N-Person Bargaining and Strategic Complexity*

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Abstract

We investigate the effect of introducing costs of complexity in the n -person unanimity bargaining game. In particular, the paper provides a justification for stationary equilibrium strategies in the class of games where complexity costs matter. As is well-known, in this game every individually rational allocation is sustainable as a Nash equilibrium (also as a subgame perfect equilibrium if players are sufficiently patient and if $n > 2$). Moreover, delays in agreement are also possible in such equilibria. By limiting ourselves to strategies that can be implemented by a machine (automaton) and by suitably modifying the definition of complexity for the purpose of analysing a single extensive form, we find that complexity costs do not reduce the range of possible allocations but they do limit the amount of delay that can occur in any agreement. In particular, we show that in any n-player game, for any allocation z, an agreement on z at any period t can be sustained as a Nash equilibrium of the machine game with complexity costs if and only if $t \leq n$. We use the limit on delay result to establish that, in equilibrium, the machines implement stationary strategies.

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†The current, March 1999, version is based primarily on one written in April 1998, and presents our results in some detail.

Finally, we also show that "noisy Nash equilibrium" with complexity costs sustain only the unique stationary subgame perfect equilibrium allocation.

1 Introduction

In many multistage games, such as repeated games and bargaining with more than two players, a large number of outcome paths can be supported as equilibria. Such outcome paths are often sustained (as equilibria) by the use of non-stationary (history dependent) strategies by the players, which induce different expectations for future play depending on the history preceding the play. These equilibria can be made to satisfy subgame perfection and other stronger refinements.

One approach for dealing with the multiplicity problem is to consider explicitly costs of human computational and storage abilities (bounded rationality) that restrict the extent to which strategies can depend on history. Modelling strategies as being implemented by finite-state machines falls within this category. Restrictions on strategies are introduced either by some exogenous bounds on the complexity of the machines or by making complexity a decision variable - complex strategies can be used but the complexity comes at a cost compared to simpler strategies. In this paper, by modelling strategies as automata, we investigate the effect of introducing complexity costs in the n-person unanimity bargaining game and thereby provide a justification in terms of complexity costs for history-independent (stationary) equilibrium strategies.[1] The bulk of the literature in this area has dealt with repeated, normal-form games. (See Kalai (1990) for a survey.) This paper is an exception, since we are concerned with a single, extensive-form game.

We now describe the object of our investigation, the unanimity bargaining game (n -player, one-cake problem) of Binmore (1985), Herrero (1985) and Shaked (1986). The game is as follows: There are n players $(1, \ldots, n)$ who have joint access to a 'cake' of size unity, if they can agree on how to share it amongst themselves. Player 1 makes the first offer at time $t = 1$, offering $x = (x_1, \ldots x_n)$,where x_i is player i 's proposed share and $\sum_i x_i = 1$. Player 2 to player n then respond sequentially, each saying either A(ccept) or R(eject) to the proposal. If all responders accept then the game ends with player i obtaining a payoff of x_i. A rejection takes the game to the next period, where player 2 now makes an offer and Players $(3, \ldots n, 1)$ sequentially respond. If one of the responders rejects at time $t = 2$, then the game goes to the next stage with player 3 making the offer and so on.[2] If the offer x is accepted by all responders in period t, the payoff is $\delta^{t-1} x_i$ to player i, where $\delta < 1$ is the common discount factor. There is no exogenously imposed limit on the duration of the game. Absence of agreement, that is bargaining forever, leads to a payoff of 0 for all players. Trivially, one can

[1] The $n = 2$ case is not the main focus of this paper; however we do prove stationarity for the two-player game under weaker conditions than for the general n-player game.

[2] Thus player k makes an offer at $t = k + \tau n$ for every non-negative integer τ.

show that any partition of the cake at any time t can be sustained as a Nash equilibrium of this game. Shaked showed (see Osborne and Rubinstein (1990) for an exposition) that for the 3-player game every allocation of the cake, including the extreme points, can be sustained as a subgame perfect equilibrium (SGPE) if $\delta > \frac{1}{2}$. Outcomes with delay can also be sustained as SGPE for 3-player games. This is in sharp contrast to the two-player bargaining model of Rubinstein (1982) where there is a unique subgame perfect equilibrium allocation $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ with no delay and the property that as $\delta \longrightarrow 1$, the allocation tends to an equal split. The indeterminacy of the three (or n) player game has aroused much interest in the literature, for example Jun (1987) and Krishna and Serrano (1996). These papers have explored alternative extensive forms that give a unique SGPE with the limiting equal split. Their models have the feature that a player is able to leave with his share before the entire bargaining process is completed, in contrast with the previously-cited work of Binmore-Shaked-Herrero, where the entire allocation has to be unanimously approved before anyone can leave.

The stationary SGPE allocation of the original extensive form has also been proposed as a "focal" equilibrium.[3] It is unique and involves no delay (Herrero, (1985)). Moreover the allocation corresponding to the unique stationary SGPE is $(\frac{1}{1+\delta+\ldots+\delta^{n-1}}, \frac{\delta}{1+\delta+\ldots+\delta^{n-1}}, \ldots, \frac{\delta^{n-1}}{1+\delta+\ldots\delta^{n-1}})$, which also tends to the equal split as $\delta \to 1$.

It might be argued, along the lines discussed above, that a player concerned with the cost of implementing complex strategies would choose a stationary strategy, where offers and responses are independent of history.[4](This would then lend some credence to the "focal" equilibrium.) This paper is an attempt to formalise this intuition by extending the framework of Rubinstein (1986), Abreu and Rubinstein (1988), Piccione and Rubinstein (1993) and others.[5] These earlier papers modelled players as finite-state automata involved in a two-player repeated game. Complexity was modelled as the (arbitrarily small) cost of maintaining an additional "machine" state. In these papers, the number of states in an automaton implementing a strategy can be regarded as a measure of the size of the parti-

[3] A general analysis of non-cooperative characteristic function bargaining primarily concerned with the properties of stationary equilibria is carried out in Chatterjee, Dutta, Ray and Sengupta, (1993). There are several other models of coalitional bargaining, almost all of which invoke stationarity to get results. See, for example, Selten (1981), which also relies on other additional principles, Moldovanu and Winter (1995), Gul (1989), Hart and Mas Colell (1996) and Perry and Reny (1994). Gul mentions computational simplicity as a reason for selecting stationary equilibria in these models, but does not elaborate on this statement.

[4] Osborne and Rubinstein (1994) argue, however, that it may be difficult to justify a stationary equilibrium by appealing to its simplicity. Equilibrium strategies, they believe, should be thought of as equilibrium in beliefs and it is not clear why players should believe that other players follow the same action at every period after a history which has involved many deviations.

[5] Another early use of the framework of finite automata to study repeated games was Neyman (1985).

tion that the strategy induces on the set of possible histories. Thus it measures the extent of history-dependence of the strategy. We will account for complexity of strategies in a similar manner. However, the new context of a single extensive form game introduces several different ways of specifying a machine and of defining a measure of the complexity of a strategy.[6] In the next section, we adopt a particular definition of an automaton and discuss the resulting interaction between complexity measures and the machine specification adopted.[7]It is important to note that, while the different specifications of machines are equivalent, in the sense of being able to implement the same set of strategies, the refining[8] effect of different complexity criteria could depend on the particular specification.

Focussing on complexity of implementation rather than of computation has its critics, for example, Papadimitriou (1992). We will not address this issue here, or more philosophical aspects of representing players as automata.

In most of the paper we consider the general n-player unanimity bargaining game. We also consider the 2-player game before embarking on the n-player game in Section 2, since this case turns out to have special features.

We now briefly describe the rest of the paper. In Section 2, we describe a game in which players choose, at the beginning of the game, machines to implement their strategies, and characterise the set of Nash equilibria in the machine game, where costs of complexity are considered lexicographically with the standard bargaining payoffs.[9] Our first set of results shows that in any n-player game, agreement at any period $t \leq n$ on any partition of the cake can be sustained as Nash equilibria of the machine game. Thus this result demonstrates that delays in agreement are possible. We also demonstrate that absence of agreement (indefinite delay) can be sustained as a Nash equilibrium in the machine game. Our second set of results show, however, that there is some bite to the simplicity requirement. We show that agreements beyond period n can not be sustained as Nash equilibria in the machine game. Thus all Nash equilibria of the machine game must entail *either* an agreement by some time period $t \leq n$ *or* indefinite delay. Further, we show that all Nash equilibrium machines must be minimal, that is, have one state. We call the set of strategies that can be implemented by one-state machines 'stationary'. [10] Thus the results of this section demonstrate

[6]Unlike the case of repeated normal-form games, there could be several different ways to specify a machine in the extensive-form context, as Piccione and Rubinstein (1992) have also pointed out.

[7]We do not follow Banks and Sundaram (1990) in considering complexity of transitions on the equilibrium path, since our complexity measure is sufficient to give us stationarity (and uniqueness) in any case.

[8]We mean refining the set of equilibria.

[9]In this paper, we do not need to assume finiteness of the machines: the number of states, the set of possible inputs and the set of possible outputs of a machine could be infinite. The structure is therefore only a small departure from that of the standard bargaining game.

[10]In our specification of the machine game, a machine changes its state after every n periods

that all strategies that can be implemented by equilibrium machines must be stationary. The two results on no agreement beyond period n and on stationary behaviour are established for 2-player games using the number of states of the machine as a measure of complexity, and for $n(\geq 3)-$player games by using a slightly stronger definition of complexity. We justify the use of this stronger requirement in the sequel.

Strategy profiles implemented by Nash equilibria in the machine game are not necessarily credible 'off the equilibrium path'. One way of introducing credibility is to extend the concept of subgame perfect equilibrium to the machine game. There is not an obvious way of doing this, since it could potentially involve changing the machine used after every period, thus making the intuitive notion of choosing to "buy" complexity at the outset of the game (as in Abreu and Rubinstein, (1988)) somewhat problematic. An alternative way of capturing credibility of strategies is to introduce noise in the system. A natural way of doing this is to consider the Nash equilibria of a machine game in which the machines make output errors with arbitrarily small probability. Section 3 introduces this version, where the machines are not error free. Such errors force the "noisy Nash equilibrium machines" to be credible on "almost" all histories. In this section, the results of section 2 on no delay beyond period n and stationary behaviour are used to show that a noisy Nash equilibrium of the machine game leads to the unique stationary subgame perfect equilibrium allocation of the bargaining game. An alternative approach, suggested by a referee, is to require that Nash equilibria in the machine game to be optimal after all histories, rather than after almost all histories. We actually prove this slightly weaker version of the result in the text, since the proof is shorter than the original proof, which can be found in Appendix B.

In Section 4, we discuss alternative formulations of the machine game as well as alternative refinements of Nash equilibrium in the machine game. Section 5 concludes.

It should be pointed out that the definitions and the methods of proof used in this paper are somewhat different from those in Abreu and Rubinstein (1988), and Piccione and Rubinstein (1993). This is for two separate reasons. First, in 2-player games, the two machines chosen by the players in equilibrium must have an equal number of states because each player faces a Markov decision problem, given the other player's behaviour, and from the basic result for Markov decision processes, a player need only use at most as many states as his opponent. In our paper, we cannot appeal to the Markov decision processes result to prove that the players will all choose the same number of states. All this result tells us is that each player will not choose more states than the product of the numbers of

of offers (after every player makes exactly one proposal). Such a sequence of n rounds of play is called a 'stage'. Thus stationary strategies in our set-up refer to strategies that behave in the same way in every 'stage', irrespective of the history in previous 'stages'.

states chosen by the other players.[11] Secondly, in this paper, as we mentioned before, we are concerned with a single extensive form game, not a repeated game.

Binmore, Piccione and Samuelson (1996) is another contemporaneous paper on complexity and bargaining.[12]Their paper is about the *two-player* Rubinstein model, and they obtain their results by using a refinement of Nash equilibrium, modified evolutionary stability. In a sense the aim of their paper is to establish the Rubinstein result with an evolutionary solution concept, which they feel has more desirable antecedents than subgame perfectness. It is therefore quite different in spirit from our model; we do consider the two-player case, but concentrate on selection among the set of (subgame perfect) equilibria in the n-player game by introducing complexity costs. It is not obvious how the framework of Binmore et al. extends to the more general n-player game considered here. Of course, there is no selection issue with subgame perfectness in two-player games; there is with *Nash.*

Baron and Kalai (1993) have considered a majority rule game played by finite automata. They use undominated Nash equilibrium as the solution concept, and are concerned with the complexity of *equilibria* rather than the complexity of individual strategies; for example, minimizing the maximum complexity of any machine used in an equilibrium.

2 The Deterministic Machine Game and Its Nash Equilibria

2.1 Basic Definitions: The Game, Machines and Complexity

Our purpose in this section is to investigate the effect of complexity costs on the equilibria of a single, extensive form game with two or more players. We first consider the formulation of the automata involved; for reasons cited in the introduction, this has to be somewhat different from that current in the existing literature. In addition, each player in our game has to play different roles, as proposer or k^{th} responder, $k \geq 1$, and thus must have different action sets corresponding to these roles.

