 \\ 0.025 & 0.45 & 0.5 & 0.025 \\ 0.05 & 0.05 & 0.05 & 0.9 \end{bmatrix},$$

which has a lot of switching between states 2 and 3. The result is that the agent who does the pricing in the economy is changed more often which produces more volatility in the stochastic discount factor.

The implied AR(1) dynamics for both shocks depend on lagged values of both shocks. Namely, in the first case:

$$z_{it} = 0.16 + 0.44 z_{it-1} + 0.4 z_{jt-1} + \epsilon_{it}$$
$$\epsilon_{it} \sim IID(0, 0.0066)$$

for $i = 1, 2$ and $j \neq i$, and in the second case:

$$z_{it} = 0.16 + 0.44 z_{it-1} + 0.4 z_{jt-1} + \epsilon_{it}$$
$$\epsilon_{it} \sim IID(0, 0.0263)$$

for $i = 1, 2$ and $j \neq i$.

5 Numerical Results

This section briefly describes obtained numerical results for the parameter values from Section 4. I calculated moments as averages over 10,000 simulated observations. Different initial conditions did not seem to affect the long-run behavior of the models.

I solved three different models. The first was "Autarky", in which there were no security markets between the two sectors. The second was "Pareto Optimum", in which security markets existed and there were no constraints on asset holdings. (Notice, that this does not correspond to Pareto optimality in the one-sector model. One and two sector formulations would be equivalent if both agents could work in both sectors and if capital could be freely allocated between sectors once the shock was realized.) The second model I solved corresponds to maximizing (28) subject to (29), but not (30) and (31). The final model was "Limited Commitment", in which (30) and (31) were also enforced.

5.1 Case I: Small Shocks

Table 2 presents selected business cycle and asset pricing moments under Autarky, Table 3 selected business cycle and asset pricing moments under Pareto Optimum, and Table 4 selected business cycle and asset pricing moments under Limited Commitment.

The lack of insurance in Autarky lead to overinvesment in capital which produces both higher means and standard deviations than Pareto Optimum. On the other hand, in Limited Commitment economy the savings constraints forced the agents to underinvest which lead to lower means.

Table 5 summarizes the asset pricing statistics in Autarky, Pareto Optimum, under Limited Commitment, and in data. The results in data taken from Cecchetti, Lam, and Mark (1993), and are based on annual US data for the period 1892–1987. While Sharpe ratio in data is 0.2564, it is 0.01875 in Autarky and 0.02778 in Pareto Optimum. This single number summarizes the difficulties that modern business cycle models have in explaining the basic features of asset prices. Under Limited Commitment, one obtains a Sharpe Ratio of 0.1719 which is substantial improvement. [7]

[7] Christiano and Fisher (1998) do worse job with respect to second moments. They obtain a standard deviation

Table 2: Business cycle and asset pricing moments in Autarky (Case I). Sharpe Ratio = 0.01875.

Variable	Mean	Standard Deviation	Correlation with Output
Output	1.0432	0.1323	1.0000
Consumption	0.7758	0.0740	0.8999
Labor	0.6115	0.0250	0.7007
Investment	0.2674	0.0732	0.8975
Return on Equity	4.2451	3.6085	0.2495
Risk-Free Rate	4.1774	1.3393	−0.5716
Return on Capital	4.2033	2.1237	0.0736
Wage Rate	1.0838	0.1295	0.8324

Table 3: Business cycle and asset pricing moments in Pareto Optimum (Case I). Sharpe Ratio = 0.02778.

Variable	Mean	Standard Deviation	Correlation with Output
Output	0.8305	0.0566	1.0000
Consumption	0.6731	0.0526	0.9942
Labor	0.5766	0.0058	−0.6992
Investment	0.1574	0.0071	0.6027
Return on Equity	4.2218	4.3654	0.2901
Risk-Free Rate	4.1006	1.0356	−0.8827
Return on Capital	8.9407	2.7461	0.4085
Wage Rate	0.9183	0.0775	0.8888

Table 4: Business cycle and asset pricing moments under Limited Commitment (Case I). Sharpe Ratio = 0.1719.

Variable	Mean	Standard Deviation	Correlation with Output
Output	0.8235	0.0534	1.0000
Consumption	0.6674	0.0515	0.9905
Labor	0.5764	0.0066	−0.8146
Investment	0.1561	0.0075	0.3140
Return on Equity	5.0853	9.2147	0.2102
Risk-Free Rate	3.5011	1.2503	−0.9156
Return on Capital	7.0026	2.3254	0.5221
Wage Rate	0.9754	0.0852	−0.9118

Table 5: Asset pricing statistics (Case I). Data from Cecchetti, Lam, and Mark (1993).

Statistic	Data	Autarky	Pareto Optimum	Limited Commitment
$E[r^e]$	7.37	4.2451	4.2218	5.0853
$E[r^f]$	2.36	4.1774	4.1006	3.5011
std$[r^e]$	19.5	3.6085	4.3654	9.2147
std$[r^f]$	5.25	1.3393	1.0356	1.2503
Equity Premium	5.00	0.06766	0.1213	1.584
Sharpe Ratio	0.2564	0.01875	0.02778	0.1719

Figures 2, 3, and 4 present nonlinear impulse-response functions for all variables. Comparison of Autarky with Pareto Optimum shows that one-period positive shock has much longer lived response in Autarky. On the other hand, the responses in Pareto Optimum and under Limited Commitment are almost identical.

Figures 5, 6, and 7 present the impulse response from a positive shock in both sectors to return on equity, risk-free rate, return on capital, and wage rate in Autarky.

Impulse response exercise shows that quantitively Pareto Optimal and Limited Commitment look very similar. In order to see the quantitative differences, I calculated the decision rules of agent 1. Since both economies have four state variables, I only chose slices of the state space in Appendix C.

For most of the variables, the decision rules were identical in Pareto Optimum and under Limited Commitment. However, the important exceptions were the behavior of investment and λ's as a function of λ presented in Figures 30–35. I show the decision rules as functions of one variable when other variables take their means values in the Limited Commitment economy.

The Figures 32–35 show that the planning weight tends to deviate from its initial value, 0.5. On the other hand, investment as a function of λ under Limited Commitment is the same as in Pareto Optimum only in a small region around 0.5. Recall that binding participation constraint calls for an increase expected utility. In the region around 0.5 the increase only through an increase in λ. Once investment deviates from this region, it behaves in opposite ways under Limited Commitment than in Pareto Optimum. In Pareto Optimum increase in λ is associated with decreased investment as the agent with higher planning can consume more. Under Limited Commitment the increase the expected utility is obtained by increasing both capital stock and planning weight.

So while the effect of the participation constraints is not pronounced in the quantities, it is reflected in the asset prices. This happens through two channels. Recall, e.g., from Campbell, Lo, and MacKinlay (1997) equation (8.1.6) page 294, that the equity premium is determined by the equity's covariance with the stochastic discount factor. The smaller the covariance, the greater is the equity's expected excess return. This covariance, obviously, can be decomposed into correlation times standard deviations of the stochastic discount factor and the return on equity.

of 37.42 for return on equity and 14.48 for risk-free rate. Their Sharpe Ratio is 0.13. Jermann (1998) has a standard deviation of 11.46 for risk-free rate. Of course, their results for business cycles are much better than mine.

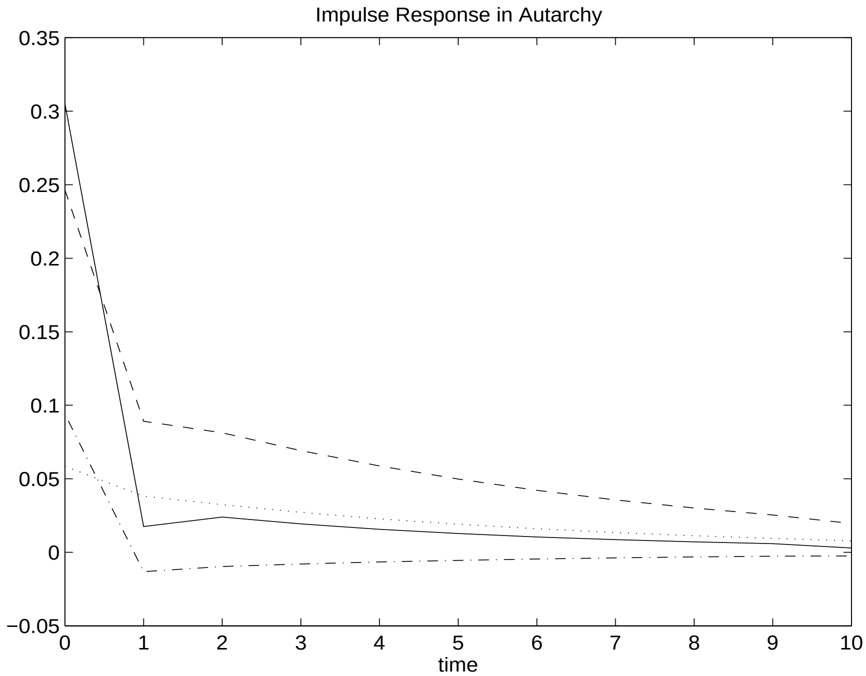

Figure 2: Impulse Response in Autarky. Solid line = output, dots = consumption, dash-dotted line = labor, and dashed line = investment

The first channel is through the switching between states 2 and 3. The result is that the agent who does the pricing in the economy is changed more often which produces more volatility in the stochastic discount factor. The second channel is through the planning weight. The planner has to make sure that the agent who would like to default wants to stay in the risk-sharing arrangement. This can obtained two ways: either by increasing the current utility by increasing current consumption and leisure or by increasing the promised about future utilities. Since the agents are risk-averse they will prefer the latter scheme as the first would increase the volatility of allocations too much. Increasing the promised utilities or planning weights allows the planner to smooth increase over several time periods. However, since the planning weight is a state variable reflecting expectations about the future distributions of wealth, its movements are reflected in the asset prices. Hence, while the quantities do not fluctuate enough to justify the movements in stock prices, the movements in this auxiliary state variable increase the volatility of the price of equity.

Numerical results also provide intuition on how solvency constraints differ between endowment and production economies. The asset-pricing mechanism is presented in Figure 8, and Figure 9 presents underlying shocks. The pricing in each period was done by the agent who had lower consumption growth (and hence a higher marginal rate of substitution). The promise of this mechanism in accounting for the behavior of asset prices is that relatively small variations in the underlying shocks generate more variability in the pricing kernel. The volatility of the pricing kernel is one necessary condition for explaining the behavior of asset prices.

