LM UNIT ROOT TEST WITH PANEL DATA; A TEST ROBUST TO STRUCTURAL CHANGES[*]

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Abstract

This paper proposes an LM test for the unit root hypothesis using panel data. The LM statistic based on the pooled likelihood function is obtained by standardizing the average of the LM statistic for individual time series. Under the null hypothesis, the statistic follows the standard normal distribution in the limit as $N, T \to \infty$ as long as N/T approaches any finite number, regardless of whether structural breaks are present. According to the Monte Carlo simulation results, the LM test is robust to the presence of structural breaks, and is more powerful than the popular test proposed by Im, Pesaran and Shin (1997) in the benchmark case where no structural breaks are involved.

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1. Introduction

Recent development of panel data-based unit root tests has attracted much attention in applied work. In particular, the tests proposed by Levin and Lin (1993) and Im, Pesaran and Shin (1997, IPS hereafter) have been widely applied. The statistics of both tests follow the standard normal distribution under the null hypothesis that all time series contain a unit root. However, the IPS test, which is obtained by standardizing the average of the augmented Dickey-Fuller (ADF) t-statistics, appears more promising. It requires a weaker condition that N/T approaches any finite number as N and $T \to \infty$; while the Levin and Lin test requires a stronger restriction that N/T goes to zero. IPS provide simulation evidence that their test performs better as compared to the Levin and Lin test.

In this paper we develop a new panel unit root test based on the Lagrangian multiplier (LM) principle. The panel LM test based on the pooled likelihood function is given as the sum of the N individual LM statistics for each time series. Since the sum of individual statistics approaches a normal random variable as N increases, we can construct a convenient standard normal test through appropriate standardization. In the basic case where the errors in individual time series are serially uncorrelated, the panel LM statistic follows a standard normal distribution under the null hypothesis as $N \to \infty$ (for fixed T), as long as the second moment of the individual LM statistic exists. When the errors are serially correlated, we conjecture, based on the results in IPS and our simulation results, that the limiting distribution of the panel LM statistic is also standard normal unless N/T diverges as $N, T \to \infty$.

Although the procedure of the panel LM test resembles that of IPS, it has an important advantage in dealing with structural breaks. Perron (1989) showed that the augmented Dickey-Fuller (ADF) unit root tests suffer from loss of power when the existing structural break is ignored. Because the IPS statistic is obtained as a linear combination of the ADF statistics of the individual time series, we would expect a similar pattern of power loss in the IPS test when the existing break in each time series is ignored. Naturally, one may suggest to modify the IPS test by including a dummy variable in each ADF regression to control for the effect of the structural break. This approach, however, relays to another difficult problem. As is well known, the limiting distribution of the ADF t-statistic depends on the location of the break when it allows for a structural break. Thus, the IPS procedure of standardizing the average of the ADF statistics requires the expected values and variances of the ADF t-statistics for all different locations of the break points in the sample.

The situation is very different when it comes to the LM approach. Schmidt and Phillips (1992) derived the LM test for the unit root hypothesis in a single time series. In a sequel, Amsler and Lee (1995) showed that the dependence of the LM statistic on the nuisance parameter indicating the position of the break disappears as $T \to \infty$. This asymptotic invariance property of the LM test for single time series turns out to carry over to the panel unit root test. When structural breaks are present, we suggest to follow the same standardization procedure using the same expected values and variances used in the basic case where no structural breaks are involved. Then there arises a subtle problem. The invariance of the individual LM statistic to the location of the break point is an asymptotic result. The dependence of the test statistic on the break point vanishes only at the order of $T^{-1/2}$, and this discrepancy of order $T^{-1/2}$ for each time series accumulates as N increases at the rate of $\sqrt{N}$. Yet, we show that the dependence of the expected values of the individual LM statistic on the location of the break point vanishes at the rate of T^{-1}. This result permits us to claim that the panel LM test remains valid as long as N/T does not diverge as $N, T \to \infty$ and the second moment of the individual LM statistic exists for all $T \geq T_0$, for some finite T_0.

The LM test is flexible. It can be applied when a structural break occurs at different time period in each time series as well as when the structural break occurs in only some of the time series.

The finite sample performance of the LM test is examined via Monte Carlo simulation. The N/T asymptotic result is generally supported in our simulation. All empirical sizes of the LM tests that properly control for the structural breaks are reasonably close to the nominal size in all N, T cases we examined. In addition, the LM test is more powerful than the IPS test in the basic case when no structural breaks occur.

The paper is organized as follows. In the next section we present the model and derive the panel LM statistic when the time series contain no structural breaks. In Section 3 we extend the test to the case when a structural break is present in each time series. In Section 4 we report simulation results. Section 5 concludes.

2. LM Test with No Structural Break

Suppose we have data y_{it}, $t = 1, 2, ..., T$, $i = 1, 2, ..., N$, generated as:

$$y_{it} = x_{it} + z_{it}, \quad x_{it} = \phi_i x_{i,t-1} + \varepsilon_{it}, \quad z_{it} = \gamma_{1i} + \gamma_{2i} t. \tag{2.1}$$

We are interested in testing the null hypothesis of unit roots $\phi_i = 1$ for all i. To do so, we express y_{it} as:

$$\Delta y_{it} = \beta_i y_{i,t-1} - \beta_i \gamma_{1i} + [1 - \beta_i (t-1)] \gamma_{2i} + \varepsilon_{it}, \; t = 1, 2, ..., T, \; i = 1, 2, ..., N, \tag{2.2}$$

where $\beta_i = -(1 - \phi_i)$. We then have the null hypothesis:

$$H_0 : \beta_i = 0 \text{ for all } i, \tag{2.3}$$

against the alternatives:

$$H_1 : \beta_i < 0, \; i = 1, 2, \ldots, N_1, \; \beta_i = 0, \; i = N_1 + 1, N_1 + 2, \ldots, N. \qquad (2.4)$$

Therefore, all or some of the time series are stationary under the alternative hypothesis.

This section is divided into two subsections. First, we study the simple case when the errors, ε_{it}, are uncorrelated. In the second subsection we deal with the serially correlated errors.

2.1. Serially Uncorrelated Errors

First we derive the LM statistic in the basic case when the errors, ε_{it}, in (2.2) are serially uncorrelated. We assume:

Assumption 2.1. $\varepsilon_{it}, i = 1, \ldots, N, t = 1, \ldots, T$, are independent normal variables with mean zero and variance σ_i^2.

We then have the following pooled log-likelihood function:

$$\ln L = \sum_{i=1} \left(-\frac{T}{2} \ln 2\pi\sigma_i^2 - \frac{1}{2\sigma_i^2} SSE_i \right), \qquad (2.5)$$

where

$$SSE_i = \sum_{t=1} \{ \Delta y_{it} - \beta_i y_{i,t-1} + \beta_i \gamma_{1i} - [1 - \beta_i (t-1)] \gamma_{2i} \}^2. \qquad (2.6)$$

If we let LM_{iT} be the LM statistic for the i-th time series, then it is straightforward to see that the LM statistic based on the pooled likelihood function (2.5) becomes

$$LM_{NT} = \sum_{i=1} LM_{iT}. \qquad (2.7)$$

Let

$$\tilde{S}_{i,t-1} = y_{i,t-1} - \tilde{\gamma}_{2i}(t-1), \qquad (2.8)$$

where $\tilde{\gamma}_{2i} = \frac{1}{T}\sum_{t=1}^{T} \Delta y_{it} = (y_{iT} - y_{i0})/T$ is the restricted maximum likelihood estimator of γ_{2i} obtained from the restricted regression:

$$\Delta y_{it} = \gamma_{2i} + \varepsilon_{it}. \qquad (2.9)$$

Schmidt and Phillips (1992) proposed an LM statistic obtained as the t-statistic testing $\beta_i = 0$ in the regression

$$\Delta y_{it} = \text{intercept} + \beta_i \tilde{S}_{i,t-1} + \text{error}, \tag{2.10}$$

which could be expressed as

$$LM_{iT} = \frac{\sqrt{T-2}\left(\tilde{S}'_{i,-1} M_{(i_T)} \Delta Y_i\right)}{\sqrt{\left(\tilde{S}'_{-1} M_{(i_T)} \tilde{S}_{-1}\right)\left(\Delta Y_i' M_{(i_T, \tilde{S}_{-1})} \Delta Y_i\right)}}, \tag{2.11}$$

where $\Delta Y_i = (\Delta y_{i1}, \Delta y_{i2}, ..., \Delta y_{iT})'$, $\tilde{S}_{i,-1} = \left(\tilde{S}_{i0}, \tilde{S}_{i1}, \tilde{S}_{i2}, ..., \tilde{S}_{i,T-1}\right)'$, $M_{(\cdot)}$ denotes the projection onto the null space of $(\cdot)$, and i_T is the $T \times 1$ vector of ones.

The distribution of LM_{NT} in (2.7) depends on N and T, but not on any other nuisance parameters under the null hypothesis. Therefore, LM_{NT} itself may be used in practice as a test statistic. However, as N increases, as long as the second moment of LM_{iT} exists, the distribution of LM_{NT} will approach a normal distribution. We denote the average of the individual LM statistic LM_{iT} in (2.11) as

$$\overline{LM}_{NT} = \frac{1}{N} \sum_{i=1}^{N} LM_{iT}, \tag{2.12}$$

and the expected value and variance of LM_{iT} defined in (2.11) under the null hypothesis as $E(L_T)$ and $V(L_T)$. More formally,

Definition 1. $E(L_T)$ and $V(L_T)$ are the mean and variance of the LM statistic obtained as the t-statistic for testing $\beta = 0$ in the regression (2.10), where y_t follows a simple random walk:

$$\Delta y_t = \varepsilon_t, \text{ with } \varepsilon_t \sim iidN\left(0, \sigma^2\right) \text{ for } t = 1, 2, ..., T, \text{ and } y_0 = 0. \tag{2.13}$$

Then, under Assumption 2.1, and under the null hypothesis,

$$\Gamma_{LM} = \frac{\sqrt{N}\left[\overline{LM}_{NT} - E(L_T)\right]}{\sqrt{V(L_T)}} \Rightarrow N(0,1) \tag{2.14}$$

as N grows (for finite T), as long as $E(L_T)$ and $V(L_T)$ exist.

Remark 1. We could not prove the existence of $E(L_T)$ and $V(L_T)$ for some finite T, and the statement (2.14) remains to be a conjecture. A similar difficulty was addressed in IPS. This technical difficulty disappears if we stick to the

LM principle. Schmidt and Phillips (1992) suggested to estimate σ_i^2 based on the unrestricted SSE_i in (2.6). However, following the LM principle, we may estimate σ_i^2 based on the restricted SSE_i or from the restricted regression (2.9), namely: $\tilde{\sigma}_i^2 = T^{-1}\sum_{t=1}^{T}(\Delta y_{it} - \tilde{\nu}_{2i})^2$. Then we have another type of LM statistic: $LM_{iT}^* = \sqrt{T}\,\tilde{S}_{i,-1}' M_{(i_T)} \Delta Y_i \Big/ \left[\sqrt{\left(\tilde{S}_{-1}' M_{(i_T)} \tilde{S}_{-1}\right)\left(\Delta Y_i' M_{(i_T)} \Delta Y_i\right)}\right]$. Note that the square of LM_{iT}^* is the familiar TR^2 statistic in the regression (2.10). Since LM_{iT}^* cannot exceed $\sqrt{T}$, it has all the finite moments for finite T. Therefore, if we replace LM_{iT} with LM_{iT}^*, the result in (2.14) is no longer a conjecture. We continue to use LM_{iT} as a building block for a panel LM test, because the difference between using LM_{iT} and LM_{iT}^* is negligible in practice; it is easily seen that $LM_{iT} = LM_{iT}^* + O_p(T^{-1})$.

