Collusion, Exclusion, and Inclusion in Random-Order Bargaining*

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Abstract

The paper examines the profitability of three types of contracts in random-order values of a cooperative game. An *exclusive* contract delays the contribution of the excluded player j until the arrival of the excluding player i. It is profitable when j is complementary to other players in the absence of i. An *inclusive* contract brings the included player j forward to player i's arrival. It is profitable when j substitutable to other players in the presence of i. Finally, a *collusive* contract between i and j can be described as a proxy agreement under which i always brings j with him. The profit from this contract is therefore the sum of profits from exclusion and inclusion, and is positive when i reduces the complementarity between j and the other players.

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1 Introduction

The effect of integration by economic agents on their bargaining share has long been studied in economic theory. While the old conventional wisdom held that size confers a bargaining advantage (see e.g. Galbraith [1952]), theoretical analysis, starting with Aumann [1973], demonstrated that this is not always the case. Since then, numerous papers have examined the bargaining effect of integration in specific settings, with applications to horizontal integration (Guesnerie [1977], Gardner [1977], Legros [1987]), vertical integration (Stole and Zwiebel [1998], Heavner [1999]), union formation in labor bargaining (Horn and Wolinsky [1988], Stole and Zwiebel [1996a]), and proxy agreements in voting games (Haller [1994]). All these papers model integration as a collusive contract that merges the resources of a group of players in the hands of one "proxy" player.

But integration does not always result in perfect collusion. For example, Hart and Moore [1990] observe that while integration unifies the ownership of physical assets, it does not affect the ownership of inalienable human assets. The owner of a physical asset has the right to exclude other agents from it, but not the right to use other agents' human assets himself. In the extreme case in which human assets are perfectly complementary to the physical asset, its acquisition is equivalent to an exclusive contract on the human assets (see e.g. Segal and Whinston [1998]), rather than to a collusive contract. Finally, Haller [1994] considers yet another kind of contracts, which he calls "associations" and we will call inclusive contracts, that give one player the right to use another player's resource, but not to exclude the latter player from using it himself.

The present paper derives simple general conditions for collusive, exclusive, and inclusive contracts to be advantageous (or disadvantageous) to a coalition of players in a transferable-utility cooperative game solved by a random-order value (Weber [1988]). In this solution, each player receives his expected marginal contribution to the set of preceding players in various orderings of players, and all players use the same probability distribution over orderings. The random-order value in which all orderings are equally likely is the Shapley value; however, we consider the more general case, which allows players to have asymmetric bargaining powers against various coalitions. We derive conditions that are necessary and sufficient for contracts to be profitable (or unprofitable) for all possible probability distributions over orderings.

The simple intuition behind our results for the case of contracting between two players, i and j, is demonstrated in Figure 1. Consider two particular orderings of players: in ordering 1 player i arrives before player j, while in ordering 2 the two players are switched. Suppose the two players write an exclusive contract giving player i the right to exclude player j. The effect of this contract is shown in Figure 1(a). In ordering 1, player j arrives after player i, hence he is not excluded, and both players' marginal contributions are the same as if no contract is written. In ordering 2, on the other hand, the contribution of player j is delayed until the arrival of player i. Thus, the marginal contribution of player j's resource is now evaluated relative to a larger coalition. This increases the total marginal contribution of the two players if and only if player j is *complementary* to other players *in the absence of player* i.

Now consider an inclusive contract between the players, under which player i can use player j's resource himself, but cannot exclude player j from using it. The effect of this contract is depicted in Figure 1(b). The contract does not affect the players' marginal contributions in ordering 2. In ordering 1, on the other hand, the contract allows player i upon arrival to bring forward player j's resource. Thus, the marginal contribution of player j's resource is now evaluated relative to a smaller coalition. This increases the total marginal contribution of the two players if and only if player j is *substitutable* to other players *in the presence of player* i.

Finally, a collusive contract transfers both players' resources into the hands of one "proxy" player, while the other player becomes a dummy. If the random-order value treats the two players symmetrically (which we assume here), it does not matter which of the two players becomes the proxy. (Moreover, for the Shapley value, the other contracting player can be eliminated, and the value of the colluding coalition can be calculated using the Shapley value for $N-1$ players.) For definiteness, let player i be the proxy player. Under this contract, player i always brings player j's resource with him, regardless of whether he arrives before or after player j. Therefore, as shown in Figure 1(c), the collusive contract is equivalent to an inclusive contract in ordering 1, and to an exclusive contract in ordering 2. The profit from this contract therefore equals to the sum of the profits from exclusive and inclusive contracts. When orderings 1 and 2 are equally likely, this observation yields a simple condition for the profitability of the collusive contract. which depends on how the complementarity of player j with other players is affected by player i. For example, when player i is a dummy, the gain from exclusion exactly offsets the loss from inclusion (or vice versa), and the collusive contract does not affect the two players' joint payoff. On the other hand, when player i reduces (increases) the complementarity between player j and other players, the gain from exclusion exceeds (falls short of) the loss from inclusion, and the collusive contract is profitable (unprofitable).[1]

The remainder of this paper formalizes the above intuition and extends it to contracts involving more than two players. It identifies simple conditions for contracts among members of a coalition $S \subset M$ to benefit or hurt M, which are robust to the choice of probability distribution over players' orderings and to the choice of contracting players from M. Section 2 introduces notation and interpretation for the cooperative game and contracting within its framework. Sections 3, 4, and 5 examine exclusion, inclusion, and collusion respectively. Section 6 demonstrates how the general results can be applied to several applied settings considered in the existing literature. In conclusion, Section 7 discusses which contracts more realistically describe integration in various applied settings, and suggests some directions for future research.

2 Setup

2.1 The Game

Consider a transferable-utility cooperative game with the set of players $N = \{1, \ldots, n\}$. The game is described by its characteristic function $v : 2^N \to \Re$. $v(S)$ is called the *worth* of coalition $S \subset N$. We adopt the standard convention that $v(\emptyset) = 0$.

The cooperative game will be interpreted as derived from an underlying economy with quasilinear preferences. Agents in this economy own resources, which can be combined to generate surplus. The worth of a coalition

[1] The closest results to ours are obtained by Haller [1994], who derives general formulae for the profits from collusion and mutual inclusion ("association") for symmetric probabilistic values (as defined in Weber [1988]). The only intersection of these values and the random-order values we consider is the Shapley value. All other symmetric probabilistic values require players to hold inconsistent beliefs about the probabilities of different orderings, which results in the players' payoffs not adding up to the worth of the grand coalition. As should be clear from the intuition in the text, it is the consistency of players' beliefs over orderings that gives rise to our sharp conditions for the profitability of contracts in random-order values.

is the maximum surplus achievable by combining its members' resources.[2] Agents' resources can be of two kinds: standard economic goods ("physical resources") and technologies or abilities that are unique to individual agents ("human resources"). The feasibility of the "integration" contracts we consider may depend on whether they involve physical or human resources: while physical resources (such as apples) can be transferred with a long-term contract, human resources (such as an agent's unique ability to produce apples or to enjoy apples) may not be. Indeed, most countries have laws against indentured servitude, which give rise to "inalienability of human capital". This point will be important for application of our results, as we will discuss in Conclusion.