The effect can be seen more clearly by comparing Figures 8 and 9 with Figures 10 and 11. Figure 10 presents solvency constraints for agent 1. Notice the asset-pricing mechanism at work

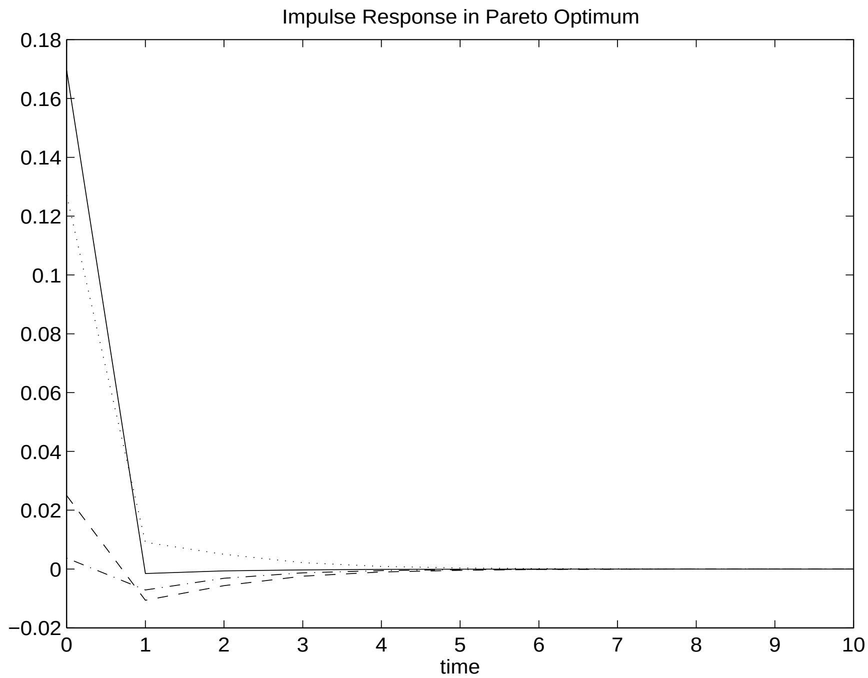

Figure 3: Impulse Response in Pareto Optimum. Solid line = output, dots = consumption, dash-dotted line = labor, and dashed line = investment

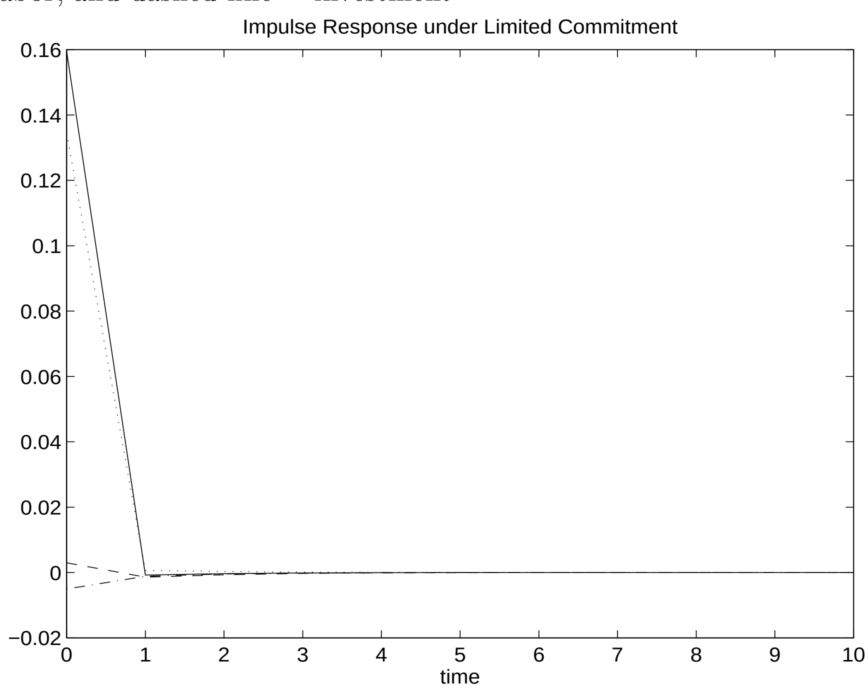

Figure 4: Impulse Response under Limited Commitment. Solid line = output, dots = consumption, dash-dotted line = labor, and dashed line = investment

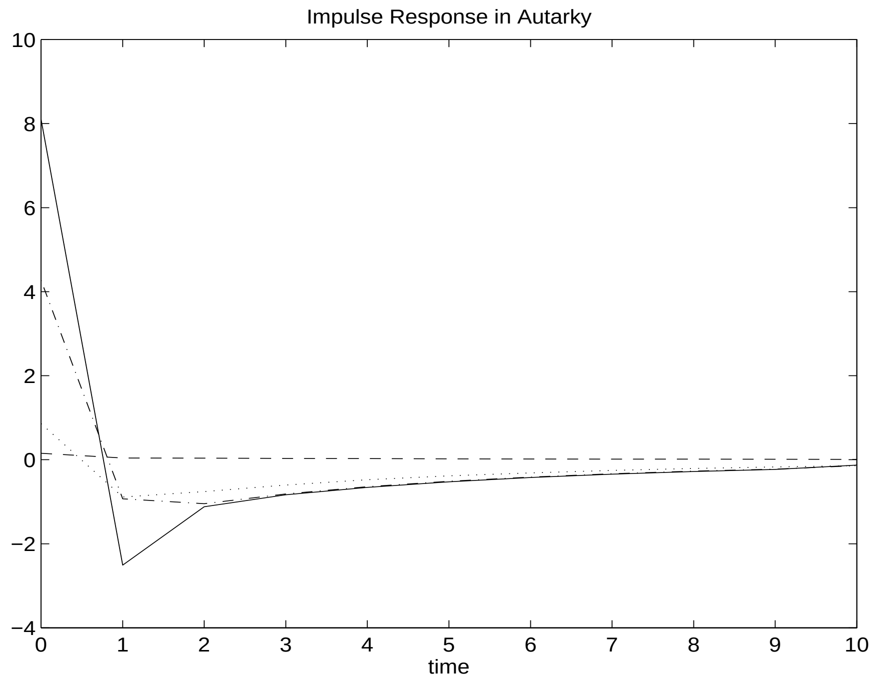

Figure 5: Impulse Response in Autarky. Solid line = return on equity, dots = risk-free rate, dash-dotted line = return on capital, dashed line = wage rate.

here: when sector 1 became more productive relative to sector 2, the consumption growth of agent 1 was higher than that of agent 2. This means that agent 1 wanted to borrow, but his borrowing constraint was binding. The unconstrained agent did the pricing. When sector 2 became more productive relative to sector 1, the effect was reversed.

Hence, borrowing constraints bring a wedge between marginal rates of substitution and asset prices. This wedge has been the key factor in explaining asset pricing anomalies in endowment economies. Results in this paper indicate that the same effect has promising results also in production economies.

Figure 10 also reveals how a production economy differs from an endowment economy. In endowment economies solvency constraints form a sequence of state-dependent borrowing constraints that may bind for at most one agent in any state. In contrast, when capital is introduced into the model, there is a sequence of savings constraints that will bind for *both agents all the time*. The mechanism is as follows. A participation constraint on the planner's problem will bind every time there is a change in exogenous state. When this is the case the corresponding solvency constraint in that state will bind for one agent. However, since the constraints on capital are for the expectation over tomorrow's states the upper bound on capital holdings (savings) will bind for both agents in every period.

The economic intuition is that the planner must prevent the agents from gaining too much wealth. Otherwise, autarky would not be too bad an outcome. Rogerson (1985) obtains a somewhat similar result in a very different context. He studies a repeated moral-hazard problem with a risk-neutral principal and a risk-averse agent and shows that for every outcome of every period the agent

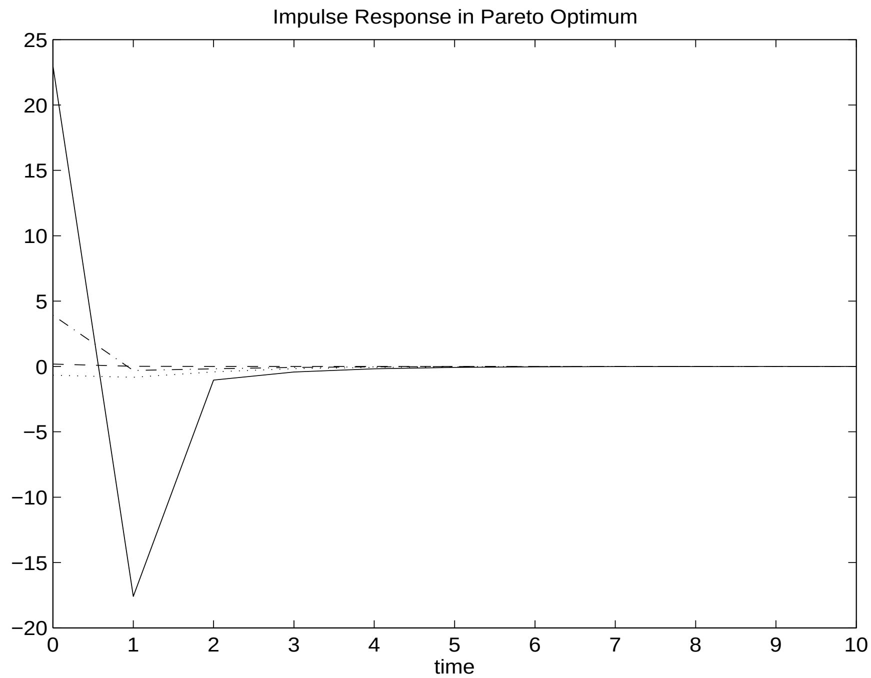

Figure 6: Impulse Response in Pareto Optimum. Solid line = return on equity, dots = risk-free rate, dash-dotted line = return on capital, dashed line = wage rate.

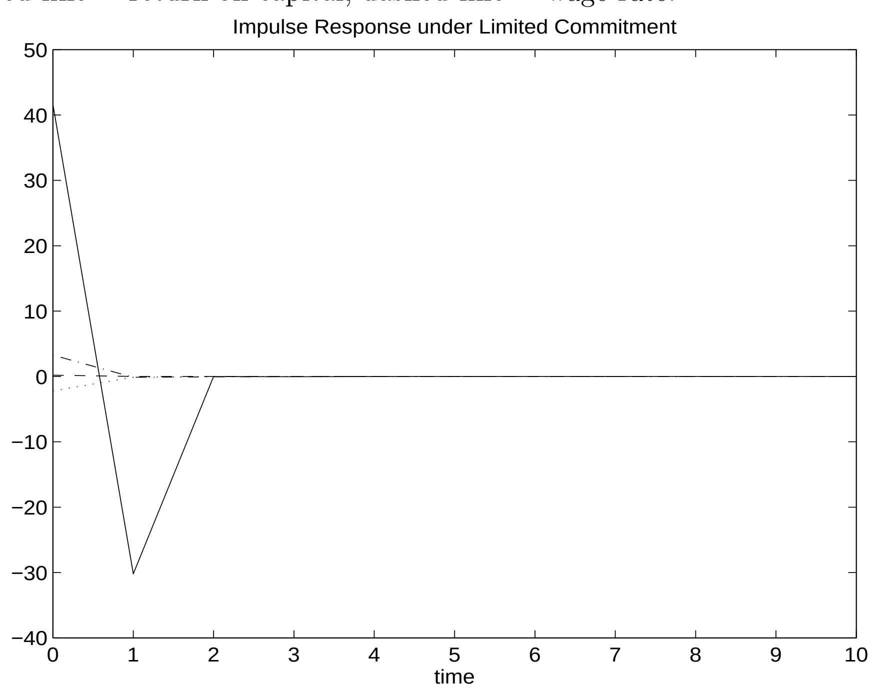

Figure 7: Impulse Response under Limited Commitment. Solid line = return on equity, dots = risk-free rate, dash-dotted line = return on capital, dashed line = wage rate.

