**The Evolution of Exchange**

by
Roberto Serrano*
Oscar Volij†

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# 1 Introduction

There has been a lot of interest lately in evolutionary game theory. In particular, there have been some attempts to use evolutionary concepts in order to make a selection among the set of Nash equilibria. One of these concepts was introduced by Foster and Young (1990) and is known as *stochastic stability.* Unlike the concept of evolutionary stability, which requires from a population to be immune to isolated random mutations, stochastic stability requires immunity against persistent random mutations. This concept was successfully applied by Kandori et al. (1993) in the analysis of symmetric $2 \times 2$ games, by Young (1993a) for weakly acyclic $n$ person games, by Young (1993b) in the analysis of bargaining, and by Vega-Redondo (1996) in the analysis of competition among firms.

As Young (1998) stresses, the stochastic stability approach can be applied to the analysis of a variety of social interactions, and not only to the adaptive playing of games. In this paper we are interested in applying the concept of stochastic stability to the simple housing market introduced by Shapley and Scarf (1974). A housing market consists of $n$ traders, each of whom is characterized by the only house he owns and by his complete, transitive and antisymmetric preference relation over the set of houses. In order to apply the concept of stochastic stability, we endow the housing market with a simple perturbed stochastic dynamic process. The unperturbed process can be described as follows. At each period a pair of traders is matched randomly and they trade their endowments if and only if the trade is mutually beneficial. The perturbation of the process consists of allowing a small probability of trade when it is not mutually beneficial. We want to see whether there is any relation between the efficient allocations in the economy and the set of stochastically stable outcomes of the process. We show that the efficient allocations are always stochastically stable.

*Dept. of Economics, Brown University, Providence, RI 02912, U.S.A. E-mail: roberto_serrano@brown.edu

†Dept. of Economics, Brown University, Providence, RI 02912, U.S.A. and Dept. of Economics, The Hebrew University of Jerusalem, 91905 Jerusalem, Israel. E-mail: Oscar_Volij@brown.edu

Further, we find that in economies with three traders, the set of stochastically stable states coincides with the set of efficient allocations. However, when there are more than three agents, there might be stochastically stable states which are inefficient.

We view the analysis of the simple housing market as a first step into the analysis of more general exchange economies. Our result though, gives little hope of finding some evolutionary foundation of efficient allocations, not to speak of Walrasian allocations.

# 2 The Model and Preliminary Results

A *house allocation problem* is a triple $\langle N, H, (\succeq_i)_{i \in N} \rangle$ where $N = \{1, \ldots, n\}$ is a finite set of individuals, $H = \{h_1, \ldots, h_n\}$ is a finite set of houses and for each individual $i \in N$, $\succeq_i$ is a complete, transitive and antisymmetric preference relations over $H$. The *size* of the problem is the number of agents in it.

Let $P$ be a house allocation problem. An *allocation* in $P$ is a one to one function $x : N \to H$ that assigns one house to each agent.

An allocation $x$ is *efficient* if there is no allocation $y$ such that $y_i \succeq_i x_i$ for all $i \in N$ and $y_i \succ_i x_i$ for some $i \in N$.

Let $x$ be an allocation in $P$. We say that individual $i$ *envies individual* $j$ *at* $x$ if and only if $x_j \succ_i x_i$. Define the *envy-graph of allocation* $x$ to be the directed graph whose vertices are the agents in the housing problem and there is an edge from agent $i$ to agent $j$ if and only if $i$ envies $j$. It is clear that allocation $x$ is efficient if and only if the corresponding envy-graph is acyclic.

The efficient allocations can also be characterized by means of the serial dictatorship mechanisms. Let $\pi : \{1, \ldots, n\} \to N$ be an ordering of the traders, i.e., $\pi(1)$ is the first trader, $\pi(2)$ is the second trader and so on. We say that allocation $x$ is the *outcome of the serial dictatorship mechanism with respect to* $\pi$ or that $x$ is induced by $\pi$, for short, if

- $x_{\pi(1)}$ is agent $\pi(1)$'s most preferred element in $H$
- for $t \in \{2, \ldots, n\}$, $x_{\pi(t)}$ is agent $\pi(t)$'s preferred element in $H \setminus \{x_{\pi(1)}, \ldots, x_{\pi(t-1)}\}$.

It is known that allocation $x$ is efficient if and only if it is the outcome of the serial dictatorship mechanism with respect to some ordering of the traders.