In this subsection, we therefore first make explicit the notion of a machine and that of a 'period' and a 'stage' of the bargaining game. Let $\Delta^n \equiv \{x = (x_1, .., x_n) \,|\sum_i x_i = 1\}$ be the $(n-1)$-dimensional simplex. A period refers to a single offer $x \in \Delta^n$ by one player and responses made sequentially and separately by the others. The

[11]Clearly, one implication of the results of this paper is that in equilibrium all machines have an equal number of states (one state), but this is not used to *establish* the results.

[12]This paper came to our notice after earlier drafts of our current paper had been written. We would like to thank Murali Agastya and Ken Binmore for drawing our attention to this work.

set of choices available to a player in the bargaining game is denoted by

$C \equiv \Delta^n \cup A \cup R$, where A and R refer to acceptance and rejection, respectively

A stage refers to n consecutive periods with player 1 making the proposal in the first period, player 2 making the proposal in the second period...and player n making the proposal in the n-th period. A stage could be terminated before n periods has elapsed if the bargaining is terminated by an agreement before the n-th period. In any given period, a player is in one of the following n roles: the proposer or the k-th responder for some $k = 1, .., (n-1)$, while in any stage the player must be prepared to play all n roles. We shall denote a history of outcomes in a stage by e. Also we shall denote the set of all such possible histories of a stage by E. Thus a particular history of a stage, $e = \{(x^1, A, A, R), (x^2, A, A, A, R), ..., (x^n, R)\} \in E$, could, for example, consist of an offer x^1 by player 1, an acceptance by the first and the second responder and a rejection by the third responder in the first period of the stage, followed by an offer x^2, an acceptance by the first three responders and a rejection by the fourth responder in the second period of the stage, .., and finally an offer x^n rejected by the first responder in the n-th period of the stage. A history of a stage is therefore a complete account of what happened in the bargaining in a given stage. Notice that a history of a stage consists of at most $n \times n$ choices (offers and responses) by the players.

We also need notation for partial descriptions of choices (partial history) within a stage. We shall denote such a partial history by s and the set of such partial histories by S. Examples of $s \in S$ could be the null set $\emptyset$, an offer x or an offer x followed by k Acceptances for some $k < n$, histories of the first k periods of the stage for some $k < n$ ..... If s is the null history, the stage is just beginning and an offer has yet to be made. Thus

$S = \emptyset \cup \{s = (c^1, ..., c^r) \in C^r \left|(s, d^1, ...d^{r'}) \in E\right.$ for some sequence of choices $(d^1, ..., d^{r'}) \in C^{r'}$ $\}$, where for any τ, C^τ is the τ-fold Cartesian product of C.

Also, we shall denote the information sets (the sets of partial histories) for player i in any stage by S_i. Thus $S_i \equiv \{s \in S \mid$ it is i's turn to play after $s\}$.

Finally, denote the set of choices available to a player i, given a partial description $s \epsilon S_i$, by $C_i(s)$. Thus

$$C_i(s) = \begin{cases} \Delta^n & \text{if } i \text{ is the proposer (if } i = 1 \text{ and } s = \emptyset \text{ or if } i > 1 \text{ and } s \text{ is a complete} \\ & \text{history of the first } i-1 \text{ periods of the stage)} \\ \{A,\ R\} & \text{if } s \text{ is such that } i \text{ is the responder to some offer } x. \end{cases}$$

An automaton consists of a set of states (not necessarily finite), an initial state, a terminal state, an output function describing the output of the machine as a function of its current state (and its current input) and a transition function determining the next state of the machine as a function of its current state and the actions of all the players in a given stage. Since, the bargaining game has a certain degree of asymmetry built in and each player has to play a different role in different periods, we can choose to specify a machine to implement a particular strategy in several different ways. In this paper, we shall assume that the states of the machine do not change during each stage of the bargaining game and transitions from a state to another state in the same player's machine take place at the end of a stage. Also it is assumed that each state of the machine would specify an action for every role of the player concerned, with the action chosen depending on s - the partial history of the stage. A referee (labelled "Master of the Game" by Piccione and Rubinstein, (1992)) would activate each player's machine when needed. We now set down the formal definition.

Definition 1 ($D1$) *A machine M_i is a five-tuple ($Q_i, q_i^1, T, \lambda_i, \mu_i$) , where*

Q_i is a set of states;

q_i^1 is a distinguished initial state belonging to Q_i;

T is a distinguished absorbing or terminating state (T for "Termination");

$\lambda_i : Q_i \times S_i \to C$, describes the output function of the machine given the state of the machine and given the partial history that has occurred during the current stage of the bargaining before $i's$ move such that $\lambda_i(q_i, s) \in C_i(s)$, $\forall q_i \in Q_i$ and $\forall s \,\epsilon\, S_i$;

$\mu_i : Q_i \times E \to Q_i \cup T$ is the transition function, specifying the state of the machine in the next stage of the bargaining as a function of the current state and the realised history of the stage.[13][14]

[13] Henceforth, we shall not always explicitly refer to the terminal state T. We are assuming that if an offer is accepted by all responders, M_i enters state T and shuts off. Thus $\mu_i\,(q_i,\ x,\ A,\ ...,A) = T$ for any state q_i and any $x \,\epsilon\, \Delta^n$. Also, unless stated otherwise, we shall drop the term 'non-terminal' from the 'non-terminal states' of the machine and simply refer to them as the states of the machine.

[14] Other specifications that we have considered are listed below. We prefer the one used in the text, because the state of a machine changes at the end of a stage, where the rest of the game "looks" just like it did at the end of the previous stage.

(D_2) The machine a player chooses to implement his/her strategy will consist of n sub-machines; each sub-machine consisting of states enabling a player to play a given role. Transitions take place from a state in each sub-machine to a state in the same sub-machine one stage (n periods) after the last choice of the sub-machine, for all sub-machines (roles). Here the action chosen by each sub-machine in any state may depend on the partial history $\overline{s}$ of the period (and not on the partial history of the stage s as in D$_1$). The "Master of the Game" activates the appropriate role for each player, as in D_1.

(D_3) As in D_1, except that the transitions now take place at the end of each period, rather than at the end of each stage. Naturally, here again the action chosen by the machine in any state also depends on the partial history $\overline{s}$ of the period.

(D_4) As in D_2, each machine is partitioned into three parts; the transitions now take place at the end of each period, rather than at the end of each stage. Since the transitions can be from a state in one sub-machine to a state in a different sub-machine for each player, the change of roles from period to period can be automated without involving the "Master of the Game". Here, as in D$_3$, the

Remark 1 *If we denote the set of strategies for a player i in any stage of the bargaining game by $\mathcal{F}_i \equiv \{f : S_i \to C \,|\, f(s) \in C_i(s)\ \forall s \in S_i\}$, then the output function λ_i in Definition 1 can be thought of as a mapping $\widetilde{\lambda}_i : Q_i \to \mathcal{F}_i$ where $\widetilde{\lambda}_i(q_i)(s) = \lambda_i(q_i, s)$.*

The basic rationale for using this specification of a machine is that the bargaining game is identical at the beginning of each stage, though strategies could differ depending on past history as encapsulated in the state. Thus the nature of the output and transition maps remain the same in each stage. Other definitions are possible, where the state changes every period, or before a player has to move, but these definitions do not have the "game-stationarity" flavour that the current one does. In Section 4 below, we shall discuss how our results need to be adapted if different specifications of a machine are used.

The complexity of a machine (or of a strategy) can be measured in many different ways. In the literature on repeated games played by automata the number of states of the machine is often used as a measure of complexity. This is because the set of states of the machine can be regarded as a partition of possible histories.

We shall denote the number of states of a machine M_i by $\|M_i\|$. This refers to the number of states in the set Q_i (we shall ignore the terminal state).

Definition 2 *(State complexity, denoted by 's-complexity') A machine M_i' is more s-complex than another machine M_i, denoted by $M_i' \succ^s M_i$, if*

$\|M_i'\| \ > \|M_i\|$.

We shall use $M_i' \succeq^s M_i$ to denote "M_i' is at least as s-complex as M_i".

Since, the states of a machine do not change during a stage of the bargaining game, counting the number of states does not fully capture the complexity of the machine during a stage, specifically the complexity of different choices following the same partial history. More formally, consider the output function $\lambda_i(.,.)$, which can be thought of (see Remark 1), as a mapping $\widetilde{\lambda}_i : Q_i \to \mathcal{F}_i$ where for each q_i, $\widetilde{\lambda}_i(q_i)$ is a mapping from the information set within a stage S_i to the set of actions C. More formally, the above definition of complexity measures for each s the cardinality of the domain of $\lambda_i(., s)$ (or that of $\widetilde{\lambda}_i(.)$), but it does not capture the cardinality of the range of the mapping $\lambda_i(\cdot, s)$. To illustrate the point further consider the following example.

Example 1 *There are two machines for player i. The first machine M_i has two states (other than the terminal state) and always proposes x in the odd stages of the bargaining game and proposes y in the even stages. Also, it always rejects any proposal. Machine M_i' also has two states and proposes x in the odd stages*

output of the machine may depend on the partial history within a period rather than that within a stage.

and y in the even stages. But it responds differently to the same proposal (by conditioning on the two states). In particular, it rejects offer z in the odd stages, accepts offer z in the even stages and rejects any other offer.

According, to the state complexity criterion ($\succ^s$), the two machines in Example 1 are of equal complexity, despite the fact that the strategy that M_i' implements has the additional complexity of different responses to the same offer. This is not a desirable property of Definition 2 ($\succ^s$). One way to capture the additional complexity of different behaviour after the same partial history would be to strengthen our definition of complexity. A minimal strengthening of $\succ^s$ would be the following.

Definition 3 *(Response-State complexity, denoted 'r-complexity') A machine $M_i' = \left\{Q_i', q_i^{1\prime}, T, \lambda_i', \mu_i'\right\}$ is more r-complex than another machine $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$, denoted by $M_i' \succ^r M_i$, if*

either (i) $M_i' \succ^s M_i$

or (ii) the machines M_i and M_i' are otherwise identical except that given some non-null partial history $s' \in S_i$, M_i always responds the same way to s' irrespective of the state of the machine and M_i' responds differently to s' depending on the state of the machine. Formally, $Q_i = Q_i', q_i^1 = q_i^{1\prime}$,$\mu_i = \mu_i'$ and there exists a non-empty partial history $s' \in S_i$ such that

$\lambda_i'(q_i, s) = \lambda_i(q_i, s) \qquad \forall s \neq s'$ *and* $\forall\ q_i \in Q_i = Q_i'$

$\lambda_i(q_i, s') = \lambda_i(q_i', s') \qquad \forall q_i, q_i' \in Q_i,$

$\lambda_i'(q_i, s') \neq \lambda_i'(q_i', s') \qquad$ *for some* $q_i, q_i' \in Q_i'$.

As before, we shall use $M_i' \succeq^r M_i$ to refer to M_i' is at least as r-complex as M_i.

Part (ii) of the above definition distinguishes between two otherwise identical machines that are such that one chooses a constant action given a partial history s' and the other does not. This minimal strengthening of the state complexity criterion is sufficient to establish our main results. In fact, it turns out that for 2-player games, the weaker definition of complexity $\succ^s$ is sufficient to obtain the stationarity of equilibria (and thus its uniqueness with noise). In the case of more than two players this is not the case. We show, by means of an example later on, that the stationarity of equilibrium strategies does not hold with the weaker definition of complexity $\succ^s$ for the case of 3-player games. However, we shall show that the stationarity result (and thus the uniqueness with 'noise') does hold for more than two players if we use r-complexity.

2.2 Definition of equilibrium in the machine game

We now describe the machine game. At time 0, players 1,..,n simultaneously and independently choose their machines. Player i's machine is denoted as M_i. At time period 1, the sequential bargaining game, described in the previous section, begins, with the prescribed order being 1,..,n.

A *Nash Equilibrium* of the machine game is defined in the usual way. Let $\pi_i(M_{i,}\ M_{\sim i}\ )$ be the expected payoff to i in the bargaining game in which the M_i are the chosen machines.

Definition 4 *A machine profile* $M = (M_1, .., M_n\ )$ *constitutes a Nash equilibrium of the machine game with complexity criterion* $l = s, r$ *(denoted by NEMl) if*

1. $\pi_i(M_i,\ M_{\sim i}) \geq \pi_i(M_i',\ M_{\sim i}), \forall M_i'$,
2. *if* $\pi_i(M_i,\ M_{\sim i}) = \pi_i(M_i',\ M_{\sim i})$, *then* $\qquad M_i' \succeq^l M_i$.

Remark 2 *Clearly, the set of NEMs contains the set of NEMr. Henceforth, whenever we refer to NEM, we mean both NEMs and NEMr.*

Note that since complexity is measured by the number of states of the machine (both in NEMs and in NEMr), a saving of a state is sufficient to destroy a Nash equilibrium. Also note that every machine must have at least one (non-terminal) state.

