Diversification, convex preferences and non-empty core

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Abstract

We show, in the Choquet expected utility model, that preference for diversification, that is, convex preferences, is equivalent to a concave utility index and a convex capacity. We then introduce a weaker notion of diversification, namely "sure diversification." We show that this implies that the core of the capacity is non-empty. The converse holds under concavity of the utility index. This property is shown to be equivalent to the notion of comonotone diversification ; notion that we introduce in the paper. Finally, in the expected utility model, all these notions of diversification are equivalent and are represented by the concavity of the utility index.

Keywords : Diversification, Choquet expected utility, Capacity, Convex preferences, Core.

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1 Introduction

Dekel [1989] made the point that having a preference for portfolio diversification is an important feature when modelling markets of risky assets. He also observed that the relationship between risk aversion and preference for diversification is trivial in the expected-utility model, and much more complicated in alternative models. More precisely, the equivalence between these two properties established in the EU framework does not hold in more general models. There, diversification implies risk-aversion but the converse is false.

This note takes up the study of diversification to the case of uncertainty, that is, non-probabilized risk, focussing on the Choquet-expected-utility model. Even though there is still no commonly accepted notion and measure of uncertainty aversion in this set-up, it is widely agreed that two properties are of special interest, namely the non-emptiness of the core and the convexity of the capacity (see Ghirardato and Marinacci [1997], Epstein [1997]).

In this paper we seek to provide characterizations in terms of diversification of these two properties, thus providing a further understanding of what exactly they mean. As could be expected, it is difficult to disentangle properties of the utility index from properties of the capacity. We establish that preference for portfolio diversification (*i.e.* convexity of the DM preferences) is equivalent to the agent having a convex capacity and a concave utility index. We then introduce a weaker notion of preference for diversification, *i.e.* preference for sure diversification. This property simply says that when indifferent between several assets, an agent should prefer a combination of these assets that yields a constant act to any of the ones used in the combination. We show that preference for sure diversification implies that the core of the capacity is non-empty. The converse holds true under the assumption that the utility index is concave.

This leads us to find conditions under which the utility index is concave. As it turns out, the concavity of the utility index is equivalent to a property we name comonotone diversification. This states that if two assets are indifferent and comonotone, then an agent prefers a combination of these assets to any of them. Comonotone diversification is therefore of a very different nature than sure diversification since no hedging at all is involved, the two assets being comonotone. Actually, these two notions of diversification are almost at each end of the spectrum, as one deals with portfolio perfectly hedged while the other one abstracts from any hedging argument. A CEU agent might exhibit preference for sure diversification but not comonotone diversification, as we make clear with an example. Conversely, it is clear that an agent exhibiting preference for comonotone diversification does not necessarily exhibit a preference for sure diversification.

A corollary to the previous result is that comonotone diversification and sure diversification is equivalent to the capacity having a non-empty core and the utility index being concave. Finally, we show that these different notions of diversification cannot be distinguished in the EU model, and are all equivalent to the concavity of the utility index.

Our contribution has some links with the recent debate around the definition and measurement of ambiguity aversion. Schmeidler [1989] provided an axiomatic definition of ambiguity aversion for his model, showing that it is characterized by the convexity of the capacity. Assuming the linearity of the utility index, Wakker [1990] and Chateauneuf [1991] derived convexity of the capacity from axioms respectively labelled pessimism-independence and strong uncertainty aversion, that are strengthenings of comonotone independence used in the derivation of CEU. Ghirardato and Marinacci [1997] defined ambiguity aversion assimilating *a priori* uncertainty neutrality with expected utility. They then show that this notion of ambiguity aversion is equivalent to non-empty core. Epstein [1997] based his definition of ambiguity aversion on the *a priori* identification of uncertainty neutrality with probabilistic sophistication. His notion of uncertainty aversion however cannot be directly linked to convexity of the capacity or non-emptiness of its core. Our analysis, by providing some axioms giving rise to these various assumptions on the capacity might help clarifying some of these issues.

We first introduce the notation and recall some definitions, before stating our main results. Proofs are gathered in an appendix.

2 Notation and definitions

There are k possible states of the world, indexed by superscript j. Let S be the set of states of the world and $\mathcal{A}$ the set of subsets of S.