Remark 2. As was noted in Schmidt and Phillips (1992), the statistic LM_{iT} is invariant numerically to the values of γ_{1i} and γ_{2i} under the null hypothesis. Therefore we do not lose generality by setting $\gamma_{1i} = \gamma_{2i} = 0$ in the data generation process of (2.13).

2.2. Serially Correlated Errors

In this subsection, we assume that the errors, ε_{it}, in (2.2) follow an autoregressive process.[1]

Assumption 2.2. $\varepsilon_{it} = \sum_{j=1}^{p_i}\rho_{ij}\varepsilon_{it-j} + e_{it}$, $i = 1, \ldots, N$, $t = 1, \ldots, T$, where e_{it} are independent normal variables with mean zero and variance σ_i^2, and all the roots of $\rho_i(z) = 1 - \sum_{j=1}^{p_i}\rho_{ij}z^j$ lie outside the unit circle.

We follow Ahn (1993) and Amsler and Lee (1995) who suggested an ADF type correction for serially correlated errors in (2.2). The LM statistic for the i-th time series is obtained as a t-statistic for $\beta_i = 0$ in the augmented regression:

$$\Delta y_{it} = \text{intercept} + \beta_i \tilde{S}_{i,t-1} + \sum_{j=1}^{p_i} \rho_{ij}\,\Delta y_{i,t-j} + \text{error}, \qquad (2.15)$$

where p_i denotes the order of augmentation for the i-th times series, and $\tilde{S}_{i,t-1}$ is defined in (2.8).[2] See Remark 3 below for more details of the construction of

[1] We do not pursue the asymptotics of the LM test when the errors are serially correlated in this paper. IPS presented the asymptotic result that the IPS test is valid in the presence of AR(p) errors as long as N/T converges to any finite number as $N, T \to \infty$. We conjecture that the LM test would be valid under the same conditions for AR(p) errors. As we discuss in Section 4, no asymptotic result for panel unit root tests is available when the errors follow moving average processes.

[2] Amsler and Lee (1995) suggested to augment $\Delta\tilde{S}_{i,t-j}$ rather than $\Delta y_{i,t-j}$. But, in the case where no structural breaks are involved, they produce numerically identical statistics.

$\hat{S}_{i,t-1}$ in practice. Let the resulting t-statistic be $LM_{iT}(p_i)$ and the average

$$\overline{LM}_{NT}(p) = \frac{1}{N}\sum_{i=1}^{N} LM_{iT}(p_i). \tag{2.16}$$

We follow the standardization procedure proposed by IPS. To do so, we define the expected value and variance of $LM_{iT}(p_i)$, when $\beta_i = 0$ and $\rho_{ij} = 0$, for $j = 1, ..., p_i$.

Definition 2. $E[L_T(p_i)]$ and $V[L_T(p_i)]$ are the mean and variance of the LM statistic obtained as the t-statistic for testing $\beta_i = 0$ in the regression (2.15), where y_{it} follows a simple random walk process:

$$\Delta y_{it} = \varepsilon_{it}, \text{ with } \varepsilon_{it} \sim iidN\left(0, \sigma_i^2\right), \text{ for } t = 1, 2, ..., T, \text{ and } y_{i0} = 0. \tag{2.17}$$

Then, the standardization of $\overline{LM}_{NT}(p)$ using $E[L_T(p_i)]$ and $V[L_T(p_i)]$ yields:

$$\Gamma_{LM}(p) = \frac{\sqrt{N}\left\{\overline{LM}_{NT}(p) - \frac{1}{N}\sum_{i=1}^{N} E[L_T(p_i)]\right\}}{\sqrt{\frac{1}{N}\sum_{i=1}^{N} V[L_T(p_i)]}}. \tag{2.18}$$

The statistic $LM_{iT}(p_i)$ is invariant numerically to the values of γ_{1i} and γ_{2i} under the null hypothesis, so that we do not lose generality by setting $\gamma_{1i} = \gamma_{2i} = 0$ in the data generation process in (2.17). (See Remark 2.) However, $LM_{iT}(p_i)$ does depend on the values of ρ_{ij} for finite T, even though this dependence disappears as T grows. Therefore, $E[L_T(p_i)]$ and $V[L_T(p_i)]$ are not the exact expected value and variance of $LM_{iT}(p_i)$. This prompts a complicated issue that under which condition the test based on $\Gamma_{LM}(p)$ is valid. We avoid this issue in this paper. But, in light of the simulation results reported in Section 4 and the results in IPS, it seems to follow that

$$\Gamma_{LM}(p) \Rightarrow N(0, 1), \tag{2.19}$$

unless N/T diverges as both N and $T \to \infty$.

The expected values and variances, $E[L_T(p)]$ and $V[L_T(p)]$, for various cases of T and p are computed via stochastic simulation based on 500,000 replications, and are provided in Table 1. (See Remark 3 for more details.) Note that Γ_{LM} in (2.14) is a special case of $\Gamma_{LM}(p)$ when $p_i = 0$, for all i.

Remark 3. Notes on the use of Table 1:
(1) The value of T in Table 1 denotes the regression dimension. For example, suppose that someone has 25 observations for the i-th time series and wants to allow $p_i = 2$. Then, the regression dimension is 22 so that the appropriate expected value and variance corresponding to $T = 22$ and $p = 2$ are -1.880 and

0.413, respectively.
(2) In computation of $E[L_T(p)]$ and $V[L_T(p)]$ (with $T+p+1$ data points available), we constructed the series $\hat{S}_{t-1}$ as $\hat{S}_{t-1} = y_{t-1} - (y_{T+p+1} - y_1)/(T+p)$. We recommend to construct the series $\hat{S}_{i,t-1}$ in the same way.
(3) When $T \geq 50$, the expected value and variance are reported for every five T. The following is an example explaining how to interpolate: Suppose one has 63 observations for each time series and wants to allow $p = 4$. The regression dimension is 58. From Table 1, the expected value for $T = 55, p = 4$ is -1.894, and -0.192 for $T = 60, p = 4$. The interpolated expected value is obtained as: $\left(\frac{2}{5}\right)(-1,894) + \left(\frac{3}{5}\right)(-1.902) = -1.899$.

3. Panel LM Test with Break

In this section we derive an LM test for unit root hypothesis in panel data where a structural break is present in each individual time series. We have two subsections. The errors are serially uncorrelated in the first subsection. Serially correlated errors are introduced in the second subsection.

3.1. Serially Uncorrelated Errors

Suppose structural shift occurs at time period $T_{B,i}$ in i-th time series. Therefore, the data are generated as:

$$y_{it} = x_{it} + z_{it}, \quad x_{it} = \phi_i x_{i,t-1} + \varepsilon_{it}, \quad z_{it} = \gamma_{1i} + \gamma_{2i} t + \delta_i D_{it}, \tag{3.1}$$

for $t = 0, 1, 2, \ldots, T$, $i = 1, 2, \ldots, N$, where

$$D_{it} = \begin{cases} 0 & t < T_{B,i} \\ 1 & t \geq T_{B,i} + 1, \end{cases} \tag{3.2}$$

which has an alternative representation:

$$\Delta y_{it} = \beta_i y_{i,t-1} - \beta_i \gamma_{1i} + [1 - \beta_i (t-1)]\gamma_{2i} + (\Delta D_{it} - \beta_i D_{it})\delta_i + \varepsilon_{it}, \tag{3.3}$$

for $t = 1, 2, \ldots, T$, $i = 1, 2, \ldots, N$, where $\Delta D_{it} = D_{it} - D_{i,t-1}$, i.e.,

$$\Delta D_{it} = \begin{cases} 1 & t = T_{B,i} + 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

From (3.3), under Assumption 2.1, we obtain the pooled likelihood function (2.5) with

$$SSE_i = \sum_{t=1} \{\Delta y_{it} - \beta_i y_{i,t-1} + \beta_i \gamma_{1i} - [1 - \beta_i (t-1)]\gamma_{2i} - (\Delta D_{it} - \beta_i D_{it})\delta_i\}^2. \tag{3.5}$$

As Amsler and Lee (1995) suggested, the LM statistic for the i-th time series can be obtained as a t-statistic for testing $\beta_i = 0$ in the regression:

$$\Delta y_{it} = \gamma_{2i} + \delta_i \Delta D_{it} + \beta \tilde{S}_{i,t-1} + \text{error}, \qquad (3.6)$$

where

$$\tilde{S}_{i,t-1} = y_{i,t-1} - \tilde{\gamma}_{2i}(t-1) - \tilde{\delta}_i D_{i,t-1}, \qquad (3.7)$$

and $\tilde{\gamma}_{2i}$ and $\tilde{\delta}_i$ are the OLS estimators of γ_{2i} and δ_i in the restricted regression:

$$\Delta y_{it} = \gamma_{2i} + \delta_i \Delta D_{it} + \varepsilon_{it}. \qquad (3.8)$$

Therefore, letting $\tilde{S}_{i,-1} = \left(\tilde{S}_{i0}, \tilde{S}_{i1}, ..., \tilde{S}_{i,T-1}\right)'$ and $\Delta D_i = (\Delta D_{i1}, \Delta D_{i2}, ..., D_{iT})'$, we have the LM statistic for the i-th time series:

$$LM^B_{iT} = \frac{\sqrt{T-3}\left(\tilde{S}'_{i,-1} M_{(i_T,\Delta D_i)} \Delta Y_i\right)}{\sqrt{\left(\tilde{S}'_{i,-1} M_{(i_T,\Delta D_i)} \tilde{S}_{i,-1}\right)\left(\Delta Y_i' M_{(i_T,\Delta D_i,\tilde{S}_{i,-1})} \Delta Y_i\right)}}. \qquad (3.9)$$

The panel LM statistic based on the pooled likelihood function is given by the sum of LM^B_{iT} so that $LM^B_{NT} = \sum_{i=1}^N LM^B_{iT}$. Let

$$\overline{LM}^B_{NT} = \frac{1}{N}\sum_{i=1} LM^B_{iT}. \qquad (3.10)$$

As Amsler and Lee (1995) showed, the distribution of LM^B_{iT} does not depend on the location of the break point

$$\lambda_i = \frac{T_{B,i}}{T} \qquad (3.11)$$

in the limit, i.e., $LM^B_{iT} - LM_{iT} = o_p(1)$. In finite sample, however, the distribution of LM^B_{iT} does depend on λ_i. If we have the exact expected value and the exact variance of LM^B_{iT} under the null hypothesis, which we denote as $E\left[L^B_T(\lambda_i)\right]$ and $V\left[L^B_T(\lambda_i)\right]$, then it follows that

$$\Gamma^{B*}_{LM} = \frac{\sqrt{N}\left\{\overline{LM}_b - \frac{1}{N}\sum_{i=1}^N E\left[L^B_T(\lambda_i)\right]\right\}}{\sqrt{\frac{1}{N}\sum_{i=1}^N V[L^B_T(\lambda_i)]}} \Rightarrow N(0,1), \qquad (3.12)$$

under the null hypothesis, as $N \to \infty$, as long as $V\left[L^B_T(\lambda_i)\right]$ exists for all i. However, the statistic Γ^{B*}_{LM}, as it stands, is not very practical since it requires $E\left[L^B_T(\lambda_i)\right]$ and $V\left[L^B_T(\lambda_i)\right]$ for all λ_i in the sample.

