

Evolution of Conventions in Endogenous Social Networks*

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Abstract

We analyze the dynamic implications of learning in a large population coordination game where both the actions of the players and the communication network evolve over time. Cost considerations of social interaction are incorporated by considering a circular model with endogenous neighborhoods, meaning that the locations of the players are fixed but players can create their own communication network.

The dynamic process describing medium-run behavior is shown to converge to an absorbing state, which may be characterized by coexistence of conventions. In the long run, when mistake probabilities are small but nonvanishing, coexistence of conventions is no longer sustainable as the risk-dominant convention becomes the unique stochastically stable state.

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1 Introduction

In a wide variety of economic and social environments, an agent's utility depends on successful coordination with other individuals. The following example, as suggested by Lewis [17], illustrates this point. Suppose that oligopolists are confronted with a change in the price of their raw material and therefore must set new prices of their product. It is to no one's advantage to set his price higher than the others set theirs, since if he does, he tends to lose his share of the market. Nor is it to anyone's advantage to set his price lower than the others set theirs, since if he does, he menaces his competitors and incurs their retaliation. Hence, each competitor must set his price close to the price he expects the others to set. Once coordination has been achieved on a certain behavior, then it is likely that this behavior will become the convention. For this reason, Lewis [17], and also Schelling [20], already stated that a convention should be considered a solution to a coordination problem. More precisely, Lewis defines a convention as a behavioral regularity such that everyone conforms to the regularity, expects others to conform, and wants to conform given that others conform.

The above example illustrates two fundamental factors important in determining optimal behavior when agents face a coordination problem. An agent's expectation about the behavior of others plays a significant role. But underlying those expectations is an interaction structure governing communication between the players. Implicit in our discussion of conventions, we find ourselves talking about localities: geographic or social. Other examples of well-known conventions are languages, currencies, product standards, codes of dress, and accounting standards.

Overview of the Model

We analyze the dynamic implications of learning in a large population coordination game where both the actions of the players and the communication network between these players evolve over time. We follow the conventional evolutionary game theoretic models on coordination problems in assuming that players use the same pure strategy against all opponents they interact with, i.e., players with whom they communicate, and we allow for this strategy to be adjusted over time. We depart from the conventional models in assuming that the interaction network itself is also subject to evolutionary pressure. Cost considerations of social interaction are incorporated by considering a circular model with an endogenous communication network, meaning

that the locations of the players are fixed but players can create their own interaction neighborhood by forming and severing links with other players. The spatial structure of the model is then used to incorporate cost considerations of communication. In fact, we assume that the larger the distance between two players on the circle, the larger the maintenance costs of the mutual link will be. As maintenance costs include invested time and effort, distance should not only be interpreted as physical distance but may also represent social distance.

Players typically react myopically to their environment by deciding about both pure strategies and links based on a best-reply dynamics. Sometimes, however, players make mistakes when implementing their decisions, or alternatively players experiment with nonoptimal replies. Whether or not these mistakes should be included explicitly in the model depends on the span of time over which we are interested in the players' behavior as predicted by the model. As explained by Binmore, Samuelson and Vaughan [5] the model corresponds to the players' long-run behavior in the absence of the perturbations representing the players' mistakes. We find that in this case, the dynamic process converges to an absorbing state. As the set of absorbing states includes states in which different kinds of behavior are observed, the population's long-run behavior is possibly characterized by coexistence of conventions. In the ultralong run, i.e., when the perturbations representing the players' mistakes are taken into account, coexistence of conventions is no longer possible. Namely, the risk-dominant convention is the unique stochastically stable convention, meaning that it will be observed almost surely when the mistake probabilities are small but nonvanishing.

Related Literature

Related to the present paper is also the literature on network formation. Dynamic models of network formation are considered by e.g. Bala and Goyal [2], Jackson and Watts [13], and Watts [21]. In these models the presence of a link however does not indicate that the involved players interact by means of playing a game. Instead there is a deterministic benefit from an emerging network. The best-known and most intuitive example is the symmetric connections model as introduced by Jackson and Wolinsky [14], which represents social communication between agents. In fact, agents communicate with all other agents they are directly or indirectly connected with. The value of the communication depends on the number of links involved in the shortest

path that connects a pair of agents. Watts [21] shows convergence to an efficient network in case costs are small and closer connections are valued more than distant connections. Note that the model is deterministic and therefore involves initial state dependence. Bala and Goyal [2] also find convergence to efficient networks in their deterministic setting. Their model differs from Watts[21] as it focuses on directed networks and players receive the same benefit from direct and indirect connections. Jackson and Watts [13] consider a stochastic model similar to Kandori, Mailath and Rob [15] and Young [22]. They find that the networks that occur with positive probability in the stationary distribution when mistake probabilities go to zero are either stable or a cycle. In their model efficiency can not be ensured.

The evolution and stability of conventions is analyzed through the population adjustment models with persistent randomness in Kandori, Mailath and Rob [15], Young [22], and Ellison [7]. With respect to coordination games these learning models identify the risk-dominant equilibrium as the unique long-run convention. Goyal and Janssen [11] focus on nonexclusive conventions in a deterministic framework to model the idea that, at the expense of some additional costs, players can remain flexible and therefore coordinate their actions more successfully. They find that the Pareto-efficient or risk-dominant equilibrium prevails depending on whether these costs are low or high, respectively. Furthermore, at intermediate cost levels, the two conventions coexist. Coexistence of exclusive conventions is also a feature of the model with noise on the margin analyzed by Anderlini and Ianni [1].

The models mentioned above deal with exogenously given patterns of interaction. In particular, Kandori, Mailath and Rob [15] and Young [22] use uniform matching rules, meaning that every player possibly interacts with all other players, while Anderlini and Ianni [1], Ellison [7], and Goyal and Janssen [11] use local matching rules, expressing that every player can only interact with a small subset of the population. However, by fixing the pattern of interaction exogenously these models ignore that players may have the desire and, at least to some extent, the ability to affect the set of players with whom they interact.

Endogenous patterns of interaction in population adjustment models can also be realized by allowing players to migrate, i.e., choose their location. Bhaskar and Vega-Redondo [4], Ely [8], Mailath, Samuelson and Shaked [18], and Oechssler [19] show that migration implies, or at least allows for, the population to coordinate on the Pareto-efficient equilibrium. In particular, Oechssler [19] shows that in his

deterministic framework, given that all conventions are initially present, the efficient one will eventually prevail throughout. Ely [8] considers a stochastic model and shows by considering the stationary distribution when mistakes probabilities become small that the efficient convention will occur independent of the initial conditions. In both models coexistence of conventions is not possible. Bhaskar and Vega-Redondo [4], in a model with asynchronous strategy and location revision opportunities, indicate the possibility of coexistence of conventions in the medium run when the game is a ‘pure’ coordination game. Also in the long run both conventions are possible, i.e., both appear with positive probability in the stationary distribution of the stochastic model, in the case of frequent play. Otherwise, only the efficient convention is stochastically stable. Finally, Mailath, Samuelson and Shaked [18] look at a quite different context in continuous time. Players of two continuum populations have to decide which location to visit. They show that if the evolutionary process is monotonic and players can avoid undesirable matching, then every locally stable configuration is efficient.

The paper is organized as follows. In Section 2 we introduce the model. Section 3 deals with the deterministic model describing the medium-run behavior of the dynamic process. Long-run behavior is analyzed in Section 4 by means of the stationary distribution of the stochastic model when mistakes are small but nonvanishing. The robustness of the obtained results with respect to the exact specification of the model is discussed in Section 5. Finally, Section 6 concludes.