2.2 Solution Concept

The cooperative game is solved by a random-order value [Weber 1988]. To define this value, we introduce the following notation. Let $\mathbf{\Pi}$ denote the set of orderings of N. We will treat the set $\mathbf{\Pi}$ as the group of permutations of $N = \{1, \ldots, n\}$, the product of two permutations being defined as their composition. (This group is called the symmetric group of degree $|N| = n$, see e.g. Kargapolov and Merzljakov [1979].) Let $\pi(i)$ denote the number of player $i \in N$ in ordering $\pi \in \mathbf{\Pi}$, in which case the player who has number k in ordering π can be described as $\pi^{-1}(k)$. Also, let $\pi^i = \{j \in N : \pi(j) \le \pi(i)\}$ denote the set of players that come before player i in ordering π, including i himself. (Note that $|\pi^i| = \pi(i)$, and that the statements $\pi(j) \le \pi(i)$, $j \in \pi^i$, and $\pi^j \subset \pi^i$ are equivalent.) Let $\Delta_{\mathbf{\Pi}} = \left\{\alpha \in \Re_+^{\mathbf{\Pi}} : \sum_{\pi \in \mathbf{\Pi}} \alpha_\pi = 1\right\}$ denote the set of probability distributions over $\mathbf{\Pi}$.

Next, we define the marginal contribution of player $k \in N$ to coalition $S \subset N$ as $\Delta_k v(S) = v(S \cup k) - v(S \backslash k)$. For future reference we also introduce higher-order differences as follows: $\Delta^2_{i,j} v(S) = \Delta_i \left(\Delta_j v(S)\right)$, and $\Delta^3_{i,j,k} v(S) =$

[2] This interpretation naturally leads to the superadditivity of the characteristic function. Indeed, for each $A, B \subset N$ with $A \cap B = \emptyset$, coalition $A \cup B$ can replicate the combinations of resources available to coalitions A and B separately, hence $v(A \cup B) \ge v(A) + v(B)$. However, other interpretations of cooperative games, such as the one in Hart and Mas-Colell [1996], allow the existence of a "public" resource (technology) that is only available to the forming coalition. In this case we need not have superadditivity, since coalition $A \cup B$ is unable to duplicate the public resource. Our general analysis is consistent with the existence of such a "public resource"; however, we do not find it realistic in economic applications, and in some examples in Section 6 we assume superadditivity to obtain sharper results.

$\Delta_i \left(\Delta^2_{j,k}\right) v(S)$. Note that these expressions do not depend on the order of taking differences (e.g., $\Delta^2_{i,j} v(S) = \Delta^2_{j,i} v(S)$), and that $\Delta^2_{i,i} v(S) = 0$.

Using this notation, for each probability distribution $\alpha \in \Delta_{\mathbf{\Pi}}$ we define a random-order value $f^\alpha(v)$, which assigns to each player $k \in N$ a payoff of

$$f^\alpha_k(v) = \sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \Delta_k v(\pi^k).$$

The joint value of a group $M \subset N$ of players will be denoted by $f^\alpha_M(v) = \sum_{i \in M} f^\alpha_i(v)$.

Weber [1988] characterizes random-order values by the linearity, dummy, monotonicity, and efficiency axioms. (Addition of the symmetry axiom yields the Shapley value, which is the random-order value in which all orderings are equally likely.) Alternatively, random-order values can be given a noncooperative foundation. Gul [1989], Hart and Mas-Colell [1996], and Stole and Zwiebel [1996b] suggest different noncooperative bargaining games that give rise to the Shapley value.[3] Asymmetric versions of their games would give rise to asymmetric random-order values.

One of the simplest noncooperative games generating random-order values can be described as follows: First nature chooses an ordering $\pi \in \mathbf{\Pi}$ at random according to the distribution α, then n bargaining stages follow, numbered in reverse order n to 1. At each stage k, agent $i = \pi^{-1}(k)$ makes an offer to the set $\pi^i \backslash i$ of preceding agents. The offer specifies a division of surplus $v(\pi^i)$ among the coalition π^i. If the offer is rejected by at least one agent from $\pi^i \backslash i$, the proposer leaves and the game proceeds to stage $k-1$, otherwise the game ends and the proposed division is implemented. In a subgame-perfect equilibrium for any realized ordering π, each player k receives $\Delta_k v(\pi^k)$. Therefore, the expected payoff of each player k is his random-order value $f^\alpha_k(v)$.

2.3 Contracts

We follow the "property rights" methodology of Hart and Moore [1990] by assuming that contracts among players do not affect the bargaining solution.

[3]Not all these games are equally realistic in the case where players own inalienable "human resources". For example, Gul's [1989] pairwise bargaining game, in which players accumulate other players' resources, is less realistic than Hart and Mas-Colell's [1996] game, in which no long-term transfers of resources are made.

The motivation for this assumption is that the bargaining procedure is only a function of the players' innate bargaining abilities.[4] What contracts do affect is the resources at the disposal of various coalitions, and thereby the characteristic function. For simplicity we will not consider contracts that split the resources initially owned by one agent, thus for expositional purposes we assume that each agent owns a single indivisible resource. A contract can then be represented by a *resource mapping* $A : 2^N \to 2^N$, where $A(S) \subset N$ is the set of agents whose resources are at the disposal of coalition $S \subset N$. This contract gives rise to a cooperative game described by the new characteristic function vA, defined by $vA(S) = v(AS)$ for all $S \subset N$.

In reality, a contract is usually accompanied by monetary side transfers. We assume that these side transfers are lump-sum and therefore do not affect the bargaining solution. If a coalition of players can use unrestricted side transfers, it will enter into a contract whenever it increases the coalition's joint value.

3 Exclusion

3.1 Bilateral Contracts

We first consider an exclusive contract between two players $i, j \in N$. The contract gives player i the right to exclude player j's resource, but not the right to use this resource. One interpretation of the contract is that player j's resource becomes "jointly owned" by the two players in the sense of Hart and Moore [1990]: both players have veto power over it.[5] The resource mapping E_i^j corresponding to this contract is therefore given by

$$E_i^j(S) = \begin{cases} S \text{ if } i \in S, \\ S \backslash j \text{ otherwise.} \end{cases}$$

The first result identifies the effect of this exclusive contract on the joint payoff of a coalition M that includes i and j. (While it is natural to focus on the coalition $M = \{i, j\}$, considering larger coalitions will enable us to apply the result to multi-agent exclusive contracts.)

[4] See, however, Aghion et al. [1994] for a model where contracts do affect the players' bargaining powers.

[5] An equivalent situation obtains when player j's resource is split into two perfectly complementary assets, say physical and human assets, and player i gains control over the physical asset, while player j retains control over the human asset.