Let $\succeq$ be the preference relation of a decision maker, defined on the set D of non-negative random variables on S. Say that two random variables C and C' are indifferent, that is $C \sim C'$, if $C \succeq C'$ and $C' \succeq C$. $C^j \in \mathbb{R}_+$ is wealth in state j.

As usual, say that an agent's preferences are

- *convex* if $\forall C, C' \in D,\ \forall \alpha \in [0,1],\quad C \succeq C' \Rightarrow \alpha C + (1-\alpha)C' \succeq C'$
- *continuous* if for all $x \in \mathbb{R}_+^k$, $\{C \in \mathbb{R}_+^k \mid C \succeq x\}$ and $\{C \in \mathbb{R}_+^k \mid x \succeq C\}$ are closed.
- *monotone* if $\forall C, C' \in D,\ C \geq C' \Rightarrow C \succeq C'$.

We focus on Choquet-Expected-Utility (Schmeidler [1989]). Preferences are then represented by the Choquet integral of a utility index U with respect to a capacity ν. The function U is cardinal *i.e.* defined up to a positive affine transformation.

A capacity is a set function $\nu : \mathcal{A} \to [0,1]$ such that $\nu(\emptyset) = 0$, $\nu(S) = 1$, and, for all $A, B \in \mathcal{A}, A \subset B \Rightarrow \nu(A) \leq \nu(B)$. We assume throughout that there exists $A \in \mathcal{A}$ such that $1 > \nu(A) > 0$.

A capacity ν is convex if for all $A, B \in \mathcal{A}$, $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$.

The core of a capacity ν is defined as follows

$$\text{core}(\nu) = \left\{ \pi \in \mathbb{R}_+^k \mid \sum_j \pi^j = 1 \text{ and } \pi(A) \geq \nu(A),\ \forall A \in \mathcal{A} \right\}$$

where $\pi(A) = \sum_{j \in A} \pi^j$. core$(\nu)$ is a compact, convex set which may be empty.

We now define the Choquet integral of $f \in \mathbb{R}^S$:

$$\int f d\nu \equiv E_\nu(f) = \int_{-\infty}^0 (\nu(f \geq t) - 1)\, dt + \int_0^\infty \nu(f \geq t)\, dt$$

Hence, if $f^j = f(j)$ is such that $f^1 \leq f^2 \leq \ldots \leq f^k$:

$$\int f d\nu = \sum_{j=1}^{k-1} [\nu(\{j, \ldots, k\}) - \nu(\{j+1, \ldots, k\})]\, f^j + \nu(\{k\})\, f^k$$

and, if we assume that an agent has wealth C^j in state j, and that $C^1 \leq \ldots \leq C^k$, then his preferences are represented by:

$$V(C) = [1 - \nu(\{2, .., k\})]\, U\left(C^1\right) + ... [\nu(\{j, .., k\}) - \nu(\{j+1, .., k\})]\, U\left(C^j\right) + ... \nu(\{k\})\, U\left(C^k\right)$$

It is well-known that when ν is convex, its core is non-empty and the Choquet integral of any random variable f is given by $\int f d\nu = \min_{\pi \in \text{core}(\nu)} E_\pi f$ (see Shapley [1967] and [1971], Rosenmueller [1972], Schmeidler [1986]) .

3 Convexity and the core

We now study the implications of different forms of diversification. We first define a natural notion of diversification (see also Dekel [1989]).

Definition 1 $\succeq$ *exhibits preference for diversification if for any* $C_1, C_2, \ldots, C_n \in D$,

$$[C_1 \sim C_2 \sim \ldots \sim C_n] \Rightarrow \sum_{i=1}^n \alpha_i C_i \succeq C_1$$

where $\alpha_i \geq 0$ *for all* i *and* $\sum_{i=1}^n \alpha_i = 1$.

For sake of completeness we recall that this notion of diversification is equivalent to convexity of preferences, that is, in our set-up, equivalent to the quasi-concavity of V.

Proposition 3.1 *Let $\succeq$ be continuous and monotone. Then, the following two assertions are equivalent:*
(i) $\succeq$ exhibits preference for diversification
(ii) $\succeq$ is convex

The following result provides a characterization of CEU agents that are diversifiers. We establish that convexity of preferences is equivalent to the capacity being convex and the utility index being concave. This generalizes results in Schmeidler [1989], Wakker [1990] and Chateauneuf [1991], where U is assumed linear.