Since $LM^B_{iT} - LM_{iT} = o_p(1)$; we consider a practical statistic using $E(L_T)$ and $V(L_T)$ defined in Definition 1 in place of $E\left[L^B_T(\lambda_i)\right]$ and $V\left[L^B_T(\lambda_i)\right]$ in (3.12) to have

$$\Gamma^B_{LM} = \frac{\sqrt{N}\left[\overline{LM}^B_{NT} - E(L_T)\right]}{\sqrt{V(L_T)}}. \tag{3.13}$$

The issue now is under which condition we have

$$\Gamma^B_{LM} \Rightarrow N(0;1). \tag{3.14}$$

The result in (3.14) holds if

$$\Gamma^B_{LM} - \Gamma^{B*}_{LM} = o_p(1). \tag{3.15}$$

We focus on the condition that guarantees (3.15). In doing so, we assume:

Assumption 3.1. The variance of L_T defined in Definition 1 and the variance of $L^B_T(\lambda)$ used in (3.12), for all values of λ; are finite for all $T \ge T_0$ for some finite T_0. (See Remark 4 below.)

Note that

$$\Gamma^B_{LM} - \Gamma^{B*}_{LM} = \frac{\sqrt{N}\left[\overline{LM}^B_{NT} - E(L_T)\right]}{\sqrt{V(L_T)}} - \frac{\sqrt{N}\left\{\overline{LM}^B_{NT} - \frac{1}{N}\sum_{i=1}^N E\left[L^B_T(\lambda_i)\right]\right\}}{\sqrt{\frac{1}{N}\sum_{i=1}^N V[L^B_T(\lambda_i)]}}$$

$$= \sqrt{N}\left(\overline{LM}^B_{NT} - \frac{1}{N}\sum_{i=1}^N E\left[L^B_T(\lambda_i)\right]\right)\frac{\sqrt{\frac{1}{N}\sum_{i=1}^N V[L^B_T(\lambda_i)]} - \sqrt{V(L_T)}}{\sqrt{\frac{1}{N}\sum_{i=1}^N V[L^B_T(\lambda_i)]}\sqrt{V(L^B_T)}} \tag{3.16}$$

$$+\frac{N^{-1/2}\sum_{i=1}^N E\left[L^B_T(\lambda_i) - L_T\right]}{\sqrt{V(L_T)}}.$$

It is shown in Appendix that

$$L^B_T(\lambda) - L_T = O_p\left(T^{-1/2}\right);\ \text{for all } \lambda. \tag{3.17}$$

Assumption 3.1 ensures that $V\left[L^B_T(\lambda)\right] - V[L_T] = O\left(T^{-1/2}\right)$ for all λ; from which we deduce $\sqrt{\frac{1}{N}\sum_{i=1}^N V[L^B_T(\lambda_i)]} - V(L_T) = O\left(T^{-1/2}\right)$: It is obvious from a simple view that the first term of the last equation of (3.16) is $o_p(1)$. Now we need to find the conditions under which the last term of (3.16) is negligible asymptotically, or $N^{-1/2}\sum_{i=1}^N E\left[L^B_T(\lambda_i) - L_T\right] = o_p(1)$: We show in the appendix that

$$E\left[L^B_T(\lambda) - L_T\right] = O\left(T^{-1}\right);\ \text{for all } \lambda. \tag{3.18}$$

Therefore, $N^{-1/2}\sum_{i=1}^N E\left[L^B_T(\lambda_i) - L_T\right] = O\left(\frac{\sqrt{N}}{T}\right)$; which is $o(1)$ as long as $\frac{\sqrt{N}}{T} \to 0$: Since $\frac{\sqrt{N}}{T} = \sqrt{\frac{N}{T}}T^{-1/2}$; it follows that $N^{-1/2}\sum_{i=1}^N E\left[L^B_T(\lambda_i) - L_T\right] = o_p(1)$ unless N/T diverges at the rate of $\sqrt{T}$ or faster as $N, T \to \infty$.

Remark 4. The uniform integrability of L_T and $L_T^B(\theta)$ is something to be verified rather than assumed. The difficulty disappears if σ_i^2 is estimated based on the restricted MLE. Then the uniform integrability condition is obviously met since both L_T and $L_T^B(\theta)$ are bounded by $\sqrt{T}$. (See Remark 1.) Following our simulation results, the second moment of $L_T^B(\theta)$ exists for T as small as 7.

Remark 5. Because the claim in (3.18) is at the center of the proof, we conduct a simulation to see how $TE\left[L_T^B(\theta) - L_T\right]$ changes as T grows. In the following Table the expected value and standard deviation of $T\left[L_T^B(\theta) - L_T\right]$ are computed based on 50,000 replications with $\theta = 0.5$:

T	100	1,000	10,000	100,000
$TE\left[L_T^B(\theta) - L_T\right]$	-0.012	0.287	-0.254	-0.331
$T\sqrt{V[L_T^B(\theta) - L_T]}$	10.118	30.910	97.795	307.480

As is seen, $TE\left[L_T^B(\theta) - L_T\right]$ fluctuates within a very narrow interval, while the standard deviation of $T\left[L_T^B(\theta) - L_T\right]$ shows an apparent tendency of creeping up at the rate of $\sqrt{T}$, which supports the claims in (3.17) and (3.18).

3.2. Serially Correlated Errors

When the errors ε_{it} in (3.3) are serially correlated, their effect can be corrected by augmenting $\Delta\tilde{S}_{i,t-j}$, as suggested in Amsler and Lee (1995). Therefore, the LM statistic for the i-th time series is obtained as a t-statistic for $\beta_i = 0$ in the augmented regression:

$$\Delta y_{it} = \text{intercept} + \delta_i \Delta D_{it} + \beta_i \tilde{S}_{i,t-1} + \sum_{j=1}^{k} \rho_{ij} \Delta\tilde{S}_{i,t-j} + \text{error}, \tag{3.19}$$

where $\tilde{S}_{i,t-1}$ is defined in (3.7). We define $LM_{iT}^B(p_i)$ as the t-statistic for $\beta_i = 0$ in regression (3.19), and its average

$$\overline{LM}_{NT}^B(p) = \frac{1}{N}\sum_{i=1}^{N} LM_{iT}^B(p_i). \tag{3.20}$$

We then follow the standardization procedure described in Section 2.2 to derive

$$\Gamma_{LM}^B(p) = \frac{\sqrt{N}\left\{\overline{LM}_{NT}^B(p) - \frac{1}{N}\sum_{i=1}^{N} E[L_T(p_i)]\right\}}{\sqrt{\frac{1}{N}\sum_{i=1}^{N} V[L_T(p_i)]}}, \tag{3.21}$$

where $E[L_T(p_i)]$ and $V[L_T(p_i)]$ are defined in Definition 2 of Section 2.2.

In view of the simulation results reported in Section 4 and the results in IPS, we conjecture that

$$\Gamma_{LM}(p) \Rightarrow N(0,1), \tag{3.22}$$

unless N/T diverges as $N, T \to \infty$.

4. Simulation Results

In this section we investigate the small sample properties of the LM test for unit root using panel data. We also compare the performance of the LM test with the IPS test when no structural breaks are present.

Three experiments are conducted. In the first experiment, we simulate the basic case where the data contain no structural breaks and the errors are serially independent. The first experiment is designed to compare the generic power of LM vis-a-vis IPS. In the second experiment we include structural breaks in the data generation process while the errors are serially independent. We investigate the consequence of ignoring existing breaks as well as the performance of the LM test that controls for structural breaks. Special attention is paid to how the asymptotic result of N/T derived in Section 3.1 works in finite samples. The third experiment investigates the case where the errors are serially correlated and a structural break is present in each time series. We consider only the LM test that controls for structural breaks.

In the first two experiments where no serial correlations are involved, we report the results for 16 combinations of $N, T = 10, 25, 50, 100.$ But, in the third experiment, where the data contain structural breaks and the errors are serially correlated, we drop $T = 10$ because the degrees of freedom problem becomes serious. Therefore, we are left with only 12 combinations of $N = 10, 25, 50, 100$ and $T = 25, 50, 100.$ In each case the results are based on 2,000 replications. In each replication we generate N independent time series of $T + 1$ data points using pseudo-iid N(0,1) random numbers from the Gauss RNDNS procedure. Therefore, the regression dimension is $T - p$. The first 50 observations are discarded to avoid a possible initial value effect. All tests are conducted using the 5% nominal size.

4.1. Experiment 1: No Breaks, No Serial Correlations

In the first experiment, each time series contains no structural break, and the errors are serially uncorrelated. N independent time series are generated as:

$$y_{it} = \phi_i y_{i,t-1} + \varepsilon_{it}, \quad t = 0, 1, \ldots, T; \quad i = 1, 2, \ldots, N; \qquad (4.1)$$

with $\varepsilon_{it} \sim \text{iid } N(0, 1).$ We set $\phi_i = 1$ for the examination of the empirical size and $\phi_i = 0.9$ for the power computation for all i. We compare the performance of the LM test based on the statistic Γ_{LM} given in (2.14) and the IPS test obtained by standardizing the average of the t-statistics from the DF regressions that include an intercept and linear trend (see IPS, 1997, equations 2.2 and 4.2). The results are reported in Table 2.

Since both the LM and IPS statistics follow the standard normal distribution as N increases (with fixed T) under the null hypothesis, we expect the empirical size reasonably close to the 5% nominal size in both tests for relatively large N.

As expected, all reported sizes are reasonably close to 5%, even for the case with $N = 10$.

Comparing the power of the tests, the LM test turns out more powerful than the IPS test. For example, when $N = 100, T = 25,$ the power of LM is $0.518,$ while the power of IPS is $0.415.$ The corresponding sizes are 0.060 and 0.063. When $N = 25, T = 50,$ the power of LM is 0.802 and the power of IPS is 0.622, while the corresponding sizes are 0.058 and $0.059,$ respectively.

It is observed that the power increases more rapidly with T than with N for both tests. A similar pattern was reported by IPS in their paper. For instance, when $T = 10$ the power is close to the size even with $N = 100$ in both tests.

