Networks of Collaboration in Oligopoly

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Abstract

In an oligopoly, prior to choosing quantities/prices, each ¯rm has an opportunity to form pair-wise collaborative links with other ¯rms. These pair-wise links lower costs of production of the ¯rms which form a link. The collection of pair-wise links de¯nes a collaboration network. We study stable and e±cient networks under di®erent types of market competition.

We ¯nd that except under extreme competition, a la Bertrand, ¯rms have an incentive to collaborate with their competitors to lower costs of production. We ¯nd that two simple architectures, the complete network, where every ¯rm has a collaboration link with every other ¯rm, and the network with a dominant group, which contains a large number of completely connected ¯rms and several isolated ¯rms, are stable under di®erent market conditions. We also observe that stable networks are often e±cient from a social point of view.

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1 Introduction

Firms often collaborate with each other to share information on market conditions/new technologies as well as to jointly bear the cost of common facilities. These collaborative arrangements typically strengthen the competitive position of the ¯rms involved in the collaboration and weaken the position of the ¯rms outside the collaboration. Thus inter-¯rm collaborations have important e®ects on the functioning of the market. In this paper we study the incentives for ¯rms to engage in collaborative arrangements and their aggregate welfare implications.

We develop a simple model to study these issues. Consider a oligopoly with symmetric ¯rm; each ¯rm has an opportunity to form pair-wise collaborative links with other ¯rms. These pair-wise links lower costs of production of ¯rms which form a link. The collection of pair-wise links de¯nes a collaboration network and induces a distribution of costs across the ¯rms in the industry. The ¯rms then compete in the market. We study the nature of stable collaboration networks under di®erent types of market competition.

The distinctive feature of our approach is that we allow for intransitive structures of collaboration. Thus, for example, it is possible that ¯rm 1 has a collaborative relationship with ¯rms 2 and 3, respectively, but that ¯rms 2 and 3 do not have any collaborative relationship. [1] Allowing for intransitive structures opens up a very rich class of collaboration arrangements and also requires novel methods of analysis.

We start with a consideration of the textbook oligopoly setting: there are n ¯rms,[2] demand is

[1] Such intransitive relationships are commonly observed in practice, both with regard to sharing common facilities as well as with regard to research and development activities. We give some examples of the latter. For instance, Raychem Corporation has collaboration relationships with General Signal Corporation (ATM Forum) and Whirlpool Corporation (NAHB Research Foundation), respectively, but General Signal Corporation and Whirlpool Corporation do not have any collaborative links. Similarly, Hubbell Inc. has collaboration relationships with Cooper Industries Inc. (NAHB Research Foundation) and Reliance Electric Co. (Corporation for Open Systems International), respectively, but Cooper Industries Inc. and Reliance Electric Co. have no collaboration relations with each other. We thank Nicholas Vonortas, for providing these, and other, instances of intransitive collaboration relationships from his NCRA-RJV database.

[2] There are $2^{n(n-1)/2}$ possible networks. Thus, if $n = 8$ then there are 2^{28} possible networks of collaboration!

linear in price, and initially ¯rms are symmetric, with zero ¯xed costs and identical marginal costs of production. We assume that pair-wise links lower this marginal cost. In this setting, we ¯nd that price competition leads ¯rms to form no collaborative links, yielding the empty network, while quantity competition leads every pair of ¯rms to form a link, thus generating the complete network. These networks are given in Figure 1. We also ¯nd that under price competition, every network with two or more fully connected ¯rms is e±cient, while with Cournot competition, the complete network is the unique e±cient network. These results suggests that the nature of market competition has an important e®ect on the type of collaboration networks that arise, and that this has a bearing on the level of welfare as well.

The above results for quantity competition are obtained under the assumption that the marginal costs of a ¯rm are declining linearly in the number of its collaboration links. We also examine the case of non-linearly decreasing marginal costs. We ¯nd that if marginal cost decrease is a decreasing function of the number of links, then connected but intransitive networks of collaboration are stable. If, on the other hand, marginal cost decrease is an increasing function of the the number of links, then the complete network is stable. In addition, networks with a large dominant group and several isolated ¯rms are also stable. Our results on intransitive stable networks rely on diminishing returns from link formation and, therefore, suggests that such patterns arise quite naturally. This ¯nding is important since one of the motivations for the study of networks is precisely the possibility of modeling intransitive relationships. The rest of the paper explores the relationship between competition and collaboration in more general settings.

We approach the general problem as follows. A set of collaborative links de¯nes a network, which in turn generates a vector of costs for the di®erent ¯rms. Given these costs, ¯rms compete in the product market. Thus for a ¯xed type of competition, we can de¯ne the corresponding payo®s of the ¯rms for any given network. We model di®erent types of competition in terms of restrictions on the payo® functions. In particular, we consider two types of competition: aggressive and moderate.

Under aggressive competition, all but the lowest cost ¯rms make zero pro¯ts. This allows for two subcases of interest: one, in which a lowest cost ¯rm makes pro¯ts only if it is the unique such firm and two, if all lowest cost firms make positive profits. The first possibility corresponds to the standard Bertrand competition under general homogeneous demand. In this case we find that, the unique stable network is the empty network (Theorem 4.1). In the latter case we provide a complete characterization of stable networks: for markets with four or more firms, a network is stable if and only if it consists of one non-singleton complete component of size $k \in \{3, 4, \cdots, n\}$ and $n - k$ singleton components (Theorem 4.2). This collaboration architecture resembles the familiar dominant cartel and fringe firms structure. Figure 2 illustrates the set of stable networks in this case, for a market with 5 firms.

Under moderate competition, all firms make positive profits, but lower cost firms make larger profits. This case accommodates quantity competition under general homogeneous or differentiated demand, and price competition under differentiated demand. In this setting, stable networks possess greater variety and richer structure and it is difficult to characterize them. We first show that every firm with the same costs must be directly linked in a stable network. Thus, every pair of firms with the same costs levels must be linked in a stable network. We develop sufficient conditions for the complete network to be the unique stable network (Theorem 4.3). We find that these conditions, though strong, are satisfied by a variety of standard models such as those mentioned above.

These results are in marked contrast to the results obtained by other authors (we discuss this literature below). One important assumption we make is that there are no spillovers across collaboration links of firms. This motivates an enquiry into the role of spillovers/externalities across collaboration links. The network structure allows us to define the `distance' between firms. We suppose that the extent of spillovers is inversely related to the `distance' between the firms in a collaboration network. Our analysis focusses on positive spillovers. Our results on aggressive competition extend to this setting easily. We, therefore, focus on the case of moderate competition. We find that the complete network is a stable network in the presence of positive spillovers. However, in the linear demand model with quantity competition, partially connected networks are also stable. A comparison of this finding with our above results suggests that, under moderate competition, spillovers may have the effect of lowering the level of collaboration.

This paper is a contribution to the study of group formation and cooperation in oligopolies. Our model of collaborative networks is inspired by the recent work on strategic models of network formation; see e.g., Aumann and Myerson (1989), Bala and Goyal (1999), Goyal (1993), Goyal and Joshi (1999a, 1999b), Jackson and Wolinsky (1996), Jackson and Watts (1998), and Kranton and Minehart (1998). To the best of our knowledge, the present paper is the ¯rst application of network games to the study of collaboration among oligopolistic ¯rms.

Issues relating to group formation and cooperation have long been a central concern of economic theory, and game theory in particular. The traditional approach to these issues is in terms of coalitions. In recent years, there has been considerable work on coalition formation in games; see e.g., Bloch (1995,1996), Ray and Vohra (1997, 1998), and Yi (1997,1998). For a survey of this work, refer to Bloch (1998). One application of this theory is to the formation of groups in oligopolies. In this literature, group formation is modeled in terms of a coalition structure which is a partition of the set of ¯rms. Each ¯rm therefore, can belong to one and only one element of the partition, referred to as a coalition.

In our paper, we consider two-player relationships. In this sense, our model is somewhat restrictive as compared to the work referred to above, which allows for groups of arbitrary size. However, the principal distinction concerns the nature of collaboration structures we examine. Our approach accommodates collaborative relations that are non-transitive. From a conceptual point of view, this distinction is substantive. It means that we allow for relationships across coalitions. Thus, we consider a class of cooperative structures which is signi¯cantly di®erent from those studied in the coalition formation literature.

The network approach also leads to quite di®erent predictions concerning the nature of collaboration among ¯rms. Bloch (1995,1996) develops a sequential coalition unanimity game in which ¯rms propose coalitions and a coalition is formed only if every member of a proposed coalition agrees to become a member. Each ¯rm's marginal cost is linearly declining in the size of the coalition of which it is a member. After coalitions are formed, the ¯rms compete as Cournot/Bertrand oligopolists in a di®erentiated market with homogeneous demand. Bloch demonstrates that generally there is a unique stable coalition structure in which ¯rms are divided into two unequal groups. By contrast, we ¯nd that the complete network, where every ¯rm has a collaborative link with every other ¯rm, is always stable. The arguments underlying these result exploit the possibility of intransitive relationships.

Yi and Shin (1995) and Yi (1998) propose a simultaneous open membership game in which all players announce their decision to form coalitions at the same time and non-members cannot be excluded from joining a coalition. They obtain the grand coalition as the stable outcome of the open membership game. Their approach is akin to a game in which the decision to join a coalition is one-sided. In such a game, in the presence of perfect spillovers, a member of a smaller group always has an incentive to join a larger group. In our paper, by contrast, link-formation is based on pair-wise incentive compatibility considerations, and it is therefore interesting to observe that a grand coalition can be obtained in such a setting also. Thus our results on complete networks (Theorems 4.2 and 4.3) provide an alternative explanation as to how a grand coalition may emerge.

Our paper is also related to the literature on cooperative R&D in oligopoly; see e.g., d`Apremont and Jacquemin (1988), Katz (1986), Leahy and Neary (1998), Suzumura (1992). This literature considers a two stage process: in the ¯rst stage, ¯rms choose the intensity of their R&D e®ort. This R&D lowers their cost of production. In the second stage, they compete in the market by choosing quantities/prices. The R&D e®ort of a ¯rm has positive spillovers: it helps in lowering the costs of all other ¯rms. Thus, these spillovers generate an externality. The literature examines the role of cooperative R&D in resolving the incentive problems arising out of the externality. In particular, existing work compares the level of R&D e®ort under two di®erent situations. The ¯rst situation is the pure non-cooperative model, where both R&D e®ort as well as the strategy in the market stage is non-cooperatively chosen. The second situation is a mixed one: the R&D is chosen in a cooperative manner so as to maximize the joint pro¯ts of ¯rms, while the strategies at the market stage are chosen in a non-cooperative manner. In the latter case, it is assumed that the ¯rms form a grand coalition. Thus, in this literature the group sizes are exogenously speci¯ed.