Theorem 3.1 *Assume $U : \mathbb{R}_+ \to \mathbb{R}$ to be continuous, differentiable on $\mathbb{R}_{++}$ and strictly increasing. Then, the following statements are equivalent*
(i) $\succeq$ exhibits preference for diversification
(ii) V is concave
(iii) V is quasi-concave
(iv) U is concave and ν is convex.

This notion of diversification might seem fairly strong and we now introduce a weaker notion.

Definition 2 *$\succeq$ exhibits preference for sure diversification if for any $C_1, C_2, \ldots, C_r \in D$, $\alpha_1, \ldots, \alpha_r \geq 0$ such that $\sum_{\ell=1}^{r} \alpha_\ell = 1$, and $b \geq 0$:*

$$\left[C_1 \sim C_2 \sim \ldots \sim C_r, \text{ and } \sum_{\ell=1}^{r} \alpha_\ell C_\ell = b\mathbf{1}_S \right] \Rightarrow b\mathbf{1}_S \succeq C_\ell \quad \forall \ell$$

Thus, sure diversification means that if the decision maker can attain certainty by a convex combination of equally desirable random variables, then he prefers certainty to any of these random variables.

Theorem 3.2 *Let a decision maker be a CEU maximizer with capacity ν and continous utility index U, differentiable on $\mathbb{R}_{++}$ and strictly increasing. Then,*
(i) $\succeq$ exhibits preference for sure diversification $\Rightarrow$ core$(\nu) \neq \emptyset$.
(ii) If U is concave, core$(\nu) \neq \emptyset \Rightarrow \succeq$ exhibits preference for sure diversification.

This theorem falls short of a complete characterization of sure diversification. Indeed, if the DM has a convex utility index, he might or might not be a sure diversifier even though $core(\nu) \neq \emptyset$. The following two examples illustrate this point. In example 3.1, the DM has a capacity with a non-empty core and a convex utility index and is not a sure diversifier. In example 3.2, the DM also has a capacity with a non-empty core and a convex utility index, but this time he is a sure diversifier.

Example 3.1 Assume there are two states. Let $\nu^1 = \nu^2 = \frac{1}{3}$ and $U(C) = C^2$. core(ν) is obviously non-empty. However, $(1, 11) \sim (11, 1)$ and $\frac{1}{2}(1, 11) + \frac{1}{2}(11, 1) = (6, 6)$ but $v(6, 6) = 36 < v(1, 11) = 41$. $\diamond$

The following example shows that a DM might be a sure diversifier even though his utility index is convex.

Example 3.2 Assume there are two states, 1 and 2. Let $U(x) = 3x + \frac{1}{1+x}$ and $\nu^1 = \nu^2 = \frac{1}{4}$. U is strictly increasing, strictly convex.

We show that the set $\mathcal{C} = \{C = (C^1, C^2) \in \mathbb{R}^2_+ \mid C \sim a\mathbf{1}_S\}$ is above the hyperplane $\mathcal{H} = \left\{C = (C^1, C^2) \in \mathbb{R}^2_+ \mid \frac{1}{2}C^1 + \frac{1}{2}C^2 = a\right\}$. We then conclude that any sure convex combination of elements of $\mathcal{C}$ is preferred to $a\mathbf{1}_S$.

In order to show that the set $\mathcal{C}$ is above the hyperplane $\mathcal{H}$, it is enough to note that the indifference curve $\mathcal{C}$ consists of two concave curves, $\mathcal{C}_1 : C^2 = g_1(C^1),\ 0 \leq C^1 \leq a$ and $\mathcal{C}_2 : C^2 = g_2(C^1),\ a \leq C^1 \leq b$, such that the slope of the tangent to $\mathcal{C}_1$ for $C^1 = 0$ is smaller than -1, and, symmetrically, the slope of the tangent to $\mathcal{C}_2$ for $C^1 = b$ is greater than -1.