Proposition 1 *For $i, j \in M \subset N$,*

$$f_M^\alpha(vE_i^j) - f_M^\alpha(v) = \sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \sum_{k \in \pi^i \backslash \pi^j \backslash M} \Delta_{j,k}^2 v(\pi^k).$$

Proof. Letting $g_M^\pi(v) = \sum_{k \in M} \Delta_k v(\pi^k)$, we can write

$$f_M^\alpha(vE_i^j) - f_M^\alpha(v) = \sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \left[g_M^\pi(vE_i^j) - g_M^\pi(v) \right]$$

For those $\pi \in \mathbf{\Pi}$ for which $\pi^i \subset \pi^j$, we have $E_i^j \pi^k = \pi^k$ for all k, hence $g_M^\pi(vE_i^j) = g_M^\pi(v)$. For those $\pi \in \mathbf{\Pi}$ for which $\pi^j \subset \pi^i$, on the other hand, we can write

$$\begin{aligned} g_M^\pi(vE_i^j) - g_M^\pi(v) &= \sum_{k \in M} \left[\Delta_k v(E_i^j \pi^k) - \Delta_k v(\pi^k) \right] = \\ &= \Delta_j v(\pi^i \backslash i) - \Delta_j v(\pi^j) + \sum_{k \in M \cap (\pi^i \backslash i \backslash \pi^j)} \left[\Delta_k v(\pi^k \backslash j) - \Delta_k v(\pi^k) \right]. \end{aligned}$$

The first term is for $k = i$, the following term for $k = j$, and the last sum collects all other nonzero terms. The expression can be further rewritten as

$$g_M^\pi(vE_i^j) - g_M^\pi(v) = \sum_{k \in \pi^i \backslash i \backslash \pi^j} \Delta_{j,k}^2 v(\pi^k) - \sum_{k \in M \cap (\pi^i \backslash i \backslash \pi^j)} \Delta_{j,k}^2 v(\pi^k) = \sum_{k \in \pi^i \backslash \pi^j \backslash M} \Delta_{j,k}^2 v(\pi^k),$$

which implies the statement. ■

The intuition behind the derived formula runs as follows. For a given ordering π of players in which player j comes before player i (i.e., $\pi^j \subset \pi^i$), consider the changes in the joint marginal contribution of coalition M as player j is moved backward by one position at a time until he comes just before player i. As player j jumps over a player $k \in M$, the joint marginal contribution of players j and k is not affected. On the other hand, when player j jumps over a player $k \notin M$, player j's marginal contribution is increased by $\Delta_j v(\pi^k) - \Delta_k v(\pi^k \backslash k) = \Delta_{j,k}^2 v(\pi^k)$. In both cases, the marginal contributions of other members of M are not affected. Adding up over all $k \notin M$ that arrive after player j but before player i, and averaging over all orderings, we obtain the formula in Proposition 1.

The term $\Delta^2_{j,k}v(\pi^k)$ in the formula describes the effect of player k's presence on the marginal contribution of player j to coalition π^k, which is equal to the effect of player j's presence of player k's marginal contribution to the same coalition. Therefore, the term reflects the complementarity of players j and k in the coalition π^k. Note that this coalition does not include player i for all orderings π that appear in the summation. Therefore, when the excluded player j is complementary [substitutable] to outside players (members of $N\backslash M$) in the absence of the excluding player i, the exclusive contract between i and j benefits [hurts] coalition M.

3.2 Robust Conditions

Now we examine contracts that give player i the right to exclude players from a group $J \subset N$. Such a contract can be thought of as a composition of bilateral exclusive contracts. The resource mapping corresponding to the contract is given by

$$E_i^J(S) = \prod_{j \in J} E_i^j(S) = \begin{cases} S \text{ if } i \in S, \\ S\backslash J \text{ otherwise.} \end{cases}$$

Instead of deriving a formula for the profitability of a *given* exclusive contract in a *given* random-order value, we offer a simple condition that is necessary and sufficient for *all* contracts in which player i excludes a subset of $M\backslash i$ to be advantageous [disadvantageous] to M for *all* random-order values. This condition captures complementarity [substitutability] between players from $M\backslash i$ and players from $N\backslash M$ using the concept of increasing differences (Topkis [1998], Milgrom and Shannon [1994]) for the set inclusion order on subsets of N. Namely, a function $w : \mathfrak{A} \times \mathfrak{B} \to \Re$ on $\mathfrak{A}, \mathfrak{B} \subset 2^N$ is said to have *increasing [decreasing] differences* in $(A, B) \in \mathfrak{A} \times \mathfrak{B}$ if $w(A', B) - w(A, B)$ is nondecreasing [nonincreasing] in $B \in \mathfrak{B}$ for all $A, A' \in \mathfrak{A}$ such that $A \subset A'$.

We can now formulate the following result:

Corollary 1 *Let $M \subset N$, and $i \in M$. If $f^\alpha_M(vE_i^j) \geq [\leq] f^\alpha_M(v)$ for all $j \in M\backslash i$ and all $\alpha \in \Delta_{\mathbf{\Pi}}$, then $v(A \cup B)$ has increasing [decreasing] differences in $A \subset M$ such that $i \notin A$ and $B \subset N\backslash M$. In turn, this implies that $f^\alpha_M(vE_i^J)$ is nondecreasing [nonincreasing] in $J \subset M\backslash i$ for all $\alpha \in \Delta_{\mathbf{\Pi}}$.*

Proof. For the first statement, suppose in negation that the increasing difference condition does not hold. In this case, we can find $k \in N\backslash M$ and

$T \subset N \backslash i$ such that $\Delta^2_{k,j} v(T) < 0$. Take an ordering $\hat{\pi} \in \mathbf{\Pi}$ such that

$$\hat{\pi}(T \backslash j \backslash k) < \hat{\pi}(j) < \hat{\pi}(k) < \hat{\pi}(i) < \hat{\pi}(N \backslash T \backslash i),$$

and let $\alpha_\pi = \begin{cases} 1 & \text{if } \pi = \hat{\pi}, \\ 0 & \text{otherwise.} \end{cases}$ Then by Proposition 1, $f^\alpha_M(vE^j_i) - f^\alpha_M(v) = \Delta^2_{k,j} v(T) < 0$, which contradicts the assumption.

For the second statement, take $J \subset M \backslash i$ and $j \in M \backslash i$. Since $v(A \cup B)$ has increasing differences in $A \subset M \backslash i$ and $B \subset N \backslash M$ and E^J_i is nondecreasing on $2^{M \backslash i}$, $vE^J_i(A \cup B) = v(E^J_i A \cup B)$ also has increasing differences in $A \subset M \backslash i$ and $B \subset N \backslash M$. Therefore, $\Delta^2_{j,k} vE^J_i(T) \geq 0$ for all $T \subset N \backslash i$. Since $E^{J \cup j}_i = E^J_i E^j_i$, by Proposition 1,

$$\begin{aligned} f^\alpha_M(vE^{J \cup j}_i) - f^\alpha_M(vE^J_i) &= f^\alpha_M((vE^J_i)E^j_i) - f^\alpha_M(vE^J_i) \\ &= \sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \sum_{k \in \pi^i \backslash \pi^j \backslash M} \Delta^2_{j,k} vE^J_i(\pi^k) \geq 0. \blacksquare \end{aligned}$$

The increasing difference condition in the proposition reflects the complementarity between the contracting players (members of M) and the other players (members of $N \backslash M$) in the absence of player i. This condition is sufficient, but not necessary, for the contract E^M_i to be profitable, except when $|M| = 2$. A weaker condition can be derived that is both necessary and sufficient for a given exclusive contract excluding more than one player to be profitable for all random-order values. We do not present it here as the condition in Corollary 1 is more intuitive and is sufficient for the applications considered in Section 6.