Our paper makes two contributions to this literature. The ¯rst contribution is the formulation of spillovers. In our model, spillovers accrue only in the event of collaboration and are therefore not industry wide, as is the case in this literature. In particular, we allow for the extent of spillovers to be related to the 'distance' between the firms in a collaboration network. The network framework permits a natural definition of the distance. This allows us to model the idea that firms that are 'far apart' receive lower spillovers as compared to firms that are 'close' in the network. The second contribution of our paper pertains to the study of stable networks. In the existing literature, the group structure is usually exogenously specified. By contrast, we allow for collaboration structures to be endogenously determined and study the nature of stable networks.

The model is presented in Section 2. In Section 3, we present two examples relating to network formation in the case of price and quantity competition, respectively. This motivates the general analysis of network formation in oligopoly, which is presented in Section 4. We discuss extensions – to allow positive spillovers, for fixed costs of link formation, and asymmetric firms – and some conceptual issues in Section 5, while Section 6 concludes.

2 The Model

We consider a setting in which a set of firms first choose their collaboration links with other firms. These collaboration agreements are pair-wise and help lower marginal costs of production. The firms then compete in the product market by choosing either quantities or prices. We are interested in the network of collaboration that emerges in this setting. We now develop the required terminology and provide some definitions.

2.1 Networks

Let $N = \{1, 2, \ldots, n\}$ denote a finite set of ex-ante identical firms. To avoid trivialities, we shall assume that $n \geq 3$. For any $i, j \in N$, the pair-wise relationship between the two firms is captured by a binary variable, $g_{i,j} \in \{0, 1\}$; $g_{i,j} = 1$ means that a direct link is established between firms i and j while $g_{i,j} = 0$ means that no direct link is formed. By definition, $g_{i,i} = 1$ and $g_{i,j} = g_{j,i}$ $\forall i, j \in N$. A network, $g = \{(g_{i,j})_{i,j \in N}\}$, is a formal description of the pair-wise collaboration relationships that exist between the firms in N. We let G

denote the set of all networks. Two special cases are the complete network, g^c, in which $g_{i,j} = 1 \ \forall i, j \in N$, and the empty network, g^e, in which $g_{i,j} = 0 \ \forall i, j \in N$, $i \neq j$. Let $g + g_{i,j}$ denote the network obtained by replacing $g_{i,j} = 0$ in network g by $g_{i,j} = 1$. Similarly, let $g - g_{i,j}$ denote the network obtained by replacing $g_{i,j} = 1$ in network g by $g_{i,j} = 0$.

Given a network g, let $N(g) = \{i \in N : \exists j \neq i \ s.t. \ g_{i,j} = 1\}$. Each firm in $N(g)$ has at least one direct link to another distinct firm in the network g. Therefore, $N(g^c) = n$ and $N(g^e) = \emptyset$. We will let $|N(g)|$ denote the cardinality of $N(g)$. A path in g connecting firms i and j is a distinct set of firms $\{i_1, \ldots, i_n\} \subset N(g)$ such that $g_{i,i_1} = g_{i_1,i_2} = g_{i_2,i_3} = \cdots = g_{i_n,j} = 1$. Given any two firms i and j, let $d_{ij}(g)$ denote the number of links in the shortest path between i and j in the network g. We refer to $d_{ij}(g)$ as the geodesic distance between firms i and j in g. We shall use the convention that $d_{ij}(g) = \infty$ if there exists no path between i and j in g. For instance, $d_{ij}(g^c) = 1$ and $d_{ij}(g^e) = \infty \ \forall i, j \in N$. We say that a network is connected if there exists a path between any pair $i, j \in N$.

Given a network, g, let $N_i(g) = \{j \in N : j \neq i \ s.t. \ g_{i,j} = 1\}$ be the set of firms with whom firm i has a direct collaboration link. Let $\eta_i(g, 1)$ denote the cardinality of $N_i(g)$. In general, let $\eta_i(g, k)$ denote the number of firms who are at a geodesic distance of k from firm i.

A network, $g' \subset g$, is a component of g if for all $i, j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j, and for all $i \in N(g')$ and $j \in N(g)$, $g_{i,j} = 1$ implies $g_{i,j} \in g'$. Generally, in a component g' with three or more agents, there will exist agents i and j such that $d_{ij}(g') \geq 2$. We shall say that a component $g' \subset g$ is complete if $g_{i,j} = 1$ for all $i, j \in N(g')$.

2.2 Collaboration Links and Cost Reduction

A collaboration link in our framework can be interpreted in different ways. One possible interpretation is that firms form collaborations to share the costs of a common facility. The facility may involve some large fixed costs and, therefore, the collaboration generates economies of scale which lowers costs of production of the collaborating firm. A second interpretation is that firms have an agreement to jointly invest in cost-reducing R&D activity.

We suppose that firms are initially symmetric, with zero fixed costs and identical marginal costs. Collaborations lower marginal costs of production. We analyze the network formation process under various specifications of the marginal cost function. In the basic model, we use the following linear function:

$$c_i(g) = \gamma_0 - \gamma \eta_i(g;1) \, ; \quad i \in N. \tag{1}$$

where γ_0 is a positive parameter representing a firm's marginal cost when it has no links. In this case, firm i's marginal costs are linearly declining in the number of direct links it has with other firms.

When cost-reducing activity takes the form of capacity-sharing agreements, gains from co-operation may decrease due to congestion as a firm forms additional links. In this case marginal cost is a decreasing convex function of the number of direct links. Alternatively, it is also possible that benefits from cooperation increase as a firm forms additional links. For instance, a larger number of links between firms aids standardization of the product with ensuing gains from network externalities. In this formulation, marginal cost is a decreasing concave function of the number of links. The following formulation accomodates these different possibilities.

$$c_i(g) = c(\eta_i(g;1)); c(\eta_i(g;1) + 1) < c(\eta_i(g;1)); \quad i \in N. \tag{2}$$

We assume throughout that the extent of cost reduction via collaborations is exogenously specified. This simplifies the analysis and allows us to focus on the structure of stable networks under oligopolistic competition. To check for robustness of our findings, we briefly examine the impact of fixed costs of link formation in Section 5.2.

2.3 Payoffs

Given a network g, firm i's cost for producing an output, q_i, is given by the following cost function showing constant returns to scale in output:

$$C_i(g; q_i) = c_i(g)q_i \tag{3}$$

where $c_i(g)$ is the marginal cost of production as a function of the network of collaboration links. To rule out uninteresting cases, we shall always suppose that $c_i(g) \geq 0$, $\forall i \in N$, $\forall g \in G$. A network g, therefore, induces a marginal cost vector for the firms which is given by $c(g) = \{c_1(g); c_2(g); \ldots; c_n(g)\}$. Given this cost vector, and the specification of the demand functions in the product market, the firms compete in the second stage as either Cournot or Bertrand oligopolists. For every network g, we assume there is a well-defined Nash equilibrium in the second stage product market game. The profits of firm i in this equilibrium are given by $\pi_i(g)$.[3]

2.4 Stable and Efficient Networks

A network g is stable if for all $i; j \in N$:

(i) $\pi_i(g) > \pi_i(g - g_{i;j})$ and $\pi_j(g) > \pi_j(g - g_{i;j})$

(ii) if $\pi_i(g + g_{i;j}) > \pi_i(g)$, then $\pi_j(g + g_{i;j}) \leq \pi_j(g)$

In words, in a stable network, any firm that is directly linked to another has a strict incentive to maintain the link and any two firms that are not directly linked have no strict incentive to form a direct link with each other. The above definition of stability is inspired by a related notion of stability presented in Jackson and Wolinsky (1996). We discuss the two different definitions in Section 5.4 below.

This definition of stability reflects the idea that a link is formed if and only if both firms forming the link. It implicitly incorporates the view that link formation may involve small costs: thus individual firms will only form a link if such a link generates strictly positive profits. The second idea is that of the absence of transfers: we suppose that there are no

[3] This implicitly assumes that there are no coordination problems of choosing across different equilibria at this stage.

transfers possible across links. Taken together with the idea of small positive costs of link formation, this implies that both firms must make strictly greater profits, by forming a link.

The requirements above are very weak and should be seen as necessary conditions for a network to be stable. One of the points of our analysis is that these weak requirements provide sufficient structure in an interesting class of network games.

In order to study efficient networks, we need to consider aggregate welfare. For any network g, this is defined as the sum of consumer surplus and aggregate profits of the n firms. We shall say that a network g^* is efficient if $W(g^*) \geq W(g)$, for all $g \in G$.

3 Homogeneous Product Oligopoly

In this section, we analyze the nature of collaboration among firms in a homogeneous product oligopoly, i.e., a market where the outputs of the firms are perfect substitutes. In particular, we restrict attention to linear inverse market demand:[4]

$$p = \alpha - \sum_{i \in N} q_i; \quad \alpha > 0 \tag{4}$$

The profits of the firms depend on the nature of market competition. In the following subsections, we will consider both price and quantity competition.

3.1 Linear Marginal Costs

In this subsection, we assume that marginal costs are linearly decreasing in the number of links that a firm has. Formally, the marginal cost structure is given by (1). To ensure that all firms make positive profits we shall assume that $\alpha > 3\gamma_0$ and $\gamma_0 \geq (n-1)\gamma$. We start with the case of Bertrand competition. Given a network g, what are the payoffs of different firms under Bertrand competition? Standard considerations (exploiting the idea of a finite

[4] We analyze the general oligopoly model in Section 4.

price grid) allow us to state that there exists an equilibrium, and in this equilibrium a firm will make profits only if it is the unique minimal cost firm in the market. In other words:

$$\pi_i(g) = 0; \text{ if } c_i(g) \geq c_j(g); \text{ for } i \neq j; \ \pi_i(g) > 0; \text{ if } c_i(g) < c_j(g); \forall j \neq i: \tag{5}$$

Since g is arbitrary, the above expression allows us to specify the payoffs for all possible networks. What are the stable networks of collaboration in this setting of extreme competition? The following result provides a complete answer to this question:

Proposition 3.1 Suppose there is price competition among the firms. If demand satisfies (4) and the marginal cost function satisfies (1), then the empty network, g^e, is the unique stable network.