Definition 5 *A machine is a minimal machine if and only if it has exactly one (non-terminal) state. A strategy in the bargaining game is said to be stationary if it can be implemented by a minimal machine.*

Clearly, in each stage a minimal machine (or a stationary strategy) will implement the same actions, following different histories of the preceding stages, provided that the partial history within the current stage is the same.Thus the behaviour of such machines at any time may depend on the past outcomes in the current stage but not on the previous history of the game before the current stage. It is in this sense that the strategy that is implemented by a minimal machine is stationary.[15] Moreover, note that if M_i is minimal then $M_i' \succeq^r M_i$ for any machine M_i' .[16]

Costs of additional states are treated lexicographically in the definition of NEM above. We could also have defined the payoff to player i as being

$$\pi_i(M_i,\ M_{\sim i}) - c \mid M_i \mid -d \sum_s \Gamma(s) \ , \tag{1}$$

where $c > 0$ and $d \geq 0$ are some fixed cost and $\Gamma(s) \equiv |\{\lambda(q,s)\,|q \in Q_i\}|$. Such a payoff function would induce at least as much economy in states (and output function) as the lexicographic criterion. Our results are also valid if complexity costs enter the payoff functions as in (1).

[15]Stationarity would have a different meaning if we specified the machines in a different way (more on this in Section 4).

[16]In the case of minimal machines condition (ii) of Definition 3 does not apply and thus all machines are as r-complex as a minimal machine.

2.3 NEM Allocations.

We now state our first proposition.

Proposition 1 *Any agreement $z \in \Delta^n$ in the first period can be sustained as a NEM.*

Proof. Consider the minimal machine M_i with one non-terminal state q_i (in addition to the termination state), with the output function defined as:

$$\left.\begin{array}{c} \lambda_i(q_i,\,\emptyset) = z, \\ \lambda_i(q_i,\, y \text{ has been offered and } y_i \geq z_i) = A, \\ \lambda_i(q_i,\, y \text{ has been offered and } y_i < z_i) = R, \end{array}\right\} \tag{2}$$

Thus machine M_i always offers z as a proposer, and rejects anything that gives player i less than z_i if called on to respond. The transitions are simple, namely: $\mu_i(q_i, e) = T$, if e involves an agreement and $\mu_i(q_i, e) = q_i$, otherwise.

The machines M_i are clearly best against each other. Since they have one state (other than T) they are minimally complex machines. ■

We can also demonstrate that delays in agreement are possible.

Proposition 2 *Any agreement $z \in \Delta^n$ can be sustained as a NEM at any period $t \leq n$.*

Proof. We exhibit minimally complex machines that constitute a Nash equilibrium and induce agreement in period $t \leq n$ of the bargaining game at some allocation $z \,\epsilon\, \Delta^n$.

Consider the n-tuple of one-state (other than T) machines $M = (M_1, ..., M_n)$ defined as follows. The one-state machine M_t always offers z as a proposer and rejects all offers as a responder. For any $i \neq t$, the one-state machine M_i always proposes unit vector $1_i \in \Delta^n$, where 1_i stands for player i receiving 1 and others receiving 0, and accepts an offer if and only if z is proposed by player t. Thus M_i accept an offer if and only if the offer is z and i is the k-th responder where k is defined by $k = \begin{cases} i - t & \text{if } i > t \\ n - t + i & \text{if } i < t \end{cases}$. Since each M_i has one state the transitions are trivial.

Clearly, if M is implemented, at any period $t' < t$ the proposal is $1_{t'}$ and is rejected by the first responder (it will also be rejected by the other responders if they are asked to move). At t the offer made is z and will be accepted by all. Since all machines have one state they are minimally complex machines. Therefore, to show that $M = (M_1, ..., M_n)$ constitute a NEM, we only need to show that the machines are best responses to each other on the equilibrium path.

At any $t' < t$, the first responder $t' + 1$ cannot make himself better off by deviating and accepting the offer $1_{t'}$ because either the other responders reject or in the game with 2 players the game ends and he will receive zero. At period t, no responder $i \neq t$ can make himself better off by deviating and rejecting z because, given the strategy of others, i can obtain at most z_i in the future.

No proposer $i \neq t$ can make himself better off by making a different offer from $1_{t'}$ because it will be rejected by the first responder. Also player t cannot make himself better off by making a proposal different from z because it will be rejected. ■

It is also possible to generate no agreement or indefinite delay.

Proposition 3 *Indefinite delay can be sustained as a NEM outcome.*

Proof. Again, we exhibit minimally complex machines that constitute a Nash equilibrium and induce no agreement.

Each one-state machine M_i for player i always offers 1_i as a proposer and rejects all offers as a responder. Clearly, the machines $(M_1, ..., M_n)$ are best against each other. Since they have one state, they are minimally complex. ■

It appears therefore that the introduction of complexity considerations really has no effect in this context. However, the next two results show that this is not true. Although complexity considerations do not restrict the equilibrium allocation of payoffs, they *do* restrict the amount of delay that can occur, if an agreement is reached in finite time.

2.4 Delay and Stationarity.

We shall first present a trivial result which will be used extensively in the rest of this section.

Lemma 1 *For any NEM profile $M = (M_1, .., M_n)$, if $\pi_i(M) = 0$ for some i then M_i is minimal (has one non-terminating state).*

Proof. Consider the machine M_i' with one (non-terminating) state q_i. Let the output map be as follows for any $s \in S_i$:

$$\lambda_i'(q_i, s) = \begin{cases} 1_i & \text{if } s \text{ is such that } i \text{ is the proposer} \\ R & \text{if } s \text{ is such that } i \text{ is the responder} \end{cases}$$

Thus M_i' asks for everything and rejects all offers. The transitions are trivial. This machine is minimal and guarantees player i a payoff of at least 0. Since, the equilibrium machine M_i also generates a payoff of zero, it follows that M_i must also have no more states than M_i'; therefore M_i must also be minimal ■.

Notice also that although players are not restricted to choose finite-state machines, in any NEM the machine chosen by each player will have a finite number of states. This follows from Lemma 1 and because, for any NEM profile $(M_i, M_{\sim i})$ resulting in an agreement at some finite time, for each i there exists a *finite* state-machine M_i' such that $(M_i', M_{\sim i})$ induces the same outcome path as $(M_i, M_{\sim i})$.

2.4.1 The Two-Player Game.

We first show our main result in this section for the two-player game, using the weaker complexity criterion $\succ^s$, with some special arguments that do not extend to the n-player (n $>$ 2) case. Thus for the 2-player case the result is stated for NEM (both NEMs and NEMr).

Proposition 4 *Consider the alternating offers, two-player Rubinstein bargaining game. Let (z, t) be an agreement reached at a time period $t < \infty$. Suppose that (z, t) is induced by a NEM. Then $t \leq 2$.*

Proof. Suppose not, then there exists a machine profile $M = (M_1, M_2)$, where $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$ for $i = 1, 2$, that constitute a NEM and results in an agreement (z, t) with $t > 2$. Let q_i^τ be the state of player i at period $\tau \leq t$ on the equilibrium path. Suppose that player i is the responder and $j \neq i$ is the proposer at t. The rest of the proof is in several steps.

Step 1: The state of the responder at t, q_i^t, could not have occurred on the equilibrium path before the last stage.

Suppose not; then q_i^t occurs on the equilibrium path at some time period $\tau = t - 2l$ for some $l > 1$. Now, consider two cases.

Case A: $z_j = 0$. Then, by Lemma 1, M_j must have just one state q_j^t. Since, M_i is in the same state in the last stage of the game as in stage $(\frac{\tau}{2})^*$, where for any real number r, r^* stands for the smallest integer greater than or equal to r, it follows that the outcome of the game is the same in the last stage of the game as in stage $(\frac{\tau}{2})^*$. Therefore, the game would end at $\tau < t$. But this is a contradiction.

Case B: $z_j > 0$. Then, player j (the proposer at t) could change his machine by making a transition to q_j^t at stage $(\frac{\tau}{2})^*$. Since i is in state q_i^t at stage $(\frac{\tau}{2})^*$, this change by j results in the same outcome in stage $(\frac{\tau}{2})^*$ as in the last stage of the original game. Thus j could obtain the payoff $z_j > 0$ sooner at $\tau < t$ as a result of this change. But this is a contradiction.

Step 2: z could not have occurred on the equilibrium path before t when i is the responder.

Suppose not; then z has been offered by $j \neq i$ on the equilibrium path at some period $\tau < t$ and the responder i must have rejected it. Now consider two cases.

Case A: $z_i = 0$. In this case, it follows from Lemma 1 that M_i must have one state. Therefore player i's behaviour must be the same in all stages. But this contradicts i rejecting z at τ and accepting it at t.[17]

Case B: $z_i > 0$. Suppose that i, the responder at τ, modifies its machine by changing its output function to accepting offer z in all states. Since this will

[17] If i is the responder in the second period of a stage, having one state implies that i always makes the same offer and this offer must be rejected on the equilibrium path before period t. So the partial histories prior to periods τ and t must be the same.

yield the same agreement z before t, it must be better for the responder i, given discounting. Therefore, the original path could not have been a Nash equilibrium.

Step 3: A contradiction .

Let s^t (it could be null) denote the partial history within the last stage prior to z being offered at t. Either (1) s^t occurs at some previous period $\tau < t$ on the equilibrium path (for example this would be the case if s^t is null),or (2) s^t does not occur elsewhere on the equilibrium path. Suppose (1) holds. Let player i deviate to a machine where q_i^t is eliminated in favor of q_i^τ, with the one change that $\lambda_i(q_i^\tau, s^t z) = A$. From the previous Steps, q_i^t and z do not appear on the equilibrium path before t when i is the responder. Therefore the new machine for i implements the same outcome path as the equilibrium machine and has one fewer state; therefore the original path could not have been an equilibrium. Suppose (2). Now s^t has not occurred before the final stage (thus, $s^t \neq \emptyset$). Player j must therefore be in the same state $q_j^t = q_j^\tau$ for some period $\tau = t - 2l$, $l > 1$; otherwise he could deviate to a machine without q_j^t and save a state.(Actions can be conditioned on the history s^t, which does not occur anywhere else.). Again, consider two cases.

Case A: $z_i = 0$. Then, by Lemma 1 i would have one state and thus, given that $q_j^t = q_j^\tau$, the game would end at $\tau < t$; a contradiction.

Case B: $z_i > 0$. Let Player i now deviate and make $q_i^\tau = q_i^t$. Then the outcome of the game in the stage containing τ would be the same as the outcome in the last stage. This would end the game at $\tau < t$, for the same agreement z, and is therefore a profitable deviation. But this is a contradiction. ■

2.4.2 The n-Player Game.

In Appendix A we provide a counter-example to show that the analogous result to Proposition 4 does not hold for more than two player games with state complexity $\succ^s$.[18] The counter-example is a 3-player game with a NEMs resulting in an agreement in 10 stages or 30 periods, with each player using four states[19].

The main result of this subsection is that Proposition 4 on no delays in agreement beyond period 2 for the 2-player game can be generalized for n-player if $\succ^r$ is the complexity criterion.

[18]Step 1 in the proof of Proposition 4 does not extend to more than 2 players. (See Appendix A for an example.)When $n = 2$, if i is a responder and q_i^t occurs at some period $\tau < t$, the proposer could speed up the agreement by offering z at τ. When $n > 2$, the proposer cannot do this, unless q_i^t occurs at τ *for all* responders i. Also Step 2 in the proof of Proposition 4 does not extend to games with more than 2 players. With $n = 2$, if z occurs at some $\tau < t$, the responder can speed up the agreement by accepting z. When $n > 2$, no responder can ensure this unilaterally, unless *every other* responder accepts the offer as well. Therefore, with $n > 2$, z can occur on the equilibrium path earlier than t, because all responders reject z before t.

[19]We could not find a counter-example within fewer stages for 3-player games.

Proposition 5 *Let $(z,\, t)$ be an agreement $z \,\epsilon\, \Delta^n$ at period $t < \infty$ in a n-player bargaining game. Suppose that (z, t) is induced by a NEMr. Then $t \leq n$.*

Proof. Suppose not. Then there exists a profile of machines $M = (M_1, ..., M_n)$, with $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}_{i=1}^n$, which constitute a NEMr and results in an agreement (z, t) with $t > n$. Suppose that on the equilibrium path the last stage of the game consists of L sequential moves (thus $n \leq L \leq n^n$). Let us denote these sequential moves by $a^1, .., a^L$ with a^1 being the first and a^L being the last move in the last stage of the equilibrium path. For any $l \leq L$, we refer to the player that makes the *l-th* move in the last stage by $l^{\#}$ and let us denote the state of $l^{\#}$ in the last stage by $q_{l^{\#}}^t$. Thus in period t, for example, player $(L - n + 1)^{\#}$ is the proposer and makes the offer $z = a^{L-n}$ and player $(L - n + i + 1)^{\#}$ is the *i-th* responder for any $i < n$ and accepts the offer ($a^{L-n+i} = A$). Also, for any $l \leq L$ define s^l to be the partial history of the equilibrium path in the last stage of the game before the *l- th* move. Thus s^1 is empty and $s^l = (a^1, .., a^{l-1})$ for all $l > 1$.

The rest of the proof of the above Proposition is in several steps.

Step 1: $\lambda_{L^{\#}}(q_{L^{\#}}, s^L) = a^L = A$ **for all** $q_{L^{\#}} \in Q_{L^{\#}}$.