2 The Model

Consider a large but finite population of players $N = \{1, \dots, n\}$ who are spatially distributed around a circle. Players are distributed around the circle in an equidistant fashion. Each discrete-time period $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ the players’ interaction consists of two stages. In the first stage players form or sever links connecting themselves to other players. In the second stage pairs of players who are linked play a coordination game and adjust their actions.

The state s_k of the dynamic process in each period k is given by a graph g_k , with its nodes representing the players and its edges capturing the established communication links, and an action profile a_k , specifying the action being played by every player $i \in N$. Let \mathcal{G} and \mathcal{A} denote the set of possible graphs and the set of possible action profiles, respectively, then $s_k = (g_k, a_k) \in \mathcal{G} \times \mathcal{A} = \mathcal{S}$.

Each period, the presence of a link between players i and j , with $i, j \in N$ and $i \neq j$, results in a maintenance cost c_{ij} to both players. The costs c_{ij} are determined by the distance between the two players on the circle. Formally, we assume that

$$c_{ij} = \min \{ \gamma |j - i|, \gamma |j - i + n| \},$$

where $\gamma \geq 0$ are the so-called unit costs.

Let $\mathcal{L}_{i,k} \subset N$ denote the set of players that player $i \in N$ is linked with after the first stage in period $k \in \mathbb{N}_0$. We refer to $\mathcal{L}_{i,k}$ as the *interaction neighborhood* of player i at time k . In addition, with slight abuse of notation, we write $a_k = (a_{i,k}, a_{\mathcal{L}_{i,k},k}, a_{-\mathcal{L}_{i,k},k})$. Note that when not causing any confusion $\mathcal{L}_{i,k}$ and $a_{i,k}$ may also be denoted by \mathcal{L}_i and a_i , respectively. In the second stage of each period k , a player i plays a coordination game with all players $j \in \mathcal{L}_{i,k}$. The gross benefits are determined by the utilities u in the 2×2 coordination game given below.

	A	B	
A	a, a	c, d	(1)
B	d, c	b, b	

The payoff to player i in period k is given by

$$\pi_i(a_{i,k}, a_{\mathcal{L}_{i,k},k}) = \begin{cases} \sum_{j \in \mathcal{L}_{i,k}} u(a_{i,k}, a_{j,k}) & \text{if } \mathcal{L}_{i,k} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_{i,k} \in \{A, B\}$, for all $i \in N$ and \mathbb{N}_0 . The net benefits of a player i can be found by subtracting all maintenance costs, $\sum_{j \in \mathcal{L}_{i,k}} c_{ij}$, from his gross benefits. For notational convenience only we assume that $a, b, c, d \in \mathbb{N} \cup \{0\}$. Furthermore, it is required that $a > d \geq 0$ and $b > c \geq 0$, implying that both (A, A) and (B, B) are strict Nash equilibria. Furthermore, the payoffs a and b are taken to be larger than or equal to γ for the model to be nontrivial, i.e., for the players to have an incentive to interact. Finally, we assume that $a + c > b + d$, implying that (A, A) is the risk-dominant equilibrium as defined by Harsanyi and Selten [12]. Note that when the two actions have equal security levels, i.e., $c = d$, (A, A) is also the Pareto-efficient equilibrium.

Now, we describe how players establish and sever links, and how they adjust their pure strategies. The process of link formation and link severance is based on the process discussed in Jackson and Watts [13]. In the first stage of each period k two

players i and j , $j \neq i$, are randomly selected with probability $\rho_{ij} > 0$. Hence, we consider a matching process with players randomly meeting each other in pairs, and time periods being identified with the encounters. Only the two players constituting that pair can alter their (potential) mutual link in period k . If their mutual link already exists, they decide whether to sever it, and otherwise they decide whether to create the link.

The part of the dynamic process that describes how the two selected players establish or sever a mutual link is modelled as follows. First, suppose player i and j are not linked and therefore have to decide whether to establish the link ij . Player i compares the payoff he would have received from playing the coordination game with player j in the previous period, i.e.,

$$u(a_{i,k-1}, a_{j,k-1}) \geq c_{ij},$$

with the maintenance costs c_{ij} of the link. Of course, player j follows exactly the same procedure. Only in case both payoffs are greater than or equal to the costs c_{ij} , the players decide to establish the mutual link. Notice that according to the deterministic part of the link formation process, two players will never establish a link if their mutual distance on the circle exceeds $\max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Namely, in that case the maintenance costs are always larger than the payoff obtained in the coordination game. Second, suppose that at the moment players i and j are linked, and that they therefore have to decide whether to sever their mutual link. In case the payoff that player i received from playing the coordination game with player j in the previous period $k-1$, i.e., $u(a_{i,k-1}, a_{j,k-1})$, or the payoff that player j received from playing with player i in that period, is strictly smaller than the maintenance costs c_{ij} , they decide to sever the link. We assume that every linking decision made by a pair of players is implemented correctly with probability $1 - 2\tau$, where $\tau \geq 0$. With probability 2τ , the two potential linking decisions are implemented with equal probability.

To describe the adjustment of the players' pure strategies, we follow Ellison [7] by assuming that players typically react myopically to their environment. In fact, we assume that in the second stage of period t , player i chooses

$$a_{i,k} \in \arg \max_{a_i} \pi_i(a_i, a_{\mathcal{L}_{i,k,k-1}}) \quad (2)$$

with probability $1 - 2\varepsilon$, where $\varepsilon \geq 0$. With probability 2ε player i chooses between the two pure strategies at random with equal probability. Player j adjusts his pure strategy in the same way.

We refer to Section 5 for a discussion of alternative specifications of the dynamic process.

For a given initial state $s_0 \in \mathcal{S}$, the dynamic process introduced above defines a Markov process $\{s_k\}_{k \in \mathbb{N}_0}$ in discrete time with the finite state space $\mathcal{S} = \mathcal{G} \times \mathcal{A}$. Let $P := P(\tau, \varepsilon)$ denote the (one-step) transition matrix of the Markov process. Then, if $s = (g, a)$, $\tilde{s} = (\tilde{g}, \tilde{a}) \in \mathcal{S}$, the entry $P_{s\tilde{s}} := P_{s\tilde{s}}(\tau, \varepsilon)$ of the transition matrix is the probability that the state is \tilde{s} at time $k + 1$ conditional on the state being s at time k , i.e.,

$$P_{s\tilde{s}} = \Pr(s_{k+1} = \tilde{s} \mid s_k = s).$$

Let $P^l := P^l(\tau, \varepsilon)$ denote the l -step transition matrix of the Markov process, then the entry $P_{s\tilde{s}}^l := P_{s\tilde{s}}^l(\tau, \varepsilon)$ of the l -step transition matrix is equal to

$$P_{s\tilde{s}}^l = \Pr(s_{k+l} = \tilde{s} \mid s_k = s).$$

We refer to the Markov process $\{s_k\}_{k \in \mathbb{N}_0}$ as *adaptive play*.

3 Adaptive Play without Mistakes

This section deals with the model introduced in section 2 when both τ and ε are equal to zero. We refer to this specific Markov process as *adaptive play without mistakes*. As discussed extensively by Binmore, Samuelson and Vaughan [5] adaptive behavior without mistakes describes the long-run behavior of the dynamic process, where the long-run refers to the time span needed for the dynamic process to reach the neighborhood of the first equilibrium near to which it will stay for a significant period of time.