4.2. Experiment 2: Structural Breaks, No Serial Correlations

In the second experiment we investigate the effect of structural breaks. A structural break occurs at $T_{B,i}$ for the i-th time series with the magnitude of shift δ_i. The data generation process follows:

$$y_{it} = x_{it} + z_{it}, \quad x_{it} = \rho_i x_{i,t-1} + \varepsilon_{it}, \quad z_{it} = \delta_i D_{it}. \tag{4.2}$$

We consider two cases. In the first case, all the structural breaks occur at the middle of the series so that $\lambda_i = T_{B_i}/T = 0.5$ for all $i.$ The structural breaks occur earlier in the second case at $\lambda_i = 0.3$ for all i. $D_{it} = 0$ for $t < \frac{T}{2}$ and 1 otherwise for the first case, and $D_{it} = 0$ for $t < \frac{T}{3}$ and 1 otherwise for the second case. In both cases, $\sigma_i = 1$ and $\delta_i = \delta_i/\sigma_i = 5$ for all $i.$

We experiment with three tests: the LM test based on the statistic Γ_{LM} defined in (2.14); the IPS test based on the statistic described in the previous subsection; and the LM test based on the statistic Γ_{LM}^B defined in (3.13), which we will refer to respectively as LM_N, IPS_N and LM_B to signify that LM_N and IPS_N ignore the existing breaks, but LM_B controls for the breaks. The results are reported in Table 3.

First, we examine the size and power property of LM_N and IPS_N. Amsler and Lee (1995, Theorem 2) showed that both the LM and DF tests are valid in single time series even for the case when existing breaks are ignored in the testing procedure. But, with T finite, ignoring structural breaks could make the actual size very different from the asymptotic size. It all depends on the values of λ, δ and T.

An obvious pattern from Table 3 is that the situation is worse in panel, and gets worse as N increases. For example, in the first panel $(\lambda = 0.5)$, when $T = 25$ and N = 10, 25, 50, 100, the respective empirical sizes of IPS_N are $0.017, 0.016, 0.007, 0.005$, and the corresponding figures of LM_N are 0.025, 0.022, 0.009, $0.004.$ Although the empirical sizes get closer to the nominal size as T increases, a similar pattern of size distortion is observed even with $T = 100.$ The sizes of IPS_N and LM_N are even worse in the second panel $(\lambda = 0.3).$ Obviously the small size distortions in individual time series accumulate in the panel as N increases.

A more serious problem of ignoring structural breaks is loss of power, as reported in Perron (1989) and Amsler and Lee (1995) for single time series. Our simulation results indicate that a similar pattern carries over to the panel unit root test. For example, in the first panel, when $T = 25; N = 100;$ the powers of IPS_N and LM_N are reported as 0.048 (size $= 0.005$ for the same T and N) and 0.050 (size $= 0.004$), respectively, while the power of LM_B for the same N and T is 0.473 (size $= 0.058$). However, the power of IPS_N and LM_N improves dramatically as T increases. For example in the first panel, when $T = 100; N = 10;$ the power of LM_N is 0.863 (size $= 0.065$).

It seems apparent that the low power of IPS_N and LM_N is largely a reflection of the downward size distortion. But, the power of LM_N is still lower than that of LM_B even after controlling for the size. When $T = 100; N = 10$ in the first panel, the power of LM_B is 0.985 (size $= 0.064$), while the power of LM_N is 0.863 (size $= 0.065$).

On the other hand, the power of LM_B remains quite close to the corresponding figures reported in Table 1. For example, when $T = N = 25$, the power of LM_B is 0.201 in the first panel (size $= 0.054$ for the same T and N), and 0.204 in the second panel (size $= 0.056$). The corresponding figure in Table 1 is 0.207 (size $= 0.054$). This result bears a quite important message to practitioners: There is almost no power loss when someone mistakenly adopt LM_B (instead of LM_N or IPS_N) when there, in fact, are no structural breaks in the time series. Also, the power of LM_B in both panels remains more or less the same for corresponding N and T cases, which supports that the power of LM_B is not affected by the value of λ.[3]

Another interesting figures are the sizes of LM_B for different combinations of the N/T ratios, since the asymptotic validity of LM_B requires that N/T should not diverge as $N, T \to \infty$. This asymptotic result seems well reflected into finite sample. In the first panel, for instance we obtained the size of LM_B; $0.083; 0.060; 0.062; 0.068,$ for $T = 10$ and $N = 10; 25; 50; 100;$ respectively. The size of all other cases remains quite stable for different N/T ratios. The N/T ratio does not seem restrictive for applying LM_B to most practical situations.

[3] When the DF regression includes a dummy variable to control for the structural break, the asymptotic distribution of the resuting DF statistic depends on λ. Therefore the IPS test requires the expected values and variances of the ADF t-statisitcs for every different value of λ in the sample. This is quite an onerous job and was excluded from our simulation. However, we simulated the IPS-type statistic constructed from the DF regression including a break dummy variable, but using incorrectly the expected values and variances reported in IPS. The size is reported as 1.0 for all the sample cases studied in our simulation.

4.3. Experiment 3: Structural Breaks, Serial Correlations

In the third experiment we allow for serial correlations as well as a structural break in each time series. Two types of serial correlations are considered:

$$AR(1) \ : \ \varepsilon_{it} = 0.3\varepsilon_{i,t-1} + e_{it}; \tag{4.3}$$

$$MA(1) \ : \ \varepsilon_{it} = e_{it} - 0.3e_{i,t-1}; \tag{4.4}$$

where $e_{it} \sim iidN(0,1)$: We maintain the same structural break pattern used in the second experiment in (4.2) with $\lambda_i = 0.5$ and $\delta_i = 5$ for all i. We simulate only the LM test based on the statistic $\Gamma^B_{LM}(p)$ in (3.21), where the number of augmented terms is fixed at $p_i = 0,1,2,3,4$; for all i in the regression (3.19).[4] When the necessary values of the expected value and variance for standardization are not provided in Table 1, they are interpolated. See Remark 3. The results are reported in Table 4.

Since N/T asymptotics is potentially an important issue and we are hoping that the test remains valid unless N/T diverges when the errors are serially correlated, special attention is paid to the movement of empirical sizes as N/T grows for different value of p.

The results for the AR(1) error are reported in the first panel. When $p = 0$ is selected, all the reported sizes are zero. The effect of selecting too small p accumulates as N grows. A similar pattern was reported for the IPS test in their paper. However, when the true model $p = 1$ is selected, the empirical sizes of all N,T cases are reasonably close to the nominal 5%. For example, when $T = 25$ and $N = 10, 25, 50, 100$; the empirical sizes are reported as 0.059, 0.064, 0.061, 0.058, respectively, which seems to support our conjecture that the test would remain valid in this case unless N/T diverges as both N and $T \to \infty$. Similar results are obtained for the cases $p = 2, 3$ or 4. Any systematic pattern of size distortion for different combinations of N/T is not observed.

The power of the test declines as larger p is selected, and the loss of power is steeper when T is relatively small. For example, when $T = 25$ and $N = 100$; the power of the test shrinks as; 0.357; 0.283; 0.207; 0.165 for $p = 1, 2, 3, 4$: But, when $T = 50$ and $N = 10$; the power of the test is 0.343, 0.299, 0.253, 0.208 for the same p.

The results for the MA(1) errors are reported in the second panel. There is no true modelling in this case. All we can hope for is that the selected p is

[4] IPS (1997, footnote 6) observed a very serious size distortion when the lag orders are selected by Akaike or Schwarz criterion. They conjectured that the information criteria, as is well known, tend to choose too few lags when T is small, and this effect accumulates in panel unit root test. A similar pattern is expected in the LM test. The data dependent method of selecting the lag order is an important issue, and it remains interesting to see the performance of the panel unit root test when the order of lags are chosen, for example, by the sequential method advocated by Ng and Perron (1995). But, in this paper we restrict ourselves to the study of fixed p for all i, leaving the data dependent selection of p to future research. By fixing p across individuals we could examine the effect of having too many or too few lagged terms.

large enough to ensure that the left over errors in the regression (3.19) are nearly uncorrelated so that the actual size of the test is reasonably close to the nominal size, and at the same time not too big so that the adverse effect on the power remains minimal. It is well known that the optimal p that serves this purpose depends on T as well as on the moving average parameter value. There is no asymptotic result available for the validity of any panel unit root tests when the errors follow moving average. Although the lack of theoretical underpinning makes it difficult to make a persuasive conjecture, it would be reasonable to expect that bigger p is needed when T is larger. Also there would be no reason to believe that the optimal p is associated with the size of N. Therefore, the problem again is which N/T condition will leave the test valid, given that the optimal p as a function of T is chosen. Although we do not have asymptotic results, we examine the effect of N on the size given that the some large enough p is selected for each time series.

Importance of the size of T in determination of the optimal p is clearly seen in the table. Using $p = 1$ always leads to a size distortion. When $T = 25$, using $p = 2$ yields the size reasonably close to the nominal 5%. But, when $T = 50$, p needs to be 3 or larger. The empirical size is tolerably close to the nominal size in all of the sample cases when $p = 3$ or 4 is selected. A systematic effect of N on the size is rather obvious when too small p is selected. But, the effect quickly becomes negligible when p is selected properly. For instance, when $T = 50$ and $p = 2$, the size rises 0.081, 0.098, 0.110, 0.138 as N increases 10, 25, 50, 100. For the same N, T case but with $p = 3$, the size is stable at 0.052, 0.055, 0.049, 0.050.

The power of the test decreases when the chosen p is larger than necessary. For example, when $T = 25$ and $N = 50$, the power drops from 0.160 to 0.066 as p changes from 2 to 3. When $T = 50$ and $N = 10$, the power decreases from 0.356 to 0.233 as p changes from 3 to 4.

5. Concluding Remarks

In this paper we developed a new unit root test using panel data based on the LM principle. The proposed test not only is robust to the presence of structural breaks, but is more powerful than the popular IPS test in the basic case where no structural breaks are involved. The former property in particular bears very important implication for empirical work since no other test has been developed yet which can handle the presence of structural shifts in a practical way. Further, as we reported in Section 4.2, since the LM test loses little power by controlling for spurious structural breaks when they do not exist, it would be a reasonable strategy to control for breaks even when they are only at a suspicious level.

We have focused on the case of one structural break in each time series in this paper, but extension to the case with multiple breaks should be straightforward. When there are more than one break in each time series, we have multiple dummy variables in the data generation process (3.1), so that in (3.6)- (3.8). We cannot see any reason why the asymptotic results presented in this paper should not extend to the case of multiple breaks. But, it requires a more involved algebra, and is relegated to future research.

A. Appendix

In this appendix we show that

$$L_T^B(\lambda) - L_T = O\left(T^{-1/2}\right), \tag{A.1}$$

$$E\left[L_T^B(\lambda) - L_T\right] = O\left(T^{-1}\right); \tag{A.2}$$

which we claimed in (3.17) and (3.18). We assume uniform integrability of $L_T^B(\lambda)$ and L_T: (See Remarks 1 and 3 in the text.) We drop subscript i for simplicity. The following lemmas will be used in the proof.