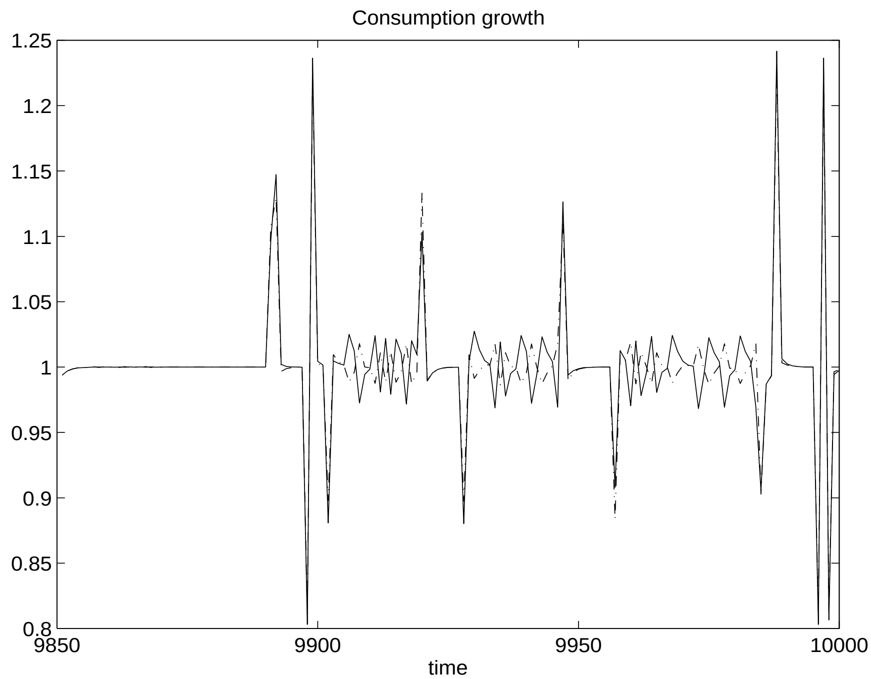

Figure 8: Consumption growth for both agents. Solid line = agent 1.

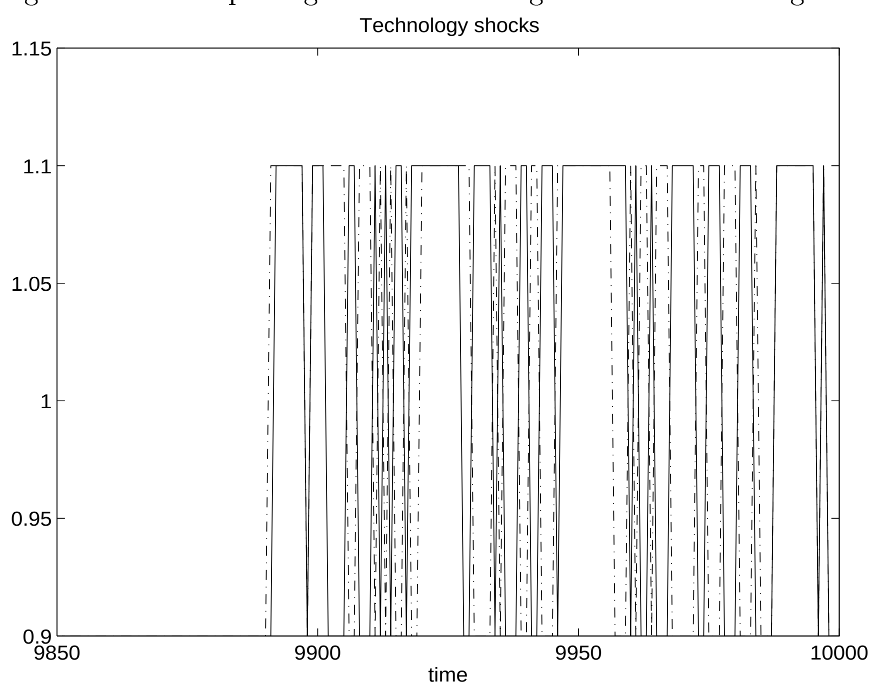

Figure 9: Technology shock in both sectors. Solid line = sector 1.

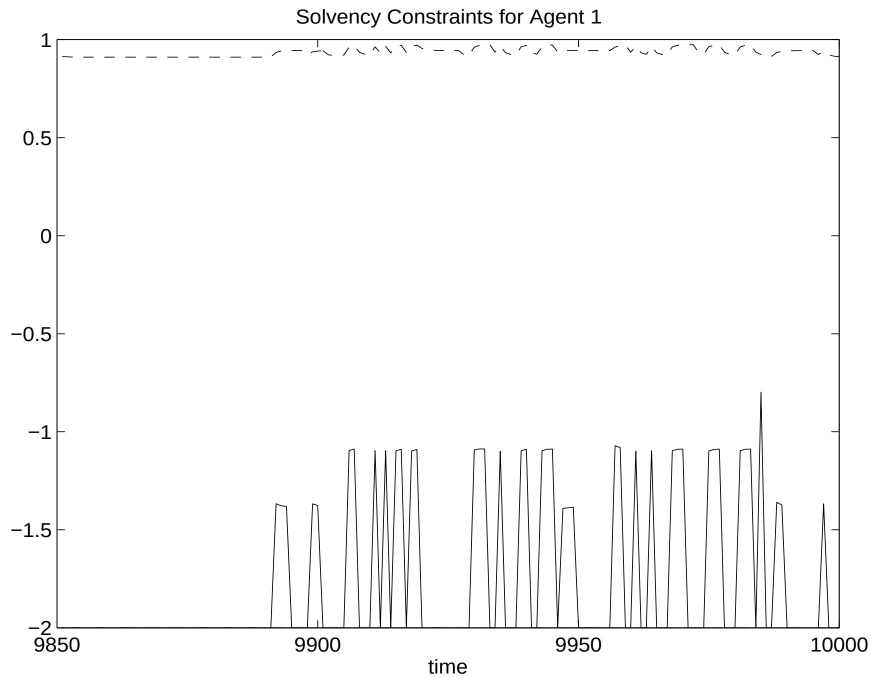

Figure 10: Solvency constraints for agent 1. Solid line = B_t^1, dashed line = C_t^1.

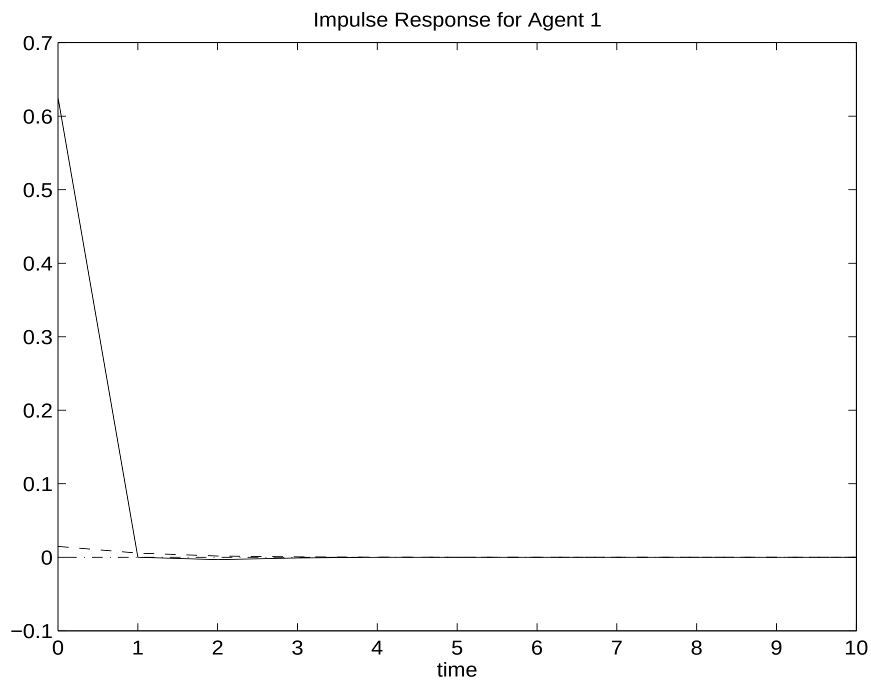

Figure 11: Impulse response on solvency constraints for agent 1. Solid line = B_t^1, dashed line = C_t^1.

Table 6: Business cycle and asset pricing moments in Autarky (Case II). Sharpe Ratio = 0.03868.

Variable	Mean	Standard Deviation	Correlation with Output
Output	1.0649	0.2672	1.0000
Consumption	0.7854	0.1479	0.8988
Labor	0.6067	0.0505	0.7068
Investment	0.2795	0.1491	0.9005
Return on Equity	4.4204	7.3195	0.2397
Risk-Free Rate	4.1373	2.6571	−0.5622
Return on Capital	4.2602	4.2728	0.0697
Wage Rate	1.0970	0.2597	0.8317

Table 7: Business cycle and asset pricing moments in Pareto Optimum (Case II). Sharpe Ratio = 0.06838.

Variable	Mean	Standard Deviation	Correlation with Output
Output	0.8323	0.1121	1.0000
Consumption	0.6754	0.1048	0.9958
Labor	0.5710	0.0133	−0.6651
Investment	0.1569	0.0124	0.6309
Return on Equity	4.4696	8.5967	0.3027
Risk-Free Rate	3.8817	2.0731	−0.9209
Return on Capital	8.9762	5.4572	0.4142
Wage Rate	0.9204	0.1540	0.8899

would choose to save some of his wage if he could. In Rogerson's setup it is Pareto improving to control as many of the agent's decisions as possible in order to gain more leverage on the incentive problem regarding the agent's effort.[8]

5.2 Case II: Large Shocks

Table 6 presents selected business cycle moments under Autarky, Table 7 selected business cycle and asset pricing moments under Pareto Optimum, and Table 8 selected business cycle and asset pricing moments under Limited Commitment.

Table 5 summarizes the asset pricing statistics in Autarky, Pareto Optimum, under Limited Commitment, and in data. The results in data taken from Cecchetti, Lam, and Mark (1993), and are based on annual US data for the period 1892–1987. Case II increases the equity premium considerably without increasing the volatility of either interest rates or returns on equity too much.

[8] See Braverman and Stiglitz (1982) for a fuller discussion of this idea.

Table 8: Business cycle and asset pricing moments under Limited Commitment (Case II). Sharpe Ratio = 0.1666.

Variable	Mean	Standard Deviation	Correlation with Output
Output	0.8280	0.1099	1.0000
Consumption	0.6719	0.1034	0.9942
Labor	0.5715	0.0132	−0.7268
Investment	0.1561	0.0132	0.5354
Return on Equity	5.9650	18.0631	0.0365
Risk-Free Rate	2.9563	2.2254	−0.9224
Return on Capital	11.3333	5.9912	0.4136
Wage Rate	0.8608	0.1426	−0.8356

Table 9: Asset pricing statistics (Case II). Data from Cecchetti, Lam, and Mark (1993).

Statistic	Data	Autarky	Pareto Optimum	Limited Commitment
$E[r^e]$	7.37	4.4204	4.4696	5.9650
$E[r^f]$	2.36	4.1373	3.8817	2.9563
$\text{std}[r^e]$	19.5	7.3195	8.5967	18.0631
$\text{std}[r^f]$	5.25	2.6571	2.0731	2.2254
Equity Premium	5.00	0.2831	0.5879	3.009
Sharpe Ratio	0.2564	0.03868	0.06838	0.1666

6 Conclusions and Further Research

This paper studies a business cycle model with heterogeneous agents that have access to complete markets but face endogenous solvency constraints. These constraints are motivated by the agents' limited commitment technology. The objective is to analyze simultaneously both asset prices and business cycles.

In the limited commitment production economy, aggregate fluctuations are close to the ones generated by Pareto Optimal (full commitment) risk-sharing arrangements. However, the always binding savings constraints force agents to underinvest in capital and increase the volatility of investment. Together with borrowing constraints, savings contraints also increase the volatilities of both the stochastic discount factor and the equity prices. This in turn increases the equity premium considerably without increasing the volatility of either interest rates or returns on equity too much. In fact, with the standard real business cycle parameters, the model is significantly closer in matching the second moments of risk-free rates and returns on equity than habit-formation models of Boldrin, Christiano, and Fisher (1995, 1999), Christiano and Fisher (1998), and Jermann (1998).

Moreover, the large equity premium is obtained without unrealistically low discount factors as in Alvarez and Jermann (1999ab) or high risk aversion parameter values as in Tallarini (1999). In order to increase the equity premium one has to make autarky more desirable. In an endowment economy, an incentive to participate in risk-sharing is very high so that only by lowering the discount factor can autarky become tempting. In a production economy, even patient agents can consider autarky as capital allows for considerable self-insurance.