Analyzing the dynamic process in the long run boils down to identifying the absorbing states of the Markov process representing adaptive play without mistakes. An absorbing state is a state that once entered cannot be left. With respect to the present model this implies that in an absorbing state no single player wants to sever any of the links he is involved in or adapt his pure strategy, and no pair of players wants to establish a mutual link. Below, we discuss what the absorbing states of adaptive play without mistakes look like, and we explore conditions which guarantee convergence to one of these absorbing states.

3.1 Absorbing States of Adaptive Play without Mistakes

As mentioned before a convention is defined by Lewis [17] as a pattern of behavior that is customary, expected, and self-enforcing, meaning that everyone conforms, expects others to conform, and wants to conform given that everyone else conforms, respectively. Since a convention is self-enforcing, it has to be an absorbing state of the Markov process describing adaptive play without mistakes. It can easily be verified that the present model has two absorbing states which are conventions. Namely, the state such that

$$\begin{cases} a_i = A, \\ j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leq \frac{a}{\gamma}, \end{cases} \quad (3)$$

for all $i \in N$ and the state such that

$$\begin{cases} a_i = B, \\ j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leq \frac{b}{\gamma}, \end{cases} \quad (4)$$

for all $i \in N$. We refer to (3) as the $(A, \frac{a}{\gamma})$ -convention or the risk-dominant convention and to (4) as the $(B, \frac{b}{\gamma})$ -convention.

The $(A, \frac{a}{\gamma})$ -convention and $(B, \frac{b}{\gamma})$ -convention are, however, not the only absorbing states of adaptive behavior without mistakes. The other absorbing states indicate that the players' long-run behavior may be characterized by coexistence of conventions. The following example shows existence of an absorbing state in which players act differently. This absorbing state therefore indicates the possibility of coexistence of conventions in the long-run, see also Bhaskar and Vega-Redondo [4] for a similar observation.

Example 1 Consider a population consisting of 12 players. Assume $\gamma = 1$ and let the payoffs of the coordination game be given by $a = 2$, $b = 1$, and $c = d = 0$. It can easily be verified that the state represented in Figure 1 is an absorbing state. Since players do not conform to the same pattern of behavior, the absorbing state is not a convention. The absorbing state does, however, show the presence of local clusters of players who conform, expect others to conform, and want to conform (given that others conform) to the same pattern of behavior. We refer to the presence of such clusters in an absorbing state as coexistence of conventions. It is not hard to see that there are typically many absorbing states which exhibit such clusters. Note that in

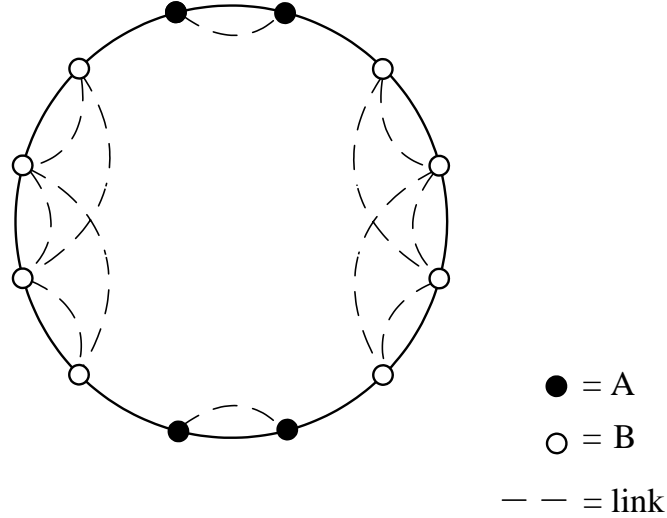


Figure 1: An absorbing state reflecting co-existence of conventions.

order for such a state to be an absorbing state the clusters of A -players and B -players should be at least of size $\frac{b}{\gamma}$ and $\frac{a}{\gamma}$, respectively.

3.2 Convergence of Adaptive Play without Mistakes

To explore the conditions under which adaptive play without mistakes converges to an absorbing state, we develop the two lemmas. We define the set $\tilde{\mathcal{S}} \subset \mathcal{S}$ by

$$\tilde{\mathcal{S}} := \left\{ s \in \mathcal{S} \mid a_i \in \arg \max_{\tilde{a}_i} \pi_i(\tilde{a}_i, a_{\mathcal{L}_i}) \text{ for all } i \in N \right\},$$

i.e., the set $\tilde{\mathcal{S}}$ contains all states $s \in \mathcal{S}$ such that all players $i \in N$ choose a pure strategy that results in the highest possible gross benefit, given the current pure-strategy profile of the players in their neighborhood. Lemma 2 specifies a condition on the size of the population under which adaptive play without mistakes becomes contained in $\tilde{\mathcal{S}}$ with positive probability in a finite period of time. Let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to x .

Lemma 2 *Assume that $\lfloor \frac{n}{2} \rfloor > \max\{a, b\}$. Then for any state $s \in \mathcal{S}$ there is a probability $p_s > 0$ that adaptive play without mistakes reaches a state $\tilde{s} \in \tilde{\mathcal{S}}$ in $K_s < \infty$ periods.*

The proof of Lemma 2 is included in Appendix A

Lemma 3 in turn states that, starting from a state which is contained in $\tilde{\mathcal{S}}$, adaptive play without mistakes converges to an absorbing state with positive probability in a finite period of time.

Lemma 3 *For any state $s \in \tilde{\mathcal{S}}$ there is a probability $q_s > 0$ that adaptive play without mistakes converges to an absorbing state in $L_s < \infty$ periods.*

The proof of Lemma 3 is included in Appendix A

Now, we are able to prove Theorem 4 which specifies the condition on the size of the population under which convergence of adaptive play without mistakes can be guaranteed almost surely.

Theorem 4 *Assume that $\lfloor \frac{n}{2} \rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Then for any initial state $s_0 \in \mathcal{S}$ adaptive behavior without mistakes converges almost surely to an absorbing state.*

Proof. Lemma 2 states that for any state $s \in \mathcal{S}$ there is a probability $p_s > 0$ that adaptive play without mistakes will be given by a state $\tilde{s} \in \tilde{\mathcal{S}}$ in $K_s < \infty$ periods, provided that $\lfloor \frac{n}{2} \rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. According to Lemma 3, for any state $s \in \tilde{\mathcal{S}}$ there is a probability $q_s > 0$ that adaptive play without mistakes converges to an absorbing state in $L_s < \infty$ periods.