Lemma 1. Consider a regression:

$$y_t = x_t\beta + \delta D_t + \text{error}; \quad t = 1; 2; \ldots; T; \tag{A.3}$$

where

$$D_t = \begin{cases} 1 & \text{for } t = \tau; \\ 0 & \text{otherwise,} \end{cases}$$

and x_t is $1 \times k$ vector. Let $(Y; X; D)$ be the $T \times (k+2)$ data matrix and $(Y_*; X_*)$ be $(T-1) \times (k+1)$ matrix obtained after eliminating the τ-th observations $(y_\tau; x_\tau)$ from $(Y; X)$: Then,

$$X' M_{(D)} X = X_*' X_*; \quad X' M_{(D)} Y = X_*' Y_*:$$

where $M_{(\cdot)}$ is the projection onto the null space of $(\cdot)$: The first $\tau - 1$ and the last $T - \tau$ OLS residuals from the regression (A.3) is identical to the residuals obtained from regression of Y_* on X_*; and the τ-th residual is zero. ∎

Lemma 2. Let $\varepsilon_t \sim iidN(0; \sigma^2)$; for $t = 1; 2; \ldots; T$; and $\varepsilon = (\varepsilon_1; \varepsilon_2; \ldots; \varepsilon_T)'$: Then

$$\sqrt{\frac{T}{\varepsilon'\varepsilon}} = \frac{1}{\sigma} + O_p\left(T^{-1/2}\right): \tag{A.4}$$

Proof: The result follows from a Nagar-type expansion: $\sqrt{\frac{T}{\varepsilon'\varepsilon}} = \sqrt{\frac{T}{E(\varepsilon'\varepsilon) + [\varepsilon'\varepsilon - E(\varepsilon'\varepsilon)]}}$: Since $\varepsilon'\varepsilon - E(\varepsilon'\varepsilon) = O_p\left(T^{1/2}\right)$, dividing both the numerator and the denominator by $\sqrt{T}$ yields the result. ∎

From Lemma 1, μ_2 (suppressing subscripts i) in regression (3:8) is the average of Δy_t after eliminating Δy_{T_B+1}; and $\xi = \Delta y_{T_B+1} - \mu_2$; namely:

$$\begin{aligned} \mu_2 &= \frac{\sum_{t=1}^{T} \Delta y_t - \Delta y_{T_B+1}}{T-1} = \frac{y_T - y_0}{T-1} - \frac{\Delta y_{T_B+1}}{T-1} \\ &= \frac{y_T - y_0}{T} + \frac{y_T - y_0}{T(T-1)} - \frac{\Delta y_{T_B+1}}{T-1}; \end{aligned} \tag{A.5}$$

$$\xi = \Delta y_{T_B+1} - \frac{y_T - y_0 - \Delta y_{T_B+1}}{T-1} = \frac{T\Delta y_{T_B+1} - (y_T - y_0)}{T-1}: \tag{A.6}$$

Therefore, if we let

$$q_{t-1} = \begin{cases} \left[\frac{y_T - y_0}{T(T-1)} - \frac{\Delta y_{T_B+1}}{T-1} \right](t-1) & \text{for } t \leq T_B; \\ \left[\frac{y_T - y_0}{T(T-1)} - \frac{\Delta y_{T_B+1}}{T-1} \right](t-1) + \frac{T\Delta y_{T_B+1} - (y_T - y_0)}{T-1} & \text{for } t \geq T_B + 1; \end{cases} \tag{A.7}$$

we have

$$\tilde{S}_{t-1} = \hat{S}_{t-1} - q_{t-1}. \tag{A.8}$$

Since $L_T^B(\lambda)$ is invariant numerically to different values of γ_1, γ_2 and δ under the null hypothesis, we assume, without loss of generality, $\gamma_1 = \gamma_2 = \delta = 0$. Therefore, in the following, y_t follows a simple random walk with 0 initial value, and is denoted as

$$S_t = \sum_{j=1}^{t} \varepsilon_t, \tag{A.9}$$

where $\varepsilon_t \sim iidN(0, \sigma^2)$. We then have, under the null hypothesis,

$$\hat{S}_{t-1} = S_{t-1} - \frac{S_T}{T}(t-1); \tag{A.10}$$

$$\Delta y_t = \varepsilon_t; \tag{A.11}$$

and

$$q_{t-1} = \begin{cases} \left[\frac{S_T}{T(T-1)} - \frac{\varepsilon_{T_B+1}}{T-1} \right](t-1) & \text{for } t \leq T_B; \\ \left[\frac{S_T}{T(T-1)} - \frac{\varepsilon_{T_B+1}}{T-1} \right](t-1) + \frac{T\varepsilon_{T_B+1} - S_T}{T-1} & \text{for } t \geq T_B + 1. \end{cases} \tag{A.12}$$

From (2.11) and (3.9), it is seen that

$$\begin{aligned} L_T &= \frac{\sqrt{T}\left(\hat{S}'_{-1} M_{(\iota_T)} \varepsilon\right)}{\sqrt{\left(\hat{S}'_{-1} M_{(\iota_T)} \hat{S}_{-1}\right)\left(\varepsilon' M_{(\iota_T, \hat{S}_{-1})} \varepsilon\right)}} + O_p\left(T^{-1}\right) \\ &= \frac{\sqrt{T}\left(\hat{S}'_{-1} M_{(\iota_T)} \varepsilon\right)}{\sqrt{\left(\hat{S}'_{-1} M_{(\iota_T)} \hat{S}_{-1}\right)(\varepsilon'\varepsilon)}} + O_p\left(T^{-1}\right); \end{aligned} \tag{A.13}$$

and

$$\begin{aligned} L_T^B(\lambda) &= \frac{\sqrt{T}\left(\tilde{S}'_{-1} M_{(\iota_T;\Delta D)} \varepsilon\right)}{\sqrt{\left(\tilde{S}'_{-1} M_{(\iota_T;\Delta D)} \tilde{S}_{-1}\right)\left(\varepsilon' M_{(\iota_T;\Delta D;\tilde{S}_{-1})} \varepsilon\right)}} + O_p\left(T^{-1}\right) \\ &= \frac{\sqrt{T}\left(\tilde{S}'_{-1} M_{(\iota_T;\Delta D)} \varepsilon\right)}{\sqrt{\left(\tilde{S}'_{-1} M_{(\iota_T;\Delta D)} \tilde{S}_{-1}\right)(\varepsilon'\varepsilon)}} + O_p\left(T^{-1}\right). \end{aligned} \tag{A.14}$$

Define

$$a_T = \hat{S}_{-1}' M_{(i_T)} \hat{S}_{-1} - S_{-1}' M_{(i_T;\Delta D)} S_{-1}; \; b_T = \hat{S}_{-1}' M_{(i_T)} \varepsilon - S_{-1}' M_{(i_T;\Delta D)} \varepsilon; \tag{A.15}$$

and

$$A_T = \frac{a_T}{\hat{S}_{-1}' M_{(i_T)} \hat{S}_{-1}}; \; B_T = \frac{b_T}{\sqrt{\left(\hat{S}_{-1}' M_{(i_T)} \hat{S}_{-1}\right)(\varepsilon'\varepsilon)}}. \tag{A.16}$$

Dividing both the numerator and the denominator of $L_T^B(\lambda)$ by $\sqrt{\left(\hat{S}_{-1}' M_{(i_T)} \hat{S}_{-1}\right)(\varepsilon'\varepsilon)}$; we have

$$\begin{aligned} L_T^B(\lambda) &= \frac{\left(\frac{\sqrt{T}\left(\hat{S}_{-1}' M_{(i_T)} \varepsilon\right)}{\sqrt{\left(\hat{S}_{-1}' M_{(i_T)} \hat{S}_{-1}\right)(\varepsilon'\varepsilon)}} - \sqrt{T} B_T\right)}{\sqrt{(1 - A_T)}} + O_p\left(T^{-1}\right) \\ &= L_T \sqrt{\left(1 + A_T + \frac{A_T^2}{1 - A_T}\right)} - \sqrt{T} B_T \sqrt{\left(1 + \frac{A_T}{1 - A_T}\right)} + O_p\left(T^{-1}\right). \end{aligned} \tag{A.17}$$

After some algebra (available from the authors upon request), we obtain

$$a_T = 2\varepsilon_{T_B+1} \left\{ -\frac{1}{T} \sum_{t=1}^{T} (t-1) S_{t-1} + \left(\frac{1}{2} + \lambda\right) \sum_{t=1}^{T} S_{t-1} + \left(\frac{1}{3} + \frac{1}{2}\lambda^2\right) T S_T \right\} + O_p(T); \tag{A.18}$$

and

$$b_T = \varepsilon_{T_B+1} \left[\frac{1}{2T} \sum_{t=1}^{T} t\varepsilon_t + \frac{1}{T} \sum_{t=1}^{T} S_{t-1} \right] + O_p(1); \tag{A.19}$$

where S_{t-1} is the partial sum process defined in (A.9), and $\lambda = \frac{T_B}{T}$. It now is obvious that $a_T = O_p\left(T^{3/2}\right)$; $b_T = O_p\left(T^{1/2}\right)$; and

$$A_T = O_p(T^{-1/2}) \text{ and } \sqrt{T} B_T = O_p(T^{-1/2}). \tag{A.20}$$

Therefore, it is seen from (A.17) that

$$L_T^B(\lambda) - L_T = \frac{1}{2} L_T A_T - \sqrt{T} B_T + O_p\left(T^{-1}\right); \tag{A.21}$$

which is $O_p(T^{-1/2})$.

Now we prove the claim in (A.2). From Assumption 3.1 of the uniform integrability of $L_T^B(\lambda)$ and L_T we have

$$E\left[L_T^B(\lambda) - L_T\right] = \frac{1}{2}E(L_T A_T) - \sqrt{T}E(B_T) + O\left(T^{-1}\right). \tag{A.22}$$

The result follows if we show $E(L_T A_T) = O(T^{-1})$ and $E(C_T) = O\left(T^{-3/2}\right)$. Combining (A.16) and (A.18) and applying Lemma 2, we have:

$$L_T A_T = \frac{2\varepsilon_{T_B+1}\hat{S}'_{-1}M_{(i_T)}\varepsilon\left[-\frac{1}{T}\sum_{t=1}^{T}(t-1)S_{t-1} + \left(\frac{1}{2}+\lambda\right)\sum_{t=1}^{T}S_{t-1} + \left(\frac{1}{3}+\frac{1}{2}\lambda^2\right)TS_T\right]}{\sigma\left(\hat{S}'_{-1}M_{(i_T)}\hat{S}_{-1}\right)^{3/2}} + O_p\left(T^{-1}\right). \tag{A.23}$$

It is straightforward to see that

$$\hat{S}'_{-1}M_{(i_T)}\hat{S}_{-1} = \sum_{t=1}^{T}S_{t-1}^2 - \frac{2S_T}{T}\sum_{t=1}^{T}(t-1)S_{t-1} + \frac{T}{12}S_T^2 - \frac{1}{T}\left(\sum_{t=1}^{T}S_{t-1}\right)^2 + S_T\sum_{t=1}^{T}S_{t-1} + O_p(T) \tag{A.24}$$

Let $\hat{S}'_{-1}M_{(i_T)}\hat{S}_{-1} = V_{1T} + V_{2T}$, where V_{1T} is the sum of all the terms that include ε_{T_B+1}, and V_{2T} is the sum of the rest of the terms. Thus, no ε_{T_B+1} is associated with V_{2T}. Then, it is straightforward to see that $V_{1T} = O_p\left(T^{3/2}\right)$ and $V_{2T} = O_p(T^2)$. For example, the term associated with ε_{T_B+1} in the leading term, $\sum_{t=1}^{T}S_{t-1}^2$, is the squared terms of order T plus the sum of the mean zero cross product terms, the order of which is $O_p\left(T^{3/2}\right)$. The order of the other terms associated with ε_{T_B+1} is similarly obtained. From the normality, ε_{T_B+1} is independent of V_{2T}.