We shall define a dynamic process according to which agents perform bilateral trades. These bilateral trades will allow us to transit from one allocation to the other. Clearly, it is not always possible to go from one allocation to another by means of a single bilateral trade. When it is possible, we say that the allocations are pairwise connected. More formally, we say that allocations $x$ and $y$ are *pairwise connected* if there is a pair $i$ and $j$ of agents such that $x_i = y_j$, $x_j = y_i$ and $x_k = y_k$ for all $k \notin \{i, j\}$. A $(x, y)$*-path* is a finite sequence of allocations $(z^0, z^1, \ldots, z^k)$ such that $z^0 = x$, $z^k = y$ and for $t = 0, 1, \ldots, k-1$, $z^t$ and $z^{t+1}$ are pairwise connected. The following lemma shows that the set of efficient allocations is "connected".

**Lemma 1** Let $P = \langle N, H, (\succeq_i)_{i \in N} \rangle$ be a house allocation problem and let $x$ and $y$ be two efficient allocations in $P$. Then, there is an $(x, y)$-path composed exclusively of efficient allocations.

**Proof** : The proof is by induction on the size of the problem. If the problem consists of one agent, there is nothing to prove because the only allocation is efficient. Assume that the claim holds for all problems of size $K$, let $\mathcal{E} = \langle N, H, (\succeq_i)_{i \in N} \rangle$ be a problem of size $K + 1$ and let $x$ and $y$ be two efficient allocations in it.

**Case 1:** There is an agent, $k \in N$, who gets his most preferred house both at $x$ and $y$. Namely, $x_k = y_k \succeq_k h$ for all $h \in H$. Then, there are orderings $\pi$ and $\sigma$ of the traders, both with trader $k$ on top of them, which induce allocations $x$ and $y$, respectively. Let $N' = N \setminus \{k\}$, $H' = H \setminus \{x_k\}$ and consider the subproblem $\mathcal{E}' = \langle N', H', (\succeq_i |_{H'})_{i \in N'} \rangle$, where $\succeq_i |_{H'}$ is the restriction of $i$'s preferences to $H'$. The allocations $x|_{N'}$ and $y|'_N$ are efficient in $\mathcal{E}'$ since they are induced by the orderings $\pi$ and $\sigma$ respectively, restricted to the agents in $S$. Since $\mathcal{E}'$ is an economy of size $K$, by the induction hypothesis, there is a path $(\hat{x}^0, \ldots \hat{x}^m)$ of efficient allocations in $\mathcal{E}'$ from $x|_{N'}$ to $y|_{N'}$. Define now the allocations $(x^0, \ldots, x^m)$ in $\mathcal{E}$ by

$$x_i^t = \begin{cases} \hat{x}_i^t & \text{if } i \in S \\ x_k & \text{if } i = k \end{cases}$$

for $t = 0, \ldots, m$. The sequence $(x^0, \ldots, x^m)$ is a $(x, y)$-path of efficient allocations in $\mathcal{E}$ since they are induced by the orderings that induce $(\hat{x}^0, \ldots \hat{x}^m)$, respectively, after adding agent $k$ to the top.

**Case 2:** There is no agent that gets his most preferred house both at $x$ and at $y$. There are orderings $\pi$ and $\sigma$ of the traders which induce allocations $x$ and $y$, respectively. Let $k$ the first agent in the order $\sigma$, namely $\sigma(1) = k$. Clearly, $k$ is not the first agent in the order $\pi$. Without loss of generality assume that $\pi$ is the natural order, $\pi(t) = t$. Therefore $\pi(1) = 1 \neq k$. Since $k$ does not get his most preferred house in $x$, there must be an agent $\ell < k$ who has $k$'s most preferred house. Namely, $x_\ell = y_k$.

**Case 2.1:** $\ell > 1$. In this case, agent $\ell$'s and agent $k$'s respective most preferred houses differ. This implies that there is an allocation $z$, which is efficient in $\mathcal{E}$, and at which both agent $\ell$ and agent $k$ get their respective most preferred houses. Since agent $\ell$ gets his most preferred house both at efficient allocation $x$ and at efficient allocation $z$, by case 1, there is a $(x, z)$-path of efficient allocations. But since $k$ gets his most preferred house both at $z$ and at $y$, by case 1 again there is a $(z, y)$-path of efficient allocations. Joining both paths, we conclude that there is a $(z, y)$-path of efficient allocations.