Note that \mathcal{S} is finite. Letting $p = \min_{s \in \mathcal{S}} p_s > 0$, $q = \min_{s \in \tilde{\mathcal{S}}} q_s > 0$, $K = \max_{s \in \mathcal{S}} K_s < \infty$, and $L = \max_{s \in \tilde{\mathcal{S}}} L_s < \infty$, it follows that there exists a positive integer $M = K + L < \infty$, and a positive probability $r = pq > 0$, such that from any initial state $s_0 \in \mathcal{S}$, the probability is at least r that adaptive play without mistakes converges to an absorbing state within M periods, provided that $\lfloor \frac{n}{2} \rfloor > \max\{a, b\}$. Obviously, both M and r are time-independent and state-independent. Hence, the probability of not reaching an absorbing state after at least mM periods is at most $(1 - r)^m$, which goes to zero as $m \rightarrow \infty$. ■

The condition $\lfloor \frac{n}{2} \rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$ states that the population is sufficiently large to ensure that for every player $i \in N$ there exist a player $j \neq i$ such that, independent

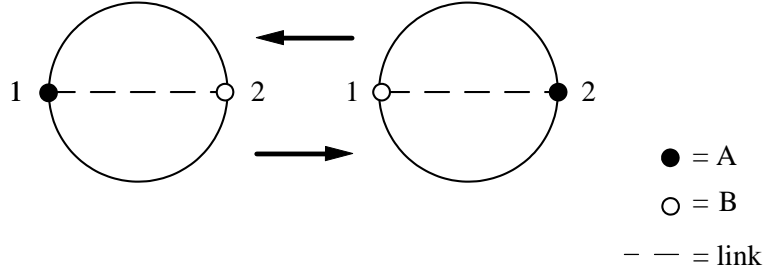


Figure 2: A cycle

of the outcome of the game, it is too costly for them to be linked. According to Theorem 4 this condition excludes the possibility that the dynamic process ends up in a cycle. The necessity of this assumption is illustrated with Example 5.

Example 5 Consider a population consisting of two players who are situated on a circle and assume that $\gamma = 1$ and $\min \{c, d\} \geq 1$, i.e., player 1 and player 2 want to be linked independent of the outcome of the game. The two players will therefore form a link (if not already present) in the first stage of period 1 and sustain it in all subsequent periods. Since the players adapt their actions myopically as specified by (2), it can easily be verified that any initial state such that the actions of player 1 and player 2 are different, results in a cycle as represented in Figure 2, where both players will adapt their pure strategies continuously. Consequently, the players' pure strategies will be different every period and adaptive behavior without mistakes will never settle down.

4 Adaptive Play with Mistakes

In this section we consider *adaptive play with mistakes*. With both kinds of mistakes as part of the model, i.e., $\tau > 0$ and $\varepsilon > 0$, the transition matrix of the Markov

process is *irreducible* and *aperiodic*. Irreducibility means that for every pair of states $s, s' \in \mathcal{S}$, there exists a time $l := l(s, \bar{s})$ such that $(P^l)_{s\bar{s}} > 0$, i.e., every state in the state space can be reached from every other state with positive probability. A Markov process is aperiodic if for every state s in the state space it holds that $P_{ss} > 0$, i.e., for every state there is a positive probability of remaining there in the next period.

A *stationary distribution* of the Markov process is a row vector $\phi := \phi(\tau, \varepsilon) \in \Delta_{|\mathcal{S}|-1}$ such that $\phi P = \phi$, where

$$\Delta_{|\mathcal{S}|-1} = \left\{ \nu \in \mathbb{R}^{|\mathcal{S}|} \mid \nu_s \geq 0 \text{ for all } s \in \mathcal{S} \text{ and } \sum_{s \in \mathcal{S}} \nu_s = 1 \right\}.$$

In Lemma 6 we summarize the standard results in the literature that an irreducible and aperiodic Markov process is ergodic, meaning that the Markov process has a unique stationary distribution, and that the process converges to this stationary distribution from any initial state. Furthermore, the distribution of realized states along any sample path approaches the stationary distribution almost surely.

Lemma 6 *Assume that the transition matrix P is irreducible and aperiodic. Then the stationary distribution ϕ is unique. Furthermore, for any $\nu \in \Delta_{|\mathcal{S}|-1}$,*

$$\nu P^k \rightarrow \phi \quad \text{as } k \rightarrow \infty.$$

Also, for all initial states $s_0 \in \mathcal{S}$,

$$\frac{1}{K} \sum_{k=1}^K X_s(s_k) \rightarrow \phi_s \quad \text{almost surely as } K \rightarrow \infty,$$

where

$$X_s(s_k) = \begin{cases} 1 & \text{if } s_k = s, \\ 0 & \text{otherwise.} \end{cases}$$

(See Theorem 1.2 and Theorem 1.3 in Chapter 3 of Karlin and Taylor [16])

As mentioned earlier, the stationary distribution is a good description of the behavior of the adaptive process in the ultralong run. Namely, the adaptive process jumps from an absorbing state to the basin of attraction of another absorbing state due to the occurrence of very unlikely realizations of the perturbations of the dynamic process, i.e., of the players' mistakes. Only given a very long period of time, these jumps will occur often enough to produce a well-defined stationary distribution, which represents an asymptotic probability distribution over the states.

To explore what the stationary distribution of adaptive play with mutations looks like, we introduce the following notation. An x -tree t on $\mathcal{S} = \mathcal{G} \times \mathcal{A}$ is a function $t : \mathcal{S} \rightarrow \mathcal{S}$ such that $t(x) = x$ and for all $s \neq x$ there exists an m with $t^m(s) = x$. The stationary distribution $\phi(\tau, \varepsilon)$ is characterized by

$$\phi_x(\tau, \varepsilon) = c(\tau, \varepsilon) \sum_{t \in H_x} \prod_{s \neq x} P_{st(s)}(\tau, \varepsilon), \quad (5)$$

where H_x is the set of x -trees on \mathcal{S} and $c(\tau, \varepsilon)$ is a continuous function in τ and ε . Notice that Foster and Young [9], Kandori, Mailath and Rob [15], and Ellison [7] use the same characterization of the stationary distribution. We refer to those papers and to Freidlin and Wentzell [10] for background material on the characterization.

Now consider the transition matrix $P(\tau, \varepsilon)$ describing adaptive behavior with mistakes. The transition probabilities $P_{s\tilde{s}}(\tau, \varepsilon)$, i.e., the entries of the transition matrix, are continuous in τ and ε . In fact, the transition probabilities are either zero or given by a polynomial in τ and ε . The constant term of such a polynomial is nonzero if and only if the transition $s \rightarrow \tilde{s}$ occurs with positive probability in the Markov process describing adaptive behavior without mistakes ($\tau = \varepsilon = 0$). Furthermore, since each time period only two players can alter their (potential) mutual link and adapt their actions, the polynomial is of τ -order and ε -order at most 1 and 2, respectively.

Because we are interested in small probabilities of τ -mistakes and ε -mistakes, we consider the asymptotic behavior of the stationary distribution $\phi(\tau, \varepsilon)$ as $\tau \rightarrow 0$ and $\varepsilon \rightarrow 0$. Henceforth, we denote the behavior of the stationary distribution $\phi(\tau, \varepsilon)$ as $\tau \rightarrow 0$ and $\varepsilon \rightarrow 0$ by $\lim_{\tau, \varepsilon \rightarrow 0} \phi(\tau, \varepsilon)$. In the analysis as stated in this section, we assume that whenever these limits are taken, τ and ε go to zero at the same rate, i.e.,

$$0 < \lim_{\tau, \varepsilon \rightarrow 0} \frac{\tau}{\varepsilon} < \infty. \quad (6)$$

The importance of this assumption will be discussed in detail in Section 5.

We write $r(z)$ for a successor of state z in the Markov process describing adaptive play without mistakes. Obviously, $r(z)$ is not uniquely determined since it depends on which pair of players can alter their (potential) mutual link and adapt their pure strategies. For this reason we write

$$D\left(A, \frac{a}{\gamma}\right) = \left\{ z \mid \lim_{m \rightarrow \infty} \Pr\left(r^m(z) = \left(A, \frac{a}{\gamma}\right)\right) = 1 \right\}$$

for the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$ -convention. Hence, the basin of attraction $D\left(A, \frac{a}{\gamma}\right)$ consists of the states that will eventually be taken to the $\left(A, \frac{a}{\gamma}\right)$ -convention by adaptive play without mistakes.