Finally, we consider contracts where more than one agent can exclude. Namely, consider a contract

$$E^J_Q = \prod_{i \in Q} E^J_i = \begin{cases} S \text{ if } Q \subset S, \\ S \backslash J \text{ otherwise.} \end{cases}$$

In this contract, any member of Q can exclude each member of J. In particular, when $Q = J$, this describes a mutually exclusive contract, which is equivalent to "joint ownership" of the players' resources, in the sense of Hart and Moore [1990]. For such contracts, we have

Corollary 2 *Let $M \subset N$. If $f^\alpha_M(vE^j_i) \geq [\leq]\ f^\alpha_M(v)$ for all $i, j \in M$ and all $\alpha \in \Delta_{\mathbf{\Pi}}$, then $v(A \cup B)$ has increasing [decreasing] differences in $A \subset M$ such that $A \neq M$ and $B \subset N \backslash M$. In turn, this implies that $f^\alpha_M(vE^J_Q)$ is nondecreasing [nonincreasing] in $Q, J \subset M$ for all $\alpha \in \Delta_{\mathbf{\Pi}}$.*

Proof. The first statement follows immediately from the first statement of Corollary 1. For the second statement, take $Q, J \subset M$ and $i, j \in M$. Since $v(A \cup B)$ has increasing differences in $A \subset M \backslash i$ and $B \subset N \backslash M$ and E_Q^J is nondecreasing on $2^{M \backslash i}$, $vE_Q^J(A \cup B) = v(E_Q^J A \cup B)$ also has increasing differences in $A \subset M \backslash i$ and $B \subset N \backslash M$. The second statement of Corollary 1 now implies that

$$\begin{aligned} f_M^\alpha(vE_Q^{J \cup j}) &= f_M^\alpha((vE_Q^J)E_Q^j) \geq f_M^\alpha(vE_Q^J), \\ f_M^\alpha(vE_{Q \cup i}^J) &= f_M^\alpha((vE_Q^J)E_i^J) \geq f_M^\alpha(vE_Q^J). \blacksquare \end{aligned}$$

4 Inclusion

4.1 Bilateral Contracts

We first consider an inclusive contract between two players $i, j \in N$. The contract gives player i the right to use player j's resource, but not the right to exclude player j from using it himself. This means that both players i and j have the right to use player j's resource. The resource mapping E_i^j corresponding to this contract is given by

$$I_i^j(S) = \begin{cases} S \text{ if } i \notin S, \\ S \cup j \text{ otherwise.} \end{cases}$$

Contracts of this kind were first considered by Haller [1994], who calls mutually inclusive contracts "associations".[6]

The first result of this section identifies the effect of the inclusive contract I_i^j on the joint payoff of a coalition M that includes i and j:

[6] One might dispute the realism of such contracts on the grounds that they give rise to "unstable" games. In the noncooperative language, players i and j have a strong incentive to race to claim player j's resource first. In the cooperative language, the resulting game vI_i^j may not be superadditive and nay have an empty core. However, stability issues are alien to the present paper. In noncooperative terms, we take the realization of the ordering of players to be exogenous. In cooperative terms, it is only for convex games that all random-order values lie in the core (see e.g. Topkis [1998, Theorems 5.2.1, 5.2.3]), and we do not restrict attention to such games. (Indeed, our results imply that in convex games, exclusion is always profitable and inclusion is always unprofitable.) I am grateful to Hans Haller for a discussion of these issues.

Proposition 2 *If $i, j \in M \subset N$, then*

$$f_M^\alpha(vI_i^j) - f_M^\alpha(v) = -\sum_{\pi\in\mathbf{\Pi}} \alpha_\pi \sum_{k\in\pi^j\backslash\pi^i\backslash M} \Delta_{j,k}^2 v(\pi^k).$$

Proof. It is instructive to derive this result from Proposition 1, exploiting a duality between exclusion and exclusion. Intuitively, i's inclusion of j into coalition S is equivalent to i's exclusion of j from the complementary coalition $N\backslash S$. Formally, letting $\overline{S} = N\backslash S$, we can write

$$\overline{I_i^j(S)} = \left\{ \begin{array}{c} \overline{S} \text{ if } i \in \overline{S}, \\ \overline{S}\backslash J \text{ otherwise} \end{array} \right\} = E_i^j(\overline{S}).$$

For any $w : 2^N \to \Re$, define $\overline{w} : 2^N \to \Re$ by $\overline{w}(S) = -w(\overline{S})$ for all $S \subset N$. Observe that

$$\Delta\overline{w}(S) = w(\overline{S\backslash k}) - w(\overline{S\cup k}) = w(\overline{S}\cup k) - w(\overline{S}\backslash k) = \Delta_k w(\overline{S}),$$

and similarly $\Delta_{j,k}^2\overline{w}(S) = -\Delta_{j,k}^2 w(\overline{S})$. Note also that $vI(S) = v(\overline{E\overline{S}}) = -\overline{v}E(\overline{S})$ for all $S \subset N$, hence $vI = \overline{\overline{v}E}$.

Let $\mu = (n, \ldots, 1) \in \mathbf{\Pi}$. Then $\overline{\pi} = \pi\mu$ is the "mirror image" of π, and $\overline{\alpha} \in \Delta_{\mathbf{\Pi}}$ given by $\overline{\alpha}_\pi = \alpha_{\overline{\pi}}$ for all $\pi \in \mathbf{\Pi}$ is the "mirror image" of α. Then

$$g_M\pi(vI) = \sum_{k\in M} \Delta_k vI(\pi^k) = \sum_{k\in M} \Delta_k\overline{\overline{v}E}(\pi^k) = \sum_{k\in M} \Delta_k\overline{v}E(\overline{\pi^k}) = g_M^{\overline{\pi}}(\overline{v}E),$$

and thus

$$f_k^\alpha(vI_i^j) = \sum \alpha_\pi g_k\pi(vI_i^j) = \sum \alpha_\pi g_M^{\overline{\pi}}(\overline{v}E_i^j) = f_k^{\overline{\alpha}}(\overline{v}E_i^j).$$

Now using Proposition 1,

$$\begin{aligned} f_k^\alpha(vI_i^j) - f_k^\alpha(v) &= f_k^{\overline{\alpha}}(\overline{v}E_i^j) - f_k^{\overline{\alpha}}(\overline{v}) = \sum_{\pi\in\mathbf{\Pi}} \alpha_{\overline{\pi}} \sum_{k\in\pi^i\backslash\pi^i\backslash M} \Delta_{j,k}^2\overline{v}(\pi^k) \\ &= -\sum_{\pi\in\mathbf{\Pi}} \alpha_{\overline{\pi}} \sum_{k\in\overline{\pi}^j\backslash\overline{\pi}^i\backslash M} \Delta_{j,k}^2 v(\overline{\pi^k}) = -\sum_{\pi\in\mathbf{\Pi}} \alpha_\pi \sum_{k\in\pi^j\backslash\pi^i\backslash M} \Delta_{j,k}^2 v(\pi^k), \end{aligned}$$

which implies the result. ■

The formula has the following intuition. For a given ordering π of players in which player j comes after player i (i.e., $\pi^i \subset \pi^j$), consider how player

j's marginal contribution changes as he is moved forward by one position at a time until he comes just after player i. As player j jumps over a player $k \in M$, the joint marginal contribution of players j and k is not affected. On the other hand, when player j jumps over a player $k \notin M$, player j's marginal contribution is increased by $\Delta_k v(\pi^k \backslash k) - \Delta_j v(\pi^k) = -\Delta^2_{j,k} v(\pi^k)$. In both cases, the marginal contributions of other members of M are not affected. Adding up over all $k \notin M$ that arrive after player i but before player j, and averaging over all orderings, we obtain the formula in Proposition 2.