We define the numerator of (A.23) to be $2\varepsilon_{T_B+1}U_T$, and divide both the numerator and the denominator of (A.23) by $(V_{2T})^{3/2}$ to have

$$L_T A_T = \frac{2\varepsilon_{T_B+1}U_T/(V_{2T})^{3/2}}{\sigma(1+V_{1T}/V_{2T})^{3/2}} = \frac{2\varepsilon_{T_B+1}U_T}{(V_{2T})^{3/2}} + O_p\left(T^{-1}\right). \tag{A.25}$$

Now let $U_T = U_{1T} + U_{2T}$, where U_{1T} is the sum of the terms associated with ε_{T_B+1} and U_{2T} is the sum of the rest. Therefore, we have;

$$E(L_T A_T) = 2E\frac{\varepsilon_{T_B+1}U_{1T}}{(V_{2T})^{3/2}} + 2E\frac{\varepsilon_{T_B+1}U_{2T}}{(V_{2T})^{3/2}} + O\left(T^{-1}\right). \tag{A.26}$$

From the normality assumption, $U_{2T}/(V_{2T})^{3/2}$ is independent of ε_{T_B+1} so that the second term of the right hand side vanishes. Also, it is straightforward to show that $U_{1T} = O_p(T^2)$. We therefore have $\varepsilon_{T_B+1}U_{1T}/(V_{2T})^{3/2} = O_p(T^{-1})$, and $E\left[\varepsilon_{T_B+1}U_{1T}/(V_{2T})^{3/2}\right] = O(T^{-1})$.

$E(B_T) = O\left[T^{-3/2}\right]$ is proved similarly. From (A.16) and (A.19), we have:

$$
\begin{aligned}
B_T &= \frac{\varepsilon_{T_B+1}\left(\frac{1}{2T}\sum_{t=1}^{T} t\varepsilon_t + \frac{1}{T}\sum_{t=1}^{T} S_{t-1}\right)}{\sqrt{\left(\hat{S}_{-1}' M_{(i_T)} \hat{S}_{-1}\right)(\varepsilon'\varepsilon)}} + O_p\left[T^{-3/2}\right] \\
&= \frac{\varepsilon_{T_B+1}\left(\frac{1}{2}\sum_{t=1}^{T} t\varepsilon_t + \sum_{t=1}^{T} S_{t-1}\right)}{\sigma\sqrt{T^3 \hat{S}_{-1}' M_{(i_T)} \hat{S}_{-1}}} + O_p\left[T^{-3/2}\right] && \text{(A.27)} \\
&= \frac{\varepsilon_{T_B+1}\left(\frac{1}{2}\sum_{t=1}^{T} t\varepsilon_t + \sum_{t=1}^{T} S_{t-1}\right)}{\sqrt{T^3 V_{2T}}} + O_p\left[T^{-3/2}\right]. && \text{(A.28)}
\end{aligned}
$$

The second equality follows from Lemma 2, and the third equality is obtained by dividing the numerator and the denominator by $\sqrt{T^3 V_{2T}}$. Let

$$W_T = W_{1T} + W_{2T} = \frac{1}{2}\sum_{t=1}^{T} t\varepsilon_t + \sum_{t=1}^{T} S_{t-1}, \qquad \text{(A.29)}$$

where W_{1T} is the sum of the terms associated with ε_{T_B+1} and W_{2T} is the sum of the rest of the terms. From the independence of ε_{T_B+1} and $W_{2T}/\sqrt{T^3 V_{2T}}$, we have

$$E(B_T) = E\frac{\varepsilon_{T_B+1}(W_{1T} + W_{2T})}{\sqrt{T^3 V_{2T}}} + O\left[T^{-3/2}\right] = E\frac{\varepsilon_{T_B+1} W_{1T}}{\sqrt{T^3 V_{2T}}} + O\left[T^{-3/2}\right]. \qquad \text{(A.30)}$$

But, $\varepsilon_{T_B+1} W_{1T} = O_p(T)$, and $\varepsilon_{T_B+1} W_{1T}/\sqrt{T^3 V_{2T}} = O_p\left[T^{-3/2}\right]$, so that we have $E(B_T) = O\left[T^{-3/2}\right]$, which completes the proof.

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Addendum to:
"LM Unit Root Test with Panel Data; A Test Robust to Structural Changes"
by Kyung So Im and Junsoo Lee

In this note we show the algebra for deriving a_T and b_T in (A.18) and (A.19) of the Appendix. Note that

$$\hat{S}'_{-1} M_{(\iota_T)} \hat{S}_{-1} = \sum_{t=1} \hat{S}^2_{t-1} - \frac{1}{T}\left(\sum_{t=1} \hat{S}_{t-1}\right)^2 . \tag{1}$$

We also have from Lemma 1

$$\begin{aligned}
\tilde{S}'_{-1} M_{(\iota_T, \Delta D)} \tilde{S}_{-1} &= \sum_{t=1} \tilde{S}^2_{t-1} - \tilde{S}^2_{T_B} - \frac{1}{T-1}\left(\sum_{t=1} \tilde{S}_{t-1} - \tilde{S}_{T_B}\right)^2 \\
&= \sum_{t=1} \hat{S}^2_{t-1} - 2\sum_{t=1} \hat{S}_{t-1} q_{t-1} + \sum_{t=1} q^2_{t-1} \qquad (2)\\
&\quad - \frac{1}{T-1}\left\{\left(\sum_{t=1} \hat{S}_{t-1}\right)^2 - 2\sum_{t=1} \hat{S}_{t-1} \sum_{t=1} q_{t-1} + \left(\sum_{t=1} q_{t-1}\right)^2\right. \\
&\quad \left. + 2\left(\hat{S}_{T_B} - q_{T_B}\right) \sum_{t=1} \left(\hat{S}_{t-1} - q_{t-1}\right) + T\left(\hat{S}_{T_B} - q_{T_B}\right)^2\right\} .
\end{aligned}$$

Plugging the results of (1) and (2) into a_T in (A.15) of the Appendix yields:

$$\begin{aligned}
a_T &= \hat{S}'_{-1} M_{(\iota_T)} \hat{S}_{-1} - \tilde{S}'_{-1} M_{(\iota_T, \Delta D)} \tilde{S}_{-1} \\
&= 2\sum_{t=1} \hat{S}_{t-1} q_{t-1} - \sum_{t=1} q^2_{t-1} + \frac{1}{T(T-1)}\left(\sum_{t=1} \hat{S}_{t-1}\right)^2 \\
&\quad - \frac{1}{T-1}\left\{2\sum_{t=1} \hat{S}_{t-1} \sum_{t=1} q_{t-1} - \left(\sum_{t=1} q_{t-1}\right)^2 + 2\left(\hat{S}_{T_B} - q_{T_B}\right) \sum_{t=1} \left(\hat{S}_{t-1} - q_{t-1}\right)\right\} \\
&\quad - \frac{T}{T-1}\left(\hat{S}_{T_B} - q_{T_B}\right)^2 .
\end{aligned}$$

It is straightforward to see that $\sum_{t=1}^{T} q^2_{t-1} = O_p(T)$; $\frac{1}{T(T-1)}\left(\sum_{t=1}^{T} \hat{S}_{t-1}\right)^2 = O_p(T)$; $\frac{1}{T-1}\left(\sum_{t=1}^{T} q_{t-1}\right)^2 = O_p(T)$; $\frac{1}{T-1}\left(\hat{S}_{T_B} - q_{T_B}\right)\sum_{t=1}^{T}\left(\hat{S}_{t-1} - q_{t-1}\right) = O_p(T)$;

and $\frac{T}{T-1}\left(\hat{S}_{T_B} - q_{T_B}\right)^2 = O_p(T)$. Therefore, we have

$$
\begin{aligned}
a_T &= 2\left(\sum_{t=1}\hat{S}_{t-1}q_{t-1} - \frac{1}{T}\sum_{t=1}\hat{S}_{t-1}\sum_{t=1}q_{t-1}\right) + O_p(T)\\
&= 2\varepsilon_{T_B+1}\left\{-\frac{1}{T}\sum_{t=1}(t-1)S_{t-1} + \left(\frac{1}{2}+\lambda\right)\sum_{t=1}S_{t-1} + \left(\frac{1}{3}+\frac{1}{2}\lambda^2\right)TS_T\right\} + O_p(T);
\end{aligned}
$$

which is what we have in (A.18) in the Appendix. Following a visual inspection, the order of a_T is $O_p\left(T^{3/2}\right)$; so that $A_T = O_p\left(T^{-1/2}\right)$.

Now we derive the expression in (A.19). Note that

$$
\begin{aligned}
\tilde{S}'_{-1}M_{(i_T;\Delta D)}\varepsilon &= \sum_{t=1}\tilde{S}_{t-1}\varepsilon_t - \tilde{S}_{T_B}\varepsilon_{T_B+1} - \frac{1}{T-1}\left(\sum_{t=1}\tilde{S}_{t-1} - \tilde{S}_{T_B}\right)\left(\sum_{t=1}\varepsilon_t - \varepsilon_{T_B+1}\right)\\
&= \sum_{t=1}\tilde{S}_{t-1}\varepsilon_t - \frac{1}{T-1}\left(\sum_{t=1}\tilde{S}_{t-1}\sum_{t=1}\varepsilon_t - \tilde{S}_{T_B}\sum_{t=1}\varepsilon_t - \varepsilon_{T_B+1}\sum_{t=1}\tilde{S}_{t-1} + T\tilde{S}_{T_B}\varepsilon_{T_B+1}\right)\\
&= \sum_{t=1}\hat{S}_{t-1}\varepsilon_t - \sum_{t=1}q_{t-1}\varepsilon_t\\
&\quad - \frac{1}{T-1}\left(\sum_{t=1}\hat{S}_{t-1}\sum_{t=1}\varepsilon_t - \sum_{t=1}q_{t-1}\sum_{t=1}\varepsilon_t - \tilde{S}_{T_B}\sum_{t=1}\varepsilon_t - \varepsilon_{T_B+1}\sum_{t=1}\tilde{S}_{t-1} + T\tilde{S}_{T_B}\varepsilon_{T_B+1}\right);
\end{aligned}
$$

and

$$
\hat{S}'_{-1}M_{(i_T)}\varepsilon = \sum_{t=1}\hat{S}_{t-1}\varepsilon_t - \frac{1}{T}\sum_{t=1}\hat{S}_{t-1}\sum_{t=1}\varepsilon_t.
$$