To determine what the stationary distribution looks like when mistake probabilities go to zero, we need the following two lemmas. Lemma 7 gives upper bounds for the number of τ -mistakes and ε -mistakes that are needed to enter the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$ -convention with positive probability.

Lemma 7 *Let $x \notin D\left(A, \frac{a}{\gamma}\right)$, $y_1, \dots, y_H \notin D\left(A, \frac{a}{\gamma}\right)$, and $y \in D\left(A, \frac{a}{\gamma}\right)$. There exists a transition from x to y via y_1, \dots, y_H such that*

$$P_{xy_1}(\tau, \varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h y_{h+1}}(\tau, \varepsilon) \cdot P_{y_H y}(\tau, \varepsilon) > 0, \quad (7)$$

where (7) is of τ -order and ε -order at most 0 and n , respectively.

The proof of Lemma 7 is included in Appendix B

Lemma 8 gives a lower bound for the total number of mistakes in a certain product of transition probabilities. All transition probabilities contained in the product refer to transitions which are specified by an x -tree t , with $x \notin D\left(A, \frac{a}{\gamma}\right)$, and which start from a state in the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$ -convention.

Lemma 8 *Let $n \geq 3$. The sum of the τ -order and ε -order of*

$$\prod_{s \in D\left(A, \frac{a}{\gamma}\right)} P_{st(s)}(\tau, \varepsilon) > 0,$$

with t an x -tree such that $x \notin D\left(A, \frac{a}{\gamma}\right)$, is at least $n + 1$.

Proof. First, consider the state $s = \left(A, \frac{a}{\gamma}\right) \in D\left(A, \frac{a}{\gamma}\right)$. Obviously, any path starting in the $\left(A, \frac{a}{\gamma}\right)$ -convention and eventually leaving $D\left(A, \frac{a}{\gamma}\right)$ contains at least one ε -mistake.

Second, consider a state s such that (i) every player $i \in N$ is linked with his two adjacent neighbors, i.e., every player $i \in N$ is linked with all players $j \neq i$ such that

$$\min\{|j - i|, n - |j - i|\} = 1,$$

and (ii) there is exactly one player i such that $a_i = B$. Obviously, such a state s is contained in $D\left(A, \frac{a}{\gamma}\right)$ when $n \geq 3$ and there exist exactly n different states that satisfy the conditions (i) and (ii). Furthermore, any path starting in such a state s and eventually leaving $D\left(A, \frac{a}{\gamma}\right)$ (or reaching another state satisfying conditions (i) and (ii)) contains at least 1 mistake (either a τ -mistake or an ε -mistake). Hence, we know that the sum of the number of τ -mistakes and ε -mistakes in $\prod_{s \in D\left(A, \frac{a}{\gamma}\right)} P_{st(s)}(\tau, \varepsilon)$ is at least $n + 1$. ■

The asymptotic stationary distribution gives the percentage of the time that the Markov process will spend in any state in the ultralong run. The following proposition states that the $\left(A, \frac{a}{\gamma}\right)$ -convention is the unique stochastically stable state, which means that when the two mistake probabilities go to zero at the same rate, adaptive play with mistakes will be in the $\left(A, \frac{a}{\gamma}\right)$ -convention almost all the time, provided that the population is sufficiently large. Notice that the structure of the proof of Theorem 9 is identical to the structure of the proof of part (a) of Theorem 1 in Ellison [7].

Theorem 9 *Let $n \geq 3$. If $0 < \lim_{\tau, \varepsilon \rightarrow 0} \frac{\tau}{\varepsilon} < \infty$, then $\lim_{\tau, \varepsilon \rightarrow 0} \phi_{\left(A, \frac{a}{\gamma}\right)}(\tau, \varepsilon) = 1$.*

The proof of Theorem 9 is contained in Appendix B.

The intuition behind the proposition above is similar to the corresponding results in Kandori, Mailath and Rob [15] and Young [22]. Namely, the $\left(A, \frac{a}{\gamma}\right)$ -convention or, equivalently, the risk-dominant convention is the state with the largest basin of attraction, meaning that the minimal numbers of η -mistakes and ε -mistakes needed to escape from its basin of attraction are larger than the minimal numbers of mistakes needed to leave the basin of attraction of any other state.

5 Discussion

We now discuss the robustness of the model presented in Section 2. In particular, we focus on alternative specifications of the stochastic and deterministic part of the dynamic process and discuss how this would influence the results.

The stochastic part of the dynamic process not only consists of the mistake probabilities, but also of the random selection of a pair of players. First, consider the selection of a pair of players. Notice that this is the only stochastic element of adaptive play without mistakes, as analyzed in Section 3. The presence of this random

element drives the convergence result as stated in Proposition 4. Without this random part of the model, i.e., with all players myopically updating links and pure strategies each period, cycles can not be excluded as shown by Ellison [7].

Random selection of a pair of players is not the only way to obtain a convergence result in a model without mistake probabilities. An alternative way to establish such a result is analyzed by Young [22], where the random element is originated in a sampling procedure by the players. Young [22] assumes that players base their decisions on limited information about the actions of other players in the recent past. More precisely, every player inspects k plays drawn without replacement from the most recent $m \geq k$ periods. A convergence result in the absence of mistake probabilities can be obtained in case the fraction $\frac{k}{m}$, which measures the completeness of the players' information, is small enough. In other words, convergence can only be ensured when the degree of randomness is sufficiently high. Results similar to those formulated in Proposition 4 and Proposition 9 could therefore still be obtained when we rephrase the present model such that it fits in the framework as analyzed by Young [22]. Alternatively, we could allow for the identified pair of players not only to alter their (potential) mutual link but also to possibly sever all other links they are involved in, without changing the results.

Second, consider the stochastic part of the dynamic process reflecting that players occasionally make mistakes when implementing their decisions. In Proposition 9, we assumed that τ and ε go to zero at the same rate. This assumption is necessary because we only showed that the total number of τ -mistakes and ε -mistakes in the numerator of (13) exceeds the number of ε -mistakes in its denominator. Recall that the denominator does not contain any τ -mistakes. In order to omit this assumption, we would have to show that both the number of τ -mistakes and the number ε -mistakes is higher in the numerator than in the denominator.

A related point of discussion is the following. The specification of the model states that we assume the two kinds of mistakes to be independent of the state of the process. Bergin and Lipman [3] show that this assumption is of 'crucial' importance, meaning that allowing for τ -mistake probabilities or ε -mistake probabilities to differ across states and in particular to go to zero at different rates would change the results significantly. Indeed, by choosing the mistake probabilities appropriately, the $\left(B, \frac{b}{\gamma}\right)$ -convention may even become the unique stochastically stable state. Van Damme and Weibull [6], however, restore the result that in 2×2 coordination games the risk-

dominant equilibrium will be selected in the long run by considering endogenously determined mistake probabilities. More precisely, they consider a model where players can control the probability of making a mistake at some costs. As players will try harder to avoid mistakes leading to larger payoff losses, the mistake probabilities depend on the state of the process. They show that mistake probabilities nevertheless go to zero at the same rate when the costs become negligible.