In endowment economies solvency constraints form a sequence of state-dependent borrowing constraints that may bind for at most one agent in any state. When capital is introduced into the model, there is a sequence of savings constraints that will bind for *both agents all the time*. The economic intuition is that the planner must prevent the agents from gaining too much wealth. Otherwise, autarky will not be too bad an outcome. Rogerson (1985) obtains a somewhat similar result but in a very different context.

While the current version of the paper is quite successful in matching the asset pricing moments, much work needs to done to improve its business cycle properties and to calibrate exogenous shocks to data. Since the observed business cycles seem to resemble more autarky than full commitment environment, the obvious direction for further research is to change the parameter values so that risk-sharing is reduced. One possibility would be an introduction of adjustment costs in capital.

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A Proofs

Proposition 1. *If autarky is not the only allocation satisfying the participation constraints then at most one agent's participation constraint is binding in each period.*

Proof. By contradiction. Suppose that for the current state $\omega = V_2^a(k_2, s)$, $V(V_2^a(k_2, s), k_1, k_2, s) = V_1^a(k_1, s)$. Next, set $c_1 = c_1^a(k_1, s)$, $c_2 = c_2^a(k_2, s)$, $\ell_1 = \ell_1^a(k_1, s)$, $\ell_2 = \ell_2^a(k_2, s)$, $k_1' = k_1^{a'}(k_1, s)$, $k_2' = k_2^{a'}(k_2, s)$, where $x^a(k, s)$ denotes the autarky decision rule when the state is (k, s). In addition, set $\omega_{s'} = V_2^a(k_2', s')$ for all s'.

Since autarky is not the only allocation satisfying the participation constraints, it must be the case that for some $\hat{s}$ $V(V_2^a(k_2', \hat{s}), k_1', k_2', \hat{s}) > V_1^a(k_1', \hat{s})$. Then it must be the case that

$$TV(\omega, k_1, k_2, s) \geq u(c_1, \ell_1) + \beta \sum_{s' \in S} V(\omega_{s'}, k_1', k_2', s')\pi(s'|s) > V_1^a(k_1, s) = u(c_1, \ell_1) + \beta \sum_{s' \in S} V_1^a(k_1', s')\pi(s'|s).$$

Contradiction. ∎

Assumption 1. *Value function is once differentiable.*

Assumption 2. *The decision rules are single-valued.*

Proposition 2. *Under Assumptions 1 and 2 if both agents are unconstrained for s' then their marginal rates of substitution are equalized. If one agent is constrained then his marginal rate of substitution will be strictly smaller than the one for the other agent.*

Proof. By inspection. Let η_1 denote the Kuhn-Tucker multiplier associated with the constraint (8), η_2 denote the Kuhn-Tucker multiplier associated with the constraint (9), and $\psi_1(s')$ and $\psi_2(s')$ denote the Kuhn-Tucker multipliers associated with the constraints (10) and (11), respectively. Then the first-order conditions with respect to c_1, c_2, and $\omega_{s'}$ will be

$$\begin{aligned} u_1(c_1, \ell_1) - \eta_1 &= 0 \\ \eta_2 u_1(c_2, \ell_2) - \eta_1 &= 0 \\ \beta V_1(\omega_{s'}, k_1', k_2', s')\pi(s'|s) + \beta\eta_2\pi(s'|s) + \psi_1(s')V_1(\omega_{s'}, k_1', k_2', s') + \psi_2(s') &= 0 \end{aligned}$$

and the envelope condition with respect to ω is

$$V_1(\omega, k_1, k_2, s) = -\eta_2.$$

Therefore,

$$\eta_2 = \frac{u_1(c_1, \ell_1)}{u_1(c_2, \ell_2)}.$$

In addition, if both agents are unconstrained in s' then

$$\eta_2 = \frac{u_1(c_1, \ell_1)}{u_1(c_2, \ell_2)} = -V_1(\omega_{s'}, k_1', k_2', s') = \frac{u_1(c_1', \ell_1')}{u_1(c_2', \ell_2')}$$

or

$$\frac{u_1(c_1', \ell_1')}{u_1(c_1, \ell_1)} = \frac{u_1(c_2', \ell_2')}{u_1(c_2, \ell_2)}.$$

Next, suppose that $\psi_1(s') > 0$. Then

$$-V_1(\omega_{s'}, k_1', k_2', s') = \frac{u_1(c_1', \ell_1')}{u_1(c_2', \ell_2')} < \eta_2 = \frac{u_1(c_1, \ell_1)}{u_1(c_2, \ell_2)}.$$

or

$$\frac{u_1(c_1', \ell_1')}{u_1(c_1, \ell_1)} < \frac{u_1(c_2', \ell_2')}{u_1(c_2, \ell_2)}.$$

Suppose that $\psi_2(s') > 0$. Then

$$-V_1(\omega_{s'}, k_1', k_2', s') = \frac{u_1(c_1', \ell_1')}{u_1(c_2', \ell_2')} > \eta_2 = \frac{u_1(c_1, \ell_1)}{u_1(c_2, \ell_2)}.$$

or

$$\frac{u_1(c_1', \ell_1')}{u_1(c_1, \ell_1)} > \frac{u_1(c_2', \ell_2')}{u_1(c_2, \ell_2)}.$$

■

Definition 2. Borrowing constraints are not too tight *if they satisfy*

$$J(A(s'), C, s') = J^a(C, s') \quad \forall s' \in S'.$$

Definition 3. The prices of Arrow securities are not too high *if the infinite sums of the form*

$$\sum_{j=1}^{\infty} \sum_{s^{t+j} \in S^{t+j}} q(s^{t+j}|s_t) x_{t+j}$$

are finite for all equilibrium objects x_{t+j}.

Proposition 3. *Given an allocation* $\{c_t^i, \ell_t^i, k_t^i\}$ *that satisfies*

1. *the feasibility condition (13) at any period and state,*
2. *the participation constraints (4) at any period and state,*
3. *intratemporal optimality condition*

$$u_2(c_t, \ell_t) = z_t f_2(k_{t-1}, 1 - \ell_t) u_1(c_t, \ell_t)$$

at any period,

4. *that the implied prices of Arrow securities are not too high, and*
5. *that the marginal utility of consumption stays finite:*

$$\lim_{t \to \infty} E_0 \beta^t u_1(c_t, \ell_t) < \infty,$$

then there exist processes $\{a_t^i, B_t^i, C_t^i, r_t^i, w_t^i, q_t\}$ such that a sequence $\{c_t^i, \ell_t^i, k_t^i, a_{t+1}^i\}$ is a competitive equilibrium given the solvency constraints $\{B_t^i, C_t^i\}$ and the prices $\{w_t^i, r_t^i, q_t\}$. In addition, the borrowing constraints are not too tight.

Proof. By construction. First, construct the equilibrium prices as follows

$$\begin{aligned}
r_t &= z_t f_1(k_{t-1}, 1-\ell_t) \\
w_t &= z_t f_2(k_{t-1}, 1-\ell_t) \\
q(s^{t+1}, s^t) &= \max_{i=1,2} \beta \frac{u_1(c_{t+1}^i, \ell_{t+1}^i)}{u_1(c_t^i, \ell_t^i)} \pi(s^{t+1}|s_t) \\
q(s^{t+j}|s_t) &= \prod_{k=t}^{t+j-1} q(s^{k+1}, s_k) \\
m_{t,t+j}\pi(s^{t+j}|s_t) &= q(s^{t+j}|s_t).
\end{aligned}$$

Next, construct the asset holdings as follows

$$a_0 = E_0\left[\sum_{t=0}^{\infty} m_{0,t}(c_t + k_t - (1+r_t-\delta)k_{t-1} - w_t n_t)\right]$$

and

$$a_t = E_t\left[\sum_{j=0}^{\infty} m_{t,t+j}(c_{t+j} + k_{t+j} - (1+r_{t+j}-\delta)k_{t+j-1} - w_{t+j}n_{t+j})\right].$$

Under condition 4 these sums are well-defined.

Finally, construct the solvency constraints as follows. If

$$u_1(c_t, \ell_t) < \beta E_t[u_1(c_{t+1}, \ell_{t+1})(1 + r_{t+1} - \delta)]$$

then set $C_t = k_t$. Otherwise set $C_t = k_t + \epsilon$.

Next, if

$$q(s^{t+1}, s^t)u_1(c_t, \ell_t) > \beta\pi(s^{t+1}|s_t)u_1(c_{t+1}, \ell_{t+1})$$

then set $B(s^{t+1}) = a(s^{t+1})$. Otherwise set $B(s^{t+1}) = -E_t[m_{t+1}y_{t+1}]$. Using the above defined prices, allocations, and solvency constraints, construct the value functions for the household, J^i's. Finally, using the value functions redefine the borrowing constraints so that for the agent i for whom the marginal rate of substitution is higher (the borrowing constraint does not bind) in period t

$$J^i(B(s^{t+1}), C_t, s^{t+1}) = J^{a,i}(C_t, s^{t+1}) \quad \forall s^{t+1} \in S^{t+1}.$$

Since $J^i(\cdot, C_t, s^{t+1})$ is stricly increasing, this has a solution. Allocations are still optimal for the same prices and solvency constraints as the feasible set is smaller, but the original plan is still feasible.

Checking the solvency and first-order conditions is trivial. Tranversality conditions

$$\lim_{t\to\infty} E_0\beta^t u_1(c_t,\ell_t)[a_t - B_t] = 0$$
$$\lim_{t\to\infty} E_0\beta^t u_1(c_t,\ell_t)[C_t - k_t] = 0$$

are satisfied assuming that

$$\lim_{t\to\infty} E_0\beta^t u_1(c_t,\ell_t) = 0,$$

which is satisfied by condition 5. ■

B Chebyshev Approximation

I first fix M, the number of exogenous states.[9] For each realization of s, V can be approximated by

$$V(\lambda, k_1, k_2; s) \approx \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} b_{ijk}(s)P_i(\lambda)P_j(k_1)P_k(k_2), \tag{32}$$

where $P_i(x)$, $i = 0, \ldots, N-1$, is a Chebyshev polynomial

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= 2x^2 - 1 \\ &\vdots \\ P_{N+1}(x) &= 2xP_N(x) - P_{N-1}(x). \end{aligned}$$

Chebyshev polynomials are orthogonal in the interval $[-1, 1]$ over the weight $(1-x^2)^{-1/2}$

$$\int_{-1}^{1} \frac{P_i(x)P_j(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, & i \neq j \\ \pi/2, & i = j \neq 0 \\ \pi, & i = j = 0. \end{cases}$$

The zeros of the polynomial are given by

$$x = \cos\left(\frac{\pi(k-1/2)}{N}\right) \quad k = 1, 2, \ldots, N,$$

and the extrema are given by

$$x = \cos\left(\frac{\pi k}{N}\right) \quad k = 0, 1, \ldots, N.$$

[9]The presentation about Chebyshev approximation follows closely Press, Teukolsky, Vetterling, and Flannery (1992, ch. 5.8).

At all of the maxima $P_i(x) = 1$ and at all of the minima $P_i(x) = -1$. Moreover, and for our purposes most importantly, Chebyshev polynomials satisfy a discrete orthogonality relation as well: If x_k $(k = 1, \ldots, K)$ are the K zeros of $P_K(x)$ and if $i, j < K$, then

$$\sum_{k=1}^{K} P_i(x_k) P_j(x_k) = \begin{cases} 0, & i \neq j \\ K/2, & i = j \neq 0 \\ K, & i = j = 0. \end{cases}$$

Since the domain of Chebyshev polynomials is $[-1, 1]$, the actual realizations of the state variables are mapped into $[-1, 1]$ by applying the following function to each state variable separately:

$$\varphi(x) = 2\left(\frac{x - \underline{x}}{\overline{x} - \underline{x}}\right) - 1,$$

where $\overline{x}$ and $\underline{x}$ are predetermined upper and lower bounds on x, respectively.