Therefore,

$$
\begin{aligned}
b_T &= \hat{S}'_{-1}M_{(i_T)}\varepsilon - \tilde{S}'_{-1}M_{(i_T;\Delta D)}\varepsilon\\
&= \sum_{t=1}q_{t-1}\varepsilon_t - \frac{1}{T(T-1)}\sum_{t=1}\hat{S}_{t-1}\sum_{t=1}\varepsilon_t\\
&\quad + \frac{1}{T-1}\left(-\sum_{t=1}q_{t-1}\sum_{t=1}\varepsilon_t - \hat{S}_{T_B}\sum_{t=1}\varepsilon_t - \varepsilon_{T_B+1}\sum_{t=1}\hat{S}_{t-1} + T\hat{S}_{T_B}\varepsilon_{T_B+1}\right).
\end{aligned}
$$

Examining each term, we have $\frac{1}{T(T-1)}\sum_{t=1}^{T}\hat{S}_{t-1}\sum_{t=1}^{T}\varepsilon_t = O_p(1)$; $\frac{1}{T-1}\tilde{S}_{T_B}\sum_{t=1}^{T}\varepsilon_t = O_p(1)$. Therefore,

$$
\begin{aligned}
\sum_{t=1}q_{t-1}\varepsilon_t &= \left(\frac{S_T}{T(T-1)} - \frac{\varepsilon_{T_B+1}}{T-1}\right)\sum_{t=1}(t-1)\varepsilon_t + \frac{T\varepsilon_{T_B+1} - S_T}{T-1}\sum_{t=T_{B+1}}\varepsilon_t\\
&= -\frac{1}{2T}\varepsilon_{T_B+1}\sum_{t=1}(t-1)\varepsilon_t + \varepsilon_{T_B+1}\sum_{t=T_{B+1}}\varepsilon_t + O_p(1);
\end{aligned}
$$

$$\frac{1}{T-1}\sum_{t=1} q_{t-1}\sum_{t=1}\varepsilon_t = \frac{1}{T-1}\left\{\frac{S_T}{T(T-1)} - \frac{\varepsilon_{T_B+1}}{T-1}\lambda\sum_{t=1}(t-1) + (1-\lambda)\frac{T\varepsilon_{T_B+1} - S_T}{T-1}\right\}\sum_{t=1}\varepsilon_t$$

$$= \left(\frac{1}{2} - \lambda\right)\varepsilon_{T_B+1}\sum_{t=1}\varepsilon_t + O_p(1);$$

$$\frac{1}{T-1}\varepsilon_{T_B+1}\sum_{t=1}\tilde{S}_{t-1} = \frac{1}{T-1}\varepsilon_{T_B+1}\sum_{t=1}\hat{S}_{t-1} + O_p(1)$$

$$= \frac{1}{T-1}\varepsilon_{T_B+1}\sum_{t=1}\left(S_{t-1} - \frac{S_T}{T}(t-1)\right) + O_p(1)$$

$$= \frac{1}{T}\varepsilon_{T_B+1}\sum_{t=1}S_{t-1} - \frac{1}{2}\varepsilon_{T_B+1}S_T + O_p(1);$$

and

$$\frac{T}{T-1}\tilde{S}_{T_B}\varepsilon_{T_B+1} = \hat{S}_{T_B}\varepsilon_{T_B+1} + O_p(1)$$

$$= S_{T_B}\varepsilon_{T_B+1} + \lambda S_T\varepsilon_{T_B+1} + O_p(1):$$

Combining these results, we obtain

$$b_T = \frac{1}{2T}\varepsilon_{T_B+1}\sum_{t=1}(t-1)\varepsilon_t - \varepsilon_{T_B+1}\sum_{t=T_B+1}\varepsilon_t + \frac{1}{T}\varepsilon_{T_B+1}\sum_{t=1}S_{t-1} - S_{T_B}\varepsilon_{T_B+1} + O_p(1)$$

$$= \varepsilon_{T_B+1}\left[\frac{1}{2T}\sum_{t=1}t\varepsilon_t + \frac{1}{T}\sum_{t=1}S_{t-1}\right] + O_p(1);$$

which is the expression in (A.19) of the Appendix. It is obvious that $b_T = O_p\left(T^{1/2}\right)$: Hence, $\sqrt{T}C_T = O_p\left(T^{-1/2}\right)$:

Table 1

Mean and Variance of L_T(p)

T	*p=0* Mean	*p=0* Var.	*p=1* Mean	*p=1* Var.	*p=2* Mean	*p=2* Var.	*p=3* Mean	*p=3* Var.	*p=4* Mean	*p=4* Var.	*p=5* Mean	*p=5* Var.	*p=6* Mean	*p=6* Var.	*p=7* Mean	*p=7* Var.	*p=8* Mean	*p=8* Var.
10	-2.012	0.518	-1.997	0.662	-1.761	0.708	-1.751	1.082										
11	-2.009	0.493	-1.994	0.604	-1.781	0.623	-1.765	0.887	-1.590	1.070								
12	-2.003	0.476	-1.991	0.561	-1.795	0.572	-1.776	0.763	-1.611	0.885	-1.604	1.218						
13	-2.002	0.464	-1.990	0.535	-1.810	0.537	-1.790	0.686	-1.635	0.779	-1.620	1.018	-1.489	1.188				
14	-1.999	0.451	-1.990	0.511	-1.823	0.509	-1.803	0.629	-1.655	0.702	-1.638	0.898	-1.511	1.016	-1.503	1.317		
15	-1.998	0.441	-1.988	0.490	-1.833	0.490	-1.814	0.591	-1.675	0.648	-1.657	0.806	-1.534	0.901	-1.521	1.123	-1.412	1.292
16	-1.995	0.435	-1.987	0.480	-1.842	0.470	-1.825	0.557	-1.693	0.605	-1.676	0.737	-1.559	0.821	-1.545	0.998	-1.439	1.116
17	-1.994	0.428	-1.985	0.464	-1.851	0.457	-1.834	0.533	-1.709	0.572	-1.689	0.684	-1.577	0.755	-1.559	0.904	-1.458	0.996
18	-1.992	0.420	-1.985	0.455	-1.858	0.444	-1.843	0.512	-1.725	0.543	-1.705	0.642	-1.598	0.704	-1.580	0.833	-1.481	0.909
19	-1.992	0.417	-1.984	0.446	-1.865	0.438	-1.851	0.498	-1.738	0.521	-1.720	0.607	-1.617	0.664	-1.599	0.779	-1.503	0.844
20	-1.990	0.411	-1.982	0.437	-1.870	0.427	-1.858	0.481	-1.750	0.502	-1.733	0.580	-1.633	0.629	-1.615	0.731	-1.524	0.793
21	-1.988	0.407	-1.983	0.429	-1.875	0.418	-1.864	0.468	-1.760	0.487	-1.745	0.558	-1.649	0.598	-1.632	0.691	-1.543	0.747
22	-1.987	0.402	-1.982	0.424	-1.880	0.413	-1.869	0.456	-1.773	0.473	-1.757	0.536	-1.666	0.574	-1.650	0.657	-1.563	0.706
23	-1.988	0.399	-1.982	0.418	-1.884	0.408	-1.875	0.449	-1.781	0.461	-1.766	0.520	-1.678	0.552	-1.661	0.626	-1.579	0.675
24	-1.987	0.397	-1.982	0.416	-1.889	0.403	-1.879	0.441	-1.789	0.453	-1.775	0.505	-1.690	0.534	-1.675	0.601	-1.595	0.645
25	-1.985	0.393	-1.980	0.410	-1.892	0.399	-1.883	0.434	-1.798	0.445	-1.785	0.494	-1.702	0.520	-1.687	0.581	-1.609	0.620
26	-1.985	0.391	-1.980	0.406	-1.895	0.395	-1.887	0.426	-1.803	0.433	-1.791	0.478	-1.712	0.504	-1.697	0.561	-1.621	0.597
27	-1.985	0.389	-1.980	0.402	-1.898	0.391	-1.890	0.421	-1.810	0.430	-1.799	0.470	-1.722	0.489	-1.709	0.542	-1.636	0.576
28	-1.984	0.386	-1.980	0.399	-1.901	0.387	-1.894	0.415	-1.818	0.420	-1.806	0.458	-1.732	0.479	-1.718	0.528	-1.646	0.557
29	-1.983	0.384	-1.977	0.396	-1.902	0.386	-1.894	0.410	-1.820	0.415	-1.809	0.450	-1.738	0.469	-1.726	0.515	-1.657	0.544
30	-1.983	0.381	-1.978	0.393	-1.906	0.384	-1.900	0.408	-1.828	0.411	-1.818	0.443	-1.748	0.460	-1.735	0.502	-1.667	0.528
31	-1.982	0.381	-1.978	0.391	-1.908	0.382	-1.902	0.403	-1.834	0.407	-1.824	0.437	-1.758	0.453	-1.746	0.494	-1.681	0.516
32	-1.982	0.379	-1.979	0.390	-1.911	0.379	-1.905	0.400	-1.838	0.402	-1.829	0.430	-1.764	0.443	-1.753	0.481	-1.689	0.504
33	-1.981	0.376	-1.977	0.387	-1.912	0.377	-1.906	0.396	-1.841	0.398	-1.833	0.426	-1.770	0.440	-1.759	0.474	-1.697	0.494
34	-1.980	0.375	-1.977	0.386	-1.913	0.375	-1.909	0.395	-1.845	0.394	-1.836	0.419	-1.776	0.432	-1.766	0.464	-1.706	0.485
35	-1.981	0.374	-1.978	0.383	-1.916	0.373	-1.911	0.392	-1.849	0.392	-1.841	0.417	-1.782	0.427	-1.771	0.459	-1.712	0.475