The term $\Delta^2_{j,k} v(\pi^k)$ in the formula reflects the complementarity of players j and k in the coalition π^k. Note that this coalition includes player i for all orderings π that appear in the summation. Therefore, when the included player j is complementary [substitutable] to outside players (members of $N \backslash M$) in the presence of the including player i, the inclusive contract between i and j hurts [benefits] coalition M.

4.2 Robust Conditions

Now we examine contracts that give player i the right to include players from a group $J \subset N$. Such a contract can be thought of as a composition of bilateral inclusive contracts. The resource mapping corresponding to the contract is given by

$$I_i^J(S) = \prod_{j \in J} I_i^j(S) = \begin{cases} S \text{ if } i \notin S, \\ S \cup J \text{ otherwise.} \end{cases}$$

The following results are given without proof. They can be easily derived from Corollaries 1,2 in the preceding section using the duality between inclusion and exclusion described in the proof of Proposition 2.

Corollary 3 *Let $M \subset N$, and $i \in M$. If $f^\alpha_M(vE^j_i) \geq [\leq]\ f^\alpha_M(v)$ for all $j \in M \backslash i$ and all $\alpha \in \Delta_{\mathbf{\Pi}}$, then $v(A \cup B)$ has decreasing [increasing] differences in $A \subset M$ such that $i \in A$ and $B \subset N \backslash M$. In turn, this implies that $f^\alpha_M(vE^J_i)$ is nondecreasing [nonincreasing] in $J \subset M \backslash i$ for all $\alpha \in \Delta_{\mathbf{\Pi}}$.*

The decreasing difference condition reflects the substitutability between the contracting players (members of M) and the other players (members of $N \backslash M$) in the presence of player i. Just as with exclusive contracts, when $|M| > 2$, there exists a weaker condition that is both necessary and sufficient for E_i^M to be profitable. The condition in Corollary 1, however, is more intuitive and is sufficient for the applications considered in Section 6. Observe that when player i is a dummy, inclusive contracts are profitable precisely when exclusive contracts are unprofitable, and vice versa.

We also examine inclusive contracts in which more than one player can exclude:

$$I_Q^J(S) = \prod_{i \in Q} I_i^J(S) = \begin{cases} S \text{ if } Q \subset N \backslash S, \\ S \cup J \text{ otherwise.} \end{cases}$$

In particular, when $Q = J$, this describes a mutually inclusive contract, which Haller [1994] called an "association".

Corollary 4 *Let $M \subset N$. If $f_M^\alpha(vI_i^j) \geq [\leq]\ f_M^\alpha(v)$ for all $i, j \in M$ and all $\alpha \in \Delta_{\mathbf{\Pi}}$, then $v(A \cup B)$ has decreasing [increasing] differences in $A \subset M$ such that $A \neq \emptyset$ and $B \subset N \backslash M$. In turn, this implies that $f_M^\alpha(vI_Q^J)$ is nondecreasing [nonincreasing] in $Q, J \subset M$ for all $\alpha \in \Delta_{\mathbf{\Pi}}$.*

5 Collusion

5.1 Bilateral Contracts

We begin with considering a collusive contract between two players $i, j \in N$. Under such a contract, one of the players, say player i, gets full control over both colluding players' resources. Formally, this contract results in the resource mapping

$$C_i^j(S) = \begin{cases} S \cup j \text{ if } i \in S, \\ S \backslash j \text{ otherwise.} \end{cases}$$

Note that collusion can be understood as a composition of exclusion and inclusion: player i can both exclude player j from using j's resource and use this resource himself. Formally, $C_i^j(S) = I_i^j\left(E_i^j(S)\right) = E_i^j\left(I_i^j(S)\right)$. The effect of collusion C_i^j in random-order values can thus be derived from those of exclusion E_i^j and inclusion I_i^j. Specifically, observe that in those orderings in which player j comes before player i, collusion is equivalent to exclusion, while inclusion has no effect. On the other hand, in those orderings in which player j comes after player i, collusion is equivalent to inclusion, while exclusion has no effect. In both cases, the gain from collusion equals to the sum of those from exclusion and inclusion. Therefore, we have

Lemma 1 *For $i, j \in M \subset N$,*

$$f_M^\alpha(vC_i^j) - f_M^\alpha(v) = \left[f_M^\alpha(vE_i^j) - f_M^\alpha(v)\right] + \left[f_M^\alpha(vI_i^j) - f_M^\alpha(v)\right].$$

Proof. Follows from the fact that

$$\begin{aligned} g_M^\pi(vC_i^j) - g_M^\pi(v) &= \begin{cases} g_M^\pi(vE_i^j) - g_M^\pi(v) & \text{when } \pi^j \subset \pi^i \\ g_M^\pi(vI_i^j) - g_M^\pi(v) & \text{otherwise} \end{cases} \\ &= \left[g_M^\pi(vE_i^j) - g_M^\pi(v)\right] + \left[g_M^\pi(vI_i^j) - g_M^\pi(v)\right]. \blacksquare \end{aligned}$$

Lemma 1 suggests that the profitability of collusion depends on the comparison of expected gains from exclusion in those orderings in which player i comes after player j to the expected losses from inclusion in the orderings in which the two players are switched. The comparison depends on the relative likelihoods of the two kinds of orderings. For this reason, we restrict attention to random-order values that are symmetric with respect to the colluding agents. Formally, let $\mathbf{\Pi}_S$ denote the group of permutations of $S \subset N$ (note that $\mathbf{\Pi}_S$ is a subgroup of Π). We say that a probability distribution $\alpha \in \Delta_{\mathbf{\Pi}}$ is S-symmetric if $\alpha_{\tau\pi} = \alpha_\pi$ for all $\tau \in \mathbf{\Pi}_S$. For example, Shapley value is the unique N-symmetric random-order value. It is easy to see that for $\{i, j\}$-symmetric values, the gain from collusion by $\{i, j\}$ does not depend on which player becomes the proxy. (With Shapley value, moreover, we can eliminate the other player, who becomes a dummy, and use the Shapley value for remaining $N - 1$ players to compute the colluding players' joint value.) This gain is given by the following result:

Proposition 3 *If $i, j \in M \subset N$ and α is $\{i, j\}$ - symmetric, then*

$$f_M^\alpha(vC_i^j) - f_M^\alpha(v) = -\sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \sum_{k \in \pi^i \backslash \pi^j \backslash M} \Delta_{i,j,k}^3 v(\pi^k).$$