First, note that if $z_L = 0$ then this step follows from Lemma 1 ($M_{L^{\#}}$ is a minimal machine) and from $L^{\#}$ accepting the offer at t . To demonstrate this step for the case in which $z_{L^{\#}} > 0$, we will first show that s^L cannot occur on the equilibrium path before the last stage. If s^L had occurred previous to the last stage, player $L^{\#}$ could have accepted the offer at the earlier stage, and would therefore have obtained a payoff at least δ^{-n} times greater than the equilibrium payoff, thus contradicting the definition of a $NEMr$. Now since s^L occurs only in the last stage and $\lambda_{L^{\#}}(q_{L^{\#}}^t, s^L) = a^L = A$, it follows that $\lambda_{L^{\#}}(q_{L^{\#}}, s^L) = A$ for all $q_{L^{\#}} \in Q_{L^{\#}}$. This is because player $L^{\#}$ could implement the same outcome path with less r-complexity (condition (ii) of Definition 3) by choosing a machine which is identical to the equilibrium machine $M_{L^{\#}}$ except that it always does A in response to the partial history s^L; since s^L occurs only once at the last stage on the equilibrium path this is feasible.

Step 2: $\forall\ l > 1$, **if for all players** $i = (l+1)^{\#}, (l+2)^{\#}, .., L^{\#}$ **we have** $\lambda_i(q_i, s^i) = a^i$ **for all** $q_i \in Q_i$ **then** $\lambda_{l^{\#}}(q_{l^{\#}}, s^l) = a^l$ **for all** $q_{l^{\#}} \in Q_{l^{\#}}$.

As in the previous step, we need to consider the two cases of $z_{l^{\#}} = 0$ and $z_{l^{\#}} > 0$ separately. In the first case, this step follows from Lemma 1 ($M_{l^{\#}}$ is minimal) and $\lambda_{l^{\#}}(q_{l^{\#}}^t, s^l) = a^l$. Now consider the case in which $z_{l^{\#}} > 0$. In this case, s^l occurs on the equilibrium only once at the last stage. This is because if s^l had occurred before the last stage, player $l^{\#}$ could have taken the action a^l at the earlier stage and obtained the payoff $z_{l^{\#}}$ earlier than at t, given that for all players $i = (l+1)^{\#}, (l+2)^{\#}, .., L^{\#}$ we have that $\lambda_i(q_i, s^i) = a^i$ for all $q_i \in Q_i$. Now since s^l occurs only in the last stage and $\lambda_{l^{\#}}(q_{l^{\#}}^t, s^l) = a^l$, it follows that $\lambda_{l^{\#}}(q_{l^{\#}}, s^l) = a^l$ for all $q_{l^{\#}} \in Q_{l^{\#}}$. This is because player $l^{\#}$ could implement the same outcome path with less r-complexity by choosing a machine which is identical to the equilibrium machine $M_{l^{\#}}$ except that it always does a^l in response to the partial history s^l; since s^l occurs only once on the equilibrium path, this is feasible.

Step 3: $\forall\ l > 1$ **we have** $\lambda_{l\#}(q_{l\#}, s^l) = a^l$ **for all** $q_{l\#} \in Q_{l\#}$.

This step follows by backwards induction from the previous two steps.

Step 4: A contradiction.

Consider separately the two cases of $z_1 = 0$ and $z_1 > 0$. In the first case, it follows from Lemma 1 (M_1 is minimal) and $\lambda_1(q_1^t, s^1) = a^1$ that player 1 always offers a^1. But this together with Step 3 imply that the outcome $(a^1, ..., a^L)$ occurs in the first stage of the game resulting in an agreement z in the first stage of the game. But this is a contradiction. Now consider the case in which $z_1 > 0$. In this case, if player 1 makes an offer a^1 at time period 1 it follows from Step 3 that the outcome $(a^1, ..., a^L)$ occurs in the first stage of the game resulting in an agreement z in the first stage of the game. Therefore player 1 could obtain the payoff $z_1 > 0$ sooner than at period $t > n$. But this contradicts the hypothesis that the original path is an equilibrium path. ■

Remark 3 *Note that the argument for proving this proposition uses the following two features of the game. First, it uses the specific feature of the bargaining game that the final payoff to a player depends on the final agreement allocation z and, through the discount factor, on how long it takes to reach an agreement ; the history of the game up to period t does not affect the payoff. Thus, if in some stage prior to the final stage (on the equilibrium path) the same partial history arises as in the final stage, and if a responder i could ensure that the game ends in an agreement z by accepting the offer, he would do so if* $z_i > 0$*. This implies, for example, that the partial history on the equilibrium path in the last stage before the last acceptance could not have occurred in an earlier stage. Second, as in other extensive-form games, only actual plays are observed, not strategies. If a state for a player and a partial history s in a stage never occur together on the equilibrium path of play, the action after this partial history by the player concerned can be arbitrary without a deviation being observed. This, together with r-complexity, ensures that if a partial history of a stage happens on the equilibrium path only once, then the action taken after such a partial history must be the same for all the states of the machine. Therefore, since the partial history in the last stage before the last acceptance could not have occurred before, it follows that the last responder's response to this partial history is always the same for all his states. In the proof of Proposition 5, by appealing to the above two features of the bargaining game (together with r-complexity) and by using a backwards induction reasoning on the sequence of moves in the last stage of the game, we show that each player's response in the last stage is the same for all states and therefore agreement would be reached in the first stage.*

Remark 4 *We also note that any outcome path with agreement* (z, t) *for any* $t < \infty$ *, can be implemented in Nash equilibrium by finite-state automata, if complexity costs are absent. To demonstrate this, first note that such an agreement* *takes place in* $(\frac{t}{n})^*$ *stages, where as before for any real number* r *the smallest integer greater than or equal to it is denoted by* r^*. *Now, consider a machine* M_i *for player* i, *which has* $(\frac{t}{n})^*$ *states* $(q_i^1, .., q_i^{(\frac{t}{n})^*})$. *Suppose that the output function of* i *is such that for all states* $q_i \neq q_i^{(\frac{t}{n})^*}$

$$\lambda_i(q_i, s) = \begin{cases} 1_i & \text{if } s \text{ is such that } i \text{ is the proposer} \\ R & \text{if } s \text{ is such that } i \text{ is the responder} \end{cases}$$

Thus in all states, other than $q_i^{\frac{t}{n}^*}$, *player* i *proposes the agreement giving player* i, *1, and the others 0; and* i *rejects all offers. Also, let the transition function be such that for all* $\tau < (\frac{t}{n})^*$, $\mu_i(q_i^\tau, e) = q_i^{\tau+1}$ *for any* e *on the equilibrium path and* q_i^1 *otherwise. Thus, on the equilibrium path player* i *is in state* q_i^τ *in the* τ*-th stage of the game and in every stage before* $\frac{t}{n}^*$, *each player asks for the whole pie and rejects all offers. Off the equilibrium path, the transitions are to the states in stage 1. In stage* $\frac{t}{n}^*$, *the states prescribe the actions appropriate for agreement at* z. *For example, if* z *is accepted in the first period of the stage,* $\lambda_1(q_1^{\frac{t}{n}^*}, \varnothing) = z, \lambda_2(q_2^{\frac{t}{n}^*}, z) = A$ *otherwise* R, ..., $\lambda_n(q_n^{\frac{t}{n}^*}, (z, A, .., A)) = A$, *and otherwise* R . *The actions in periods beyond* t *can be specified in any arbitrary way, since on the path such periods will never be reached. Clearly, the above construction constitutes a Nash equilibrium in machines without complexity cost.*

The last two propositions demonstrate that the machine game must therefore entail agreement by $t = n$ either if $n = 2$ or if $\succ^r$ is used as a measure of complexity. We now show that any equilibrium with agreement at $t \leq n$ or no agreement can be implemented by one-state machines.

Lemma 2 *Suppose* $M = (M_1, \ldots, M_n)$ *constitute a NEM. Then each machine* M_i *has only one state if either* M *results in an agreement at some* $t \leq n$ *or if* M *results in no agreement.*

Proof. Let us consider each case.

Case 1: Agreement is reached at some $t \leq n$.

The essential idea in this case is that if player i had more than one state, only one of them would be used along the equilibrium path (in the first stage) and the others could be dropped.

Case 2: No agreement is reached in equilibrium.

In this case, for each i, $\pi_i(M_1, ..., M_n) = 0$. Therefore, it follows from Lemma 1 that M_i has only one state. ■

Finally, we can state the main result of this section.

Proposition 6 *In* n*-player games if the profile* $M = (M_1, \ldots, M_n)$ *is a NEMr then each machine* M_i *has only one state. In 2-player game if the profile* $M = (M_1, M_2)$ *is a NEM then each machine* M_i *has only one state.*

Proof. This follows immediately from Propositions 4 and 5 and Lemma 2. ■

The results of this section have demonstrated that the Nash equilibrium of the machine game must involve machines each with a single state[20] and the strategies that they implement are "stationary". However, Nash machines do not generate uniquely the stationary **subgame perfect allocation** of the bargaining game. All partitions of the cake and delayed agreements up to $t = n$ can also be generated. The arguments in the proof of Propositions 2 and 3 explain why this is the case. For example, in games with more than 2 players, if there are two rejections anticipated for any non-equilibrium offer, it does not pay any single proposer or responder to deviate unilaterally from the equilibrium path. (In 2-player games, the same is true if 'off the equilibrium path' the players always ask for the whole pie and always reject all offers). However, if the responders make mistakes, so that a rejection could be an acceptance sometimes, the preceding reasoning would not hold. Thus the result is a consequence of the assumption that the machines are error-free. In the next section, we allow the levers of the machine to tremble when performing their tasks (there will be errors in the machine output, but we will keep transitions deterministic)and thereby introduce 'credibility'.

3 Noisy Machines

We now introduce machines that make mistakes with a small probability. We shall throughout assume that the cost of an additional state is of greater order than the error probability in a sense to be made clearer in what follows.

Let $M_i(\epsilon_k)$ represent the following five-tuple $(Q_i, q_i^1, T, \lambda_i, \mu_i)$ with the only change from the previous section being in the definition of λ_i, which is now given by

$$\lambda_i : Q_i \times S \to \Delta_{\epsilon_k}(C_i(s)),$$

where, as before, $C_i(s)$ is the set of actions corresponding to s, that is A or R for a responder or Δ^n for a proposer. The set $\Delta_{\epsilon_k}(C_i(s))$ is the set of completely mixed probability distributions on $C_i(s)$ with ϵ_k being the probability mass associated with non-desired choices. That is, if a player's machine chooses a particular action, this is played only with probability $1 - \epsilon_k$. We will consider the effects of taking an infinite sequence of k, such that $\epsilon_k \to 0$.

More explicitly, where $C_i(s) = \{A, R\}, \Delta_{\epsilon_k}(C_i(s))$ is the set of probability vectors that accord at least ϵ_k probability to each choice. Of course, $1 - \epsilon_k > \epsilon_k$,which is supposed to be "small". If $C_i(s)$ describes the set of offers, then $\Delta_{\epsilon_k}(C_i(s))$ consists of probability distributions F$(\cdot)$ with an absolutely

[20]As stated earlier, the termination state T is not being considered explicitly in our analysis, but $(n-1)$ acceptances automatically lead to a transition to this state.

continuous part with specified density $f(x) > 0, \forall x$ in the feasible set, and $\int_{\sum x_i=1} f(x)dx_1dx_2dx_3...dx_n = \epsilon_k$ and a spike of mass $1-\epsilon_k$ at a particular value $x \in \Delta^n$.

Thus all machines $M_i(\epsilon_k)$ make errors of magnitude at most ϵ_k, whenever generating an output. Transitions are error- free. *Note that the errors made by the machine in each period are independent of each other- independent of errors made by other machines and independent of errors made by the same machine in other time periods.* This specification seems natural; a machine can make errors of a certain magnitude, but whether it makes an error at some stage or not does not depend on whether other errors were made.

We now discuss convergence of the $M_i(\epsilon_k)$ as $k \to \infty$. We shall require that each $\mathrm{M}_i(\epsilon_k)$ be a best response to $\mathrm{M}_{\sim i}(\epsilon_k)$ along the sequence, *in the payoff sense only , without considering costs of complexity.* Thus convergence of machines will be identical with convergence of behavioural strategies, given the specification that the errors are independent. One can interpret this construction as implying that the fixed cost of an additional state (see comment after Definition 6) is greater than $(\epsilon_k)(1)$, where 1 is the largest payoff possible with any number of additional states.[21] This means that for ϵ_k sufficiently small, the size of the optimal "noisy" machine will be the same as the size of the limiting machine.

The behavioural strategies involved are those of a proposer or one of the responders. The proposer has to put a positive density on all feasible offers, but a mass of only $\epsilon_{k.}$ The remaining $1-\epsilon_k$ can be put on any offer or any set of offers. Convergence as $k \to \infty$ is then used in the sense of weak convergence of the probability distributions of offers. It will be seen later (Appendix B) that the mass $1-\epsilon_k$ will be placed on an offer that is the largest(from the point of view of the proposer) offer acceptable with "high" probability, so that the entire machinery of weak convergence need not be brought into play. For the responders, an offer x is accepted or rejected with at least ϵ_k probability. As $\epsilon_k \to 0$,the acceptance probability for a given x converges to some value (we shall see it to be either 1 or 0).