Proof Consider some non-empty network g. There are two possibilities. First, there is some firm $i \in N$ which is the unique lowest cost firm. But this implies that firm i must have at least two links since all firms are ex-ante identical. However, since firm i is the unique lowest cost firm, all other firms make zero profits. In particular, consider $j \neq i$ such that $g_{i;j} = 1$. For this firm, condition (i) of stability is violated since $\pi_j(g) = \pi_j(g - g_{i;j}) = 0$. Hence, firm i cannot be uniquely minimal cost in a stable network.

The second possibility, given that links are bilateral, is that one or more pairs of firms have minimal cost. Let $i; j \in N$ be two firms with minimal costs. Under price competition both firms make zero profits. If these firms would delete their links they would still make zero profits. Thus $\pi_i(g) = \pi_i(g - g_{i;j}) = 0$. This once again violates condition (i) of stability.

Thus the only candidate for a stable network is g^e. Condition (i) is trivially satisfied since there are no links to sever. In the network $g^e + g_{i;j}$, there are two lowest cost firms, i and j. From (5), it follows that both firms will get a payoff of zero. Thus condition (ii) is satisfied. This completes the proof.

$\square$

The arguments in this proof are very general; in particular, we do not make use of the linear structure of the demand or the cost function. This suggests that the absence of collaborative links is likely to obtain in general settings where competition is extreme (see Section 4.1).

Next, we turn to Cournot competition between the firms. We start by defining the payoffs in the quantity competition game. Given any network g, the Cournot equilibrium output can be written as:

$$q_i(g) = \frac{(\alpha - \gamma_0) + n\gamma\eta_i(g;1) - \gamma\sum_{j\neq i}\eta_j(g;1)}{(n+1)}; \quad i \in N \tag{6}$$

This implies that aggregate Cournot output, for a given g, is:

$$Q(g) = \sum_{i\in N} q_i(g) = \frac{n(\alpha - \gamma_0) + \gamma\sum_{i\in N}\eta_i(g;1)}{(n+1)} \tag{7}$$

The second stage Cournot profits for firm $i \in N$ are given by $\Pi_i(g) = q_i^2(g)$. In our study of stable networks, we will find it convenient to use a positive monotone transform of the firm's profits to write the payoffs as follows:

$$\pi_i(g) = (\alpha - \gamma_0) + n\gamma\eta_i(g;1) - \gamma\sum_{j\neq i}\eta_j(g;1); \quad i \in N \tag{8}$$

Our restrictions on the parameters ensures that each firm produces a positive quantity in the Cournot game. We can now characterize the stable collaboration networks under quantity competition.

Proposition 3.2 Suppose there is quantity competition among the firms. If demand satisfies (4) and the marginal cost function satisfies (1), then the complete network, g^c, is the unique stable network.

Proof We first show that g^c is stable. In g^c; $\eta_i(g^c; 1) = n - 1$; $\forall i \in N$. Therefore, firm i has a marginal cost of $\gamma_0 - \gamma(n-1)$ and payoff of:

$$\pi_i(g^c) = (\alpha - \gamma_0) + \gamma(n-1) \tag{9}$$

There are no links to add so condition (ii) of stability is automatically satisfied. We check condition (i) next. Suppose we set $g_{i;j} = 0$ for some pair i and j. In the ensuing network, $g^c - g_{i;j}$, the payoff to i is given by:

$$\pi_i(g^c - g_{i;j}) = \alpha - (n-1)[\gamma_0 - \gamma(n-2)] + (n-2)[\gamma_0 - \gamma(n-1)] = (\alpha - \gamma_0) \tag{10}$$

The payoff to firm j is identical. There is no incentive to delete link $g_{i;j} = 1$ since $\pi_i(g^c) - \pi_i(g^c - g_{i;j}) = \gamma(n-1) > 0$.

We now show that g^c is the unique stable network. Consider a stable network $g \neq g^c$. Then, there exists a pair of firms $i; j \in N$ with $g_{i;j} = 0$. We show that both i and j are strictly better off by forming a link. In the network, $g + g_{i;j}$, the payoff to firm i is given by:

$$\pi_i(g + g_{i;j}) = (\alpha - \gamma_0) + n\gamma\eta_i(g + g_{i;j}; 1) - \gamma\eta_j(g + g_{i;j}; 1) - \gamma \sum_{k \neq i;j} \eta_k(g + g_{i;j}; 1)) \tag{11}$$

Note that $\eta_l(g + g_{i;j}; 1) = \eta_l(g; 1) + 1$ for $l = i; j$ and $\eta_k(g + g_{i;j}; 1) = \eta_k(g; 1)$ for $k \neq i; j$. Therefore, $\pi_i(g + g_{i;j}) - \pi_i(g) = \gamma(n-1) > 0$. An identical argument establishes that for firm j, $\pi_j(g + g_{i;j}) - \pi_j(g) = \gamma(n-1) > 0$. Thus, condition (ii) is violated and g is not stable, a contradiction.

$\square$

The intuition behind this result is as follows. First note that if two firms form a link then the costs of all other firms are unaffected, while the cost advantage to both firms forming a link is the same under (1). An inspection of the profit expression in (8) reveals that the positive effects on the profits of a firm i from a link with another firm j is given by $n\gamma$, while the negative effects are given by γ. Thus link formation is clearly profit enhancing. This argument shows that any network other than the complete network cannot be stable. To see why the complete network is stable note that no further links can be added, while the deletion of a link by a firm i, with (say) firm j only increases the costs of firm i and j but leaves the costs of all other firms unaffected, lowering profits of firm i by $(n - 1)\gamma$. Thus it is not profitable to delete links either. This completes the argument.

It is interesting to compare our result with that of Bloch [3, Proposition 2] who, under a similar specification of demand and marginal cost, derives a stable coalition structure consisting of two asymmetrically-sized coalitions in which the number of firms in the larger coalition is the integer closest to $3(n + 1)/4$. To explain this sharp difference in our results, consider some firm i who belongs to a complete component, g', with $k \geq 2$ firms. In the network framework, since collaboration links can be intransitive, firm i can initiate a link with some firm $j \notin N(g')$ if it is profitable to do so. Given (1) and (4), Proposition 3.2 shows that i and j always establish a link because it yields a net gain of $(n - 1)\gamma > 0$.

In the coalition framework of Bloch [3, 4], however, collaboration links are, by assumption, transitive; therefore, i can form a link with j if and only if all other k firms in the same coalition as i agree to merge with the singleton coalition $\{j\}$. However, it may be no longer profitable for firm i to have a collaboration link with j in the coalition framework where, under a merger of coalitions, all other k firms will have a link with j as well. In this case, each of the $k \neq i$ firms in the coalition experiences a reduction in marginal cost of γ. Further, firm j experiences a reduction of $(k + 1)\gamma$ in its marginal costs because of its $k + 1$ additional links after the merger. Therefore, the payoff to firm i changes by $(n - 1)\gamma - 2k\gamma$. If i is already a part of a large coalition ($k > (n - 1)/2$), then it may not want to be part of a merger with the singleton $\{j\}$.

3.2 Non-linear Marginal Costs

In this section we examine non-linear reductions in marginal cost. For price competition, the empty network continues to be the unique stable network even if we allow for a general marginal cost function. Therefore, in the following analysis, we restrict attention to Cournot competition. Here the analysis is more interesting, since the relative decrease in costs matters in the incentives to form links. Our analysis suggests that incomplete and intransitive networks arise naturally in this setting.

Recall that under Cournot competition, when the inverse demand is given by (4), the Cournot equilibrium output for a generally specified marginal cost vector, $c(g)$ is given by:

$$q_i(g) = \frac{\alpha - nc_i(g) + \sum_{k \neq i} c_k(g)}{(n+1)}; \quad i \in N \tag{12}$$

The Cournot profits of firm i are $q_i^2(g)$. Under a monotonic transform, we can write the payoff of i as $\pi_i(g) = (n+1)q_i(g)$.

We consider the following simple case: a firm gains from forming collaboration links only if it has a small number of links, after a critical number of links have been reached, there are no further gains from forming additional links.[5]

$$\begin{aligned} c_i(g) &= \gamma_0 - \gamma\eta_i(g;1); \quad \eta_i(g;1) \leq k^* \\ c_i(g) &= \gamma_0 - \gamma k^*; \quad \eta_i(g;1) > k^* \end{aligned} \tag{13}$$

Note that (1) is a special case of (13) for $k^* \geq n-1$. Therefore, in the following we assume that $k^* < n-1$. We can now characterize the structure of stable networks.

Proposition 3.3 Suppose there is quantity competition among the firms and demand and marginal costs are specified by (4) and (13) respectively. (a) In the class of connected networks, every stable network is incomplete. A connected network in which every firm has exactly k^* links is stable. (b) If $n/2 \leq k^* < n-1$, then a stable unconnected network can consist of at most two components. The network with one complete component with exactly

[5] The costs functions we consider in this section are analogous to those considered by Bloch [3, Assumption 3]. Our results hold under somewhat more general specifications. But using analogous specifications helps us in clarifying the implications of the network approach.

$k^* + 1$ firms and one complete component with $n - (k^* + 1)$ firms is stable. (c) If $k^* < n/2$, then a stable unconnected network can consist of at most $[n/(k^* + 1)]$ components, where $[x]$ denotes the smallest integer exceeding a real number x.

The proof is given in Appendix A. Figure 3 gives examples of stable networks under convex costs. The first part of the result is particularly interesting, since it illustrates that intransitive networks can be stable. In fact, in the set of connected networks, since $k^* < n - 1$ by assumption, the first part of the above proposition implies that that all stable networks must be intransitive. The intuition behind this result is simple: each firm lowers costs for every additional link if and only if it has fewer than k^* links. Deleting a link lowers profits; this follows from arguments in Proposition 3.2, while forming additional links is at best worthless, since there is no cost reduction from links over and above k^* links.[6]

Next, we consider the case where cost reduction requires a certain minimum number of links. We specify the following functional form:

$$\begin{aligned} c_i(g) &= \gamma_0 - \gamma; & \eta_i(g;1) \leq k^* \\ c_i(g) &= \gamma_0 - \gamma(\eta_i(g;1) - k^* + 1); & \eta_i(g;1) \geq k^* \end{aligned} \tag{14}$$

Note that (1) follows as a special case of (14) when $k^* = 0$. So, in the following discussion, we assume that $k^* \geq 1$. The next result provides a complete characterization of stable networks for concave costs.