Finally, we want to make two remarks concerning the deterministic part of the dynamic process. First, instead of assuming that a player can observe the pure strategies of all potential links independently of whether or not he is linked with them, we could alternatively consider the case that a player is only able to observe the pure strategies of the players he is currently linked with. The decision to establish a new link could then be based on comparing the maintenance costs with the expected payoff in the coordination game, where the expected payoff is given by a player's average gross benefit in the last period. We conjecture that coexistence of conventions in the long run will no longer be possible when this specification of the deterministic part of the dynamic process is used. To motivate this point, notice that the state illustrated in Figure 1 will no longer be a steady state of adaptive play without mistakes. Namely, adjacent A -players and B -players will establish a link if given the opportunity. As can be concluded from the proof of Proposition 9, the result that the $(A, \frac{a}{\gamma})$ -convention is the unique stochastically stable state of adaptive play with mistakes remains true for this alternative specification.

Second, the deterministic part of the process is characterized by sequential decision making of the players. In the first stage of a period players are concerned with the network formation process and in the second stage players decide upon their pure strategies in the coordination game. Considering players who decide on links and pure strategies simultaneously would, however, not significantly change any of the results. Convergence to an absorbing state of adaptive play without mistakes can still be obtained because simultaneous decision making does not decrease the degree of randomness in the model. Furthermore, adaptive play with mistakes will still converge to the risk-dominant convention as the size of the basins of attraction does not change significantly.

6 Concluding Remarks

We have analyzed a large population coordination game where not only the actions of the players but also the communication network is subject to evolutionary pressure. Cost considerations of social interaction are incorporated by exploiting the spatial structure of the model. The costs of interacting increase in the distance between two players on the circle.

The Markov process describing medium-run behavior is shown to converge to an absorbing state, which may be characterized by coexistence of conventions. In the long run, when mistake probabilities are small but nonvanishing, coexistence of conventions is no longer possible as the risk-dominant convention is the unique stochastically stable state.

Interesting topics for future research are as follows. First, considering a larger class of games, see e.g. Kandori, Mailath and Rob [15] and Young [22], in the framework of the present paper. Second, extending the model to also allow for players who are only indirectly connected to play a game as is common practice in the literature on network formation.

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Appendices

A Proof of Lemmas

The proof of Proposition 4 is conducted through two lemmas. Recall that $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x . Furthermore, we define

$$\tilde{S} := \left\{ s = (g, a) \in \mathcal{S} : a_{i,k} \in \arg \max_{a_i} \pi_i(a_i, a_{\mathcal{L}_{i,t,k}}) \text{ for all } i \in N \right\},$$

i.e., the set \tilde{S} contains all states $s \in \mathcal{S}$ such that all players $i \in N$ choose an action that results in the highest possible payoff given the *current* actions of the players in their neighborhood.

Lemma 2 *Assume that $\lfloor \frac{n}{2} \rfloor > \max \left\{ \frac{a}{\gamma}, \frac{b}{\gamma} \right\}$. Then for any state $s \in \mathcal{S}$ there is a probability $p_s > 0$ that adaptive play without mistakes reaches a state $\tilde{s} \in \tilde{S}$ in $K_s < \infty$ periods.*

Proof. Consider a state $s \in \mathcal{S}$ and a player $i \in N$. With positive probability player i is matched with a player $j \in N$ such that $i \neq j$ and $c_{ij} > \max \left\{ \frac{a}{\gamma}, \frac{b}{\gamma} \right\}$. Notice that for every player $i \in N$, existence of such a player j is guaranteed because we assume that $\lfloor \frac{n}{2} \rfloor > \max \left\{ \frac{a}{\gamma}, \frac{b}{\gamma} \right\}$. Obviously, players i and j will never establish a mutual link. Furthermore, if a mutual link was already present, then it will definitely be severed. After that, both players will adapt their pure strategy and, as they no longer belong to each other's neighborhood, end up choosing a pure strategy that results in the highest possible benefit given the current pure strategies of the players in their neighborhood.

Consider the positive probability event of successively matching all players $i \in N$ in the way specified above. Notice that player i changing his pure strategy may imply that players in his neighborhood, who have already undergone the above procedure, no longer choose a pure strategy that results in the highest possible benefit given the current pure-strategy profile in their neighborhood. We restore this characteristic for the involved players by means of the positive probability event of once again matching all of them in the same pairs as before. Obviously, this may in turn result in other players, who have already undergone the above procedure, no longer playing optimally given the current pure-strategy profile in their neighborhood, and so on. It therefore remains to be shown that such a restoration process will terminate in finite time.

Suppose player i changes his pure strategy and becomes an A -player. As the restoration process does not include any link formation or link severance and $a - c > d - b$, the change by player i makes pure strategy A more attractive for all players involved in the restoration process. Consequently, all involved A -players will definitely

remain A -players, while involved B -players possibly become A -players. Since we have a finite number of players such a restoration process terminates in finite time. A similar argument holds when player i becomes a B -player. ■

Lemma 3 *For any state $s \in \tilde{S}$ there is a probability $q_s > 0$ that adaptive play without mistakes converges to an absorbing state in $L_s < \infty$ periods.*

Proof. Consider a state $s \in \tilde{S}$ and a player $i \in N$. Suppose player i is an A -player, i.e., $a_i = A$. With positive probability player i is successively matched in pairs with all players $j \in N$ such that $j \in \mathcal{L}_{i,k}$, $a_j = B$, and $c_{ij} > \frac{c}{\gamma}$. Since the costs c_{ij} are strictly larger than the payoff $\frac{c}{\gamma}$ to player i , all links between player i and these players j will be severed. Because $b > c$ this dynamic process of link severance implies that $\pi_i(A, a_{\mathcal{L}_{i,k,k-1}})$ decreases less than $\pi_i(B, a_{\mathcal{L}_{i,k,k-1}})$, causing player i to remain an A -player. Furthermore, all players j remain B -players (when their link with player i is severed) due to the fact that $a > d$ causes $\pi_j(B, a_{\mathcal{L}_{j,k,k-1}})$ to decrease less than $\pi_j(A, a_{\mathcal{L}_{j,k,k-1}})$.

We continue with the positive probability event that player i is successively matched in pairs with all players j such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} > \frac{a}{\gamma}$. Since costs c_{ij} exceed payoffs $\frac{a}{\gamma}$ for these pairs of players, all links between player i and players j will be severed. Notice that at some point during this dynamic process of link severance, player i may become a B -player because $a > d$, i.e., $\pi_i(A, a_{\mathcal{L}_{i,k,k-1}})$ decreases more than $\pi_i(B, a_{\mathcal{L}_{i,k,k-1}})$. First, consider the case that player i is an A -player when he severs the link with a player j . Since $a > d$, the severance of this link may cause player j to become a B -player. This in turn may cause that A -players in the neighborhood of player j also want to become B -players, and so on. It can easily be verified that successively matching all the involved players in pairs, which is a positive probability event, results in a dynamic process only consisting of players changing their pure strategy from A to B and possibly severing their mutual link. Obviously, such a dynamic process will terminate in a finite number of steps and result in a state contained in \tilde{S} . Notice that player i may also be one of the players who changes his pure strategy from A to B in the above dynamic process.

Second, consider the case that player i , because of one of the two reasons mentioned above, becomes a B -player while he is severing links with the players j . With positive probability player i is successively matched in pairs with all players $j \in N$ such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} > \frac{d}{\gamma}$. Since $a > d$, this includes all players j such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} > \frac{a}{\gamma}$. All these links will be severed because the costs c_{ij} are strictly larger than the payoff $\frac{d}{\gamma}$ to player i . Furthermore, player i and players j who are involved in this dynamic process will not change their pure strategy because $a > d$ and $b > c$, respectively.