Proof. Using Lemma 1 and Propositions 1 and 2, and letting $(i, j) \in \mathbf{\Pi}$ denote the transposition of players i and j, we have

$$\begin{aligned} f_M^\alpha(vC_i^j) - f_M^\alpha(v) &= \left[f_M^\alpha(vE_i^j) - f_M^\alpha(v)\right] + \left[f_M^\alpha(vI_i^j) - f_M^\alpha(v)\right] \\ & \sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \sum_{k \in \pi^i \backslash \pi^j \backslash M} \Delta_{j,k}^2 v(\pi^k) - \sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \sum_{k \in \pi^j \backslash \pi^j \backslash M} \Delta_{j,k}^2 v(\pi^k) \\ &= \sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \sum_{k \in \pi^i \backslash \pi^j \backslash M} \Delta_{j,k}^2 v(\pi^k) - \sum_{\pi \in \mathbf{\Pi}} \alpha_{(i,j)\pi} \sum_{k \in \pi^i \backslash \pi^j \backslash M} \Delta_{j,k}^2 v(((i,j)\pi)^k) \end{aligned}$$

$$= \sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \sum_{k \in \pi^i \backslash \pi^j \backslash M} \left[\Delta^2_{j,k} v(\pi^k) - \Delta^2_{j,k} v(\pi^k \cup i)\right]$$
$$= -\sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \sum_{k \in \pi^i \backslash \pi^j \backslash M} \Delta^3_{i,j,k} v(\pi^k),$$

since $\alpha_{(i,j)\pi} = \alpha_\pi$ and $((i,j)\pi)^k = \pi^k \backslash j \cup i$ when $k \in \pi^i \backslash \pi^j \backslash M$. ■

To understand the result intuitively, recall from the previous sections that the exclusive contract E_i^j is profitable when the excluded player j is complementary to outside players *in the absence of player* i, while the inclusive contract I_i^j is profitable when the included player j is substitutable to outside players *in the presence of player* i. The net effect depends on how the complementarity between player j and outside players is affected by player i, which is captured by the third-difference term $\Delta^3_{i,j,k} v(\pi^k)$. Another interpretation of this term is that it reflects how the complementarity between the colluding players i and j is affected by the outside player k.

5.2 Robust Condition

Now we examine collusive contracts in which player i becomes a proxy for a group of players $J \subset N$. Such a contract can be thought of as a composition of bilateral collusive contracts. The resource mapping corresponding to the contract is given by

$$C_i^J(S) = \prod_{j \in J} C_i^j(S) = \begin{cases} S \cup J \text{ if } i \in S, \\ S \backslash J \text{ otherwise.} \end{cases}$$

To formulate a robust condition for collusion by members of a coalition M to benefit or hurt the coalition for all random-order values, we will use the concepts of super- and submodularity (Topkis [1998], Milgrom and Shannon [1994]) for the set inclusion order on subsets of N. Namely, a function $w : \mathfrak{A} \to \Re$ on $\mathfrak{A} \subset 2^N$ is said to be *supermodular [submodular]* in $A \in \mathfrak{A}$ if $w(A \cup A') + w(A \cap A') \geq [\leq]\ w(A') + w(A)$ for all $A, A' \in \mathfrak{A}$. We can now formulate the following result:

Corollary 5 *Let $M \subset N$. If for all $i, j \in M$, $f_M^\alpha(vC_i^j) \geq [\leq]\ f_M^\alpha(v)$ for all $\{i,j\}$-symmetric $\alpha \in \Delta_{\mathbf{\Pi}}$, then the function $\Delta_k v(A \cup B)$ is submodular [supermodular] in $A \subset M$ for all $B \subset N \backslash M$ and all $k \in N \backslash M$. In turn, this implies that $f_M^\alpha(vC_i^J)$ is nondecreasing [nonincreasing] in $J \subset M$ for all $i \in M$ and all M-symmetric $\alpha \in \Delta_{\mathbf{\Pi}}$.*

Proof. For the first statement, suppose in negation that $\Delta_k v(A \cup B)$ is not submodular, in which case $\Delta^3_{i,j,k} v(A \cup B) > 0$ for some $i, j \in M$, $A \subset M$, $B \subset N \backslash M$. Take an ordering $\hat{\pi} \in \mathbf{\Pi}$ such that

$$\hat{\pi}(B \backslash k) < \hat{\pi}(A \backslash \{i,j\}) < \hat{\pi}(j) < \hat{\pi}(k) < \hat{\pi}(i) < \hat{\pi}(N \backslash B \backslash A \backslash \{i,j\}).$$

Let $\alpha_\pi = \begin{cases} 1/2 & \text{if } \pi = \hat{\pi} \text{ or } \pi = (i,j)\,\hat{\pi}, \\ 0 & \text{otherwise.} \end{cases}$ which is $\{i,j\}$-symmetric by construction. By Proposition 3,

$$f^\alpha_M(vC^j_i) - f^\alpha_M(v) = -\frac{1}{2}\Delta^3_{i,j,k} v(A \cup B) > 0,$$

which contradicts the assumption.

For the second statement, take any $J \subset M$ and $i, j \in M$. Since $\Delta_k v(A \cup B)$ is submodular in $A \subset M$ and C^J_i is a nondecreasing mapping from M into itself, $\Delta_k v C^J_i(A \cup B) = \Delta_k v(C^J_i A \cup B)$ is also submodular $A \subset M$ for all $B \subset N \backslash M$ and all $k \in N \backslash M$, which implies that $\Delta^3_{i,j,k} v C^J_i(T) \leq 0$ for all $T \subset N$ and all $k \in N \backslash M$. Since $C^{J \cup j}_i = C^J_i C^j_i$, by Proposition 3,

$$\begin{aligned} f^\alpha_M(vC^{J\cup j}_i) - f^\alpha_M(vC^J_i) &= f^\alpha_M((vC^J_i)C^j_i) - f^\alpha_M(vC^J_i) \\ &= -\sum_{\pi \in \mathbf{\Pi}} \alpha_\pi \sum_{k \in \pi^i \backslash \pi^j \backslash M} \Delta^3_{i,j,k} v C^J_i(\pi^k) \geq 0. \blacksquare \end{aligned}$$

One interpretation of the obtained condition for [un]profitability of collusion is that the complementarity of the colluding players is reduced [enhanced] in the presence of more outside players. Just as in previous sections, a weaker condition exists that is necessary and sufficient for collusion by a given coalition M with $|M| > 2$ to be profitable for all M-symmetric random-order values. However, the condition in Corollary 5 is more intuitive and sufficient in applications of interest.

6 Applications

In this section we apply our general results to five (partially overlapping) settings that have been studied in the literature.

6.1 Three-player games

The case of $N = \{i, j, k\}$ covers, for example, vertical contracting in the presence of a second seller (see e.g. Segal and Whinston [1998], Heavner's [1999]).[7] In this case,

$$\Delta^2_{j,k} v(\{j,k\}) = [v(\{j,k\}) - v(j)] - [v(k) - v(\emptyset)] = v(\{j,k\}) - v(j) - v(k) \geq 0$$

whenever v is superadditive. In words, in a superadditive game, two players are always complementary to each other in the absence of the third player. Therefore, by Proposition 1, an exclusive contract between any two players is advantageous to them. This generalizes the finding of Segal and Whinston [1998] (earlier discovered by Aghion and Bolton [1987] in a setting with asymmetric information) that exclusive contracts can be used to extract surplus from third parties (entrants).

Next we examine the profitability of collusive contracts. In the simple case where $v(i) = v(j) = v(k) = 0$, we have

$$\Delta^3_{i,j,k} v(\{i,j,k\}) = v(\{i,j,k\}) - [v(\{i,j\}) + v(\{i,k\}) + v(\{j,k\})].$$

By Proposition 3, collusion between any pair of players is profitable [unprofitable] when the expression is negative [positive].