Proposition 3.4 Let $1 \leq k^* < n - 1$. Suppose there is quantity competition among the firms. If demand satisfies (4) and marginal costs satisfy (14) respectively, then: (a) In the set of connected networks, the only stable network is the complete network. (b) In the set of unconnected networks, the empty network is stable. Further, all other stable unconnected networks are of the following kind: there is one complete component with at least $k^* + 1$ firms and all other firms constitute singleton components.

[6] This result does not depend on the specific functional form we have assumed. Similar results hold under more general convex specifications. The details of this derivation are available from the authors upon request.

The proof is given in Appendix A. Figure 4 shows the set of stable networks in a market with $n = 5$ and $k^* = 3$. Given that $k^* < n - 1$, the stability of the complete network follows from arguments in Proposition 3.2. The stability of the empty network follows by noting that forming only one link is not worthwhile. Consider non-empty but incomplete networks next. First, note that since there are no benefits to having fewer than k^* links, every firm in a non-singleton component must have at least k^* links. This implies one, that every non-singleton component must have at least $k^* + 1$ firms. Second, it implies from arguments in Proposition 3.2 that every pair of firms in a non-singleton component must be linked, i.e. the component must be complete. These arguments constitute the proof of the proposition. The second part of the proof shows how the delimited set of networks is stable.

3.3 Efficient Networks

In this section, we study efficient networks under price and quantity competition. On the demand side, we restrict attention to the linear specification given by (4). On the cost side, however, the analysis is relatively general and accommodates the various specifications of marginal cost listed under (1), (13) and (14).

We now examine the nature of efficient networks, under price competition. Let $\underline{c}$ be the minimum cost attainable by a firm in any network. Under (1), and (14), this is achieved when a firm has $(n - 1)$ links, while under (13) it is achieved if a firm has at least k^* links.[7] The following result provides a complete characterization of efficient networks.

Proposition 3.5 Suppose there is price competition among the firms. If demand satisfies (4) and the marginal cost function satisfies (2), then a network g is efficient if and only if there are two or more firms which attain the minimum cost, $\underline{c}$.[8]

[7] The arguments given below can be extended in a straightforward manner to the case of spillovers, which are discussed later in section 5.1.

[8] This result also holds in the case of convex and concave decreasing costs considered in section 3.2 above. For expositional simplicity, we do not mention these cases in the proof.

Proof Fix some network g. Let firm i be a minimum cost firm in this network and let its cost be given by $c_i(g) > \underline{c}$. Let equilibrium price be given by $p(g)$. Under price competition, it follows that $p(g) \geq c_i(g)$. Hence the consumer surplus is given by $1/2[\alpha - p(g)]^2$, while the profits of firms are bounded above by $[p(g) - c_i(g)][\alpha - p(g)]$. Thus social welfare in a network g is bounded above by the expression:

$$\hat{W}(g) = \frac{[\alpha - p(g)]^2}{2} + [p(g) - c_i(g)][\alpha - p(g)] \tag{15}$$

It is easily seen that this expression is strictly declining with respect to $p(g)$ so long as $p(g) > c_i(g)$. Thus for a network g, the potential social welfare is bounded above by the expression, $[\alpha - c_i(g)]^2/2$.

It is easily checked that this maximum potential social welfare is decreasing in $c_i(g)$ and is, therefore, maximized when the price in the market is equal to $\underline{c}$. Thus social welfare is maximized when the product is produced and sold at the minimum marginal cost, $\underline{c}$.

Note that if there is only one firm with this minimum cost, then under price competition it will charge a price higher than $\underline{c}$, and earn positive profits in equilibrium. If there are two or more firms with this minimum cost, then price competition will force the firms to charge this minimum cost. Thus two or more firms are necessary as well as sufficient for the market price to be equal to the minimum cost level. This completes the proof.

$\square$

In the case of price competition, we observe a conflict between stability and efficiency in networks. The stability result indicates that, irrespective of the specification of marginal cost, no firm has any incentive to form a link with another. Efficiency, on the other hand, dictates a connected network under (1) and (14); under (13), efficiency requires either a connected network, or an unconnected network in which at least one component g' satisfies $|N(g')| \geq k^*$.

We now consider the nature of efficient networks under quantity competition. Let $c(0)$ denote the marginal cost of a firm with no links and $c(n-1)$ the marginal cost with $(n-1)$

links. To ensure that all firms produce a strictly positive output in the Cournot equilibrium corresponding to any network, we will maintain the restriction that $\alpha > 3nc(0)$. Social welfare is defined as:

$$W(g) = \frac{1}{2}Q^2(g) + \sum_{i \in N} q_i^2(g) \tag{16}$$

We shall consider a general class of marginal cost functions that satisfy (2). This formulation accommodates the linear specification of marginal cost given by (1). Further, as long as each firm has at least k^* links, it also covers concave marginal costs specified by (14). For all these cases, the following proposition shows that adding a link in any arbitrary $g \neq g^c$ strictly increases social welfare implying thereby that g^c is uniquely efficient.[9]

Proposition 3.6 Suppose there is quantity competition. If demand and cost satisfy (4) and (2) respectively, then the complete network is the unique efficient network.

Proof Consider any network $g \neq g^c$ with $g_{i,j} = 0$ for some $i, j \in N$. Letting $\Lambda \equiv \sum_{k \in \{i,j\}} \{c_k(g) - c_k(g+g_{i,j})\}$, it follows that $\Delta q_l(g) \equiv q_l(g+g_{i,j}) - q_l(g) = \{c_l(g) - c_l(g+g_{i,j})\} - \Lambda/(n+1)$, $l \in \{i,j\}$. Further, for any firm $k \notin \{i,j\}$, $\Delta q_k(g) = -\Lambda/(n+1)$. Therefore, the change in aggregate profits is given by:

$$\sum_{h \in N}\left[q_h^2(g+g_{i,j}) - q_h^2(g)\right] = \sum_{l \in \{i,j\}}\left[2q_l(g) + \{c_l(g) - c_l(g+g_{i,j})\} - \frac{\Lambda}{(n+1)}\right]\Delta q_l(g) - \sum_{k \notin \{i,j\}}\left[2q_k(g) - \frac{\Lambda}{(n+1)}\right]\Delta q_k(g) \tag{17}$$

[9] The case of concave marginal costs needs to be qualified because if all firms have less than k^* links, then adding another link may not strictly increase social welfare. But such networks cannot be efficient because social welfare will strictly increase with each additional link by virtue of Proposition 3.6 once all firms have at least k^* links. If $(n-1)$ firms have k^* or more links and firm i has less than k^* links, then any link of i with some $j \neq i$ will only reduce the marginal cost of i. This case can be covered similar to Corollary 3.1 which follows Proposition 3.6. We have bunched the concave case with specification (1) because, in contrast to the convex case, it has the complete network as uniquely efficient.

The change in consumer surplus, $\Delta CS(g) \equiv CS(g + g_{i;j}) - CS(g)$, is given by:

$$\Delta CS(g) = \frac{1}{2}\left[2Q(g) + \frac{\Lambda}{(n+1)}\right]\left[\frac{\Lambda}{(n+1)}\right] \tag{18}$$

Therefore, to show that $W(g + g_{i;j}) - W(g) > 0$, it suffices to show that:

$$Q(g)\frac{\Lambda}{(n+1)} + 2\sum_{l \in \{i;j\}} q_l(g)\left[\{c_l(g) - c_l(g + g_{i;j})\} - \frac{\Lambda}{(n+1)}\right] - 2\sum_{k \notin \{i;j\}} q_k(g)\frac{\Lambda}{(n+1)} > 0 \tag{19}$$

After some manipulation, we can simplify (19) to:

$$\sum_{l \in \{i;j\}} q_l(g)\left[(2n+2)\{c_l(g) - c_l(g + g_{i;j})\} - \Lambda\right] > \sum_{k \notin \{i;j\}} q_k(g)\Lambda \tag{20}$$

Theoretically, the lowest output produced by any firm is when it is a singleton and all other firms belong to a complete component. This is given by:

$$\underline{q} = \frac{\alpha - nc(0) + (n-1)c(n-1)}{(n+1)} \tag{21}$$

Similarly, theoretically the largest possible output that can be produced by any firm is when its marginal cost is minimum at $c(n-1)$ and all other firms are singletons with the highest marginal cost of $c(0)$. This is given by:

$$\overline{q} = \frac{\alpha - nc(n-1) + (n-1)c(0)}{(n+1)} \tag{22}$$

Therefore, (20) holds for any arbitrary network if it holds when $q_l(g) = \underline{q}$ for $l \in \{i;j\}$ and $q_k(g) = \overline{q}$ for $k \notin \{i;j\}$. Substituting (21) and (22) into (20), it follows that aggregate welfare strictly increases with the addition of the link $g_{i;j} = 1$ if $(2n - \eta_i(g;1) - \eta_j(g;1))\underline{q} > (n - \eta_i(g;1) - \eta_j(g;1) - 2)\overline{q}$. For this, it suffices to show that $\alpha(n+2) > [3n^2 - 3n - 2(\eta_i(g;1) + \eta_j(g;1)) + (2 + \eta_i(g;1) + \eta_j(g;1))]c(0)$ which is true under our assumption that $\alpha > 3nc(0)$.

△

In the case where marginal cost is specified by (13), it is possible that an additional link only reduces the marginal cost of one firm (which has less than k^* links) and not the collaborator (which has more than k^* links). The following corollary adds to Proposition 3.6 by demonstrating that social welfare increases strictly when an additional link strictly decreases the marginal cost of just one firm while leaving all other $(n - 1)$ marginal costs unaffected.

Corollary 3.1 Suppose demand is specified by (4) and marginal cost is specified by (13). Under quantity competition, any network in which each firm has at least k^* links is efficient.

Proof Consider a network $g \neq g^c$ in which $g_{i;j} = 0$ for some $i; j \in N$. If the link $g_{i;j} = 1$ strictly decreases the marginal cost of both firms, then Proposition 3.6 implies that $W(g + g_{i;j}) > W(g)$. Now suppose that $c_i(g + g_{i;j}) < c_i(g)$ but $c_k(g + g_{i;j}) = c_k(g)$ $\forall k \neq i$. Letting $\Lambda \equiv c_i(g) - c_i(g + g_{i;j})$, an argument identical to the one in Proposition 3.6 establishes that $W(g + g_{i;j}) > W(g)$ if $(2n + 1)q_i(g) > \sum_{k \neq i} q_k(g)$. Recalling (21) and (22), it follows that $W(g + g_{i;j}) > W(g)$ for any arbitrary network g if $(2n + 1)\underline{q} > (n - 1)\overline{q}$. This is equivalent to showing $\alpha(n + 2) + c(n - 1)[(2n + 1)(n - 1) + n(n - 1)] > c(0)[(2n + 1)n + (n - 1)^2]$ which is true under our parametric restriction $\alpha > 3nc(0)$.