Equation (32) approximates the value function by using a tensor product basis. To reduce the number of coefficients $b_{ijk}(s)$ to be estimated, I use a "complete polynomial" basis:

$$V(\lambda, k_1, k_2; s) \approx \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-i} \sum_{k=0}^{N-1-i-j} b_{ijk}(s) P_i(\varphi_\lambda(\lambda)) P_j(\varphi_{k1}(k_1)) P_k(\varphi_{k2}(k_2)). \tag{33}$$

That is, instead using all the terms in tensor product, I use only the terms that correspond to the multivariate Taylor approximation. When N is, say, 5, this reduces the number of estimated coefficients per each exogenous state from 125 to 35.

C Decision Rules in Case I

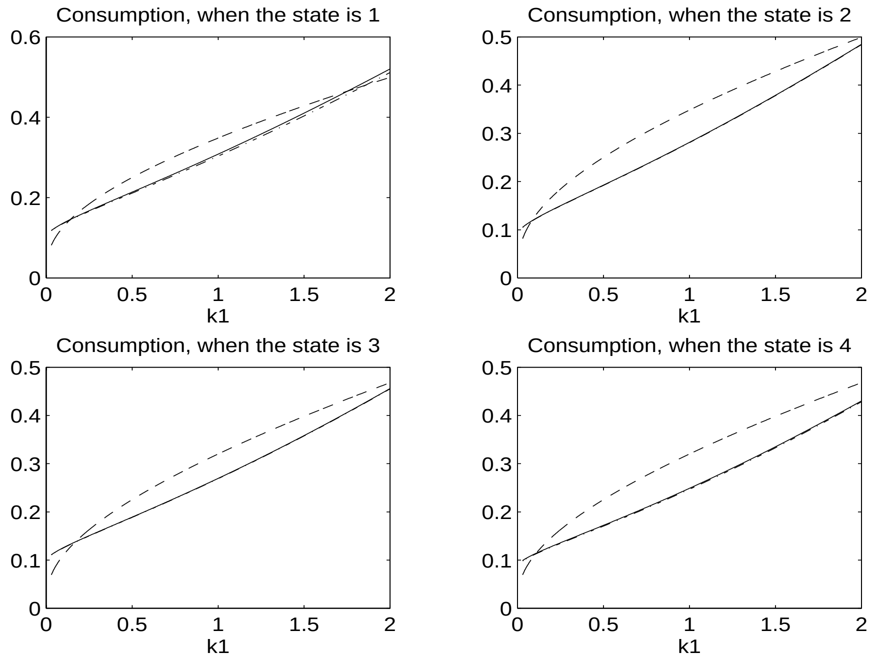

Figure 12: Consumption of agent 1 as a function of k_1 when $k_2 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

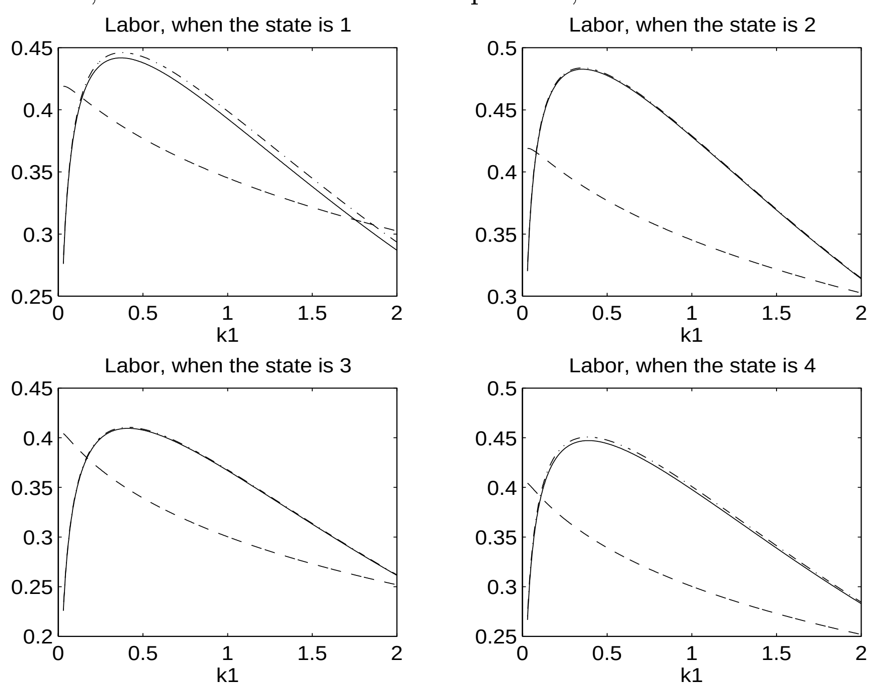

Figure 13: Labor of agent 1 as a function of k_1 when $k_2 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

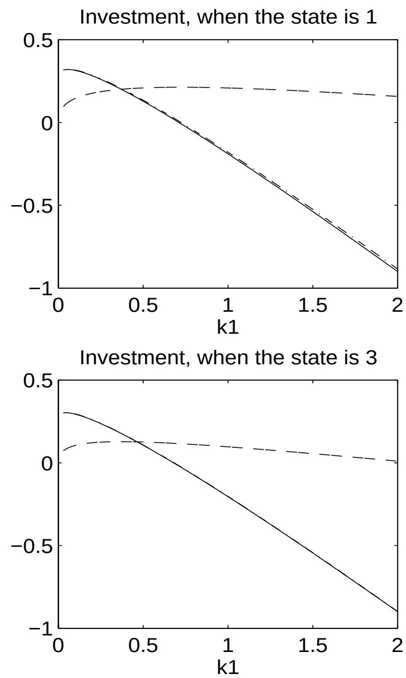

Figure 14: Investment of agent 1 as a function of k_1 when $k_2 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

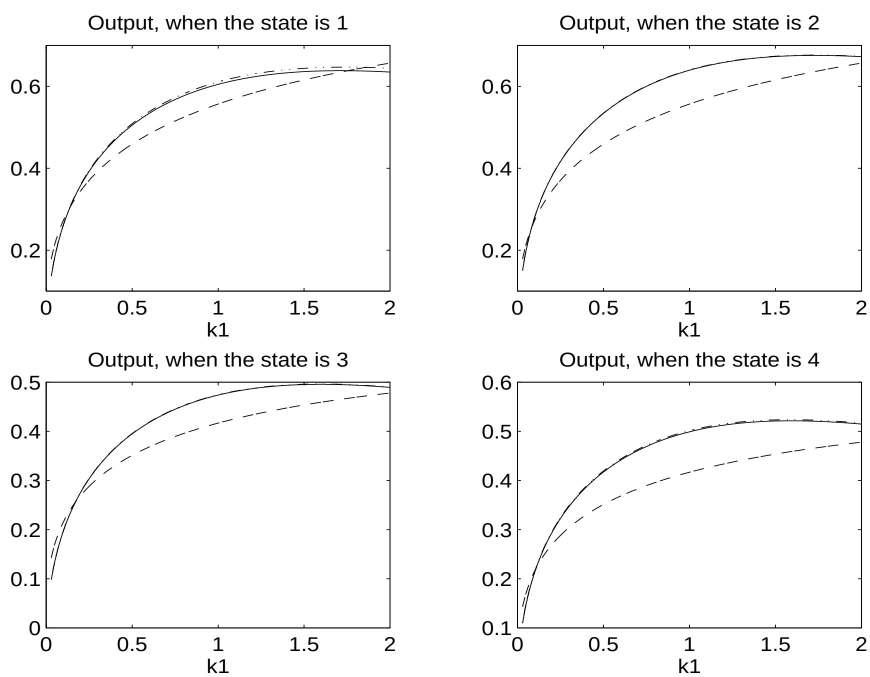

Figure 15: Output of agent 1 as a function of k_1 when $k_2 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

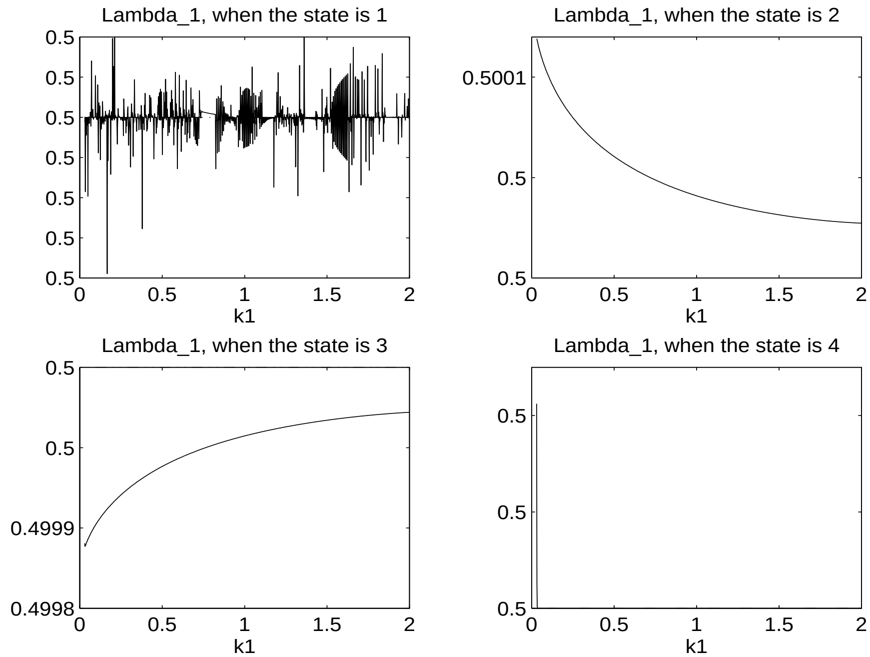

Figure 16: λ_1 as a function of k_1 when $k_2 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

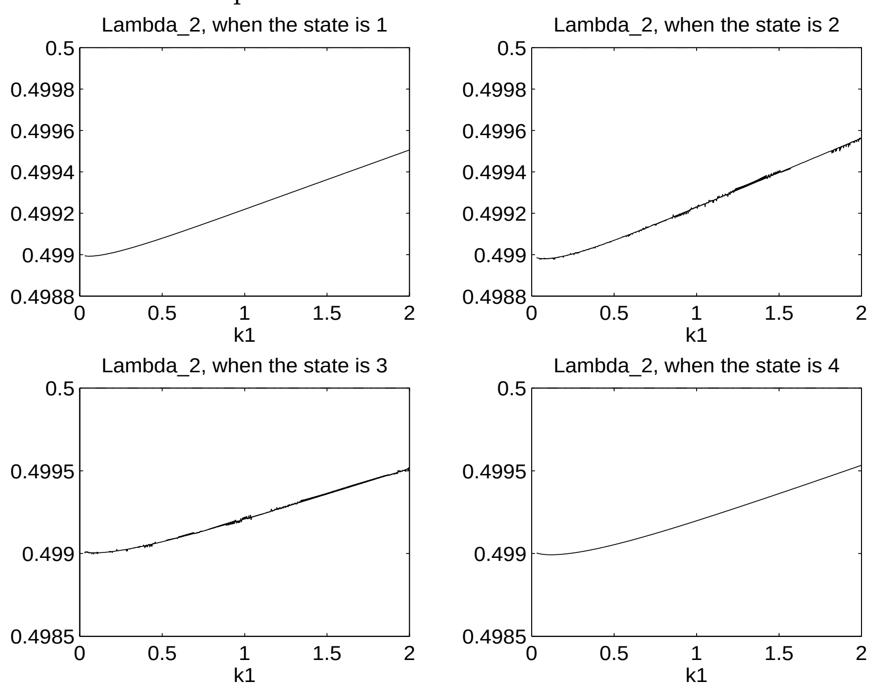

Figure 17: λ_2 as a function of k_1 when $k_2 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