The above argument shows that a player i , who is initially an A -player, will

become an A -player such that

$$\left\{ \begin{array}{l} \left[a_j = A \text{ and } c_{ij} > \frac{a}{\gamma} \right] \Rightarrow j \notin \mathcal{L}_i, \\ \left[a_j = B \text{ and } c_{ij} > \frac{c}{\gamma} \right] \Rightarrow j \notin \mathcal{L}_i, \end{array} \right. \quad (8)$$

for all $j \in N$ with $i \neq j$, or a B -player such that

$$\left\{ \begin{array}{l} \left[a_j = A \text{ and } c_{ij} > \frac{d}{\gamma} \right] \Rightarrow j \notin \mathcal{L}_i, \\ \left[a_j = B \text{ and } c_{ij} > \frac{b}{\gamma} \right] \Rightarrow j \notin \mathcal{L}_i, \end{array} \right. \quad (9)$$

for all $j \in N$ with $i \neq j$, with positive probability and in finite time. A similar argument can be used to show that this also holds for a player i who is initially a B -player. In fact, starting from any state $s \in \tilde{S}$, we attain a state $\tilde{s} \in \tilde{S}$ such that all players $i \in N$ simultaneously satisfy either (8) or (9) with positive probability and in finite time. Such a state can actually be attained by successively applying the above dynamic process to all players $i \in N$.

Notice that the interactive effects may cause that players at some point in time no longer satisfy (8) or (9) even though the dynamic process has already been applied to them. For these players the dynamic process has to be repeated. However, due to the fact that the dynamic process does not include any link formation and, in case no links are severed, only consists of players changing their pure strategy in the same way, this can only happen a finite number of times.

Now consider a state $\tilde{s} \in \tilde{S}$ such that all players satisfy either (8) or (9) and there exists a player $i \in N$ such that $a_i = A$. With positive probability player i is successively matched in pairs with all players $j \in N$ such that $j \notin \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} \leq \frac{a}{\gamma}$. Obviously, not only will all these pairs of players establish a link, but also will none of the involved players change his pure strategy. We continue with the positive probability event that player i is successively matched in pairs with all players $j \in N$ such that $j \notin \mathcal{L}_{i,k}$, $a_j = B$, and $c_{ij} \leq \frac{c}{\gamma}$. Again, all pairs of players will establish a link, which may possibly cause players to change their pure strategy. In fact, players i and players j may become B -players and A -players, respectively. As explained before, however, every time that link severance, or link formation for that matter, causes a player to change his pure strategy, the subsequent dynamic process of changing pure strategies by other players will terminate in finite time with positive probability. In case player i actually becomes a B -player, we continue with the positive probability event that player i is successively matched in pairs with all players j such that $j \notin \mathcal{L}_{i,k}$, $a_j = B$, and $c_{ij} \leq \frac{b}{\gamma}$, and all players j such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $d < \gamma \cdot c_{ij} \leq a$, which obviously only results in link formation and link severance, respectively.

The above argument shows that a player i , who is an A -player in a state $\tilde{s} \in \tilde{S}$,

will become an A -player such that

$$\begin{cases} a_j = A \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leq \frac{a}{\gamma} \right], \\ a_j = B \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leq \frac{\varepsilon}{\gamma} \right], \end{cases} \quad (10)$$

for all $j \in N$ with $i \neq j$, or a B -player such that

$$\begin{cases} a_j = A \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leq \frac{d}{\gamma} \right], \\ a_j = B \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leq \frac{b}{\gamma} \right], \end{cases} \quad (11)$$

for all $j \in N$ with $i \neq j$, with positive probability and in finite time. A similar argument can be used to show that this also holds for a player i who is a B -player in a state $\tilde{s} \in \tilde{S}$. Furthermore, starting from a state $\tilde{s} \in \tilde{S}$, we attain a state such that all players $i \in N$ simultaneously satisfy (10) or (11) with positive probability and in finite time. In that case, no single player wants to adapt his pure strategy or sever any of the links he is involved in, and no pair of players wants to establish a mutual link, which implies that we have attained an absorbing state of the Markov process.

Like before it may happen that in order to attain an absorbing state from a state $\tilde{s} \in \tilde{S}$, the dynamic process described above has to be applied repeatedly to some players. However, because of the following two reasons this can only happen a finite number of times. First, the part of the dynamic process dealing with link severance does not result in players changing their pure strategies. Second, parts of the dynamic process that do consist of players changing their pure strategies will always terminate in finite time with positive probability as they only involve changes in the same direction. ■

B Proof of Proposition 9

The proof of Proposition 9 is based on a characterization of the stationary distribution as used by e.g. Ellison [7], Foster and Young [9], and Kandori, Mailath and Rob [15].

Remark again that $P_{s\tilde{s}}(\tau, \varepsilon)$ either equals zero or is given by a polynomial in τ and ε whose constant term is nonzero if and only if the transition $s \rightarrow \tilde{s}$ occurs with positive probability in the Markov process describing adaptive behavior without mistakes ($\tau = \varepsilon = 0$). Write $r(z)$ for a successor of state z in the Markov process describing adaptive behavior without mistakes. Obviously, $r(z)$ is not uniquely determined since it depends on which pair of players can alter their (potential) mutual link and adapt their actions. For this reason we write

$$D\left(A, \frac{a}{\gamma}\right) = \left\{ z \mid \lim_{m \rightarrow \infty} \Pr\left(r^m(z) = \left(A, \frac{a}{\gamma}\right)\right) = 1 \right\}$$

for the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$ -convention. Hence, the basin of attraction $D\left(A, \frac{a}{\gamma}\right)$ consists of the states that will eventually be taken to the $\left(A, \frac{a}{\gamma}\right)$ -convention by adaptive play without mistakes. The proof of Proposition 9 is conducted through two lemmas.

Lemma 7 *Let $x \notin D\left(A, \frac{a}{\gamma}\right)$, $y_1, \dots, y_H \notin D\left(A, \frac{a}{\gamma}\right)$, and $y \in D\left(A, \frac{a}{\gamma}\right)$. There exists a transition from x to y via y_1, \dots, y_H such that*

$$P_{xy_1}(\tau, \varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h y_{h+1}}(\tau, \varepsilon) \cdot P_{y_H y}(\tau, \varepsilon) > 0, \quad (12)$$

where (12) is of τ -order and ε -order at most 0 and n , respectively.

Proof. To prove Lemma 7 it suffices to show that $D\left(A, \frac{a}{\gamma}\right)$ can be reached from any state with positive probability using at most n ε -mistakes. First, let the population size n be even. Consider the positive probability event of successively matching players i and $i+1$ in pairs, where $i = 2j - 1$ and $j = 1, \dots, \frac{n}{2}$. Every time a player has the opportunity to update his pure strategy, he chooses A . Obviously, this requires at most n ε -mistakes and leaves us with a state $y \in D\left(A, \frac{a}{\gamma}\right)$.