**Case 2.2:** $\ell = 1$. In this case $x$ awards agent $\ell$ the house that is most preferred by both $\ell$ and $k$. Consider an ordering $\mu$ of the agents in which agent $\ell$ is first and agent $k$ is last and let $z$ be the efficient allocation induced by that ordering. Since agent $\ell$ gets his most preferred house both at $x$ and at $z$, by case 1, there is a $(x, z)$-path of efficient allocations. Let $z'$ be the allocation that is obtained from $z$ after agents $\ell$ and $k$ switch houses. Allocation $z'$ is efficient because it is induced by the ordering that is obtained from $\mu$ after $\ell$ and $i^*$ switch places. Therefore, $z$ and $z'$ are two pairwise connected efficient allocations. Clearly, $z'$ awards agent $k$ his most preferred house. Therefore, by case 1 again, there is a $(z', y)$-path of efficient allocations. We have built then a path of efficient allocations that connects $x$ with $y$. □

# 3 The Dynamic Process

Given a house allocation problem we want to define a perturbed Markov process as defined in Young (1998) where the states are the allocations of the housing problem. In each period one pair of agents is selected at random, where all pairs are equally likely to be chosen. Consider a pair of individuals, say $i$ and $j$. The probability that they trade depends on the degree of advantageousness of the trade. If the trade is mutually beneficial, then it takes place with high probability, say 1. If the trade is not mutually beneficial, then it takes place with a very low probability. Specifically, assume that if the trade is advantageous for only one trader, the trade takes place with probability $\epsilon$ and if the trade is disadvantageous for both traders, it takes place with probability $\epsilon^2$.

It can be checked that a state of the unperturbed process is absorbing if and only if there is no pair of agents that envy each other. It turns out that the absorbing states constitute the only recurrent classes of the process.

**Proposition 1** The recurrent classes of the unperturbed process are the singletons containing the absorbing states.

**Proof** : It is clear that a singleton containing an absorbing state is a recurrent class. Conversely, assume that $x$ is an allocation where there are at least two agents that envy each other. Then, with positive probability they will meet and trade. The number of pairs of agents that envy each other will be reduced by at least one as a result of the trade. If the resulting allocation is an absorbing state, we are done. Otherwise, there is a positive probability that two individuals that envy each other meet and trade, thus reducing the number of pairs that envy each other. Continuing in this fashion, we see that there is a positive probability that an absorbing state is reached which shows that $x$ does not belong to a recurrent class. □

We are interested in the stochastically stable states of the perturbed Markov process defined above. For any two allocations $x$ and $y$, define the *resistance* of the transition $x \to y$ as follows: if $x$ and $y$ are pairwise connected, then the resistance is the number of agents (0, 1, or 2) that find the bilateral trade unprofitable. Otherwise define the resistance to be $\infty$. Similarly, let $\xi = (z^i, \ldots, z^j)$

be an $(x, y)$-path. The resistance of the path $\xi$ is the sum of the resistances of its transitions.

Let $Z^0 = \{z^1, \dots z^n\}$ be the set of absorbing states of the unperturbed process and consider the complete directed graph with vertex set $Z^0$, which is denoted by $\Gamma$. We want to define the resistance of each one of the edges in this graph. For this, let $z^i$ and $z^j$ be two elements of $Z^0$. The resistance of the edge $(z^i, z^j)$ in $\Gamma$ is the minimum resistance over all the resistances of the $(z^i, z^j)$-paths. Let $z^i$ be an absorbing state. A $z^i$-*tree* is a tree with vertex set $Z^0$ such that from every vertex different from $z^i$, there is a unique directed path in the tree to $z^i$. The resistance of the $z^i$-tree is the sum of the resistances of the edges that compose it. The *stochastic potential* of the absorbing state $z^i$ is the minimum resistance over all the $z^i$-trees. The *stochastically stable states* are those states with minimum stochastic potential. The following lemma will be useful.

**Lemma 2** Let $x$ be a stochastically stable state and let $y$ be an absorbing state such that the edge $(x, y)$ has resistance 1. Then, $y$ is a stochastically stable state.