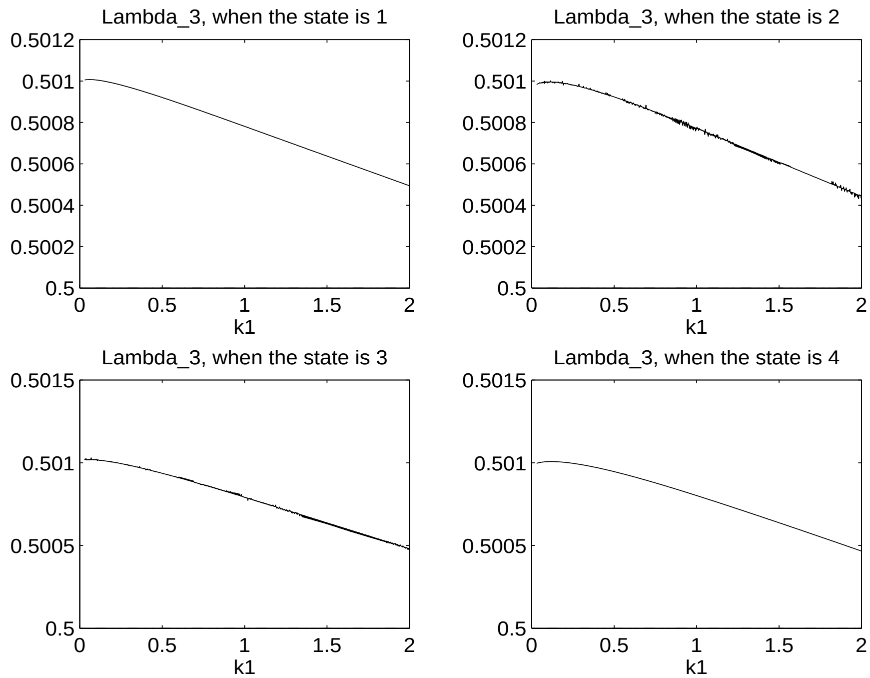

Figure 18: λ_3 as a function of k_1 when $k_2 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

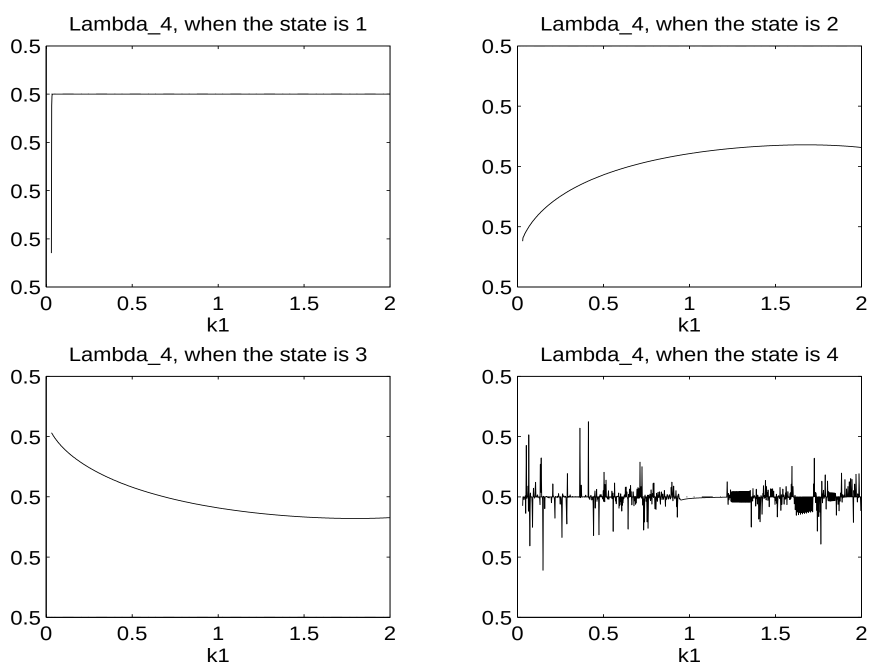

Figure 19: λ_4 as a function of k_1 when $k_2 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

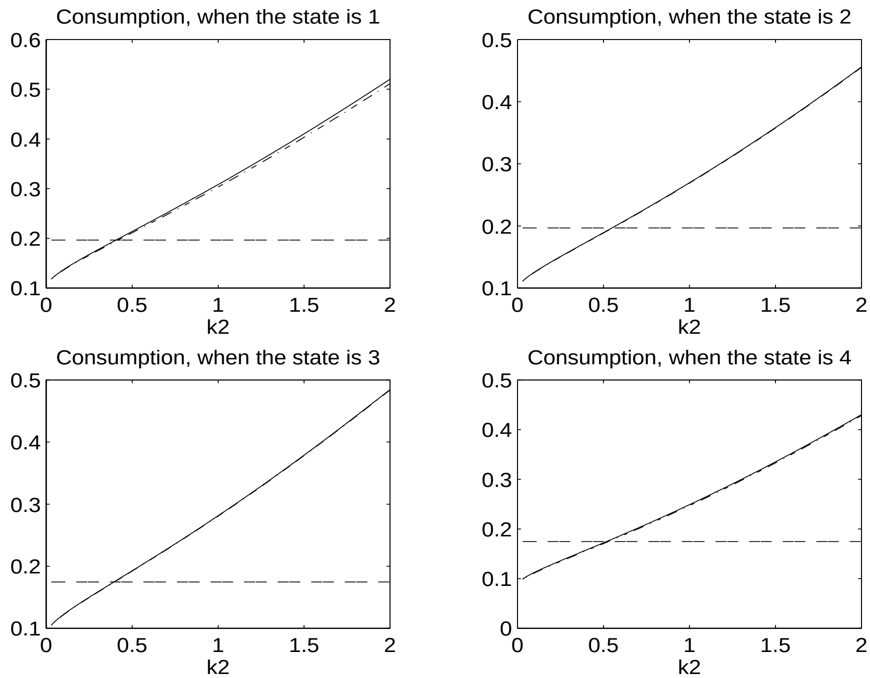

Figure 20: Consumption of agent 1 as a function of k_2 when $k_1 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

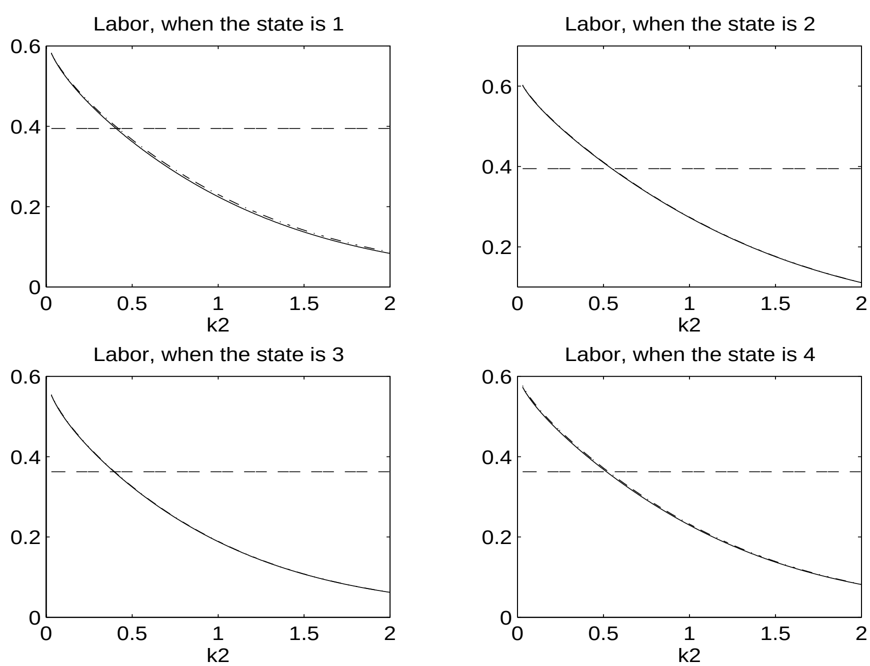

Figure 21: Labor of agent 1 as a function of k_2 when $k_1 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

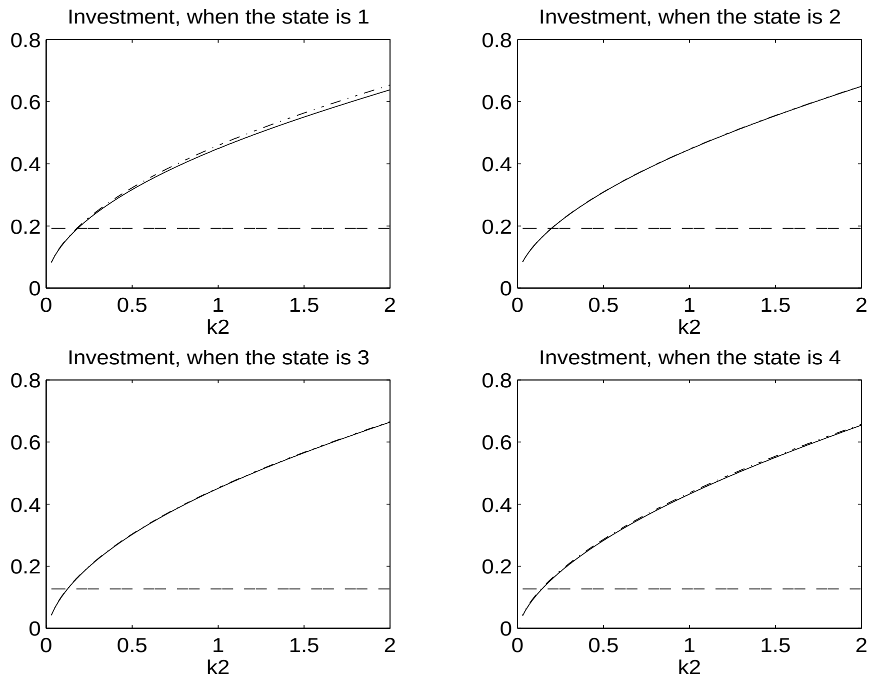

Figure 22: Investment of agent 1 as a function of k_2 when $k_1 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

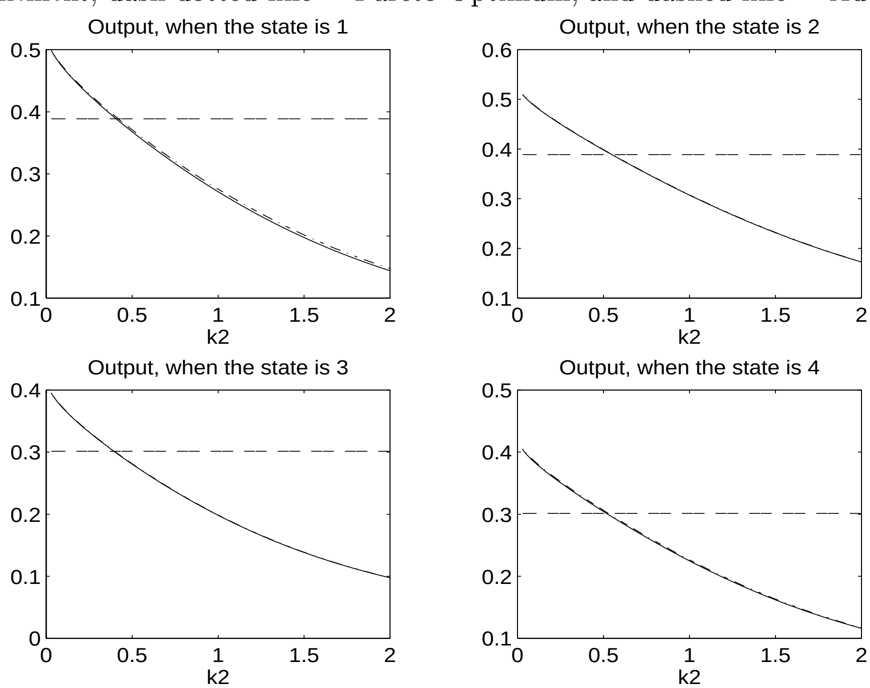

Figure 23: Output of agent 1 as a function of k_2 when $k_1 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