It is easily shown that if the $\mathrm{M}_i(\epsilon_k)$ are best responses to the $\mathrm{M}_{\sim i}(\epsilon_k)$ for all ϵ_k, the limiting machine M_i is a best response to the limiting machines $\mathrm{M}_{\sim i}$. We therefore have a Nash equilibrium in payoffs and now invoke complexity costs on the limiting machines. We write down now a definition of a "noisy" Nash equilibrium of the machine game (or a "shaking lever" equilibrium).

Definition 6 *For any $l = s, r$ a profile of machines $(M_i,\ M_{\sim i})$ is said to be a noisy Nash equilibrium of the machine game with complexity cost $\succ^l$ (NNEMl) if*

[21] With a completely mixed strategy being used by each player, the notion of states being used on the equilibrium path has to be extended to "states used with a high probability on the path". Comparing the cost of an additional state, c, with the maximum expected benefit (a probability of ϵ multiplied by a maximum payoff of 1) gives us the basis for the statement in the text.

1. *For some sequence $\epsilon_k \to 0$, there exists a machine profile $(M_i(\epsilon_k),\ M_{\sim i}(\epsilon_k))$ s.t. $M_i(\epsilon_k)$ is a best response to $M_{\sim i}(\epsilon_k)$ for all i, and all k, and $M_i(\epsilon_k) \to M_i$ as $k \to \infty$ (and as $\epsilon_k \to 0$).*

2. *If $\pi_i(M_i,\ M_{\sim i}) = \pi_i(M_i',\ M_{\sim i})$, then $\left|M_i'\right| \succeq^l |M_i|$.*

The noise introduced into the definition above affects behavioural strategies. This is natural, since the machine is generating output at each stage of the game. Part (1) of the definition captures "extensive form perfectness" and part (2) the cost of complexity. Note that a "noisy Nash equilibrium", is also a Nash equilibrium of the machine game; this follows directly from the definition and the properties of perfectness.

Since a "noisy Nash equilibrium" is a Nash equilibrium, the results of the previous section can be applied. Therefore, each equilibrium machine must be minimal and agreement (if any) must take place by $t = n$ (in the limiting case where $\epsilon = 0$) if either $n = 2$ or if $\succ^r$is the complexity criterion.

Proposition 7 *In n-player games, any NNEMr must have an agreement at $t = 1$ (no delay) and give players $1, 2, ..., n$ the following payoffs*

$$\frac{1}{1+\delta+\delta^2+...+\delta^{n-1}}, \frac{\delta}{1+\delta+\delta^2+...+\delta^{n-1}}, ..., \frac{\delta^{n-1}}{1+\delta+\delta^2+...+\delta^{n-1}}$$

respectively. When $n = 2$, the same result holds for any NNEM.

Proof. See Appendix B ■

The above result demonstrates that the unique noisy NEM is the stationary subgame perfect equilibrium allocation, and agreement occurs in the first period.

The main reason for the preceding construction, involving noise, is to ensure that the strategies used by players are credible off the equilibrium path. However, since the number of possible paths is uncountable (the number of offers is not countable), in any noisy NEM it is not necessary for machines to choose optimal responses for all possible subgames. It is only required that the set of subgames after which non-optimal behaviour occurs is of measure zero. Thus such errors in outputs of the machines ensure that credibility is restored for *almost* all subgames. This is sufficient to obtain our uniqueness result in Proposition 7. A more direct way of introducing credibility and obtaining our uniqueness result is to require the strategies induced by any NEM to be a subgame perfect equilibrium (SGPE) of the bargaining game. Clearly, such a requirement ensures optimal behaviour after every history. We give below the proof that Nash equilibrium machines with subgame perfect behaviour in the bargaining game implement the stationary, subgame perfect allocation. This is similar to the proof for noisy Nash equilibrium, but more straightforward to describe. Before stating this result, we would like to make the following two points. Firstly, we are not considering subgame perfect equilibria of the machine game. (See Section 4 below.) Secondly, we prefer the 'noisy' NEM interpretation of credibility to the more direct interpretation of requiring any NEM to be SGPE because in the latter interpretation complexity cost and the payoff in the game are not treated in a consistent fashion (they are not formally inconsistent, however); complexity is only relevant at the beginning whereas strategies are required to be optimal in every bargaining subgame. Noise in the output of the machines of an order less than the fixed cost of a state seems a natural as well as a coherent way of inducing (almost) credibility when we use NEM as the solution concept.

Proposition 8 *In n-player games, the unique payoff allocation that is both a NEMr and a SGPE of the original bargaining game is* $(\frac{1}{\sum_{i=1}^{n}\delta^{i-1}}, \frac{\delta}{\sum_{i=1}^{n}\delta^{i-1}}, ..., \frac{\delta^{n-1}}{\sum_{i=1}^{n}\delta^{i-1}})$. *This allocation is obtained without delay at* $t = 1$. *When* $n = 2$, *the same result holds for payoff allocations that are both NEM and SGPE.*

Proof. Consider any NEMr (or any NEM for the case of $n = 2$) profile $M = (M_1, ..., M_n)$ that is also a SGPE of the bargaining game. Since the profile M is a NEMr (or a NEM when $n = 2$), each M_i has one state. Therefore, at the beginning of each stage, the outcome path induced by M in each stage is unique. Let the payoffs associated with this outcome path be $(v_1, v_2, .., v_n)$ respectively.

Our first step in the proof is to show that the (continuation) equilibrium payoff profile to the players at the beginning of the each period $t \leq n$ (end of stage 1) is also unique and involves an agreement. This is done by backward induction as follows. In any period $t \leq n+1$, if it is reached, suppose that for each player i the continuation equilibrium payoff is unique (is independent of history) and is equal to v_i^t. We now want to show that at $t-1$ the continuation equilibrium payoff of player i is also unique and it involves an agreement. To do this, consider period $t-1$ and a responder i in this period. Suppose an offer $x = (x_1, ..., x_n) \in \Delta^n$ is made and is accepted by all responders before i. Also, assume that all responders after i, if any , also accept the offer. Then, it is clear that in this subgame i will accept the offer if and only if $x_i \geq \delta v_i^t$.

Therefore, it follows from above that the last responder at $t-1$, player $t-2$, accepts any offer x if and only if $x_{t-2} \geq \delta v_{t-2}^t$. By backward induction on the set of responders, it follows that any offer x such that $x_i \geq \delta v_i^t$ for all $i \neq t-1$ will be accepted by all responders i. Also, by the same reasoning, all offers such that $x_i < \delta v_i^t$ for some $i \neq t-1$ will be rejected by some responder.

Now, consider the proposer at $t-1$. If his offer is rejected he will receive $\delta v_{t-1}^t = \delta - \sum_{i \neq t-1} \delta v_i^t$. If he offers $y \in \Delta^n$ such that $y_i = \delta v_i^t$ for all $i \neq t-1$, it will be accepted and he will receive $y_{t-1} = 1 - \sum_{i \neq t-1} \delta v_i^t$. Since $\delta < 1$, it follows that at $t-1$ the optimal decision for the proposer is to make the offer y. This will be accepted and each player $i's$ (continuation) equilibrium payoff at $t-1$ is unique and is equal to

$$v_i^{t-1} = \begin{cases} \delta v_i^t & \text{if } i \neq t-1 \\ 1 - \sum_{i \neq t-1} \delta v_i^t & \text{otherwise} \end{cases} \tag{3}$$

Since, the equilibrium payoff profile at the beginning of the second stage (period $t+1$) is unique and equal to v_i for each i, it follows by backwards induction from above that the game must end in agreement in period 1 and the payoff to player i at the beginning of each period $t \leq n+1$ is unique and is equal to v_i^t satisfying condition (3).

It also follows from the stationarity of the equilibrium machines that $v_i = v_i^1 = v_i^{n+1}$. This together with condition (3) imply that

$$v_1 = \frac{1}{1+\delta+\delta^2+...\delta^{n-1}}, \quad .., \quad v_i = \frac{\delta^{i-1}}{1+\delta+\delta^2+..\delta^{n-1}}, \quad .., \quad v_n = \frac{\delta^{n-1}}{1+\delta+\delta^2+..\delta^{n-1}}$$

This completes the proof. ■

The above result is identical and the nature of the proof is similar to the noisy NEM case. In fact, if we restricted the set of proposals of the machines to be finite then any strategy profile induced any 'noisy' NEM would be subgame perfect in the bargaining game. Thus, in this case, Proposition 7 follows from Proposition 8.

The Shaked argument that generates every individually rational agreement as a subgame-perfect equilibrium does not work in the setup of this paper, primarily because of the result of Section 2 that a Nash equilibrium machine has one state.[22] This result and the addition of credibility off the equilibrium path (noise or SGPE) is sufficient to rule out all but the unique stationary subgame perfect equilibrium.

4 Alternative specifications

In obtaining our results we took a particular specification of the automata in the bargaining game and we also considered a specific refinement of Nash equilibrium (noisy NEM) in Section 3. Let us consider each of these issues further.

4.1 Alternative machine specifications

As we mentioned before, since the bargaining game has a certain degree of asymmetry built in, we can choose to specify a machine to implement a particular strategy in several different ways.[23] It can be shown that *all* the results of this paper hold if we change the specification of the machines so that the *transitions* take place at the end of each *period,* rather than at the end of each stage. In this specification, for more than 2-player games, to show that machines are minimal one also needs a stronger definition of complexity (analogous to r-complexity in Definition 3) than counting the number of states. Also notice that with this

[22]Osborne and Rubinstein (1990), in their exposition of the Shaked result, use automata with four states per player. They do not attempt to determine the NEM profiles for the game, but it is clear from the construction that in general one needs at least four states to implement Shaked's type of strategy.

[23]Notice that all strategies that can be implemented with one specification can be implemented with any of the others.

specification, a minimal machine, and thus stationary behaviour, has a different meaning from that in Section 2. Here stationary behaviour means choices in each period depend on the partial history in the period and not on the history of the game prior to the period.

In Chatterjee and Sabourian (1997), we also consider another specification in which the machine a player chooses to implement his strategy consists of n sub-machines; each sub-machine composed of states enabling a player to play a given *role*. Transitions take place from a state in each sub-machine to a state in the same sub-machine one stage (n periods) after the last choice of the sub-machine, for all sub-machines (roles). Here the action chosen by each sub-machine in any state could be made to depend on the partial history of the stage or on the partial history of the period. The results in this paper are also valid for this new specification. However, it is worth mentioning that with this specification r-complexity is needed to obtain our results on no delays beyond period n and on stationarity only for games with $n > 3$. In the above paper and in Appendix D, we show that, with this specification, s-complexity is sufficient to obtain our results in games with *three* players as well as for those with two. However, by means of a counter-example (Appendix E) we can show that this is not true for games when $n = 4$.

The above specifications of machines are similar to that of Piccione and Rubinstein (1993). In these specifications, changes in the state of the machine are allowed after each stage (or after each period) of the game and the output of the machine in each state depends on the partial history within the stage (or period). Yet another different specification (pointed out by one of the referees) of machines would be to allow changes in the states to occur also *within* each stage of the bargaining, after every piece of new information. Thus in this specification the output function is only a function of the state and not of any partial history, and transitions take place just before a player is required to move. For this last specification, we can show (see Appendices G and H) that the result on stationarity (and uniqueness with 'noise') for two-player games continues to hold with s-complexity, but s-complexity is insufficient for three or more players, just as in the definitions used in this paper.[24]

4.2 Alternative refinements of NEM

An alternative definition of equilibrium with machines is contained in Rubinstein (1986). A machine profile is a semi-perfect equilibrium (SPE) if the two NEM conditions, in Definition 4, *hold after every history* induced by the machine profile. This means that a player's machine has to be the least complex one implementing the equilibrium payoff after every history reached *on the equilibrium path.* Thus states that are not used in the future can be dropped and other

[24]Since output, in this last specification, is a function only of the state, r-complexity and s-complexity are equivalent; strengthening the s-complexity measure to obtain our results for the case of $n > 2$ with this specification requires considering complexity of transitions.

states combined, provided the equilibrium path is unchanged. A refinement of SPE would be to require subgame perfectness in the choice of machines. This would require that the NEM conditions are satisfied after every history, not just those induced on the candidate equilibrium path. (Neme and Quintas (1995) explore this refinement of Rubinstein's solution concept.) The results for n-player games on no delay in agreement beyond period n and stationary behaviour can be obtained with state complexity if we consider SPE rather than NEM. The relevant proposition is stated and proved in Appendix C. Therefore, a Noisy SPE will also give the unique stationary subgame perfect equilibrium as in Proposition 7.

A referee has enquired why we did not consider *normal form* trembling hand perfectness in a strategy space consisting of one-state machines. It is harder to specify the space of probability distributions on the set of such machines than to specify trembles in proposals or in responses, because the set is larger and more complicated to describe.[25]With this setup, there is still a positive probability of offers that would test suboptimal response rules, and results similar to the ones we have currently should be obtained. Our specification is simpler, while still representing the intuition of mistakes in the actions taken.