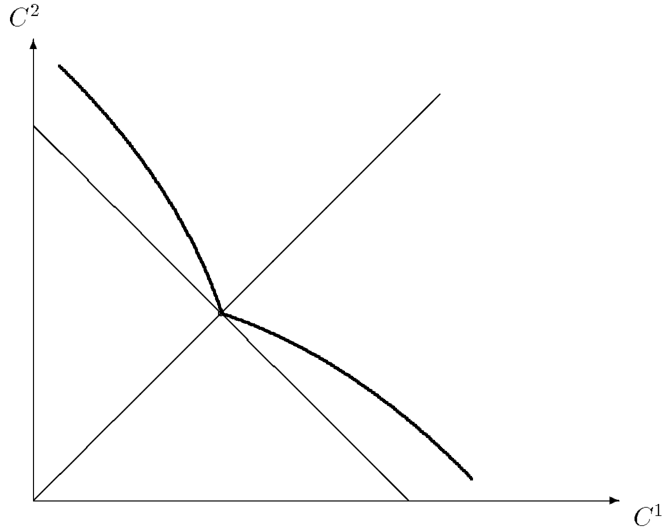
Notice that the existence of b and c such that $(0, c)$ and $(b, 0)$ belong to $\mathcal{C}$ follows from strict increasingness, continuity and unboundedness of U; concavity of g_1 and g_2 comes from concavity of U. Finally, straightforward computations yield that $g_1'(0) = \frac{-6}{U'(c)}$ and $g_2'(0) = \frac{-1}{6}U'(b)$. Since $U'(x) \leq 3\ \forall x \in \mathbb{R}_+$, it comes $g_1'(0) \leq -2$ and $g_2'(0) \geq \frac{-1}{2}$, and hence $g_1'(0) \leq -1$ and $g_2'(0) \geq -1$. Figure 1 illustrates this example. $\diamond$

Now, the concavity of the utility index can be shown to be equivalent to a different form of diversification, from which any hedging is eliminated.

To define this notion of diversification, we first need to recall the definition of comonotony of random variables. Say that two random variables x and x' are comonotone if there is no s and s' such that $x(s) > x(s')$ and $x'(s') > x'(s)$. Observe that two random variables that are comonotonic cannot be used to hedge against each other.

Definition 3 *A decision maker exhibits preference for comonotone diversification if for all comonotonic C and C' such that $C \sim C'$ one has $\lambda C + (1-\lambda) C' \succeq C$ for all $\lambda \in (0, 1)$.*

Figure 1:



Hence, comonotone diversification is nothing but convexity of preferences restricted to comonotone random variables. Note that any hedging (in the sense of Wakker [1990]) is prohibited in this diversification operation.

This type of diversification turns out to be equivalent, in the CEU model, to the concavity of U.

Theorem 3.3 *Let a decision maker be a CEU maximizer with capacity ν and continuous utility index U, differentiable on $\mathbb{R}_{++}$ and strictly increasing. Then, the following two assertions are equivalent:*

(i) $\succeq$ exhibits preference for comonotone diversification.

(ii) U is concave.

Corollary 3.1 *Let a decision maker be a CEU maximizer with capacity ν and continuous utility index U, differentiable on $\mathbb{R}_{++}$ and strictly increasing. Then, the following two assertions are equivalent:*

(i) $\succeq$ exhibits preference for comonotone and sure diversification.

(ii) U is concave and core(ν) is non-empty.

We end this note by discussing the implications of the different forms of diversification in the (subjective) expected utility model. It is well-known (although may be not in the finite case, for which the proof is more intricate, see Debreu and Koopmans [1982] and Wakker [1989]) that diversification (*i.e.* preference convexity) is equivalent to the concavity of the utility index. One can also deduce from theorem 3.3 that comonotone diversification is equivalent to the concavity of the utility index in the EU model as well. Finally, sure diversification is also equivalent, in the EU model, to concavity of the utility index.

Proposition 3.2 *Let a decision maker be an EU maximizer with utility index U, C^2 on $\mathbb{R}_{++}$, strictly increasing and continuous on $\mathbb{R}_+$. Then, the following assertions are equivalent:*

(i) $\succeq$ exhibits preference for diversification

(ii) $\succeq$ exhibits preference for sure diversification

(iii) $\succeq$ exhibits preference for comonotone diversification

(iv) U is concave

In the EU model, the two forms of diversification we introduced, namely sure and comonotone diversification, are both represented by concavity of the utility index and consequently cannot be distinguished. Furthermore, they cannot be distinguished from the usual notion of diversification (*i.e.* convexity of the preferences).