△

Our results on quantity competition in networks do not indicate the sharp divergence between stability and efficiency that is exhibited under price competition in networks as well as in the literature on formation of coalitions under price and quantity competition. When marginal costs are linear, then the complete network is both uniquely stable and uniquely efficient in the class of all networks. With concave marginal costs, the complete network is stable and uniquely efficient; therefore, the set of efficient networks is a proper subset of the set of stable networks. When marginal costs are convex, the set of stable networks and the set of efficient networks have in common all networks in which each firm has exactly k^* links.

4 Network Formation under General Payoffs

Our analysis of the Bertrand and Cournot models of market competition under homogeneous linear inverse demand suggests that the nature of market competition has a major influence on the structure of networks that we should expect to see. We now analyze the robustness of this finding under more general conditions on the nature of demand, the cost function, and types of market competition.

4.1 Collaboration under Aggressive Competition

In this subsection, we characterize the structure of stable networks under aggressive competition. The notion of aggressive competition should be seen as a generalization of Bertrand competition for a homogeneous good. We shall say that competition among firms is aggressive if all but the lowest cost firms make zero profits. There are two sub-cases: one, the lowest cost firm makes positive profits only if it is the unique such firm, and two, all the lowest cost firms make positive profits. The former case is written as follows:

Assumption B Fix some g. If $c_i(g) \geq c_j(g)$, then $\pi_i(g) = 0$, while if $c_i(g) < c_j(g)$ for all $j \in N \setminus \{i\}$ then $\pi_i(g) > 0$.

This specification generalizes the Bertrand competition of Section 3 to allow for general demand functions and also general cost reduction functions. We can now state our first general result on the architecture of stable networks in an oligopoly.

Theorem 4.1 Suppose the marginal cost function satisfies (2) and the payoff function satisfies (B). Then no collaboration links are formed by firms and the unique stable network is the empty network, g^e.

The proof of this result is essentially the same as the proof of Proposition 3.1 and therefore omitted.

We now analyze the case where all lowest cost firms make positive profits. This case may be formally written as follows:

Assumption AC Fix some g. If $c_i(g) > c_j(g)$, then $\pi_i(g) = 0$, while if $c_i(g) \leq c_j(g)$ for all $j \in N \setminus \{i\}$ then $\pi_i(g) > 0$.

By way of motivation, consider a set of firms that are competing to apply for a patent for a cost reducing process technology. Suppose that each of the firms has some useful complementary knowledge. If they collaborate then this knowledge can be jointly used to lower costs. Moreover, only the lowest cost technology is patented. Once the patent is available, it is randomly allotted to one of the firms who have the lowest cost technology. Price competition then ensures that only this firm makes profits. The positive profits mentioned above then should be seen as the (ex-ante) expected profits from collaboration.

In our analysis we shall use the following symmetry assumption with respect to the lowest cost firms.

Assumption SY1 Fix some g. Suppose that for a pair of firms i and j, $c_i(g) = c_j(g) = \min_{k \in N} c_k(g)$. (i) If $g_{i,j} = 0$ then $\pi_i(g + g_{i,j}) > \pi_i(g) > 0$ and $\pi_j(g + g_{i,j}) > \pi_j(g) > 0$. (ii) If $g_{i,j} = 1$ then $\pi_i(g - g_{i,j}) < \pi_i(g)$ and $\pi_j(g - g_{i,j}) < \pi_j(g)$.

In words, the first condition says that if in g two firms have minimum costs and they are not connected directly, then they get strictly greater payoffs if they form a direct link. It is immediate that such a direct link will lower the costs of only these two firms and thus improve their competitive position relative to the rest of the firms. It seems natural then that their payoffs should also increase. Hence the two firms that form a link will still remain the minimum cost firms and will also gain competitive advantage since their costs will go down more as compared to the others firms, who may be linked to them directly or indirectly.

The second condition says that in a network g, if two minimum cost firms have a link then this link is strictly advantageous, in the sense that deleting this link will strictly lower the payoffs of the firms. The reasoning behind this condition is analogous to the first condition.

Symmetry in the presence of aggressive competition has strong implications for collaboration. This is demonstrated in the following result.

Theorem 4.2 Let $n \geq 4$. Suppose (AC) and (SY1) hold and marginal cost is specified by (2). Then a network is stable if and only if it has the following structure: there is a complete component with $k \in \{3, 4, \cdots, n\}$ firms and all the other $n - k$ firms constitute singleton components.

The number of stable networks is very small as compared to the number of total networks. For example, when n is 3, 4, 5 or 6, the total number of networks is given by 8, 64, 1024 and 32768. By contrast, the number of stable networks is given by 3, 5, 16, and 42. Thus the two simple requirements of stability lead to a strong restriction on the class of networks.

The argument in the proof of this theorem proceeds as follows: first we show that any non-singleton component in a stable network must be complete. In proving this property, we also establish that all firms in a non-singleton component must have the same costs and that these costs must be the minimum in the given network. Second, we show that there can be at most one non-singleton component in a stable network. These two properties reduce the set of candidates for stable networks dramatically. The last step then completes the characterization. The proof builds on two lemmas.[10]

Lemma 4.1 Let g be a stable network. Then every non-singleton component in g is complete.

Proof Suppose that g is a stable network and $g' \subset g$ is a non-singleton component of g. We show that g' must be complete. We know that no unique firm can have the lowest cost in g'; this follows from an argument as in the first part of Proposition 3.1. Thus, there must

[10] The above result is stated for $n \geq 4$. It is easily seen that in case of $n = 3$ an analogous result obtains: a stable network is either complete or has two components, one component with two firms and the other component with a singleton firm. We have stated the result for $n \geq 4$ as it allows for a simpler statement.

exist at least a pair of firms $i, j \in N$ such that $c_i(g) = c_j(g) = \min_{k \in N} c_k(g)$. Consider any other firm $l \in N(g')$, $l \neq i, j$. If such a firm has $c_l(g) > c_i(g)$, then under (AC), clearly this cannot be uniquely optimal for the firm. For instance, firm l can delete a link $g_{l,k} = 1$ and retain zero profits. Hence, all firms in g' must have the same costs, and these costs must be minimum. Thus, $c_j(g) = \min_{k \in N(g)} c_k(g)$ $\forall j \in N(g')$. Finally, if $i, j \in N(g')$ are not connected, then under Assumption SY1(i), they can do strictly better by forming a direct link. Thus g' must be complete.

$\Box$

We next characterize the number of non-singleton components.

Lemma 4.2 In a stable network g, there can be at most one non-singleton component.

Proof Suppose there are two non-singleton components, g' and g'' and let firm $i \in N(g')$ and that firm $j \in N(g'')$. From the proof of Lemma 4.1 we know that firms i and j are minimum cost firms. It now follows from Assumption SY1(i), that these firms can do strictly better by forming a link. This violates condition (ii) in the definition of stability. Thus g is not a stable, a contradiction. This shows that a stable network cannot have more than one non-singleton component.

$\Box$

We have shown that in a market with four or more firms there can be at most one non-singleton component, and that it is complete. This means that the only candidates for stable networks are networks of the following form: there is a complete component with $k \geq 1$ firms and there are $n - k$ singleton components. The proof of the theorem shows that networks with $k = 1$ and $k = 2$ are not stable, while the networks with $k \geq 3$ are stable.

Proof of Theorem 4.2 The candidates for stable networks can be parameterized in terms of the size of the non-singleton component, k. Given the ex-ante symmetry of firms, Assumption SY1(i) immediately implies that a network with $k = 1$ cannot be stable. Next consider $k = 2$. This is a network with one component with 2 firms and (since $n \geq 4$) at least 2 singleton components. Given speci¯cation (2), it follows that if the two singleton ¯rms form a link then they have will have the same costs as the two ¯rms already in the 2 ¯rm component. Under Assumption (AC) this yields them positive payo®s, violating requirement (ii) in the de¯nition of stability. Thus any network g with k = 2 is not stable. We are left with networks where k ¸ 3. In such a network every ¯rm i in the non-singleton component is a minimum cost ¯rm, with (say) marginal cost $c_i(g)$. Under speci¯cation (2), it follows that $c_i(g) < c_j(g)$, for all ¯rms j which are singleton components. Thus under assumption (AC), $¼_i(g) > 0$ and $¼_j(g) = 0$. Now suppose a ¯rm j forms a link with another ¯rm i. Then the marginal cost of the former ¯rm will fall still further and under (2) will remain below the marginal cost of ¯rm j. Thus ¯rm j has no incentive to form such a link. Since k ¸ 3, and competition is speci¯ed by assumption (AC), it is also clear that two singleton component ¯rms j and k do not have an incentive to form a link either. Finally, using assumption (SY1(ii)), it follows that ¯rms in the non-singleton component have no incentive to delete a link. We have thus shown that both requirements (i) and (ii) are satis¯ed for any network with the structure: a non-singleton complete component with k ¸ 3 ¯rms and n ¡ k singleton ¯rms. This completes the proof.

4

4.2 Collaboration under Moderate Competition

We now consider a market in which competition is such that all ¯rms, irrespective of their costs, make positive pro¯ts. However, lower cost ¯rms make higher pro¯ts. Such a situation is described as moderate competition. Formally, this situation is captured in the following assumption:

Assumption MC Fix some g. $¼_i(g) > 0$ for all i 2 N; $¼_i(g) = ¼_j(g)$ if $c_i(g) = c_j(g)$, while $¼_i(g) > ¼_j(g)$ if $c_i(g) < c_j(g)$.

The next assumption concerns the payo®s of similar cost ¯rms and is a stronger version of Assumption SY1, stated in the previous section.

Assumption SY2 Fix some g. Suppose that for a pair of firms i and j, $c_i(g) = c_j(g)$. (i) If $g_{i,j} = 0$ then $\pi_i(g + g_{i,j}) > \pi_i(g) > 0$ and $\pi_j(g + g_{i,j}) > \pi_j(g) > 0$. (ii) If $g_{i,j} = 1$ then $\pi_i(g - g_{i,j}) < \pi_i(g)$ and $\pi_j(g - g_{i,j}) < \pi_j(g)$.