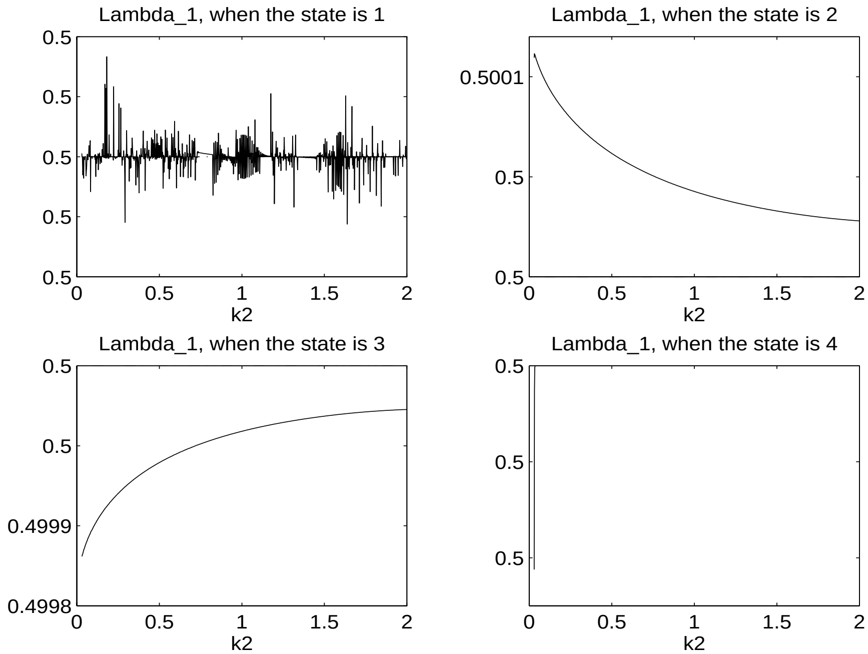

Figure 24: λ_1 as a function of k_2 when $k_1 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

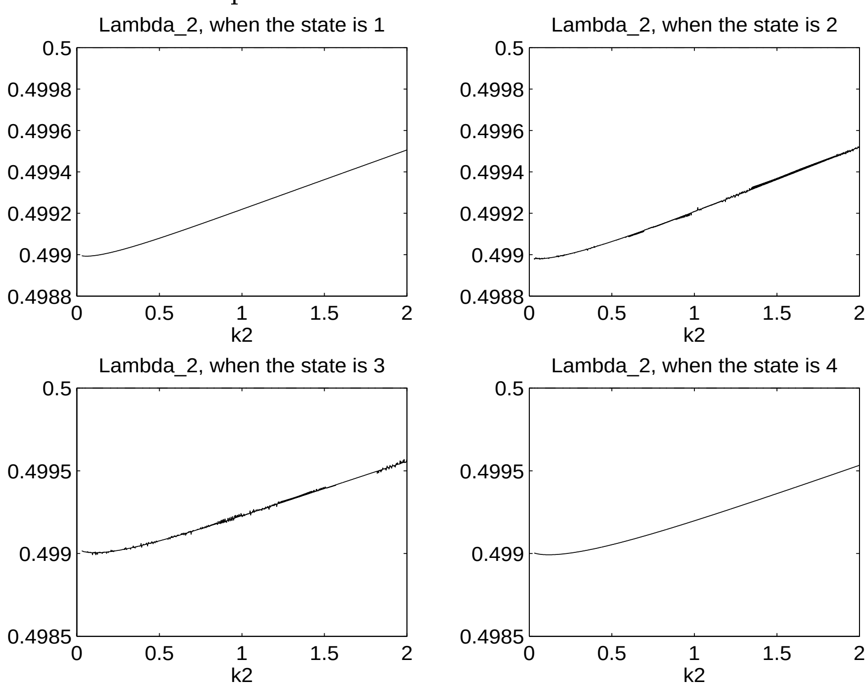

Figure 25: λ_2 as a function of k_2 when $k_1 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

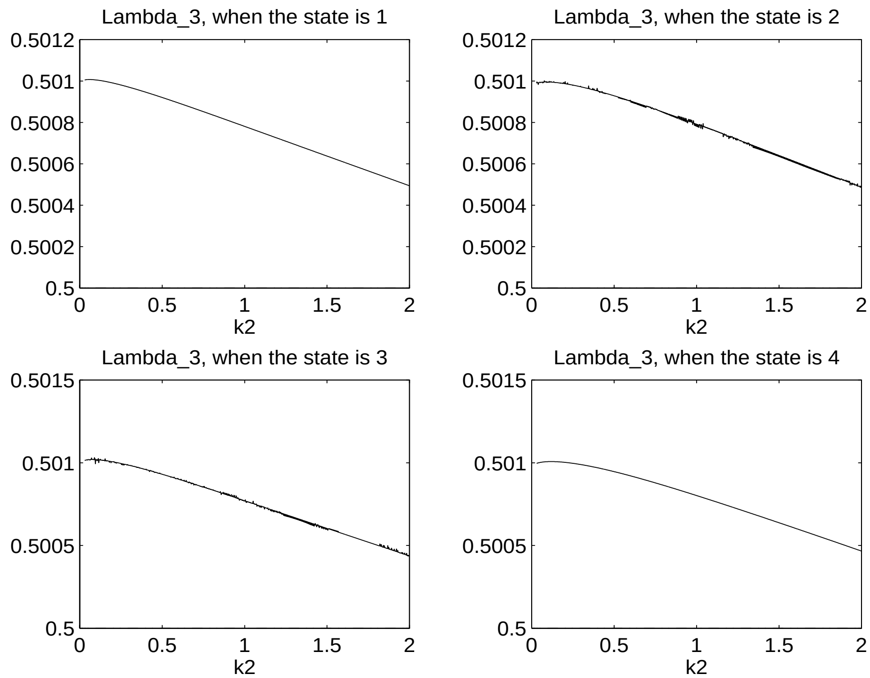

Figure 26: λ_3 as a function of k_2 when $k_1 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

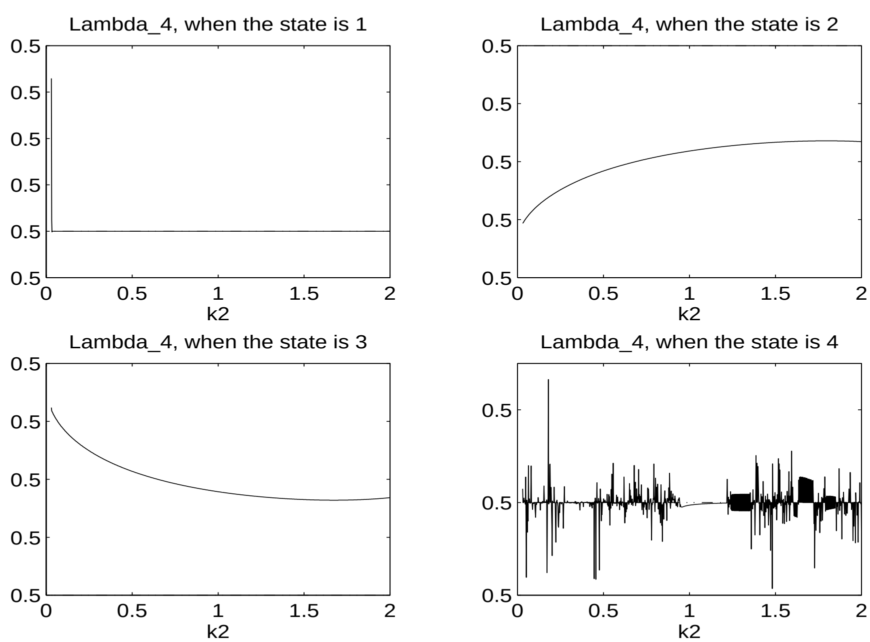

Figure 27: λ_4 as a function of k_2 when $k_1 = 0.78$ and $\lambda = 0.5$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

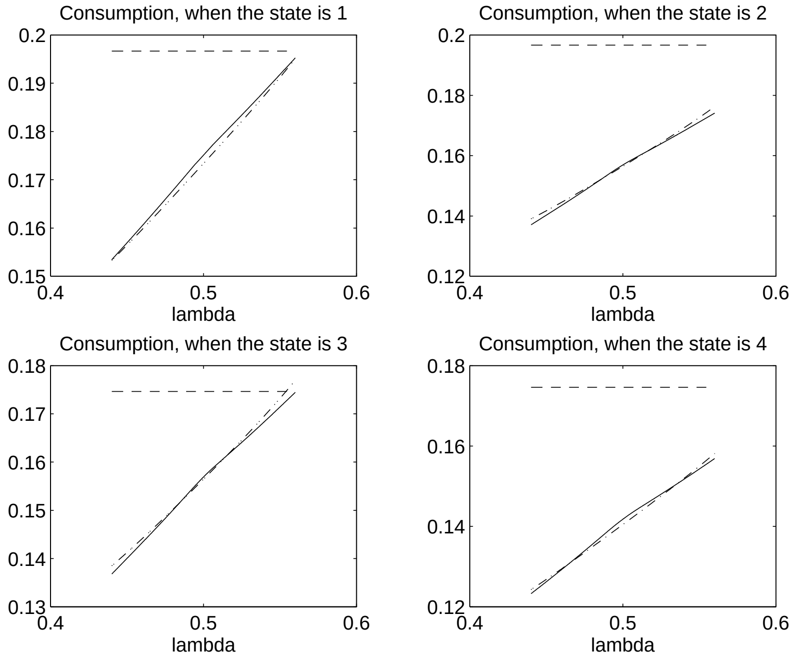

Figure 28: Consumption of agent 1 as a function of λ when $k_1 = k_2 = 0.78$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

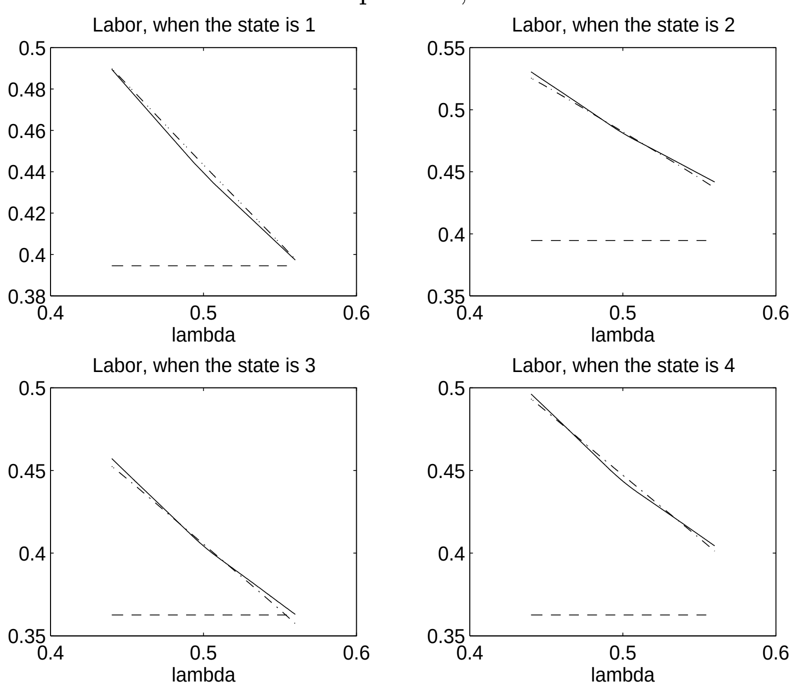

Figure 29: Labor of agent 1 as a function of λ when $k_1 = k_2 = 0.78$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