Appendix : Proofs

Proof of proposition 3.1:

$(ii) \Rightarrow (i)$ Let $C_i \in D$, $i = 1, \ldots, n$ be such that $C_1 \sim \ldots \sim C_n$, and let us prove by induction on n that $\sum_i \alpha_i C_i \succeq C_1$. The result is straightforwardly true for $n = 2$. Assume it holds true for $n \geq 2$, and let us show it is true for $n+1$. Let $C_1 \sim \ldots \sim C_n \sim C_{n+1}$ and $\alpha_i > 0$, $i = 1, \ldots, n+1$, $\sum_{i=1}^{n+1} \alpha_i = 1$. Define $\beta_i = \frac{\alpha_i}{1-\alpha_{n+1}}$, $i = 1, \ldots, n$. From the induction hypothesis, $\sum_{i=1}^{n} \beta_i C_i \succeq C_1$ and hence $\sum_{i=1}^{n} \beta_i C_i \succeq C_{n+1}$. Now, $\succeq$ convex implies $(1-\alpha_{n+1})(\sum_{i=1}^{n} \beta_i C_i) + \alpha_{n+1} C_{n+1} \succeq C_{n+1}$ that is $\sum_{i=1}^{n+1} \alpha_i C_i \succeq C_1$.

$(i) \Rightarrow (ii)$ What remains to be proved is that

$$C \succ C' \Rightarrow \alpha C + (1-\alpha) C' \succeq C' \quad \text{where } \alpha \in [0,1]$$

$\{\alpha \mid 0 \leq \alpha \leq 1, C' \succeq (1-\alpha)C\} \neq \emptyset$ since $C' \geq 0$ implies $C' \succeq 0$ by monotonicity. Let $\varepsilon \in \mathbb{R}_+$ be defined by $\varepsilon = \inf\{\alpha, 0 \leq \alpha \leq 1, C' \succeq (1-\alpha)C\}$. $\varepsilon > 0$ since $C \succ C'$. Let us show now that $(1-\varepsilon)C \sim C'$. Let (ε_n) be a stricly increasing sequence converging towards ε. From the definition of ε, $(1-\varepsilon_n)C \succ C'$, and from continuity $(1-\varepsilon)C \succeq C'$. Therefore, $(1-\varepsilon)C \sim C'$. Applying (i) gives $\alpha(1-\varepsilon)C + (1-\alpha)C' \succeq C'$ and hence by monotonicity $\alpha C + (1-\alpha)C' \succeq C'$. □

Proof of theorem 3.1:

$(i) \Leftrightarrow (iii)$ follows from proposition 3.1.

$(ii) \Rightarrow (iii)$ is well-known.

We now establish that $(iii) \Rightarrow (iv)$. We first show V quasi-concave implies ν convex. Convexity of ν is equivalent (see Shapley [1971]) to:

$$\forall A, B, E \in \mathcal{A} \text{ s.th. } B \subset A \text{ and } E \cap A = \emptyset, \quad \nu(A \cup E) - \nu(A) \geq \nu(B \cup E) - \nu(B) \qquad (1)$$

Assume (1) is false, and let $A, B, E \in \mathcal{A}$ be such that:

$B \subset A$, $E \cap A = \emptyset$, and $(\nu(A \cup E) - \nu(A)) - (\nu(B \cup E) - \nu(B)) + \alpha < 0$ for some $\alpha > 0$.

Let $c \in \mathbb{R}_{++}$ be such that $U'(c) > 0$ and let $a, b \in \mathbb{R}_+$ satisfy $a < c < b$. Finally, let $F, E, A \setminus B, B$ be a partition of S and consider the following random variables:

	F	E	$A \setminus B$	B
C	a	$c - \varepsilon\alpha_1$	$c + \varepsilon\alpha_2$	b
C'	a	$c + \varepsilon\beta_1$	$c - \varepsilon\beta_2$	b

where $\varepsilon > 0$ is sufficiently small so that a and b are respectively the smallest and the largest value of C and C', and where

$$\alpha_1 = \nu(A) - \nu(B) \qquad \alpha_2 = \nu(A \cup E) - \nu(A) + \alpha$$
$$\beta_1 = \nu(A \cup E) - \nu(B \cup E) + \alpha \qquad \beta_2 = \nu(B \cup E) - \nu(B)$$

Let us assume, w.l.o.g., that $U(a) = 0$ and $U(b) = 1$. A straightforward computation yields, knowing that U is strictly increasing:

$$V\left(\frac{C+C'}{2}\right) < \left(\nu\left(A \cup E\right) - \nu\left(B\right)\right) U\left(c\right) + \nu\left(B\right)$$