Essentially we require the conditions mentioned in the earlier assumption to hold for all symmetrically located firms and not just the minimum cost firms. We note that this assumption implicitly incorporates the idea of moderate competition: for example, part (i) cannot be satisfied under aggressive competition, for a pair of high cost firms. Symmetry in the presence of moderate competition implies the following property of stable networks.

Proposition 4.1 Suppose that (SY2) and (2) hold. Consider a stable network, g. If $\eta_i(g; 1) = \eta_j(g; 1)$, then $g_{i,j} = 1$.

Proof Let g be stable. If $\eta_i(g; 1) = \eta_j(g; 1) = n$, then by definition $g_{i,j} = 1$. Therefore, consider the case where $\eta_i(g; 1) = \eta_j(g; 1) < n$ and $g_{i,j} = 0$. Under (2) the costs of i and j are identical if $\eta_i(g; 1) = \eta_j(g; 1)$. Under assumption (SY2)(i), it follows that $\pi_i(g + g_{i,j}) > \pi_i(g)$ and $\pi_j(g + g_{i,j}) > \pi_j(g)$. This violates requirement (ii) of stability and contradicts the hypothesis that g is stable.

$\square$

Remark: We note here that (SY1) is not sufficient for the conclusion of Proposition 4.1. In our earlier result on completeness of components, Lemma 4.1, we used assumption (AC) in addition to (SY1).

Proposition 4.1 has several interesting implications for the nature of stable networks. The first implication is that a stable network cannot have two or more singleton components. This implies in particular that the empty network cannot be stable. The second implication is that the star/hub-spokes network is not stable. This is because in all these networks, there are at least two firms i and j who have the same number of direct links but $g_{i,j} = 0$. By Proposition 4.1, such firms have an incentive to form a direct link. A third implication of this result is that if a stable network contains two or more complete components then they must be of unequal size.

In general many networks can be stable. We now examine some properties of stable networks. The first question pertains to the set of symmetric stable networks. The result above implies that if all firms have the same cost, then every pair of firms must be directly linked; thus, the only candidate for stability is the complete network.

Corollary 4.1 Suppose that (SY2) and (2) hold. Then the unique symmetric stable network is the complete network, g^c.

The above results leave open the issue of existence of stable networks. The next result shows that the set of stable networks is non-empty. It also provides conditions under which there is a unique stable network.

Theorem 4.3 Suppose that hypotheses (MC) and (SY2) hold. Then the complete network, g^c, is stable. If in addition, for every network g and any link $g_{i,j} = 0$ it is true that $\pi_i(g + g_{i,j}) > \pi_i(g)$ and $\pi_j(g + g_{i,j}) > \pi_j(g)$ then the complete network, g^c, is the unique stable network.

Proof We provide a proof of the first statement. The second statement is immediate and a proof is omitted. In g^c, $\eta_i(g^c; 1) = n - 1$; $\forall i \in N$. Therefore, all firms have the same cost and this is the minimum cost. There are no links to add so requirement (ii) of stability is automatically satisfied. We check requirement (i) next. Suppose we set $g_{i,j} = 0$ for some pair i and j. In the ensuing network, $g^c - g_{i,j}$, assumption (SY2)(ii) implies that both firms i and j loose strictly. This implies that requirement (i) is also satisfied. Thus g^c is stable.

$\square$

The additional monotonicity condition in Theorem 4.3 may seem strong. However, it is satisfied by Cournot oligopoly under fairly general demand conditions. Suppose that the inverse demand, $p(Q)$, satisfies the following general specification: $p(Q)$ is a twice continuously differentiable function with $p'(Q) < 0$ and $p''(Q) \le 0$. We show that if inverse demand satisfies this condition, then the additional monotonicity condition on profits of the firms is also satisfied. The details are in Appendix B. It is easily verified that this condition is also satisfied by the standard model of a differentiated oligopoly with linear demand and linearly reducing costs (as in(1)).[11]

Finally, we note that the monotonicity condition in Theorem 4.3 is also satisfied in the case where each of the firms is a monopoly in its own market. This is true since the only `costs' of forming links in our model arise out of the greater competitiveness of a firm whose costs are lowered. However, if the other firms are in unrelated markets then there is no `cost' to forming additional links while there are benefits in terms of of lowering marginal costs of production. It is then immediate that in such a case every pair of firms has an incentive to form links and thus the unique stable network is the complete network. This finding supports the general argument in the paper: collaboration among firms is easier when market competition is mild.

We briefly comment on the number of components under general costs conditions. We know from Proposition 4.1 that complete components in stable network are of unequal size. This allows us to derive an upper bound on the number of complete components in a market with a fixed number of firms. The idea here is that minimum number of firms needed to support k unequal components is given by $k(k+1)/2$. This implies that for a fixed number of firms, n, the maximum number of complete components possible in a stable network is given by the largest number k that satisfied the inequality $k(k+1)/2 \leq n$. This implies, for instance, that in a market with 10 firms there are at most 4 complete components.

5 Discussion

In this section we briefly discuss the role of some assumptions in our analysis.

[11] The calculations are available from the authors upon request.

5.1 Spillovers

In the analysis so far, we have restricted attention to the case where there are no spillovers across the collaborative links of firms. We found that the nature of stable collaborative arrangements differ considerably from the findings in the literature on coalition formation. An important assumption in our analysis has been the absence of spillovers and in this section we examine if this is crucial for our results. The analysis is brief and our results are quite incomplete. However, they serve to illustrate two points: one, that the complete network is stable so long as spillovers are positive but imperfect and two, that incomplete networks can also be stable in the presence of spillovers.

In principle, it is possible that the collaborative links of a collaborator will also have some influence on the benefits that a firm can expect from the joint R&D activity. These indirect effects can be negative (when resources can be diverted into competing collaborations) or positive (if there are cross-collaboration knowledge spillovers). We follow the literature (for instance, d'Aspremont and Jacquemin [7], Kamien, Muller and Zang [13], Suzumura [21] and Leahy and Neary [16]) in considering the case of positive spillovers in our analysis.

Let $\delta \in [0,1]$ be a parameter measuring the extent of spillovers. The case of $\delta = 0$ corresponds to zero spillovers. The extent of spillovers is increasing in δ and is perfect when $\delta = 1$. The effects of indirect collaborations are inversely related to the distance between two firms, in a network. An example of a simple cost function which reflects this is:

$$c_i(g) = \gamma_0 - \gamma \left[\eta_i(g;1) + \delta \eta_i(g;2) + \dots + \delta^{n-2} \eta_i(g;n-1) \right], \quad i \in N \tag{23}$$

If there is no path between two firms in a given network, then the distance between them is ∞, and there are thus no spillovers. In our specification, spillovers only occur if two firms i and j are either directly or indirectly connected.

It is relatively straightforward to extend the arguments for Bertrand competition and more generally, aggressive competition, to cover the case of positive spillovers. The same results on stable and efficient networks obtain. In what follows, we will therefore focus on the case of quantity competition. In this setting, the e®ects of spillovers are substantive. The complete network remains stable under positive spillovers; however, even in the basic linear demand model with linear cost reduction, other networks can be stable. Broadly speaking, this suggests that spillovers have a negative e®ect on the incentives for collaborative relationships.

We start by stating a fairly general result on the stability of the complete network under moderate competition and in the presence of positive spillovers.

Proposition 5.1 Suppose (SY2) holds and marginal cost is speci¯ed by (23). Then the complete network, g^c, is stable.

The proof of this proposition follows along the lines of Theorem 4.3, and is omitted.

We have been unable to obtain a complete characterization of stable networks in the presence of spillovers. To get some intuition into the e®ects of spillovers, we consider an example. This example illustrates that positive spillovers can lead to less collaboration under moderate competition. Recall from Proposition 3.2 that the complete network is the unique stable network when demand is linear and marginal cost is speci¯ed by (1). We now show that with positive spillovers speci¯ed quite generally by (23), in addition to the complete network, some incomplete networks can also be stable.

Example: The impact of spillovers Let $n = 10$. Suppose the demand is linear as in Section 3, and let the cost reduction function satisfy (23). Moreover, ¯rms compete in quantities. We show that a network g with two complete components, one with 8 ¯rms and another with 2 ¯rms is stable. Let the ¯rms in the ¯rst component be numbered from 1 to 8 while ¯rms 9 and 10 belong to the second component. Figure 5 below gives an example of such a network.

We begin by showing that no ¯rm in this network has an incentive to delete links. The condition of all ¯rms in component 1 is symmetric. The payo® to ¯rm 1 in network g is given by $[(® ¡ 10c_1(g) + \mathbf{P}_{j \neq 1} c_j(g))=11]^2$. The payo® to ¯rm 1, from the network $g ¡ g_{1;2}$ is given by $[(® ¡ 10c_1(g ¡ g_{1;2}) + \mathbf{P}_{j \neq 1} c_j(g ¡ g_{1;2}))=11]^2$. Using the fact that these components in g are complete, it follows that ¯rm 1 looses payo® by deleting the link $g_{1;2}$. Identical arguments apply in the case of ¯rms 9 and 10. Thus requirement (i) is satis¯ed by g. We now check the incentives of ¯rms to form additional links.

Suppose without loss of generality that ¯rms 1 and 9 form link. The payo® to ¯rm 1 is given by $[(\alpha - 10c_1(g+g_{1;9}) + \sum_{j \neq 1} c_j(g+c_{1;9}))/11]^2$. Consider, for the sake of argument, the case of perfect spillovers, i.e., where $\delta = 1$. In this case, the payo® to ¯rm 1 in the network $g + g_{1;2}$ is given by $[(\alpha - (\gamma_0 - 9\gamma))/11]^2$. The payo® of ¯rm 1 under g is given by $[(\alpha - (\gamma_0 - 19\gamma))/11]^2$. It is then immediate that ¯rm 1 looses payo® by forming the link $g_{1;2}$. Since payo®s are continuous with respect to the spillover parameter, δ, the strict inequality also obtains for δ close to 1. Given the symmetry of ¯rms location in component 1, no ¯rm in this component has an incentive to form a link with a ¯rm in component 2. Thus requirement (ii) is also satis¯ed and the network g is stable.

4

5.2 Fixed Costs of Link Formation

In our analysis, we have assumed that link formation does not involve any direct costs. Our de¯nition of stability implicitly allows for small costs of forming links, but signi¯cant costs are ruled out. In this section, we discuss the nature of stable networks when every ¯rm has to incur a ¯xed cost, denoted by F, for every link it forms with another ¯rm. This ¯xed cost can be interpreted as the contribution to joint research or as the individual ¯rm's share of the cost of a common facility created by the collaboration between the ¯rms.