Second, let the population size n be odd. Consider the positive probability event of successively matching players i and $i+1$ in pairs, where $i = 2j - 1$ and $j = 1, \dots, \frac{n-1}{2}$. Again, all players involved in this positive probability event choose pure strategy A whenever they have the opportunity to update their strategy. This requires at most $n - 1$ ε -mistakes and leaves us with a state such that $a_i = A$ for all $i = 1, \dots, n - 1$. Now, we continue with the positive probability event of matching player n with an A -player $i^* \in \{1, \dots, n - 1\}$, where player i^* is an A -player who is linked to at least one other A -player. Notice that existence of such an A -player can always be established without the need for any additional mistakes. Then, at most one more ε -mistake is needed for player n to become an A -player, while player i will remain an A -player. Again, we have reached a state $y \in D\left(A, \frac{a}{\gamma}\right)$ using at most n ε -mistakes. ■

Lemma 7 gives a lower bound for the total number of mistakes in a certain product of transition probabilities. All transition probabilities contained in the product refer to transitions which are specified by an x -tree t , with $x \notin D\left(A, \frac{a}{\gamma}\right)$, and which start from a state in the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$ -convention.

Lemma 8 *Let $n \geq 3$. The sum of the η -order and ε -order of*

$$\prod_{s \in D(A, a)} P_{st(s)}(\eta, \varepsilon) > 0,$$

with t an x -tree such that $x \notin D\left(A, \frac{a}{\gamma}\right)$, is at least $n + 1$.

The asymptotic stationary distribution gives the percentage of the time that the Markov process will spend in any state in the ultralong run. The following proposition states that the $\left(A, \frac{a}{\gamma}\right)$ -convention is the unique stochastically stable state, which means that when the two mistake probabilities go to zero at the same rate, adaptive play with mistakes will be in the $\left(A, \frac{a}{\gamma}\right)$ -convention almost all the time, provided that the population is sufficiently large. Note that the proof is basically identical to the proof of part (a) of Theorem 1 of Ellison [7].

Proposition 9 *Let $n \geq 3$. If $0 < \lim_{\eta, \varepsilon \rightarrow 0} \frac{\eta}{\varepsilon} < \infty$, then $\lim_{\eta, \varepsilon \rightarrow 0} \phi_{(A, a)}(\eta, \varepsilon) = 1$.*

Proof. The characterization as specified by (5) allows us to express the quantity $\frac{\phi_x(\tau, \varepsilon)}{\phi_{\left(A, \frac{a}{\gamma}\right)}(\tau, \varepsilon)}$ as a ratio of polynomials in τ and ε for any state x . To prove Proposition 9 it is sufficient to show that for n sufficiently large, i.e., $n \geq 3$, it holds that

$$\lim_{\tau, \varepsilon \rightarrow 0} \frac{\phi_x(\tau, \varepsilon)}{\phi_{\left(A, \frac{a}{\gamma}\right)}(\tau, \varepsilon)} = 0$$

for all $x \neq \left(A, \frac{a}{\gamma}\right)$. This will follow if we demonstrate that for any x -tree t ($x \neq \left(A, \frac{a}{\gamma}\right)$) such that $\prod_{s \neq x} P_{st(s)}(\tau, \varepsilon) > 0$, we have

$$\lim_{\tau, \varepsilon \rightarrow 0} \frac{\prod_{s \neq x} P_{st(s)}(\tau, \varepsilon)}{\sum_{t' \in H_{\left(A, \frac{a}{\gamma}\right)}} \prod_{s \neq \left(A, \frac{a}{\gamma}\right)} P_{st'(s)}(\tau, \varepsilon)} = 0.$$

This in turn follows if we show that there exists an (A, a) -tree t' such that

$$\prod_{s \neq \left(A, \frac{a}{\gamma}\right)} P_{st'(s)}(\tau, \varepsilon) > 0$$

and

$$\lim_{\tau, \varepsilon \rightarrow 0} \frac{\prod_{s \neq x} P_{st(s)}(\tau, \varepsilon)}{\prod_{s \neq \left(A, \frac{a}{\gamma}\right)} P_{st'(s)}(\tau, \varepsilon)} = 0. \quad (13)$$

We show that (13) holds by distinguishing two cases.

First, assume that $x \in D\left(A, \frac{a}{\gamma}\right)$. Define t' by

$$t'(z) = \begin{cases} r(z) & \text{if } z \in D\left(A, \frac{a}{\gamma}\right), \\ t(z) & \text{otherwise.} \end{cases}$$

Notice that t' is an $\left(A, \frac{a}{\gamma}\right)$ -tree because for any state z the path described by t' initially coincides with t and hence eventually enters $D\left(A, \frac{a}{\gamma}\right)$. From the first point at which $t'^m(z) \in D\left(A, \frac{a}{\gamma}\right)$, the tree maps every point to a successor according to adaptive play without mistakes and hence reaches $\left(A, \frac{a}{\gamma}\right)$. In this case the ratio specified in (13) equals

$$\frac{P_{(A, \frac{a}{\gamma})t((A, \frac{a}{\gamma}))}(\tau, \varepsilon) \prod_{s \in D(A, \frac{a}{\gamma}) - \{(A, \frac{a}{\gamma}), x\}} P_{st(s)}(\tau, \varepsilon)}{P_{xr(x)}(\tau, \varepsilon) \prod_{s \in D(A, \frac{a}{\gamma}) - \{(A, \frac{a}{\gamma}), x\}} P_{sr(s)}(\tau, \varepsilon)}.$$

The above expression converges to 0 as $\tau, \varepsilon \rightarrow 0$ because $P_{st(s)}(\tau, \varepsilon) / P_{sr(s)}(\tau, \varepsilon)$ is bounded, $P_{(A, \frac{a}{\gamma})t((A, \frac{a}{\gamma}))}(\tau, \varepsilon) \rightarrow 0$, and $P_{xr(x)}(\tau, \varepsilon) \rightarrow \xi > 0$.

Second, assume that $x \notin D\left(A, \frac{a}{\gamma}\right)$. Define t' by

$$t'(z) = \begin{cases} r(z) & \text{if } z \in D\left(A, \frac{a}{\gamma}\right), \\ t(z) & \text{if } z \notin D\left(A, \frac{a}{\gamma}\right), z \neq x, \text{ and } z \neq y_1, \dots, y_H, \\ y_1 & \text{if } z = x, \\ y_2 & \text{if } z = y_1, \\ \dots & \dots \\ y_H & \text{if } z = y_{H-1}, \\ y & \text{if } z = y_H, \end{cases}$$

where $y_1, \dots, y_H \notin D\left(A, \frac{a}{\gamma}\right)$ and $y \in D\left(A, \frac{a}{\gamma}\right)$ are as specified in Lemma 5.5. Obviously, t' is again an $\left(A, \frac{a}{\gamma}\right)$ -tree. In this case the ratio specified in (13) equals

$$\frac{\prod_{s \in D(A, \frac{a}{\gamma})} P_{st(s)}(\tau, \varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h t(y_h)}(\tau, \varepsilon)}{P_{xy_1}(\tau, \varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h y_{h+1}}(\tau, \varepsilon) \cdot P_{y_H y}(\tau, \varepsilon) \cdot \prod_{s \in D(A, \frac{a}{\gamma}) - \{(A, \frac{a}{\gamma})\}} P_{sr(s)}(\tau, \varepsilon)}.$$

Because of Lemma 7 and the fact that $P_{sr(s)}(\tau, \varepsilon) \rightarrow \mu > 0$, the denominator is of τ -order 0 and of ε -order at most n . According to Lemma 8, the numerator is of a higher total order if $n \geq 3$. Consequently, the above expression converges to 0 as long as $0 < \lim_{\tau, \varepsilon \rightarrow 0} \frac{\tau}{\varepsilon} < \infty$. \blacksquare