In Section 3, strategies implementable by NEM machines are made credible by introducing arbitrarily 'small' noise in the output of the machines. The noise does not affect transitions. The reason for not introducing errors in transitions is the following . If the equilbrium machines have a finite number of states then, at the end of any stage, there are only a finite number of states to make transitions to, and even a completely mixed transition would generate at most a finite number of offers. This is not sufficient to induce credibility in almost all subgames. In particular, since Nash equilbrium in machines are each one-state, noise in transitions would not refine NEM[26]

[25]For example, consider a 3-player game. If S_i^0 denotes the set of partial histories within a stage prior to player i making an offer, S_i^1 the set of partial histories prior to his making a response as a first responder and S_i^2 the set of partial histories prior to his choosing as a second responder, a one-state machine for i would consist of output specifications: $\lambda_i(s) \in \Delta^3$ for all $s \in S_i^0$ and $\lambda_i(s) \in \{A, R\}$ for all $s \in S_i^1 \cup S_i^2$. A completely mixed strategy over the set of such machines would need to specify a distribution of mass ϵ over a product set. For example, if $i's$ offer is independent of the partial histories and if the acceptance/rejection decisions are taken in the following special way: as first responder accept if and only if $i's$ share is at least r_1; as second responder accept if and only if $i's$ share is at least r_2. Here, r_1 and r_2 are numbers in [0,1]. Then the set of such machines for i, *which is a subset of the actual set of one-state machines*, is $\Delta^3 \times [0,1] \times [0,1]$.

[26]Also, transitions to different states are made in order to play different actions in different stages, after identical partial stage histories. Therefore, noise in transitions has the effect of introducing a limited degree of noise in the output; noise directly placed on output would include the possible effects of noisy transitions.

5 Conclusions

This paper has attempted to explore the consequences on multiperson bargaining of incorporating costs of complexity. Essentially, such restrictions on complexity, modelled along the lines proposed in the papers of Abreu and Rubinstein and Piccione and Rubinstein (but not identical to these), yield the result that the machines will be stationary ones. This is not sufficient to give the stationary, subgame perfect equilibrium allocation as the sole equilibrium; this requires an additional property of robustness against errors. It is interesting to note that, except for two-player games, not all definitions of complexity and specification of machines give us stationarity; the intuition behind stationarity relies on a plausible kind of response complexity and not just on the number of states of the machines.

It would be of interest to explore the issue of stationarity in other contexts where multiplicity of equilibria is a theoretical problem. This might be true especially in other bargaining games, which share the special feature that no payoffs are obtained unless there is an agreement.

6 Appendix A: Counter example for three players with state-complexity

As mentioned earlier in the paper, the state complexity measure, while sufficient to obtain stationarity in 2-player games , is not strong enough to rule out non-stationary equilibria and delays beyond period n, when $n > 2$. We sketch below a counter-example that constructs a NEMs in a three-player game in which each player has four states, denoted by a, b, c, d , and an agreement $z'' \gg 0$ is reached in 30 periods. The construction is itself of some independent interest, in bringing out some of the considerations that make $n-$player ($n > 2$) bargaining games played by machines different from the 2-player case.

The equilibrium consists of 29 periods of offers followed by rejection by the first responders and an agreement z'' in the 30th period (the last period of the 10-th stage). Table 1 below describes *the equilibrium path* of the offers and the responses for the 30 periods and the states (and thus the transitions among the states) for different players in the 10 different stages, *along the equilibrium path*. In this table, $(x^a, x^b, x^c, x^d), (y^a, y^b, y^c, y^d, y', y'')$ and $(z^a, z^b, z^c, z^d, z', z'')$ represent different offers made by players $1, 2$ and 3 respectively; 'A' and 'R' represent acceptance and rejection respectively.

In order to ensure the path described in Table 1 can be induced as as NEM*s* , we have to specify other actions as follows.

We require that any responder in any state to reject all offers unless the partial history and the state of the responder are those that occur on the equilibrium path in period 30 when the agreement is reached. Thus, player 3 rejects all offers; player 1 rejects all offers unless, as the first responder, he is in state d and the partial history is that on the equilibrium path in period 30 before $1's$ move - namely (x^d, R, y'', R, z'');and player 2 rejects all offers unless, he, as the second responder, is in state c and the partial history is that on the equilibrium path in period 30 before $2's$ last move - namely (x^d, R, y'', R, z'', A). If a partial history is observed in any stage that is not on the equilibrium path for a proposer, given his state, he chooses an offer that is not on the equilibrium path at any stage, and thus creates an off-equilibrium outcome e.

The above ensures at any period $t \leq 30$ no player can improve his payoff by deviating unless the deviation can induce a sequence of actions that results in an agreement z'' being concluded earlier than in period 30.This is the phenomenon of "speedup", which is not important in repeated games, but *is* important in the context of a single extensive-form bargaining game.

Speedup can happen directly, if any two of the three players have the same states in any two stages, one earlier than the other. (From Table 1, we see this is not the case.) The third player can then deviate from his state in the earlier stage to his state in the later stage, and thus cause intermediate stages to be skipped, hence "speeding up". There could also be an indirect way of speeding up; a deviation is designed to cause the play of the game to go off the equilibrium path (by which we mean both the outcome path and the machine states that gave rise to the outcome path) and to rejoin it at a later stage, so as to reduce the total number of stages before the final agreement is reached. In order to make such deviations unprofitable, out-of-equilibrium transitions are constructed (see below) to have the property that a subgame following a deviation can rejoin the equilibrium path only in the first stage.

Table 1

period	stage	1's state	1's action	2's state	2's action	3's state	3's action
1	1	a	x^a	a	R	a	-
2	1	a	-	a	y^a	a	R
3	1	a	R	a	-	a	z'
4	2	a	x^a	b	R	b	-
5	2	a	-	b	y^b	b	R
6	2	a	R	b	-	b	z'
7	3	a	x^a	c	R	c	-
8	3	a	-	c	y^c	c	R
9	3	a	R	c	-	c	z'
10	4	a	x^a	d	R	d	-
11	4	a	-	d	y^d	d	R
12	4	a	R	d	-	d	z'
13	5	b	x^b	b	R	a	-
14	5	b	-	b	y'	a	R
15	5	b	R	b	-	a	z^a
16	6	b	x^b	c	R	b	-
17	6	b	-	c	y'	b	R
18	6	b	R	c	-	b	z^b
19	7	b	x^b	d	R	c	-
20	7	b	-	d	y'	c	R
21	7	b	R	d	-	c	z^c
22	8	b	x^b	a	R	d	-
23	8	b	-	a	y'	d	R
24	8	b	R	a	-	d	z^d
25	9	c	x^c	c	R	a	-
26	9	c	-	c	y''	a	R
27	9	c	R	c	-	a	z''
28	10	d	x^d	c	R	d	-
29	10	d	-	c	y''	d	R
30	10	d	A	c	A	d	z''

Consider any out-of-equilibrium history of actions in a stage e' that results from a deviation by say player 1, and the transitions for Player 2 and 3. For any $i = 2$,3 and for any state q_i, if $(q_i, e^{'})$ does not occur on the equilibrium path for both $i = 2$ and 3 then go to the beginning of the game: $\mu_i(q_i, e^{'}) = a$. If $(q_i, e^{'})$ does not occur on the equilibrium path for some $i \neq 1$ and does occur for the other player on the equilibrium path then $\mu_i(q_i, e^{'})$ is constructed so that either both 2 and 3 are in states a in the next stage or the final outcome of the next stage , $e^{''}$, and the states of i in the next stage, do not occur on the equilibrium path.Thus the players who see histories that are not on the equilibrium path for them generate transitions that ultimately lead back to the beginning of the game. Such a construction implies that any agreement reached as a result of a deviation ultimately does not succeed in accelerating the agreement.We omit details and simply give an example.

Suppose player 1 considers a deviation by offering x^b instead of x^a in stage 2 (period 4) at which the states of the players are (a, b, b). Now (x^b, R) occurs on the equilibrium path for player 2 in his state b, at period 13 (in stage 5). Therefore player 2 must play the same way in period 5, after observing (x^b, R), as in period 14 on the equilibrium path and thus he plays y' and player 3 rejects. Player 3 then faces the history $(\mathrm{x}^b, R, y^{'}, R)$ and is in state b. This partial history and state b occur for player 3 on the equilibrium path in stage 6 (period 17), and he responds by a play of z^b,as the equilibrium path action in period 18. The history $e^{'} \equiv (x^b, R, y^a, R, z^b, R)$ does not appear anywhere else on the equilibrium path for player 2 in state b. Then suppose $\mu_2(b, e^{'}) = b$. Since $e^{'}$ does occur on the equilibrium path for player 3 in state b, player 3 makes an equilibrium path transition to state c (as if going to stage 7). Thus in stage 3 players 2 and 3 are in states b and c respectively. Notice that this pair of states for players 2 and 3, never appears on the equilibrium path. Also notice that the only offers by 1 that appear on the equilibrium path with either state b for player 2 or with state c for player 3 are x^a or x^b. Thus any other offer in stage 3 by player 1 will take the game back to the beginning of the game by the construction in the previous paragraph.. Clearly, player 1 does not want to offer x^a (no speeding up will take place). Player 1 could repeat x^b in stage 3, inducing an offer z^c from player 3, again a surprise (i.e. not on the equilibrium path) for player 2 in state b. In the next stage, suppose player 2 will be in state b again; also the equilibrium transitions takes player 3 to state d. Given that in this stage (stage 4) players 2 and 3 are in states (b,d) respectively, for any offer of player 1, we can describe transitions (we omit details) in the same way, consistent with the equilibrium path, that take players 2 and 3 to states (a, a) respectively in the next stage (stage 5). Thus player 2's transitions are designed, using player 3's different offers, to delay agreement sufficiently to deter this possible deviation from player 1. The checking that other deviations are unprofitable could be done in the same way, *essentially* because the example has been constructed with distinct outcomes for distinct stages.

Finally we have to check that no player can save a state. Note that each state is used to make a specific offer by each player. Player 1 uses states 1,2,3,4 to make distinct offers in stages 1-4, 5-8, 9 and 10 respectively. Player 2 makes distinct offers in stages 1,2,3 and 4 following identical histories (an offer x^a followed by rejection) in every stage. Distinct states are needed for player 2, because the stage histories preceding his distinct offers are identical. Similarly player 3 needs distinct states to make distinct offers in stages 5, 6, 7 and 8. Note that in these stages, player 1 uses the same state to make the same offer and player 2 uses different states and conditioning on player 1's offer of x^b to make the same offer, so that the partial histories preceding player 3's distinct offers are the same. So every state of every player is essential - saving a state results in an 'off-the-equilibrium' outcome.

7 Appendix B: Proof of Proposition 7 on Noisy Nash Equilibrium Machines

Proof. Consider any NNEMr machine (or any NNEM if $n = 2$) profile $M = (M_1, ..., M_n)$. Consider also some sufficiently small $\epsilon_k > 0$ and machines $\mathrm{M}_i(\epsilon_k)$, defined as machines that play the equilibrium action with probability $1 - \epsilon_k$ and distribute mass ϵ_k over all other feasible actions. Clearly, M is a NEMr (or a NEM if $n = 2$). Therefore, it follows from Proposition 6 that M_i is minimal and therefore stationary for any i. Thus the limiting equilibrium payoffs to the players at the beginning of the each stage are uniquely determined. Let these be $v_i, i = 1, 2, ...n$.

Our first step in the proof is to show that the limiting (continuation) equilibrium payoff profile to the players (payoffs when M is chosen) at the beginning of the each period $t \leq n$ (end of stage 1) is unique and involves an agreement. This is done by backward induction as follows.

In any period $t \leq n + 1$ (the beginning of stage 1), if it is reached, suppose the limiting continuation equilibrium payoff of player i is unique (is independent of history) and is equal to v_i^t. We now want to show that at $t - 1$ the limiting continuation equilibrium payoff of player i is also unique and it involves an agreement. To do this, let $v_i^t + \overline{\zeta}_i^t(\epsilon_k)$ be the supremal equilibrium payoff for player i in t; let $v_i^t + \underline{\zeta}_i^t(\epsilon_k)$ be the corresponding infimal equilibrium payoff. The uniqueness of the limiting equilibrium payoff implies that

$$\lim_{\epsilon_k \to 0} \overline{\zeta}_i^t(\epsilon_k) = \lim_{\epsilon_k \to 0} \underline{\zeta}_i^t(\epsilon_k) = 0, \forall i \tag{B.1}$$

Consider period $t - 1$ and suppose an offer $x \in \Delta^n$ is made, with player $i's$ share being x_i. Suppose at period $t-1$ player i is a responder and all responders before i accept x. Also, assume that all responders after i place probability $1 - \epsilon_k$ on accepting the offer and the complementary probability on rejecting it. Then if player i places probability $1 - \epsilon_k$ on accepting, his payoff will be bounded below by:

$$(1-\epsilon_k)^{q+1}x_i + (1-(1-\epsilon_k)^{q+1})(\delta v_i^t + \underline{\zeta}_i^t(\epsilon_k)), \tag{B.2}$$

where q is the number of responders left after player i (note that q may equal zero in which case i is the last responder and thus $i = t-2$).