We study the nature of stable networks in the linear demand model with no spillovers presented in Section 3.1. Apart from an additional cost for every link formed by ¯rm, the payo®s of a ¯rm are as speci¯ed before.

Recall that under price competition, even in the absence of ¯xed costs of link formation, the unique stable network was the empty network. The introduction of ¯xed costs of link formation can only make the prospects of link formation less sanguine. It is easily checked that the empty network is the unique stable network under price competition. In what follows we therefore focus on the case of quantity competition.

The first step in the analysis is to note the following interesting property of stable networks:

Lemma 5.1 Consider the linear demand model with quantity competition. Suppose that (1) and (4) hold. Let i and j be two distinct firms. Then any stable network satisfies the following property: if there exists a firm k such that $g_{i;k} = 1$ and a firm l such that $g_{j;l} = 1$, then it must also be true that $g_{i;j} = 1$.

Proof The proof is by contradiction. Suppose that $g_{i;j} = 0$. Since g is stable it follows that $\pi_i(g) - \pi_i(g - g_{i;k}) > F$. Using the expressions for profits stated in section 3, we can rewrite this condition as follows:

$$\left[\frac{\alpha - \gamma_0 + n\eta_i(g;1)\gamma - \sum_{m \neq i} \eta_m(g;1)\gamma}{n+1}\right]^2 - \left[\frac{\alpha - \gamma_0 + n[\eta_i(g;1) - 1]\gamma - \sum_{m \neq i;k} \eta_m(g;1)\gamma - [\eta_k(g;1) - 1]\gamma}{n+1}\right]^2 > F. \tag{24}$$

It is convenient to define:

$$T(g) = \alpha - \gamma_0 + n(\eta_i(g;1)\gamma) - \sum_{m \neq i} \eta_m(g;1)\gamma). \tag{25}$$

Then we can rewrite the above inequality as follows:

$$\left[\frac{T(g)}{n+1}\right]^2 - \left[\frac{T(g) - (n-1)\gamma}{n+1}\right]^2 > F. \tag{26}$$

Next we observe that the additional (gross) payoff to firm i from forming a link with firm j is given by $\pi_i(g + g_{i;j}) - \pi_i(g)$. This can be written as follows:

$$\left[\frac{\alpha - \gamma_0 + n[\eta_i(g;1) + 1]\gamma - \sum_{m \neq i;j} \eta_m(g;1)\gamma - [\eta_j(g;1) + 1]\gamma}{n+1}\right]^2 - \left[\frac{\alpha - \gamma_0 + n\eta_i(g;1)\gamma - \sum_{m \neq i;j} \eta_m(g;1)\gamma - \eta_j(g;1)\gamma}{n+1}\right]^2 > F. \tag{27}$$

Using the above definition of $T(g)$ this can be rewritten as follows:

$$\left[\frac{T(g) + (n - 1)\gamma}{n + 1}\right]^2 - \left[\frac{T(g)}{n + 1}\right]^2. \tag{28}$$

From the above calculations, it is immediate that

$$\pi_i(g + g_{i,j}) - \pi_i(g) > \pi_i(g) - \pi_i(g - g_{i,k}). \tag{29}$$

Since the right hand side term is larger than F, it follows that firm i has an incentive to form a link with firm j. The only property we have used is that firm i has a link with some other firm. In this respect the situation of firm j is similar. Hence, using identical arguments, we can show that firm j has an incentive to form a link with firm i. This shows that g is not stable, a contradiction which completes the proof.

$\square$

The lemma says that, in a stable network, if a pair of firms have any links at all then they must also be linked with each other. The proof exploits the convexity of the profit function with respect to the level of costs. The lemma has some interesting implications: one, it implies that every component in a stable network must be complete; two, it implies that in a stable network, there will be at most one non-singleton component. Thus this lemma sharply restricts the set of possible networks that can be stable. The following proposition summarizes these observations and also shows that stable networks always exist. Define $F^* = [(\alpha - \gamma_0 + (n - 1)\gamma)/(n + 1)]^2 - [(\alpha - \gamma_0)/(n + 1)]^2$

Proposition 5.2 Consider the linear demand model with quantity competition. Suppose that demand satisfies (4) and the marginal cost function satisfies (1). Then there is at most one non-singleton component in a stable network and this component is complete. The complete network is stable if and only if $F < F^*$, while the empty network is stable if and only if $F > F^*$.

Proof The payoff to a firm i from the complete network is given by,

$$\left[\frac{\alpha - \gamma_0 + (n-1)\gamma}{n+1}\right]^2 - (n-1)F. \tag{30}$$

The payoff to a firm i if it were to delete one of its $(n-1)$ links is given by,

$$\left[\frac{\alpha - \gamma_0 + n(n-2)\gamma - (n-2)(n-1)\gamma - (n-2)\gamma}{n+1}\right]^2 - (n-2)F. \tag{31}$$

Thus the change in payoff to firm i by deleting a link with another firm j, given that all the other firms are directly linked is given by,

$$-\left[\frac{\alpha - \gamma_0 + (n-1)\gamma}{n+1}\right]^2 + \left(\frac{\alpha - \gamma_0}{n+1}\right)^2 + F = F - F^*. \tag{32}$$

This expression is negative if and only if $F < F^*$. Similarly, we can check the conditions for the empty network to be stable.

□

5.3 Asymmetries and Stable Networks

In the analysis we have assumed that all firms are ex-ante symmetric with respect to initial costs. Moreover, they also have the same costs reduction function. In this section we briefly discuss the role of this symmetry assumption with the help of a simple example. The main point of this example is to illustrate that intransitive networks – such as stars – can arise when firms are ex-ante asymmetric. The general model of asymmetric firms is quite complicated and its analysis lies outside the scope of the present paper.

Example: Asymmetries and intransitive networks Suppose there are three firms, 1,2 and 3 with initial costs given by $c_1(0) = \gamma$, $c_2(0) = 2\gamma$, and $c_3(0) = 2\gamma$, respectively. Let the inverse demand be given by $P = \alpha - Q$. Assume that direct collaborative links lower costs of the linking ¯rms by °, unless these costs are already zero. In the latter case, there are no further cost reductions possible via the formation of additional links. Suppose that there are no knowledge spillovers. Assume that ¯rms compete by setting quantities. Finally let ® > 6°. This last requirement ensures that all ¯rms make positive pro¯ts in equilibrium, given any network g.

In this setting, there are four possible network architectures: the empty network g^e, the network with one pair of ¯rms linked and one ¯rm isolated, the star network, g^s, and the complete network, g^c. We claim that the star network g^s, with ¯rm 2 or ¯rm 3 at the center of the star, are the only two stable networks. Figure 6 depicts these networks.

To see the incentives of the ¯rms suppose that 2 is the center of the star. In this star network ¯rms 2 and 1 have both moved down to a cost level of 0, while ¯rm 3 still has a a positive cost level. It follows that ¯rm 3 will in principle be interested in forming a link with ¯rm 1. But ¯rm 1 will not gain anything in terms of costs since its costs are already zero, while a link with ¯rm 3 will lower the costs of ¯rm 3, making it more competitive, thereby lowering the pro¯ts of ¯rm 1. It is easily checked that this star is stable. Similar calculations prove that the star with 3 at the center is also stable. Direct comparisons of payo®s show that (apart from the star with 3 at the center) there is no other stable network.

4

This example also shows that intransitive networks arise quite naturally with asymmetric ¯rms. This provides a motivation for a further study of network games, since this approach allows for intransitive relationships, unlike the earlier approach based on coalitions.

5.4 Notion of Stability

We have employed a notion of stability which requires that in a network each link formed must generate strictly greater payo®s for the concerned ¯rms and secondly that there are no unformed links with this property. In an earlier paper, Jackson and Wolinsky [11], proposed a related notion of stability. Their definition of stability – which we shall term JW-stability – had two requirements. A network g is JW-stable if for all $i, j \in N$,

(i) $\pi_i(g) \geq \pi_i(g - g_{i,j})$ and $\pi_j(g) \geq \pi_j(g - g_{i,j})$

(ii) if $\pi_i(g + g_{i,j}) > \pi_i(g)$, then $\pi_j(g + g_{i,j}) < \pi_j(g)$

The first requirement says that every link in g must yield non-negative benefits. The second requirement says that every link not in g must have the property that if one firms prefer to have it then the other firm must strictly loose from it. What are the JW-stable networks in oligopoly and what is their relationship to the stable networks we have identified? We will not provide a characterization of JW-stable networks here. Instead we will briefly compare the requirements on stability and then say a few words about what this implies about the set of stable networks.

The first requirement in the two definitions pertains to the incentives for having the existing links. In our definition, we require that all such links yield strictly greater payoffs to the firms that form such links, while in their definition, Jackson and Wolinsky require only that existing links yield non-negative additional payoffs. This suggests that our first requirement is stronger. This has important effects on the results. For instance, take the Bertrand example of Section 3. In this setting, we showed that the unique stable network was the empty network. On the other hand, it can be verified that under price competition, the complete network is JW-stable!

The second requirement pertains to potential links which are not formed in a given network. We require that such unformed links not be strictly profitable for the firms individually. By contrast, Jackson and Wolinsky require that if such a link is strictly profitable to one firm then it should not be unprofitable for the other firm. In this case, our requirement is milder than their requirement. This difference again has important effects. Take for example the case of aggressive competition covered by Theorem 4.2. We showed that for markets with 4 or more firms a stable network has the following structure: there is a non-singleton complete component of $k \in \{3, 4, ..., n\}$ firms and $n - k$ singleton components. It can be seen that the disconnected networks of this type, i.e., where k · n ¡ 1, are not stable under requirement (ii) of Jackson and Wolinsky. This is because a ¯rm in the non-singleton component will typically have an incentive to lower its costs further while a singleton ¯rm will be indi®erent between forming or not forming a link. Thus a network of the type we have identi¯ed above will violate requirement (ii) of Jackson and Wolinsky. The complete network satis¯es both their requirements.

6 Conclusion

In this paper, we have examined the endogenous formation of networks in an oligopoly with either price or quantity competition. We have characterized the set of stable networks and compared them with e±cient networks. Our results suggest that except under extreme competition, a la Bertrand, ¯rms have an incentive to collaborate with their competitors to lower costs of production. Stable networks of collaboration possess simple architectures, which can be characterized under a variety of circumstances. In particular, the complete network, where every ¯rm has a collaboration link with every other ¯rm, and the network with a dominant group, which contains a large number of completely connected ¯rms and several isolated ¯rms, appear to be stable under di®erent competitive environments. Finally, we observe that stable networks are often e±cient, from a social point of view. Our ¯ndings are very di®erent from those derived by other authors who have used a coalition formation approach to study these questions.