**Proof** : Let $T$ be an $x$-tree with minimum resistance over all the $x$-trees. Let $s(y)$ denote the immediate successor of $y$ in the unique path that connects $y$ to $x$. Build a new tree $T'$ by deleting from $T$ the edge $(y, s(y))$ and adding the edge $(x, y)$. It can be seen that $T'$ is a $y$-tree. Indeed, if there was a directed path in $T$ from $z$ to $y$, the same path connects $z$ to $y$ in $T'$. And if there was a directed path in $T$ from $z$ to $x$ that did not go through $y$, now the path that is obtained from that path by adding the edge $(x, y)$, is a directed path in $T'$ that connects $z$ to $y$. $T'$ is a $y$-tree that is obtained from $T$ by adding an edge of resistance 1 and deleting one edge of resistance greater or equal 1. Therefore, the resistance of $T'$ is less or equal the resistance of $T$. But since $T$ is an $x$-tree with minimum resistance over all the $x$-trees and since $x$ is a stochastically stable state, the resistance of $T'$ equals the resistance of $T$ and therefore $y$ is a stochastically stable state. □

**Corollary 1** If there is an efficient allocation that is stochastically stable, then all efficient allocations are stochastically stable.

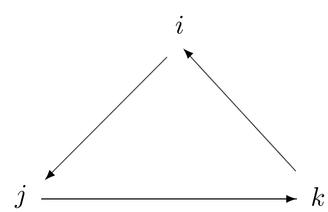
**Proof** : Let $x$ be an efficient allocation that is stochastically stable and let $y$ be another efficient allocation. By Lemma 1, there is a $(x, y)$-path of efficient allocations. Every edge along this path has resistance 1. By Lemma 2, all the efficient allocations along this path, and in particular allocation $y$, are stochastically stable. □

We are now ready to state one of the main results of the paper.

**Proposition 2** Let $P$ be a problem with 3 agents. The stochastically stable states of the corresponding perturbed Markov process are all the efficient allocations.

**Proof** : Let $Z^0$ be the set of absorbing states. The set $Z^0$ can be written as $Z^0 = Z^E \cup Z^I$ where $Z^E$ is the set of efficient allocations and $Z^I$ is the set of absorbing states that are not efficient. It is known that $Z^E \neq \emptyset$ but $Z^I$ may be empty.

**Case 1**: $Z^I \neq \emptyset$.
This means that there is an inefficient allocation at which no two agents envy each other. Therefore the envy graph associated to this allocation must look as follows:




Namely each agent envies one and only one agent and no two agents envy each other. But in this case, by executing the advantageous trilateral trade, each agent ends up with its most preferred house. Consequently, this resulting allocation is the only efficient one. Therefore $Z^0$ consists of 2 states: the efficient one, $z^e$ and the inefficient one, $z^i$.

In order to show that the efficient allocation is the only stochastically stable state, we shall show a path from the inefficient state to the efficient one that has resistance 1 and we shall show that every path from the efficient state to the inefficient one must have a resistance of at least 3. Starting from the inefficient allocation, execute the advantageous trilateral trade by letting agents $i$ and $k$ trade first and then letting agents $i$ and $j$ trade. The first trade has a resistance 1 and the second has a resistance 0 (since it is mutually beneficial for both $i$ and $j$). Therefore the corresponding path from the inefficient state to the efficient one has resistance 1 and since any path from one recurrent class to another must have a resistance of at least one, the stochastic potential of the efficient allocation is 1.

We shall show now that any path from the efficient state to the inefficient one has a resistance of at least 3. To see this, note that in order for each of the three traders to end up with a house that is less preferred than the initial one, each of them must execute at least one disadvantageous trade. Consequently, the stochastic potential of the inefficient absorbing state must be at least 3. The efficient state minimizes the stochastic potential over the absorbing states and therefore it is the only stochastically stable state.

**Case 2**: $Z^I = \emptyset$. In this case, since the set of stochastically stable states is nonempty, it must contain at least one efficient allocation. Then, by Corollary 1, all the efficient allocations are stochastically stable. $\square$

Unfortunately, Proposition 2 cannot be generalized to economies with more than 3 agents, as the following example shows. Consider the following four agent economy:

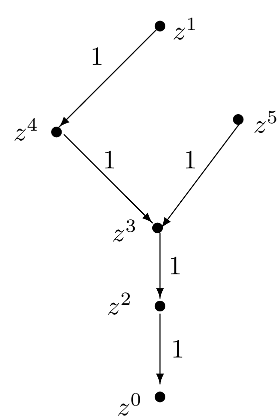
$$h_1 P_1 h_2 P_1 h_3 P_1 h_4$$
$$h_1 P_2 h_3 P_2 h_2 P_2 h_4$$
$$h_4 P_3 h_3 P_3 h_1 P_3 h_2$$
$$h_1 P_4 h_2 P_4 h_4 P_4 h_3$$