If i places probability ϵ_k on accepting, his payoff will be at most

$$(\epsilon_k)(1-\epsilon_k)^q x_i + (1-\epsilon_k(1-\epsilon_k)^q)(\delta v_i^t + \overline{\zeta}_i^t(\epsilon_k)). \tag{B.3}$$

It is clear from conditions () and (B.3) that in the above case player i prefers to place probability $(1-\epsilon_k)$ on accepting offers $x_i \geq \delta v_i^t + \overline{\eta}_i^t(\epsilon_k)$ where $\overline{\eta}_i^t(\epsilon_k) = \frac{1}{(1-\epsilon_k)^q}(\overline{\zeta}_i^t(\epsilon_k) - \underline{\zeta}_i^t(\epsilon_k)) - \epsilon_k\overline{\zeta}_i^t(\epsilon_k) + (1-\epsilon_k)\underline{\zeta}_i^t(\epsilon_k)$. By the same reasoning i will prefer to place probability ϵ_k on accepting offers $x_i < \delta v_i^t + \underline{\eta}_i^t(\epsilon_k)$ where $\underline{\eta}_i^t(\epsilon_k) = -\frac{1}{(1-\epsilon_k)^q}(\overline{\zeta}_i^t(\epsilon_k) - \underline{\zeta}_i^t(\epsilon_k)) - \epsilon_k\underline{\zeta}_i^t(\epsilon_k) + (1-\epsilon_k)\overline{\zeta}_i^t(\epsilon_k)$. Clearly, $\overline{\eta}_{jt}(\epsilon_k)$ and $\underline{\eta}_{jt}(\epsilon_k)$ are such that

$$\lim_{\epsilon_k\to 0} \overline{\eta}_i^t(\epsilon_k) = \lim_{\epsilon_k\to 0} \underline{\eta}_i^t(\epsilon_k) = 0, \forall i \tag{B.4}$$

Any particular offer x that is not in equilibrium will occur with probability 0, and responders can take sub-optimal actions without affecting their expected payoffs from the game. But they can take sub-optimal actions only for sets of offers of measure zero. The offers x such that $x_i \geq \delta v_i^t + \overline{\eta}_i^t(\epsilon_k)$ have positive measure and the responder i must choose optimally and accept almost all such offers with probability $(1-\epsilon_k)$, if all responders following i also accept such offers with the same probability. Therefore, the last responder at $t-1$, player $t-2$, accepts almost any offer x such that $x_{t-2} \geq \delta v_{t-2}^t + \overline{\eta}_{t-2}^t(\epsilon_k)$ with probability $(1-\epsilon_k)$. By backward induction on the set of responders, it follows that almost any offer x such that $x_i \geq \delta v_i^t + \overline{\eta}_i^t(\epsilon_k)$ for all $i \neq t-1$ will be accepted with probability $(1-\epsilon_k)$ by all responders i. Also, by the same reasoning almost all offers $x_i < \delta v_i^t + \overline{\eta}_i^t(\epsilon_k)$ must be rejected. Thus as $\epsilon_k \to 0$, it follows from (B.4) that almost any offer with $x_i \geq \delta v_i^t$ for all $i \neq t-1$ will be accepted by all responders.

Now, consider the proposer at $t-1$. If his offer is rejected he will receive at most

$$\delta v_{t-1}^t + \overline{\zeta}_{t-1}^t(\epsilon_k) = \delta - \sum_{i\neq t-1} \delta v_i^t + \overline{\zeta}_{t-1}^t(\epsilon_k) \tag{B.5}$$

If he offers $y \in \Delta^n$ such that $y_i(\epsilon_k) = \delta v_i^t + \overline{\eta}_i^t(\epsilon_k)$ for all $i \neq t-1$ and if it is accepted he will receive

$$y_{t-1}(\epsilon_k) = 1 - \sum_{i\neq t-1} \left\{\delta v_i^t + \overline{\eta}_i^t(\epsilon_k)\right\} \tag{B.6}$$

Since $\delta < 1$, it follows from (B.1) and (B.4) that for sufficiently small values of ϵ_k the payoff in (B.5) is less than the payoff in (B.6). Also, since almost all offers x such that $x_i \geq \delta v_i^t + \overline{\eta}_i^t(\epsilon_k)$ for all $i \neq t-1$ are accepted, it follows that at $t-1$, for sufficiently small values of ϵ_k, the proposer's machine $M_{t-1}(\epsilon_k)$ puts probability $1-\epsilon_k$ on making an acceptable offer $y(\epsilon_k)$. Therefore, in the limit as $\epsilon_k \to 0$, if period $t-1$ is reached, the proposer's offer is $\lim_{\epsilon_k \to 0} y(\epsilon_k)$. This is accepted and each player $i's$ (continuation) equilibrium payoff is unique and is equal to

$$v_i^{t-1} = \begin{cases} \delta v_i^t & \text{if } i \neq t-1 \\ 1 - \sum_{i \neq t-1} \delta v_i^t & \text{otherwise} \end{cases} \tag{B.7}$$

Since, the limiting equilibrium payoff profile at the beginning of the second stage (period $t+1$) is unique and equal to v_i for each i, it follows by backward induction from above that in the limit:

(i) the game must end in agreement in period 1 and therefore the stationary equilibrium with perpetual disagreement can not be a noisy NEM

(ii) the payoff to player i at the beginning of each period $t \leq n+1$ is unique and is equal to v_i^t satisfying condition (B.7).

It also follows from the stationarity of the equilibrium machines that $v_i = v_i^1 = v_i^{n+1}$. This together with condition (B.7) can be used to show in a straightforward way that

$$v_1 = \frac{1}{1+\delta+\delta^2+...\delta^{n-1}}, \quad .., \quad v_i = \frac{\delta^{i-1}}{1+\delta+\delta^2+..\delta^{n-1}}, \quad .., \quad v_n = \frac{\delta^{n-1}}{1+\delta+\delta^2+..\delta^{n-1}}$$

This completes the proof. ■

8 Appendix C: The results for SPE with state complexity

Proposition 9 *Let (z,t) be an agreement reached in a SPE*[27]*. Then $t \leq n$, or $t = \infty$.*

Proof. We demonstrate this result by showing that no equilibrium machine can have more than one non-terminal state. Suppose $t > n$, and some machine M_i has more than one state in equilibrium. Consider the last stage of the game, the stage that contains period t. At the beginning of this stage, suppose M_i is in state q_i^t. Define an alternative machine $M_i^{'}$, with one non-terminal state q_i and output and transition functions determined as follows: (we use primes for machine $M_i^{'}$).

$\lambda_i^{'}(q_i, s) = \lambda_i(q_i^t, s)$,for all s on the equilibrium path and $\lambda_i^{'}(q_i, s) = R$, for all s off the equilibrium path for which Player i is a responder, and $\mu_i^{'}(q_i, e) = q_i, \forall e \in E$.

The machine M_i replicates equilibrium play for the last stage for Player i, and does so with one state. Therefore, any M_i with two or more non-terminal

[27] We recall that a machine profile is a semi-perfect equilibrium if the NEM conditions are satisfied at every point along the outcome path induced by the machine profile.

states can not be a NEMs at the beginning of the last stage. Therefore, such a machine M_i can not be part of a SPE, which requires the NEM conditions hold at every period.

Since the minimal machine must have one state, it follows that a SPE machine is minimal. Therefore, either $t \leq n$, or $t = \infty$. ■

Using this result and the reasoning of Section 3, it is clear that a Noisy SPE will also give the unique stationary subgame perfect equilibrium.

9 Appendix D.The results for the D$_2$ specification of the machines.

We give the formal definition below. (An informal definition was given in a footnote in Section 2.1.)

Definition 7 $\Big(D_2)$ *A machine M_i consists of n-submachines, one for each role, $M_{ik} = (Q_{ik}, q^1_{ik}, T, \lambda_{ik}, \mu_{ik}\ )$ for $k = 0, 1, ..., n-1$[28], where Q_{ik} is a finite set of non-terminal states used by the machine M_i in the k-th role,*

$q^1_{ik} \in Q_{ik}$ is a distinguished initial state in the k-th role,

T is a distinguished absorbing or terminating state,

$\lambda_{ik} : Q_{ik} \times \overline{S}_{ik} \rightarrow C$, describes the output function of the machine in the k-th role given the state of the machine in that role and given a partial history that has occurred during the current stage of the bargaining, where $\overline{S}_{ik} \subset \overline{S}_i$ is the set of partial histories in a period at which i is in the k-th role

$\mu_{ik} : Q_{ik} \times E \rightarrow Q_{ik} \cup T$,is the transition function specifying the state of the k-th sub-machine one stage (n periods) after the current move of the sub-machine as a function of the current state and the realized endpoint of the stage.

Note the following three points concerning the above definition. Firstly, the output function λ_{ik} can be thought of as a mapping $\widetilde{\lambda}_{ik} : Q_i \rightarrow \mathcal{F}^k_i$ where $\mathcal{F}_{ik} = \{a : \overline{S}_{ik} \rightarrow C \mid a(\overline{s}) \in C_i(\overline{s})\ \forall \overline{s} \in \overline{S}_{ik}\}$ and $\widetilde{\lambda}_{ik}(q_i)(\overline{s}) = \lambda_{ik}(q_i, \overline{s})$ for all $q_i \in Q_{ik}$ and for all $\overline{s} \in \overline{S}_{ik}$. Thus for each $q_i \in Q_{ik}$, $\widetilde{\lambda}_{ik}(q_i) \in \mathcal{F}_{ik}$ specifies a mapping from histories within a period in the k-th role, $\overline{S}_{ik}$, to set of actions C. Secondly, the number of states $\|M_i\|$ in the above definition refers to the sum of all states in the different roles $\sum_k \left|Q^k_i\right|$. And thirdly, note that a machine in the above specification has to be able to play in each role, so it needs a minimal n states.

In this Appendix, we demonstrate that the results of the paper on stationarity of NEM and uniqueness of NNEM hold for 3-player games when counting number of states criterion $\succ^a$ is used if machines are specified as in D$_2$. Henceforth, in this appendix we assume there are 3 players and machines and equilibrium in machines refers to the D$_2$ specification.

[28]Note that $k = 0$ refers to the proposal role and $k > 0$ refers to the k-th responder role.

As we mentioned before, a minimal machine for the D_2 specification in 3 player games has 3 (non-terminal) states - one for each role. Our first result is an analogous result to Proposition 3 for the D_2 specification.

Proposition 10 *For any Nash equilibrium in the machine game $M = (M_1, M_2, M_3)$, if $\pi_i(M) = 0$ for some i then M_i is minimal (has one non-terminating state for each role), irrespective of the complexity criterion used.*

The proof of the above is similar to that of the analogous result in the text.

Proposition 11 *Let $(z,\, t)$ be an agreement $z \in \Delta^2$ at period $t < \infty$ for a 3-player game. Suppose that (z, t) is induced by a Nash equilibrium of the machine game $M = (M_1, M_2, M_3)$ with the D_2 specification. Then $t \leq 3$.*

Proof. Suppose not. Then there exists some $t > 3$, for which this is not true. W.l.o.g let Player 3 be the second responder in period t. Also for each player i, let q_i^τ be the state of i at period τ. The rest of the proof is in several steps.

Step 1: M_3 has one state as the second responder: $|Q_{32}| = 1$.

If $z_3 = 0$ then this step follows from the previous Proposition. So let us consider the case in which $z_3 > 0$.

We note that the partial history (z, A) cannot have appeared on the equilibrium path in any period $t-3\tau, \tau \geq 1$. If (z, A) had occurred previous to t, Player 3 could have chosen a machine $M_3^{'}$ such that $\lambda_3(q_3, z, A) = A$, for all $q_3 \in Q_{32}$, and would therefore have obtained a payoff δ^{-3} times greater than the equilibrium payoff, without adding states, thus contradicting the definition of a NEM.

For any other partial history $s^\tau \in S_{32}$ such that it occurs on the equilibrium path at period τ, it must be the case that $\lambda_3(q_3^\tau, s^\tau) = R$, where $q_3^\tau \in Q_{32}$ is the state of the machine as a second responder at τ, This is because if Player 3's move as a second responder appears on the equilibrium path in these periods, the action chosen must be R. (An A ends the game at a period before t,contrary to hypothesis.) Therefore, if Q_{32} contains two or more states, Player 3 can replace the candidate equilibrium machine M_3 by M_3', where M_3' differs from M_3 only in that $Q_{32}^{'}$ has one state (in addition to the terminal state T). Let this one state be q'; let

$$\begin{aligned}
\lambda_3^{'}(q', s) &= A \text{ if } s = (z, A) \\
\lambda_3^{'}(q', s) &= R \text{ if } s \neq (z, A) \\
\mu_3^{'}(q', e) &= q', \text{for all } e \neq (z, A, A).
\end{aligned} \tag{4}$$

Clearly, $M_3^{'}$ is less complex and reproduces the same equilibrium path as before given no other changes in the other players' machines. This contradicts M_3 being part of a NEM. Therefore Q_{32} must have one state.

Step 2: z is not offered by player 1 before t.