We have assumed that the e®ort level in collaboration arrangements is ¯xed and exogenously speci¯ed. Collaboration agreements create possibilities for free-riding and the case of endogenous e®ort levels merits closer attention. A second avenue for further research is the dynamics of network formation. We have examined a static model. There are several incentive issues that seem to be related to the timing of collaboration. This requires a dynamic model of network formation, which we hope to study in future work.

7 Appendix A

Proof of Proposition 3.3 a. The first claim follows from the hypothesis that $k^* < n - 1$. A network in which all firms have k^* links is stable because, if $g_{i,j} = 0$, then $\pi_l(g + g_{i,j}) = \pi_l(g)$ for $l = i, j$ implying no incentive to form an additional link. If $g_{i,j} = 1$, then $\pi_l(g - g_{i,j}) < \pi_l(g)$ implying no incentive to sever a link.

b. Suppose that a stable network, g, has three or more components. If g' and g'' are the two smallest components, then $|N(g')| = l \leq k = |N(g'')| < n/2$. In this case, it is easily verified that for any $i \in N(g')|$ and $j \in N(g'')|$, $\pi_i(g + g_{i,j}) > \pi_i(g)$ and $\pi_j(g + g_{i,j}) > \pi_j(g)$ contradicting the stability of g. The proof of the latter part of this statement is immediate.

c. The proof is similar to that in part 2 and is, therefore, omitted.

$\triangle$

Proof of Proposition 3.4 a. We first show that any non-singleton component of stable network must be complete. Fix a stable network and consider a non-singleton component g'. By virtue of connectedness, there exist $i, j \in N$ such that $g_{i,j} = 0$ and $1 \leq \eta_i(g; 1) \leq \eta_j(g; 1) < (n - 1)$. If $k^* \leq \eta_i(g; 1) \leq \eta_j(g; 1)$, then $\pi_l(g + g_{i,j}) > \pi_l(g)$ for $l = i, j$ contradicting the stability of g. Now suppose that $\eta_i(g; 1) < k^* \leq \eta_k(g; 1)$. Consider some firm $k \in N(g)$, $k \neq i$, such that $g_{i,k} = 1$. Then, $\pi_i(g - g_{i,k}) = (>)\pi_i(g)$ if $\eta_k(g; 1) \leq (>)k^*$ contradicting the stability of g. This argument also covers the case where $\eta_i(g; 1) \leq \eta_j(g; 1) < k^*$, establishing the result.

Now observe that if a stable network is connected then it has only one component, and the above argument implies that it must be complete. Next we show that g^c is stable. There are no links to add, and it is easily verified that $\pi_l(g^c - g_{i,j}) < \pi_l(g^c)$ for any $l = i, j$.

b. Note that g^e is stable because there are no links to sever, and given that $k^* \geq 1$, adding a link between any pair of firms leaves profits unaffected. Note that no firm in the component has any incentive to delete a link because this will reduce profits. Further, if $i \in N(g')$, and j is a singleton, then $\pi_i(g + g_{i,j}) > \pi_i(g)$ but $\pi_j(g + g_{i,j}) < \pi_j(g)$.

To show that such networks are the only stable unconnected networks (except for g^e), note first of all that if a component is a non-singleton, then from the proof of part 1, it must have at least $k^* + 1$ firms and must be complete. Second, there cannot be two (or more) such non-singleton components $g', g'' \subset g$ for otherwise $i \in N(g')$ and $j \in N(g'')$ will profit from forming a link. This proves the result.

$\triangle$

8 Appendix B

Assumption D $p(Q)$ is a twice continuously differentiable function with $p'(Q) < 0$ and $p''(Q) \leq 0$.

Proposition Suppose there is quantity competition. Let $p(Q)$ satisfy Assumption D and marginal cost be specified by (1). In any network $g \neq g^c$, if $g_{i,j} = 0$, then $\pi_i(g + g_{i,j}) > \pi_i(g)$ and $\pi_j(g + g_{i,j}) > \pi_j(g)$.

Proof Consider any arbitrary network $g \neq g^c$. Given g, each firm $k \in N$ chooses its output, q_k to maximize $p(Q)q_k - (\gamma_0 - \gamma\eta_k(g;1))q_k$ while taking the output profile of the other firms as fixed. The first order conditions are:

$$p(Q(g)) + p'(Q(g))q_k(g) - \gamma_0 + \gamma\eta_k(g;1) = 0; \quad k \in N \tag{33}$$

We start by showing that a firm with a larger number of direct links in g has a larger Cournot output. Let $\eta_i(g;1) > \eta_j(g;1)$. Then:

$$q_i(g) - q_j(g) = -\frac{\gamma(\eta_i(g;1) - \eta_j(g;1))}{p'(Q(g))} > 0 \tag{34}$$

Now note that in any $g \neq g^c$, there exist i, j such that $g_{i,j} = 0$ and some $m \neq i, j$ such that $q_i(g) \leq q_j(g) \leq q_m(g)$. In the network $g + g_{i,j}$, the first order condition for any $k \in N$ is:

$$p(Q(g + g_{i,j})) + p'(Q(g + g_{i,j}))q_k(g + g_{i,j}) - \gamma_0 + \gamma\eta_k(g + g_{i,j};1) = 0 \tag{35}$$

Note that $\eta_l(g + g_{i,j}; 1) = \eta_l(g; 1) + 1$ for $l = i, j$. Let $\Delta q_l(g) \equiv q_l(g + g_{i,j}) - q_l(g)$. Therefore, subtracting (33) from (35) yields for $l = i, j$:

$$p(Q(g+g_{i,j})) - p(Q(g)) + p'(Q(g+g_{i,j}))\Delta q_l(g) + [p'(Q(g+g_{i,j})) - p'(Q(g))]q_l(g) + \gamma = 0 \quad (36)$$

On the other hand, $\eta_k(g + g_{i,j}; 1) = \eta_k(g; 1)$ for $k \neq i, j$. Therefore:

$$p(Q(g+g_{i,j})) - p(Q(g)) + p'(Q(g+g_{i,j}))\Delta q_k(g) + [p'(Q(g+g_{i,j})) - p'(Q(g))]q_k(g) = 0 \quad (37)$$

Let $\Delta Q(g) \equiv Q(g + g_{i,j}) - Q(g)$. From the intermediate value theorem:

$$\begin{aligned} p(Q(g + g_{i,j}) - p(Q(g)) &= p'(\hat{Q}(g))\Delta Q(g), \\ p'(Q(g + g_{i,j}) - p'(Q(g)) &= p''(\tilde{Q}(g))\Delta Q(g) \end{aligned} \quad (38)$$

for some $\hat{Q}(g)$ and $\tilde{Q}(g)$. Let $\lambda_k \equiv [p'(\hat{Q}(g)) + p''(\tilde{Q}(g))q_k(g)]/p'(Q(g + g_{i,j}))$, $k \in N$. Note from (D) that $\lambda_k > 0$. Using (38), we can now rewrite (36) and (37) as:

$$\Delta q_l(g) = -\lambda_l \Delta Q(g) - \frac{\gamma}{p'(Q(g + g_{i,j}))}, \quad l = i, j \quad (39)$$

$$\Delta q_k(g) = -\lambda_k \Delta Q(g), \quad k \neq i, j \quad (40)$$

Summing up (39) and (40) and letting $\lambda = \sum_{k=1}^{n} \lambda_k$, we get:

$$\Delta Q(g) = -\frac{2\gamma}{(1 + \lambda)p'(Q(g + g_{i,j}))} > 0 \quad (41)$$

Therefore, if firms i and j establish a collaboration link, then aggregate output is greater in the new Cournot equilibrium. Substituting (41) in (40) shows that $\Delta q_k(g) < 0$, i.e. firms $k \neq i, j$ produce a lower output in the new Cournot equilibrium. This implies that $\Delta q_i(g) + \Delta q_j(g) > 0$. We now show that $\Delta q_i(g) > 0$ and $\Delta q_j(g) > 0$. Consider firm i and substitute (41) in (39) to get:

$$\Delta q_i(g) = \frac{2\gamma\lambda_i}{(1 + \lambda)p'(Q(g + g_{i,j}))} - \frac{\gamma}{p'(Q(g + g_{i,j}))} \quad (42)$$

Therefore, $\Delta q_i(g) > 0$ if and only if $\xi_i < 1 + \sum_{k \neq i} \xi_k$. But this is true since in g there is some $m \neq i; j$ such that $q_i(g) \leq q_m(g)$ and, therefore, $\xi_i \leq \xi_m$. Similarly for j.

We now turn to the change in profits for firms i and j. Recalling (38), note that:

$$\begin{aligned} \pi_i(g + g_{i;j}) - \pi_i(g) &= p(Q(g + g_{i;j}))q_i(g + g_{i;j}) - p(Q(g))q_i(g) \\ &\qquad - \{\gamma_0 - \gamma\eta_i(g;1)\}\Delta q_i(g) + \gamma q_i(g + g_{i;j}) \\ &= [p(Q(g)) - \{\gamma_0 - \gamma\eta_i(g;1)\}]\Delta q_i(g) \\ &\qquad + p'(\hat{Q}(g))\Delta Q(g)q_i(g + g_{i;j}) + \gamma q_i(g + g_{i;j}) \end{aligned} \quad (43)$$

Note from (33) that $p(Q(g)) - \{\gamma_0 - \gamma\eta_i(g;1)\} = -p'(Q(g))q_i(g) > 0$, and we have already shown that $\Delta q_i(g) > 0$. Therefore, substituting for (41) in (43), to show that $\pi_i(g + g_{i;j}) - \pi_i(g) > 0$, it suffices to show that:

$$p'(\hat{Q}(g)) \left[\frac{-2\gamma}{(1 + \xi)p'(Q(g + g_{i;j}))} \right] q_i(g + g_{i;j}) + \gamma q_i(g + g_{i;j}) > 0 \quad (44)$$

Some simple manipulation shows that (44) is equivalent to:

$$p'(Q(g + g_{i;j})) + (n - 2)p'(\hat{Q}(g)) + \sum_{k \in N} p''(\breve{Q}(g))q_k(g) < 0 \quad (45)$$

However, (45) is true by virtue of (D). Similarly, $\pi_j(g + g_{i;j}) > \pi_j(g)$.

$\square$

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