As one example, consider the setting of vertical contracting where player i is the indispensable buyer, i.e., $v(S) = 0$ when $i \notin S$, and players j and k are two sellers. Then the expression above becomes

$$\Delta^3_{i,j,k} v(\{i,j,k\}) = v(\{i,j,k\}) - v(\{i,j\}) - v(\{i,k\}).$$

Therefore, bilateral collusion is profitable when the two sellers are substitutable and unprofitable when they are complementary. The latter result is consistent with Heavner's [1999] finding that a seller's collusion with a buyer is unprofitable in the presence of an upstream (perfectly complementary) seller.

6.2 Games with an indispensable player

Suppose that player $p \in N$ is indispensable, i.e., $v(T) = 0$ whenever $p \notin T$. This setting covers the labor bargaining model of Stole and Zwiebel [1996a],

[7] Along with the "bargaining effect", Segal and Whinston [1998] and Heavner [1999] also examine investment effects of contracts, which are not considered here.

in which the indispensable player p is the firm, and the remaining players are workers. Alternatively, the indispensable player p could be a monopolistic seller who bargains with many buyers. The intuition of "countervailing power" (Galbraith [1952]) suggests that integration of "dispensable" players - members of $M = N \backslash p$ - increases their joint bargaining surplus. We examine whether this is indeed the case for the three kinds of contracts considered in the paper.[8]

Note first that the marginal contribution of the indispensable player p to a coalition $A \subset M$ equals

$$\Delta_p v(A \cup p) = v(A \cup p) - v(A) = v(A \cup p) \tag{1}$$

When v is monotonic, this marginal contribution is nondecreasing in A. In words, members of M are always complementary to the indispensable player. This implies the increasing difference conditions in Corollaries 1,2, and 3,4, and therefore exclusive contracts by members of M always benefit M, while inclusive contracts by members of M always hurt M.

Finally, using (1) and Corollary 5, collusion among members of M benefits [hurts] M when the function $v(A \cup p)$ is submodular [supermodular] in $A \subset M$, i.e., when the members of M are substitutable [complementary] with each other. This result has been previously noted by Stole and Zwiebel [1996a], who demonstrate that substitutable workers benefit from collusion in labor bargaining while complementary workers do not. (Horn and Wolinsky [1988] make a similar observation in a different bargaining game.)

6.3 Games with two jointly indispensable players

Suppose now we have two players $p_1, p_2 \in N$, who are jointly indispensable in the sense that $v(T) = 0$ whenever $T \cap \{p_1, p_2\} = \emptyset$. This setting covers the labor bargaining model of Stole and Zwiebel [1998], in which the jointly indispensable players p_1, p_2 are firms, and the remaining players $W = N \backslash \{p_1, p_2\}$ are workers who cannot produce without at least one firm present.[9]

Consider first exclusive contracts within $M = \{p_1, p_2\}$. Note that for $B \subset W$, $\Delta_{p_i} v(p_i \cup B) = v(p_i \cup B) - v(B) = v(p_i \cup B)$ is nondecreasing in B when v is monotonic. In words, each firm is always complementary

[8] For another approach to modeling "countervailing power," see Snyder [1996]

[9] Along with the "bargaining effect", Stole and Zwiebel [1998] also examine the employment effect of merger, which is not considered here.

to workers in the absence of the other firm. Therefore, by Corollaries 1,2, exclusive contracts within M (namely, $E_{p_1}^M$, $E_{p_2}^M$, and the joint ownership contract E_M^M) are always profitable.

Now consider the collusive contract between p_1 and p_2. For $k \in B \subset W$,

$$\Delta^3_{p_1,p_2,k} v(p_1 \cup p_2 \cup B) = \Delta_k v(p_1 \cup p_2 \cup B) - \Delta_k v(p_1 \cup B) - \Delta_k v(p_2 \cup B).$$

In the model of Stole and Zwiebel [1998], this third difference is positive, due to two more primitive assumptions. The first assumption is that the set W of workers is partitioned into two subsets W_1 and W_2, so that a worker from W_i can only contribute in the presence of firm p_i: $\Delta_k v(p_i \cup B) = 0$ when $k \notin W_i$. Intuitively, this means that firms do not compete for workers. The second assumption is that workers are complementary to firms, i.e., $\Delta_k v(A \cup B)$ is nondecreasing in $A \subset M$. Under these two assumptions, for $k \in W_i$ and $B \subset W$ we have

$$\Delta^3_{p_1,p_2,k} v(M \cup B) = \Delta_k v(M \cup B) - \Delta_k v(p_i \cup B) \geq 0.$$

Proposition 3 establishes that in this case collusion between the firms is unprofitable, which Stole and Zwiebel [1998] find as well.

Note, however, that each of the two above assumptions may be reversed. First, a worker may be able to contribute to either firm, i.e., firms may compete for workers. Second, cooperating firms may utilize a labor-saving technology that is not economical for an individual firm, in which case $\Delta_k v(M \cup B) < \Delta_k v(p_i \cup B)$. In either case, the third difference $\Delta^3_{p_1,p_2,k} v(M \cup B)$ may be negative, in which case Proposition 3 establishes that the firms will find a collusive merger profitable.

6.4 Two-sided markets with CES technology

Suppose that the set N of players is partitioned into two sets, M_1 and M_2, and that each player from M_i holds one unit of factor x_i ($i = 1, 2$). The worth of a coalition is a function of the total quantities of the two factors available. Formally, if $A_1 \subset M_1$ and $A_2 \subset M_2$, then $v(A_1 \cup A_2) = \phi(|A_1|, |A2|)$. Suppose furthermore that the production function ϕ is smooth and exhibits diminishing marginal productivity and constant returns to scale. Horizontal mergers (i.e., mergers by owners of one factor) in such a model have been considered by Gardner [1977] and Legros [1984].

Note first that under our assumptions, the two factors must be Edgeworth substitutes, i.e., $\partial^2\phi(x_1,x_2)/\partial x_1\partial x_2 \geq 0$.[10] Hence, $v(A_1 \cup A_2)$ has increasing differences in $A_1 \subset M_1$ and $A_2 \subset M_2$, and Corollaries 1-4 establish that horizontal exclusion contracts are profitable, while horizontal inclusion contracts are not.

As for horizontal collusion, its profitability depends on the shape of the production function ϕ. To see this, we follow Gardner [1977] in assuming that ϕ has a constant elasticity of substitution:

$$\phi(x_1,x_2) = \left(x_1^{-\beta} + x_2^{-\beta}\right)^{-1/\beta},$$

where $\beta > -1$. The elasticity of substitution is then given by $\sigma = 1/(1+\beta)$.

According to Proposition 3, the profitability of collusion among owners of factor x_1 depends on the third derivative $\partial^3\phi(x_1,x_2)/\partial x_1^2\partial x_2$. A simple calculation shows that

$$sign\frac{\partial^3\phi(x_1,x_2)}{\partial x_1^2\partial x_2} = sign\left[\frac{\beta}{1+\beta} - \left(\frac{x_1}{x_2}\right)^{\beta}\right].$$

When $\beta \leq 0$ (which corresponds to $\sigma \geq 1$), the sign is always negative, hence, according to Corollary 5, collusion by owners of x_1 is always profitable. On the other hand, when $\beta > 0$ (i.e., $\sigma < 1$), the sign depends on the ratio x_1/x_2, hence a robust conclusion on the profitability of collusion among members of M_1 cannot be made. These findings are consistent with those of Gardner [1977].