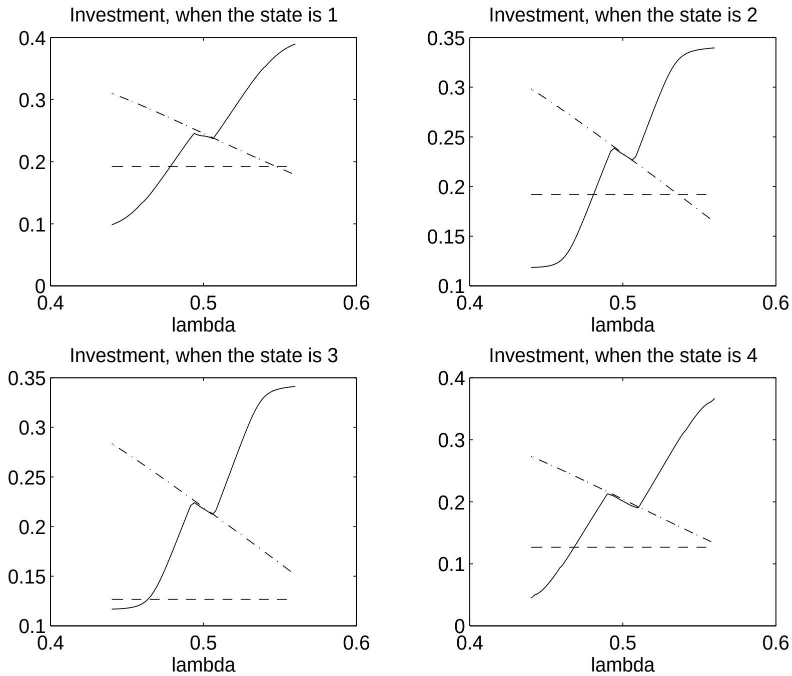

Figure 30: Investment of agent 1 as a function of λ when $k_1 = k_2 = 0.78$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

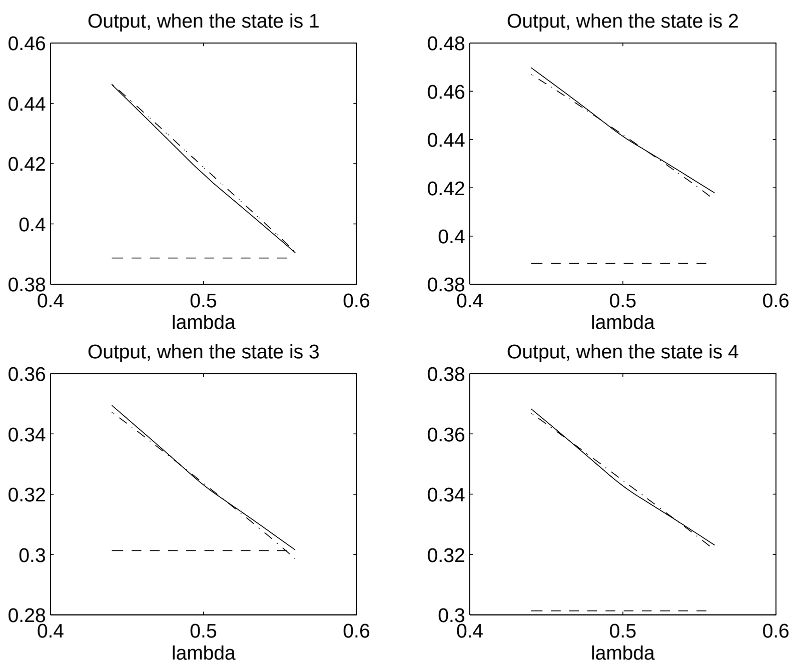

Figure 31: Output of agent 1 as a function of λ when $k_1 = k_2 = 0.78$. Solid line = Limited Commitment, dash-dotted line = Pareto Optimum, and dashed line = Autarky.

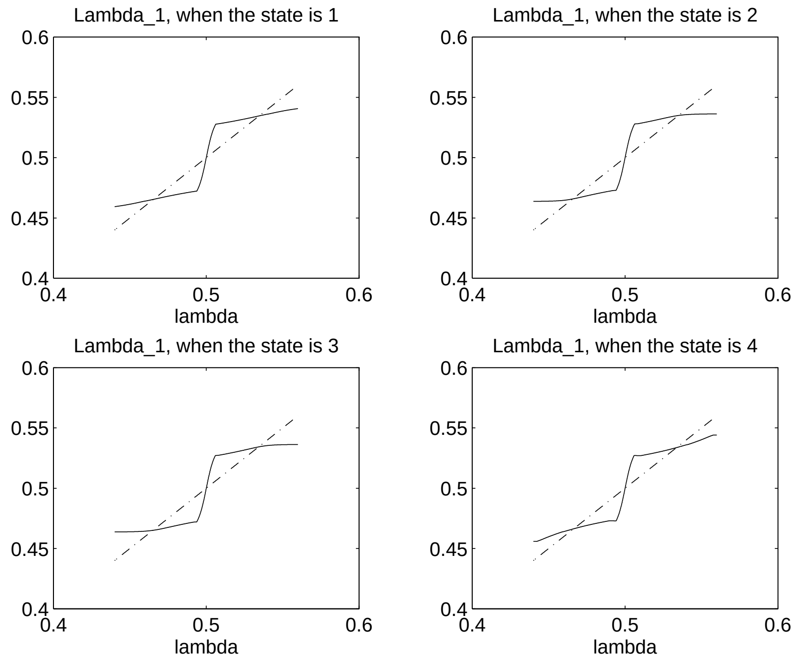

Figure 32: λ_1 as a function of λ when $k_1 = k_2 = 0.78$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

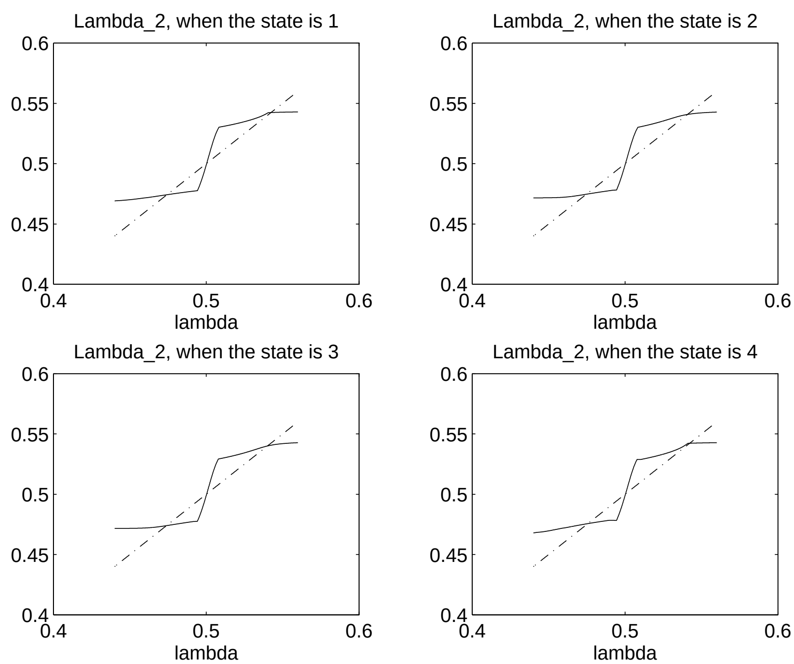

Figure 33: λ_2 as a function of λ when $k_1 = k_2 = 0.78$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

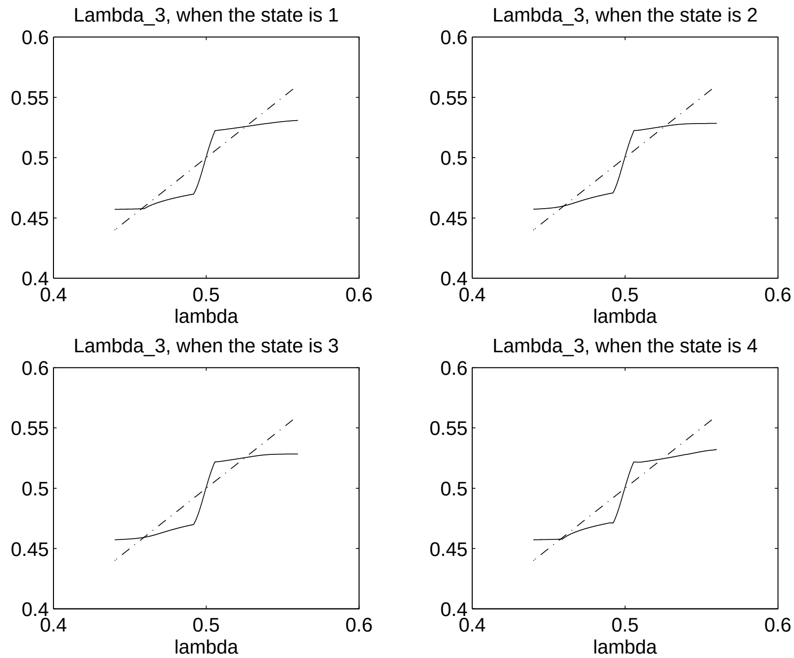

Figure 34: λ_3 as a function of λ when $k_1 = k_2 = 0.78$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.

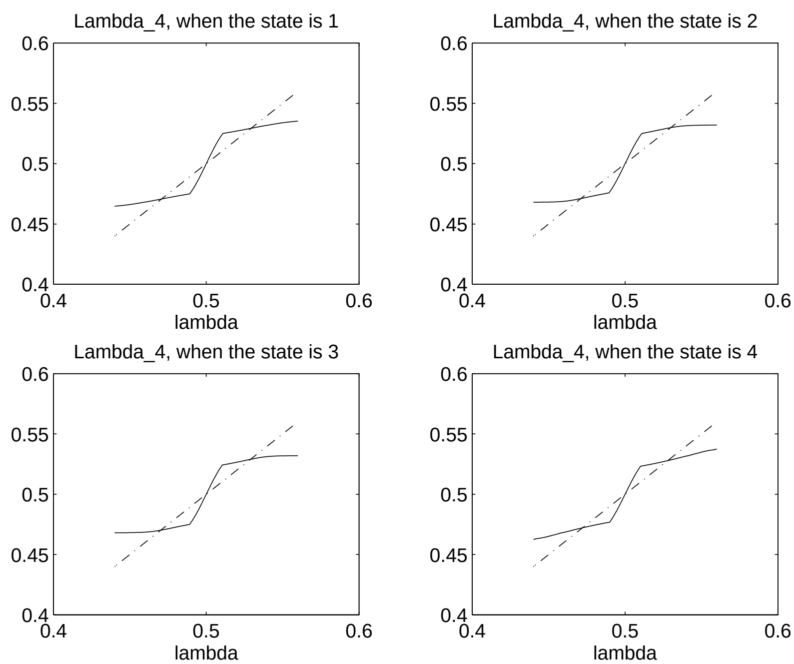

Figure 35: λ_4 as a function of λ when $k_1 = k_2 = 0.78$. Solid line = Limited Commitment and dash-dotted line = Pareto Optimum.