In this economy there are six absorbing states, five of which are efficient allocations.

| | Agents | | | |
|---|---|---|---|---|
| Allocation | 1 | 2 | 3 | 4 |
| $z^0$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
| $z^1$ | $h_1$ | $h_3$ | $h_4$ | $h_2$ |
| $z^2$ | $h_2$ | $h_1$ | $h_3$ | $h_4$ |
| $z^3$ | $h_2$ | $h_1$ | $h_4$ | $h_3$ |
| $z^4$ | $h_2$ | $h_3$ | $h_4$ | $h_1$ |
| $z^5$ | $h_3$ | $h_1$ | $h_4$ | $h_2$ |

**Table 1: Absorbing states**

We will show that the inefficient allocation $z^0$ is stochastically stable by showing a $z^0$-tree with a resistance of 5, which is the minimum resistance that a $z$-tree with 6 vertices can possibly have. This will mean that the inefficient allocation has minimum stochastic potential and therefore it is stochastically stable. One of the $z^0$-trees with minimum resistance is the following:



To see that each of the directed edges $z^i \to z^j$ in the above tree have a weight of 1, note that one can move from allocation $z^i$ to allocation $z^j$ by means of one bilateral trade with resistance 1.

We have seen that if an efficient allocation is stochastically stable, so are all the efficient allocations. One may ask whether it is possible that no efficient allocation is stochastically stable. The following proposition shows that this is impossible.

**Proposition 3** The set of stochastically stable allocations contains all the efficient allocations.

**Proof** : Given Corollary 1, it is enough to show that there is one efficient allocation that is stochastically stable. Pick a stochastically stable allocation, $x$. If $x$ is efficient we are done, so assume it is not efficient. If we show that there is a path from $x$ to an efficient allocation, such that each of its transitions has resistance less or equal 1, we are done, because by Lemma 2 the efficient allocation will be stochastically stable. The existence of the required path is an immediate consequence of the following:

**Lemma 3** Let $x$ be an absorbing state that is not efficient and let $m \geq 3$ be the number of agents that belong to some cycle of $x$'s envy-graph. Then, there is an absorbing state $y$ such that the edge $(x, y)$ has resistance 1 and such that the number of agents who belong to some cycle of $y$'s envy graph is less than $m$.

**Proof** : Let $A_1$ be the set of agents that are allocated their most preferred house under allocation $x$ and let $B_1$ be the complement of $A_1$:

$$\begin{aligned} A_1 &= \{i \in A : x_i \succeq_i x_j \quad \forall j \in N\} \\ B_1 &= N \setminus A_1. \end{aligned}$$

Define recursively the following set of agents: for $k = 1, 2, \ldots$

$$\begin{aligned} A_{k+1} &= \{i \in B_k : x_i P_i x_j \qquad \forall j \in B_k\} \\ B_{k+1} &= B_k \setminus A_{k+1}. \end{aligned}$$

Let $A = \cup_{k=1}^{\infty} A_k$. It is immediate that the agents in $A$ do not belong to any cycle of the envy-graph of $x$. Since $x$ is not efficient, $B = N \setminus A \neq \emptyset$. $B$ is the set of agents that belong to a cycle of the envy-graph of $x$. Therefore $m = |B|$. It is also clear that no agent in $A$ envies anybody in $B$. Let $i \in B$. Then there is an agent in $B$ who is envied by $i$. Let $j$ be the agent who owns the $\succeq_i$-maximal house in the set of houses that belong to agents in $B$. That is, $x_j \succeq_i x_t$ for all $t \in B$. Let $x'$ be the allocation that is obtained from $x$ after $i$ and $j$ trade. At this allocation, no agent in $A \cup \{i\}$ envies anybody in $B_k \setminus \{i\}$. Therefore, the number of agents that belong to a cycle in the envy-graph of $x'$ is less or equal $|B_k \setminus \{i\}|$ which is less than $|B_k| = m$. If $x'$ is absorbing, then we are done. Otherwise, there is a $(x', y)$-path from $x'$ to some absorbing state $y$ with resistance 0. Clearly, the number of agents that belong to a cycle of the envy graph of $y$ is less that $|B_{k^*}| = m$. $\square$

This completes the proof of the proposition. $\square$

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