We can make a more definitive predictions for the case $\beta > 0$ if we restrict attention to the Shapley value, and to the case in which individual players are infinitesimal. Recall the insight of Aumann and Shapley [1974] that the Shapley value for non-atomic games puts all weight on coalitions in which the types of agents are represented proportionally, i.e. $x_1/x_2 = |M_1|/|M_2|$. Substituting in the above expression for $sign\,[\partial^3\phi(x_1,x_2)/\partial x_1^2\partial x_2]$, we can use Proposition 3 to see that collusion by members of M_1 is profitable when $|M_1|/|M_2| > (1+1/\beta)^{-1/\beta}$, and unprofitable when the inequality is reversed. Legros [1987] considers the fixed-proportion (Leontieff) technology, which corresponds to the limiting case where $\beta \to \infty$ (and $\sigma \to 0$). In this case, the above inequality becomes $|M_1|/|M_2| > 1$, which is consistent with Legros's

[10] Indeed, differentiation of the Euler identity: $x_1\partial\phi/\partial x_1 + x_2\partial\phi/\partial x_2 \equiv \phi$ with respect to x_2 yields $\partial^2\phi/\partial x_1\partial x_2 = -(x_2/x_1)\,\partial^2\phi/\partial x_2^2 \geq 0$.

finding that collusion is profitable on the abundant side of the market and unprofitable on the short side of the market. Intuitively, agents on one side of the market are substitutable when the market is perfectly balanced, and not substitutable when their side is either abundant or scarce. Thus, the substitutability of agents on the abundant side of the market is increased by having more agents on the other side, and therefore according to Corollary 5 they will find collusion profitable. Substitutability of agents on the scarce side is increased by having *fewer* agents on the other side, therefore according to Corollary 5 they will find collusion unprofitable.

6.5 Symmetric games

Here we consider games that are symmetric with respect to players. This implies that the worth of each coalition is only a function of its size and not of its members' identities, i.e., there exists a function $\phi : \{1, \ldots, n\} \to \Re$ such that $v(S) = \phi(|S|)$ for all $S \subset N$. Our analysis implies that in such games, exclusion is robustly profitable [unprofitable] if and only if $\phi(k)$ is convex [concave] on $k \leq N-1$. Inclusion is robustly profitable [unprofitable] if and only if $\phi(k)$ is concave [convex] on $k \geq 1$. Finally, collusion is robustly profitable [unprofitable] if and only if all the third differences of $\phi(k)$ are nonpositive [nonnegative] everywhere.

As an example, consider the one man-one vote voting games with quorum q considered by Haller [1994], in which ϕ is a step function given by $\phi^q(k) = \begin{cases} 1 & \text{if } k \geq q, \\ 0 & \text{otherwise.} \end{cases}$ Consider first the case of $q = n$, in which the game becomes the unanimity game. Since ϕ^n is a convex function, exclusion is robustly profitable and inclusion is robustly unprofitable. Intuitively, this happens because players in the unanimity game are always weakly complementary to each other. Also, the only nonzero third difference of ϕ^n occurs at $k = n$, where it equals 1, which implies that collusion in the unanimity game is robustly unprofitable. Intuitively, this happens because players in the unanimity game are complementary in the grand coalition, and not complementary in smaller coalitions.

On the other hand, when $q < n$, the step function ϕ^q is neither concave nor convex. Intuitively, players in this game are complementary in coalitions of size q, and substitutable in coalitions of size $q + 1$. Similarly, the third differences of ϕ^q also change signs. Because of this, robust statements on the profitability of exclusion/inclusion/exclusion are impossible. On the other hand, the profitability of a contract *for a given random order value* can be easily calculated using the formulas in Propositions 1-3. For the case of the Shapley value, this simple exercise can be used to verify the results of Haller [1994] on the profitability of collusion and "association" (mutual inclusion) in voting games.

7 Conclusion

While previous research has examined the profitability of collusion in several specific instances of bargaining, it has failed to produce an intuitive general condition for when collusion is profitable and when it is not. This paper provides such a condition by representing collusion as a composition of exclusion and inclusion contracts. Exclusion is profitable when excluded players are complementary to outside players in the absence of the excluding player, while inclusion is profitable when included players are substitutable with outside players in the presence of the including player. The gain from collusion obtains as a sum of those from exclusion and inclusion, and depends on a third derivative: it is positive when colluding players are less complementary in the presence of more outside players.

However, we do not view the analysis on exclusion and inclusion as merely auxiliary to the analysis of collusion. Rather, the difference in the results invites us to take a closer look at the forms "integration" takes in reality. One important observation is that the feasibility of various forms of integration depends on whether economic agents own physical or human assets. As one example, consider the labor bargaining framework with one indispensable player - the firm. Stole and Zwiebel [1996a] and Horn and Wolinsky [1988] model a labor union as a collusive agreement among workers, and find it to be advantageous when workers are substitutable and disadvantageous when they are complementary. These findings are consistent with our general characterization of the profitability of collusion. However, in reality workers cannot relinquish control over their inalienable human assets to the union, as a collusive agreement would require. Instead, a workers' union is better described as an exclusive agreement, under which the union can prevent its members from working, but cannot force them to work against their will. As shown in the Section 6, such an agreement always benefits workers as a whole, regardless of whether they are complementary or substitutable.

The same point applies to the analysis of firm mergers. Stole and Zwiebel

[1998] and Heavner [1999] find that a "collusive" merger is disadvantageous to firms in their respective settings. Their findings are consistent with our general analysis of collusion. However, a collusive agreement that transfers all the assets of one firm to another firm is not feasible when the assets in question are inalienable human assets. In the extreme case where the human asset of the acquired firm's manager is perfectly complementary to its physical assets, the merger can be described as an exclusive contract on the human asset. Our analysis implies that such an exclusive contract is profitable in the settings considered by Stole and Zwiebel [1998] and Heavner [1999], which reverses the two papers' conclusions. In reality, the acquired firm's human and physical assets may be partially but not perfectly complementary. Analysis of such cases requires consideration of contracts that split the resources owned by a single player, which we have avoided for the sake of simplicity. Nevertheless, our analysis indicates that the profitability of mergers depends on the degree of specificity of the manager's human capital.

One interesting extension of our work concerns the bargaining effects of more complex contracts, such as message-contingent resource-mapping contracts. To take a simple example, player j may be given the option to opt out of an exclusive arrangement with player j by paying a pre-specified penalty. If the option is exercised after player j observes the realized ordering of players, the contract may be used to implement exclusivity that is contingent on the ordering. More generally, such contingent contracts might be used to implement "selective intervention", under which player j's resource is transferred to player i only in those orderings in which this raises the two players' joint marginal contribution. However, the consideration of noncooperative game forms in the cooperative game framework introduces conceptual problems that are far beyond the scope of this paper. For some issues that arise, see Ray and Vohra [1997].

Another interesting direction for future research concerns the set of contracts to emerge in equilibrium. While the question seems related to the literature on endogenous coalition formation (see e.g. Aumann and Myerson [1988]), there are some important differences. First, contracting in our setting does not affect the value of the grand coalition, hence it is a zero-sum game. Second, while an incomplete graph of links can replicate an exclusive contract between players i and j (e.g. consider the graph in which player j is connected to other players only through player i but not directly), they cannot replicate contracts in which players can include other players' resources, such as inclusive and collusive contracts.

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