Now, one gets the following expression for $V(C)$:

$$\begin{aligned}
V\left(C\right) &= \left(\alpha_2 - \alpha\right) U\left(c - \varepsilon\alpha_1\right) + \alpha_1 U\left(c + \varepsilon\alpha_2\right) + \nu\left(B\right) \\
&= \left(\alpha_2 - \alpha\right)\left[U\left(c\right) - \varepsilon\alpha_1 U'\left(c\right) + \varepsilon\alpha_1\left(\varepsilon\right)\right] + \alpha_1\left[U\left(c\right) + \varepsilon\alpha_2 U'\left(c\right) + \varepsilon\alpha_2\left(\varepsilon\right)\right] + \nu\left(B\right) \\
&= \left(\nu\left(A \cup E\right) - \nu\left(B\right)\right) U\left(c\right) + \nu\left(B\right) + \varepsilon\left[U'\left(c\right)\alpha_1\alpha + \alpha_3\left(\varepsilon\right)\right]
\end{aligned}$$

where $\alpha_i\left(\varepsilon\right) \to 0$ as $\varepsilon \to 0$ for $i = 1, 2, 3$. $\alpha_1 > 0$ since if it were not then (1) would be true by monotony of ν. Hence, $U'\left(c\right)\alpha_1\alpha > 0$, and therefore:

$$V\left(C\right) > \left(\nu\left(A \cup E\right) - \nu\left(B\right)\right) U\left(c\right) + \nu\left(B\right)$$

for ε small enough.

A similar argument would establish the same inequality for $V(C')$, and therefore we get:

$$V\left(\frac{C+C'}{2}\right) < \min\left(V\left(C\right), V\left(C'\right)\right)$$

that is, V not quasi-concave, a contradiction. We conclude that ν is convex.

Let us now show that V quasi-concave implies that U is concave. Recall first theorem 2 in Debreu and Koopmans [1982]:

> Let I and J be open intervals in $\mathbb{R}$, f and g functions that are non-constant on I and J and such that $F : I \times J \to \mathbb{R}$ defined by $F(x, y) = f(x) + g(y)$ is quasi-convex. Then, at least one of the two functions f or g is convex.

Let $a > 0$ and $A \in \mathcal{A}$ be chosen such that $0 < \nu(A) < 1$. Let $I \equiv]0, a[$ and $J \equiv]a, +\infty[$ and define F on $I \times J$ by $F(x, y) = V\left(x\mathbf{1}_{A^c} + y\mathbf{1}_A\right)$.

Clearly, F is quasi-concave and $F(x, y) = (1 - \nu(A))\, U(x) + \nu(A)\, U(y)$. Therefore, U is concave on $]0, a[$ or on $]a, +\infty[$ for all $a > 0$, hence on $]0, +\infty[$ since U is differentiable, and on $\mathbb{R}_+$ since U is continuous.

Finally, $(iv) \Rightarrow (ii)$. Indeed, $V(C)$ is then equal to $\min_{Q \in \text{core}(\nu)} \int U(C)\, dQ$ and is therefore concave being the minimum of a family of concave functions. $\square$

Proof of theorem 3.2:

(i) Recall first (see Shapley [1967]) that core$(\nu) \neq \emptyset$ is equivalent to

$$\left[\sum_{\ell=1}^{r} a_\ell \mathbf{1}_{A_\ell} = \mathbf{1}_S,\ a_\ell \geq 0\right] \Rightarrow \sum_{\ell=1}^{r} a_\ell \nu\left(A_\ell\right) \leq 1, \quad \text{where } A_\ell \in \mathcal{A}$$

Let $A_\ell \in \mathcal{A}$, $a_\ell \geq 0$ be such that $\sum_{\ell=1}^{r} a_\ell \mathbf{1}_{A_\ell} = \mathbf{1}_S$. W.l.o.g., assume $a_\ell > 0$.