Table 1 Continued

T	*p=0*		*p=1*		*p=2*		*p=3*		*p=4*		*p=5*		*p=6*		*p=7*		*p=8*	
	Mean	Var.	Mean	Var.	Mean	Var.	Mean	Var.	Mean	Var.	Mean	Var.	Mean	Var.	Mean	Var.	Mean	Var.
36	-1.980	0.373	-1.977	0.381	-1.918	0.373	-1.913	0.389	-1.853	0.391	-1.845	0.414	-1.787	0.421	-1.777	0.452	-1.719	0.469
37	-1.979	0.371	-1.977	0.381	-1.919	0.372	-1.914	0.388	-1.856	0.387	-1.849	0.409	-1.793	0.420	-1.783	0.447	-1.728	0.463
38	-1.980	0.371	-1.977	0.379	-1.919	0.369	-1.915	0.383	-1.858	0.384	-1.852	0.405	-1.797	0.412	-1.788	0.439	-1.734	0.454
39	-1.979	0.370	-1.977	0.377	-1.922	0.368	-1.918	0.382	-1.863	0.382	-1.856	0.403	-1.803	0.408	-1.795	0.434	-1.742	0.447
40	-1.978	0.368	-1.974	0.375	-1.921	0.367	-1.917	0.381	-1.864	0.380	-1.858	0.399	-1.806	0.405	-1.798	0.429	-1.747	0.441
41	-1.979	0.367	-1.976	0.375	-1.924	0.365	-1.920	0.378	-1.868	0.377	-1.862	0.395	-1.810	0.401	-1.802	0.425	-1.752	0.436
42	-1.979	0.367	-1.977	0.374	-1.926	0.364	-1.921	0.376	-1.871	0.376	-1.865	0.392	-1.815	0.398	-1.807	0.419	-1.758	0.430
43	-1.979	0.366	-1.977	0.373	-1.927	0.364	-1.924	0.377	-1.874	0.375	-1.868	0.391	-1.820	0.396	-1.812	0.417	-1.764	0.428
44	-1.978	0.364	-1.975	0.371	-1.927	0.362	-1.923	0.374	-1.875	0.371	-1.870	0.388	-1.822	0.392	-1.815	0.412	-1.768	0.422
45	-1.977	0.363	-1.975	0.370	-1.928	0.361	-1.924	0.372	-1.877	0.370	-1.872	0.386	-1.825	0.390	-1.818	0.408	-1.772	0.418
46	-1.978	0.364	-1.975	0.369	-1.928	0.360	-1.925	0.371	-1.879	0.369	-1.874	0.384	-1.828	0.388	-1.821	0.407	-1.776	0.415
47	-1.977	0.362	-1.974	0.368	-1.928	0.359	-1.925	0.370	-1.880	0.368	-1.875	0.381	-1.831	0.384	-1.824	0.403	-1.780	0.411
48	-1.977	0.362	-1.974	0.367	-1.930	0.360	-1.927	0.369	-1.882	0.365	-1.878	0.380	-1.835	0.382	-1.830	0.399	-1.787	0.407
49	-1.978	0.361	-1.976	0.368	-1.933	0.360	-1.929	0.369	-1.886	0.366	-1.881	0.379	-1.838	0.381	-1.832	0.398	-1.789	0.405
50	-1.976	0.360	-1.974	0.365	-1.932	0.357	-1.930	0.366	-1.888	0.363	-1.884	0.377	-1.841	0.379	-1.835	0.394	-1.794	0.402
55	-1.976	0.358	-1.974	0.363	-1.936	0.355	-1.933	0.363	-1.894	0.360	-1.891	0.371	-1.854	0.371	-1.850	0.385	-1.812	0.390
60	-1.975	0.355	-1.974	0.359	-1.939	0.353	-1.937	0.360	-1.902	0.357	-1.899	0.366	-1.864	0.365	-1.859	0.377	-1.824	0.379
65	-1.975	0.352	-1.973	0.356	-1.941	0.351	-1.939	0.356	-1.907	0.351	-1.904	0.360	-1.872	0.359	-1.868	0.370	-1.835	0.372
70	-1.974	0.351	-1.972	0.355	-1.943	0.349	-1.941	0.353	-1.912	0.349	-1.909	0.357	-1.880	0.357	-1.876	0.366	-1.846	0.367
75	-1.974	0.350	-1.973	0.353	-1.945	0.347	-1.943	0.351	-1.914	0.347	-1.913	0.354	-1.885	0.353	-1.882	0.361	-1.854	0.361
80	-1.973	0.348	-1.971	0.351	-1.945	0.347	-1.944	0.350	-1.918	0.347	-1.916	0.352	-1.890	0.350	-1.887	0.358	-1.861	0.358
85	-1.973	0.347	-1.972	0.349	-1.947	0.344	-1.946	0.347	-1.921	0.343	-1.920	0.349	-1.895	0.347	-1.892	0.354	-1.869	0.355
90	-1.973	0.348	-1.971	0.349	-1.949	0.344	-1.948	0.348	-1.925	0.345	-1.923	0.349	-1.900	0.348	-1.898	0.352	-1.875	0.352
95	-1.973	0.346	-1.971	0.347	-1.949	0.342	-1.948	0.345	-1.926	0.342	-1.925	0.346	-1.904	0.345	-1.902	0.350	-1.880	0.349
100	-1.973	0.346	-1.972	0.347	-1.951	0.343	-1.950	0.346	-1.929	0.343	-1.928	0.347	-1.908	0.346	-1.905	0.350	-1.885	0.350
200	-1.970	0.337	-1.969	0.337	-1.960	0.336	-1.959	0.337	-1.949	0.334	-1.949	0.335	-1.938	0.333	-1.938	0.336	-1.927	0.334

See Definition 2 and Remark 3 in the text.

Table 2
Size and Power of Panel Unit Root Tests
(Experiment 1: No Structural Breaks, No Serial Correlation)

		T = 10		*T* = 25		*T* = 50		*T* = 100	
		Size	Power	Size	Power	Size	Power	Size	Power
N=10	LM	0.065	0.086	0.068	0.149	0.068	0.484	0.051	0.990
	IPS	0.063	0.062	0.062	0.124	0.059	0.357	0.050	0.961
N=25	LM	0.064	0.080	0.054	0.207	0.058	0.802	0.060	1.000
	IPS	0.051	0.066	0.051	0.162	0.059	0.622	0.055	1.000
N=50	LM	0.050	0.072	0.056	0.325	0.053	0.974	0.058	1.000
	IPS	0.046	0.071	0.051	0.261	0.047	0.899	0.057	1.000
N=100	LM	0.051	0.083	0.060	0.518	0.052	1.000	0.042	1.000
	IPS	0.051	0.072	0.063	0.415	0.054	0.997	0.049	1.000

Table 3
Size and Power of Panel Unit Root Tests
(Experiment 2: Structural Breaks, No Serial Correlation)

		T = 10		*T* = 25		*T* = 50		*T = 100*	
		Size	Power	Size	Power	Size	Power	Size	Power
					l = 0.5				
N=10	LM_N	0.008	0.008	0.033	0.055	0.049	0.226	0.065	0.863
	IPS_N	0.001	0.002	0.017	0.038	0.033	0.144	0.047	0.745
	LM_B	0.083	0.094	0.074	0.160	0.072	0.460	0.064	0.985
N=25	LM_N	0.001	0.002	0.021	0.051	0.042	0.379	0.067	0.997
	IPS_N	0.001	0.001	0.016	0.040	0.033	0.272	0.044	0.989
	LM_B	0.060	0.074	0.054	0.201	0.067	0.775	0.075	1.000
N=50	LM_N	0.001	0.001	0.013	0.050	0.036	0.622	0.043	1.000
	IPS_N	0.000	0.000	0.007	0.043	0.023	0.486	0.030	1.000
	LM_B	0.062	0.086	0.064	0.297	0.060	0.968	0.059	1.000
N=100	LM_N	0.000	0.000	0.006	0.050	0.028	0.870	0.043	1.000
	IPS_N	0.000	0.000	0.005	0.048	0.017	0.763	0.036	1.000
	LM_B	0.068	0.106	0.058	0.473	0.059	1.000	0.050	1.000
					l = 0.3				
N=10	LM_N	0.006	0.007	0.029	0.046	0.057	0.199	0.056	0.851
	IPS_N	0.000	0.000	0.006	0.007	0.025	0.046	0.035	0.534
	LM_B	0.070	0.081	0.070	0.144	0.068	0.452	0.061	0.986
N=25	LM_N	0.001	0.000	0.019	0.048	0.046	0.353	0.046	0.998
	IPS_N	0.000	0.000	0.002	0.003	0.009	0.058	0.016	0.877
	LM_B	0.068	0.085	0.056	0.204	0.060	0.781	0.072	1.000
N=50	LM_N	0.000	0.000	0.009	0.032	0.033	0.536	0.048	1.000
	IPS_N	0.000	0.000	0.000	0.000	0.003	0.077	0.017	0.995
	LM_B	0.061	0.087	0.056	0.291	0.056	0.961	0.060	1.000
N=100	LM_N	0.000	0.000	0.004	0.028	0.024	0.810	0.040	1.000
	IPS_N	0.000	0.000	0.000	0.000	0.000	0.085	0.007	1.000
	LM_B	0.070	0.104	0.051	0.459	0.054	1.000	0.050	1.000

See Section 4.2 for the definition of LM_N, IPS_N and LM_B.

Table 4
Size and Power of Panel LM Unit Root Tests
(Experiment 3: Structural Breaks, Serial Correlation)

T	*N*	*p = 0* Size	*p = 0* Power	*p = 1* Size	*p = 1* Power	*p = 2* Size	*p = 2* Power	*p = 3* Size	*p = 3* Power	*p = 4* Size	*p = 4* Power
						AR(1) Error: *r* = 0.3					
25	10	0.000	0.000	0.059	0.111	0.061	0.105	0.064	0.098	0.070	0.088
	25	0.000	0.000	0.064	0.163	0.086	0.160	0.075	0.135	0.070	0.110
	50	0.000	0.000	0.061	0.229	0.075	0.180	0.061	0.144	0.075	0.130
	100	0.000	0.000	0.058	0.357	0.077	0.283	0.072	0.207	0.073	0.165
50	10	0.000	0.000	0.061	0.343	0.067	0.299	0.065	0.253	0.059	0.208
	25	0.000	0.000	0.065	0.632	0.079	0.540	0.073	0.462	0.073	0.361
	50	0.000	0.000	0.063	0.883	0.068	0.810	0.069	0.703	0.069	0.584
	100	0.000	0.000	0.066	0.993	0.073	0.972	0.070	0.916	0.059	0.823
100	10	0.000	0.024	0.066	0.954	0.076	0.913	0.066	0.859	0.067	0.808
	25	0.000	0.027	0.060	1.000	0.061	1.000	0.061	0.998	0.059	0.991
	50	0.000	0.032	0.044	1.000	0.050	1.000	0.047	1.000	0.056	1.000
	100	0.000	0.044	0.063	1.000	0.070	1.000	0.069	1.000	0.076	1.000
						MA(1) Error: *q* = -0.3					
25	10	0.880	0.961	0.165	0.267	0.063	0.106	0.043	0.073	0.042	0.052
	25	1.000	1.000	0.237	0.441	0.059	0.132	0.040	0.063	0.030	0.051
	50	1.000	1.000	0.334	0.656	0.052	0.160	0.024	0.066	0.021	0.053
	100	1.000	1.000	0.540	0.890	0.058	0.237	0.021	0.078	0.019	0.053
50	10	0.974	1.000	0.232	0.705	0.081	0.356	0.052	0.233	0.044	0.186
	25	1.000	1.000	0.410	0.965	0.098	0.658	0.055	0.431	0.042	0.330
	50	1.000	1.000	0.620	1.000	0.110	0.884	0.049	0.677	0.042	0.493
	100	1.000	1.000	0.859	1.000	0.138	0.994	0.050	0.898	0.029	0.732
100	10	0.992	1.000	0.304	1.000	0.096	0.967	0.065	0.890	0.054	0.810
	25	1.000	1.000	0.542	1.000	0.137	1.000	0.074	0.999	0.061	0.992
	50	1.000	1.000	0.774	1.000	0.162	1.000	0.066	1.000	0.051	1.000
	100	1.000	1.000	0.956	1.000	0.211	1.000	0.054	1.000	0.035	1.000

All the tests are conducted based on the statistic $\Gamma_{LM}^{B}(p)$ defined in (3.14) of the text. See Section 4.3.