Consider two cases. If $z_2 = 0$, then by the previous Proposition and player 2 accepting z at t, 2 will always accept z. From the previous step, Player 3 would choose A in response to (z, A). Therefore, if z happens before t an agreement would have taken place before t. But this is a contradiction. Now consider the case in which $z_2 > 0$. If the offer z occurs on the equilibrium path at $t - 3\tau, \tau \geq 1$, for the same reason as before Player 2 must reject it in equilibrium (because from the previous step, Player 3 would choose A in response to (z, A)). Player 2 should therefore deviate and accept z, thus ending the game at least three periods earlier and obtaining a payoff greater by a factor of δ^{-3}. Therefore, z must not have appeared in equilibrium in periods $t - 3l, l \geq 1$.

Step 3: $q_2^t = q_2^\tau$ for some $\tau < t$.

Suppose that this step is false. Then, consider a machine M_2' where M_2' is the same machine as M_2 without the state q_2^t and the output function is changed to

$$\lambda_2'(q_2, s) = \begin{cases} A & \text{if } s = z \\ \lambda_2'(q_2, s) & \text{if } s \neq z \end{cases}$$

Since z does not occur on the path before t and $q_2^t \neq q_2^\tau$ for all $\tau < t$, this construction does not affect the equilibrium path. Since, M_2' has one less state than M_2 the above construction contradicts the hypothesis of the machines M being a NEM.

Step 4: z will be agreed to at some period $\tau < t$ of the game, resulting in a contradiction.

We have already seen that (z, A) will be followed by an acceptance by Player 3. Also, $z_1 > 0$ because if $z_1 = 0$ then M_1 would be minimal and would always offer z as a proposer; but this contradicts the second step above.

Now by the previous step $q_2^t = q_2^\tau$ for some $\tau < t$.Therefore, if z appears in period τ, it will be accepted and the game will end.

Consider finally Player 1, the proposer in period $t - 3k, k \geq 0$. Player 1 will deviate in period τ and offer z, which will be accepted by both responders, thus ending the game before t. Therefore (z, t) cannot be a NEM outcome for $\infty > t > 3$,for $z >> 0$.

■

Once again, this result and the addition of noise give us uniqueness at the stationary, subgame perfect allocation.

10 Appendix E: Counter-example to stationarity with state complexity with four players for the sub-machine specification (D2).

This is a counter-example for four players, showing that for the definition of a machine as the combination of four sub-machines, (one for each role), state complexity is not sufficient to lead to stationarity. The equilibrium path has been constructed so as to make all states essential (no saving of a state is possible without inducing an 'out–of-equilibrium outcome'). The Table below describes the equilibrium path of actions and states. The states for each player in each stage are listed in brackets in each entry in the table; note there are four sub-machines per player, since each player must play four roles. Note also that the sub-machine that implements each player's behaviour as the third responder contains one state, whereas each of the other three sub-machines, for each player, contains two states (denoted by 1 &2). All offers are distinct, unless explicitly noted. Transitions take place from sub-machine to sub-machine just before a player has to move in the role corresponding to the sub-machine. Such *transitions* could, of course, depend on the entire history of the game since the last transition for that sub-machine. The *actions* depend on partial history within a period.

Period	1	2	3	4	5	6	7	8
Player 1	x^1(2)	(1)	R(1)	A(2)	x^1(2)	R(1)	A(2)	A(1)
2	A(2)	x^2(2)	(1)	R(1)	A(1)	x^2(2)	R(1)	A(2)
3	R(1)	A(2)	x^3(2)	(1)	A(2)	A(1)	x^3(2)	R(1)
4	(1)	R(1)	A(2)	x^4(2)	R(1)	A(2)	A(1)	x^4(2)

9	10	11	12	13	14	15	16
z^1(1)	(1)	R(2)	R(2)	z^1(1)	(1)	(1)	A(1)
A(2)	z^2(1)	(1)	(2)	R(1)	z^2(1)	(1)	A(1)
R(2)	A(2)	z^3(1)	(1)	(1)	R(1)	z^3(1)	A(1)
(1)	R(2)	A(2)	z^4(1)	(1)	(1)	R(1)	z^4(1)

States are made essential for each player because each sub-machine, on the equilibrium path, is required to play different actions, after the same history, in different periods where a player has the same role.

As in the paper, deviations from the above equilibrium path are deterred by

(i) constructing responses that reject all offers not on the equilibrium path

(ii) constructing transition functions that go to the beginning of the game if any out-of-equilibrium behaviour is observed. For example, if out-of equilibrium behaviour is observed by player 1, his sub-machines in the roles of the proposer, first responder and second responder will move to state 2, 1 and 1 respectively (the states of the sub-machines in period 1,2 and 3 respectively).

An example is given below of how such a construction ensures that no player would deviate from the equilibrium path.

If player 1, for example, attempts to speed up the game by offering z^1 in period 5, then player 2, in state 1, rejects this offer (because player 2 in this state rejects z^1 on the equilibrium path at period 13). Thus the history of actions in the last 4 periods, before period 6, will be $e = (x^2, A, R; x^3.A, R; x^4, A, R; z^1, R)$. Since this history does not occur on the equilibrium path, by construction the other players move back to the states in their appropriate roles at the beginning of the game. Thus in period 6, the players 2,3 and 4 move to the states (2,2,1) respectively (to the states on the equilibrium path at period 2). This makes the deviation by 1 in period 5 unprofitable. The unprofitability of other deviations can be checked in the same way.

Note that every sequence of four periods has a path distinct from that of any other corresponding four periods (each sequence beginning with periods where players have the same role). This means any attempt to substitute an action by one occurring at a later period will be out-of-equilibrium and be detected; the transitions for out-of-equilibrium histories then ensure that such deviations do not increase a player's payoff.

11 Appendix F: Counter-example to stationarity with three players and s-complexity, for definition D3.[29]

In this counter-example to stationarity, the states of each machine are allowed to change every period and the output function $\lambda_i(.)$ is a function of the state of the machine and the partial history within a period.

Each machine has two states 1 and 2 and an agreement z^2 is reached in period 6. The actions and transitions on the equilibrium path are as shown above, with the states for each player as shown in brackets in each entry. The offers, apart from z^2, are such that the second responder prefers waiting until period 6 to obtain the equilibrium payoff rather than accepting the current offer and ending the game.All the offers are distinct. (We assume $z^2 \gg 0$.)

	1	2	3	4	5	6
Pl.1	$x^1(1)$	R(1)	A(1)	$x^2(2)$	R(2)	A(2)
2	A(1)	$y^1(1)$	R(1)	A(2)	$y^2(2)$	A(2)
3	R(1)	A(1)	$z^1(1)$	R(2)	A(2)	$z^2(2)$

Out of equilibrium, the actions and transitions are as follows.

For any offer x, we define the output function λ_i out of equilibrium as:

$$\lambda_i(q_i^k, x) = R, \text{ if } (q_i^k, x) \text{ does not occur on the equilibrium path}$$

Also

$$\lambda_i(q_i^k, x, A) = R, \text{ if } (q_i^k, x, A) \text{ does not occur on the equilibrium path}$$

For any state-history pair (q_i, e) that does not occur on the equilibrium path in a given period, the transition functions are constructed as follows

$$\mu_i(q_i, e) = \begin{cases} q_i^{'} & \text{if i is the proposer next period} \\ q_i & \text{if i is the first responder next period} \\ q_i^{'} & \text{if i is the second responder next period} \end{cases}$$

where $q_i^{'} \neq q_i$ (the other state being used on the path).

In this example the two states of each machine are essential because each is required to make a separate offer (e.g. x^1, x^2). Thus any attempt to save states results in an out-of-equilibrium outcome. Given the above transition and response functions any deviation resulting in an out-of-equilibrium outcome makes the player worse off by delaying the agreement.

[29] In this definition, states could change every *period*, instead of at every stage.

Consider, as an example, a deviation by player 1, who drops state 1 from his machine, and thus begins the game with x^2. This is rejected by player 2, who moves to state 2, while 3 stays in state 1. In the next period, the outcome is (y^2, R), and in period 3, player 3 offers z^2, which is accepted by player 1 and rejected by player 2 (who has moved back to state 1). The outcome (z^2, A, R) is not on the equilibrium path, and player 3 moves back to state 1, while player 2 stays in state 1, so the game moves back to the first period making the initial deviation unprofitable for player 1.

12 Appendix G: Stationarity result for two-player game with another alternative specification[30].

Here we consider a specification, which we shall not write out formally, where each state corresponds to the output of an offer, except for a termination state T (equivalent to accepting an existing offer), and transitions can take place after every piece of new information is available. We obtain the result that any Nash equilibrium machine must be stationary (one-state) (and any agreement must occur by period 2). State-complexity is sufficient for this.

Here is an outline of the argument.

Clearly, if there is an agreement in the first two periods or if there is no agreement the equilibrium machine must have one state (exactly analogous to the treatment in the paper -Lemma 2).

Suppose now that there is an agreement $(z, t), t > 2, z \gg 0$.

1. All states for both players must be used on the equilibrium path. If not, a player could save a state by dropping one that is unused, and this contradicts the definition of equilibrium.

2. No state for any player i can occur twice on the equilibrium path. Suppose otherwise, so that $q_i^\nu = q_i^\tau$ (the states at ν and τ are the same), where $\tau < \nu \leq t$. Then the other player can deviate, by playing the game from period τ as if he/she is at ν, on the equilibrium path, thus eliminating the states between period τ and period ν . This will speed up the game by $\nu - \tau$ periods, and make players better off (because of discounting and $z_i > 0$).

3. Player 1 must have one state, other than T. Suppose not; then if Player 1 has two states, he can drop q_1^1 (that occurs only in period 1 by the previous step) and replace it with the other non-terminal state. Player 2 cannot punish, because the worst punishment (using one of the distinct states used on the equilibrium path) would be to make a transition to period 2, and this is what would have happened on the original equilibrium path.

4. Player 2 must have one state, other than T. Suppose not; then if player 2 has two states, he can drop q_2^2 (that occurs only in period 2) and replace it with the other non-terminal state. Player 1 cannot punish, because he has only one state.

The case of $z_i = 0$, for some i, is exactly analogous to the treatment in the paper.

[30] A referee suggested we consider this specification, which is similar to Binmore, Piccione and Samuelson (1998).

13 Appendix H: Counter-example to stationarity with state complexity, with three players, for the machine specification of Appendix F .

This three-player counter-example involves an agreement on a partition x^1 at period 4. (Here too, all offers are distinct).

M_1 has two states, q_1^P and q_1^R; M_2 has three, q_2^P, q_2^R, q_2^A; and M_3 has three states q_3^P, q_3^R, q_3^A.

The output maps are defined as follows:

$$\begin{aligned}\lambda_i(q_i^P) &= x^i.\\ \lambda_i(q_i^R) &= R,\\ \lambda_i(q_i^A) &= A.\end{aligned}$$

The transitions are as follows. For any state q_1 for player 1,

$$\mu_1(q_1, e) = \begin{cases} q_1^R & \text{if } e \text{ is such that player 1 is a responder} \\ q_1^P & \text{otherwise} \end{cases}$$

where e is the new information preceding a transition and contains enough information to indicate the role that each player is going to play in his or her next move. Also,

$$\mu_2(q_2^P, e) = q_2^R$$

$$\mu_2(q_2^R, e) = \begin{cases} q_2^P & \text{if } e \text{ is such that player 2 is the proposer} \\ q_2^A & \text{if } e \text{ is such that } x^1 \text{ has been offered by player 1} \\ q_2^R & \text{otherwise} \end{cases}$$

$$\mu_3(q_3^P, e) = \begin{cases} q_3^P & \text{if } e \text{ is such that player 1 has just made an offer} \\ q_3^R & \text{otherwise} \end{cases}$$

$$\mu_3(q_3^R, e) = \begin{cases} q_3^P & \text{if } e \text{ is s.t. player 3 is the proposer} \\ q_3^A & \begin{array}{l}\text{if } e \text{ is s.t. player 3 is the second responder} \\ \text{and an offer has been accepted by the first responder}\end{array} \\ q_3^R & \text{otherwise} \end{cases}$$

The initial states are (q_1^P, q_2^P, q_3^P).

The outcome path in this example will be

Period			
1	2	3	4
x^1	x^2	x^3	x^1
R	R	R	A
			A

The states for player 1 on the equilibrium path in the first four periods will be q_1^P , q_1^R, q_1^R, q_1^P . The transitions of player 2 on the equilibrium path will be: start initially in state q_2^P, change to q_2^R after receiving the offer x^1 in period 1, switch back to q_2^P after rejecting the offer in the first period, change to q_2^R after making the offer x^2 in period 2, stay in q_2^R until period 4, then switch to q_2^A in period 4 after receiving the offer x^1. The states of player 3 on the equilibrium path are initially q_3^P, switching to q_3^R after player 2's response in period 1 and staying in this state until period 3, switching to q_3^P at the beginning of period 3, switching back to q_3^R after making his offer x^3 in this period, and finally switching to q_3^A in period 4 after player 2 accepts the offer x^1.

Note that no saving of states is possible, since all states are used on the equilibrium path to take different actions. The offer x^1 appears at $t = 1$, but is rejected by player 2, because an acceptance by player 2 would lead to a rejection by player 3 (here player 3 is in state q_3^R).

Clearly, no deviation can make any player better off.

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