Assume there exists $x > 0$ such that $\sum_\ell a_\ell \nu\left(A_\ell\right) > 1 + x$. $\mathcal{L}$ will denote the set $\{\ell \,|\, \nu(A_\ell) > 0\}$. Let $a > 0$ be such that $U'(a) > 0$ and choose $\varepsilon > 0$ such that

$$\varepsilon\left(1 + x\right) \leq a \tag{2}$$

Define now the following positive random variables:

$$D_{\ell,\varepsilon} = \left[a - \varepsilon\nu\left(A_\ell\right)\right]\mathbf{1}_{A_\ell^c} + \left[a + \varepsilon\left(1 + x - \nu(A_\ell)\right)\right]\mathbf{1}_{A_\ell}$$

Let $\alpha_\ell = \dfrac{a_\ell}{\sum_\ell a_\ell}$. A straightforward computation (recall that $\sum_{\ell=1}^{r} a_\ell \mathbf{1}_{A_\ell} = \mathbf{1}_S$) yields: $\sum_\ell \alpha_\ell D_{\ell,\varepsilon} = d\left(\varepsilon\right)\mathbf{1}_S$ where

$$d\left(\varepsilon\right) = a + \frac{\varepsilon}{\sum_\ell a_\ell}\left(1 + x - \sum_\ell a_\ell \nu\left(A_\ell\right)\right) < a$$

If $\ell \notin \mathcal{L}$, $\nu(A_\ell) = 0$ and clearly $V(D_{\ell,\varepsilon}) = U(a)$.

If $\ell \in \mathcal{L}$, a computation similar to the one of theorem 3.1 yields:

$$V\left(D_{\ell,\varepsilon}\right) = U\left(a\right) + U'\left(a\right)\varepsilon\left(x\nu\left(A_\ell\right) + \alpha_\ell\left(\varepsilon\right)\right)$$

where $\alpha_\ell\left(\varepsilon\right) \to 0$ as $\varepsilon \to 0$. By assumption, $U'\left(a\right) > 0$. Hence, $V\left(D_{\ell,\varepsilon}\right) > U\left(a\right)$ for all $\ell \in \mathcal{L}$ if $\varepsilon > 0$ is sufficiently small.

Let ε_0 be such an ε. For all $\ell \in \mathcal{L}$, consider the following random variables:

$$D'_{\ell,t_\ell} = D_{\ell,\varepsilon_0} - t_\ell \mathbf{1}_S$$

where $t_\ell \geq 0$ is chosen such that $D'_{\ell,t_\ell} \geq 0$. Let

$$g_\ell\left(t_\ell\right) \equiv V\left(D'_{\ell,t_\ell}\right) = \left(1 - \nu(A_\ell)\right)U\left(a - t_\ell - \varepsilon_0\nu(A_\ell)\right) + \nu\left(A_\ell\right)U\left(a - t_\ell + \varepsilon_0\left(1 + x - \nu(A_\ell)\right)\right)$$

g_ℓ is continuous, strictly decreasing, and $g_\ell\left(0\right) = V\left(D_{\ell,\varepsilon_0}\right) > U\left(a\right)$ as previously shown.

Let us now prove that there exists $t_\ell \geq 0$ such that $D'_{\ell,t_\ell} \geq 0$ and $g_\ell(t_\ell) \leq U(a)$. It is enough to show that there exists t_ℓ satisfying:

$$a - t_\ell - \varepsilon_0\nu\left(A_\ell\right) \geq 0 \tag{3}$$

$$a - t_\ell + \varepsilon_0\left(1 + x - \nu(A_\ell)\right) \geq 0 \tag{4}$$

$$-t_\ell + \varepsilon_0\left(1 + x - \nu(A_\ell)\right) \leq 0 \tag{5}$$

Since (3) implies (4), it is enough to note that there exists $t_\ell \geq 0$ satisfying (3) and (5), *i.e.*, $t_\ell \geq 0$ such that $\varepsilon_0 (1 + x - \nu(A_\ell)) \leq t_\ell \leq a - \varepsilon_0 \nu(A_\ell)$. This proves to be true from (2).

Hence, for all $\ell \in \mathcal{L}$, there exists $\bar{t}_\ell > 0$ such that $D'_{\ell,\bar{t}_\ell} \geq 0$ and $D'_{\ell,\bar{t}_\ell} \sim a\mathbf{1}_S$. Let $C_\ell = D_{\ell,\varepsilon_0}$ if $\ell \notin \mathcal{L}$, and $C_\ell = D'_{\ell,\bar{t}_\ell}$ if $\ell \in \mathcal{L}$. Then $C_\ell \sim a\mathbf{1}_S$ for all ℓ, and $\sum_\ell \alpha_\ell C_\ell = b\mathbf{1}_S$, where $b \geq 0$, and $b = d(\varepsilon_0) - \sum_{\ell \in \mathcal{L}} \alpha_\ell \bar{t}_\ell < a$, a contradiction.

(ii) Suppose U concave and assume core$(\nu) \neq \emptyset$. Let C_ℓ, $\ell = 1, \ldots, r$ be such that $C_1 \sim C_2 \sim \ldots \sim C_r$ and $\sum_{\ell=1}^r \alpha_\ell C_\ell = b\mathbf{1}_S$, $\alpha_\ell \geq 0$, $\sum_{\ell=1}^r \alpha_\ell = 1$. Let $\pi \in$ core(ν). Then, $\int U(C_\ell)\, \mathrm{d}\nu \leq E_\pi U(C_\ell)$ for all ℓ (see, *e.g.*, proposition 2.1 in Chateauneuf, Dana and Tallon [1997]). Hence,

$$\sum_{\ell=1}^r \alpha_\ell \int U(C_\ell)\, \mathrm{d}\nu \leq \sum_{\ell=1}^r \alpha_\ell E_\pi U(C_\ell) \leq E_\pi U\left(\sum_{\ell=1}^r \alpha_\ell C_\ell\right) = U(b)$$

Therefore, $U(b) \geq \int U(C_\ell)\, \mathrm{d}\nu$ for all ℓ, *i.e.* $b\mathbf{1}_S \succeq C_\ell$ for all ℓ. □

Proof of theorem 3.3:

$[(i) \Rightarrow (ii)\ ]$ The same argument as in the end of the proof of $(ii) \Rightarrow (iii)$ of theorem 3.1 applies, since the random variables considered there, *i.e.* $x\mathbf{1}_{A^c} + y\mathbf{1}_A$ are comonotone.

$[(ii) \Rightarrow (i)\ ]$ Let C and C' be two comonotone random variables such that $C \sim C'$, and $\lambda \in (0,1)$. Then, $\lambda E_\nu U(C) + (1-\lambda) E_\nu U(C') = E_\nu [\lambda U(C) + (1-\lambda) U(C')]$. This last expression is less than $E_\nu U(\lambda C + (1-\lambda) C')$ by concavity of U and hence $\lambda C + (1-\lambda) C' \succeq C$. □

Proof of proposition 3.2:

The following implications are straightforward, $[(iv) \Rightarrow (i)]$, $[\ (i) \Rightarrow (ii)]$. $[(iii) \Rightarrow (iv)]$ follows from theorem 3.3.

What remains to be proved is $[(ii) \Rightarrow (iv)]$. To that effect, suppose U is not concave on $\mathbb{R}_+$. Hence, there exists $x_0 \in \mathbb{R}_{++}$ such that $U"(x_0) > 0$, and therefore there exist $a, b \in \mathbb{R}_{++}$, $a < b$, such that $U"(x) > 0$ on $[a,b]$. U is hence strictly convex on $[a,b]$.

Let A and A^c be events with probability π and $1-\pi$ such that $0 < \pi \leq 1 - \pi$.

Now, since U is strictly increasing, continuous and $\pi \leq 1/2$, there exists $a' \in \mathbb{R}_+$, $b > a' \geq a$ such that

$$\pi U(a) + (1-\pi) U(b) = \pi U(b) + (1-\pi) U(a')$$

Consider now the following two acts $C_1 = a\mathbf{1}_A + b\mathbf{1}_{A^c}$ and $C_2 = b\mathbf{1}_A + a'\mathbf{1}_{A^c}$. Notice that $C_1 \sim C_2$.

Let $\alpha = \frac{b-a'}{b-a'+b-a} \in (0,1)$. A straightforward computation gives:

$$\alpha C_1 + (1-\alpha)C_2 = k\mathbf{1}_S \quad \text{with} \quad k = \frac{b^2 - aa'}{2b - a - a'} \in \mathbb{R}_{++}$$

But $U(k) = E(U(\alpha C_1 + (1-\alpha)C_2)) < \alpha E(U(C_1)) + (1-\alpha)E(U(C_2))$ by strict convexity of U on $[a,b]$. $E(U(C_1)) = E(U(C_2))$ then implies $C_1 \succ k\mathbf{1}_S$, a contradiction. $\square$

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