FISCAL POLICY AND THE MATURITY STRUCTURE WITH NON-CONTINGENT DEBT*

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The literature on optimal fiscal policy, following Lucas & Stokey [1983], has relied heavily on the assumption of state-contingent debt, while in reality this is mostly non-contingent. Hence the uneasy gap between theory and reality: *Is the existing theory as irrelevant as it is silent about optimal fiscal policy with non-contingent debt?* The resolution we offer is reassuring: Long maturity makes the debt burden *effectively* state-contingent, thanks to the *equilibrium* fluctuation of the term structure of interest rates. The government can *manipulate* this so as to sustain with non-contingent debt *almost every* policy that would be sustainable with contingent debt. In particular, we show non-contingent bonds to essentially complete the markets for *any generic* policy. This is good news for existing theory as regards the relevance of the complete-markets paradigm. Besides, social welfare monotonically increases with market completeness. And finally, the maturity structure and the level of debt at the Ramsey optimum are invariant over the business cycle, a result contrasting both common wisdom and martingale models à la Barro [1979].

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I. Introduction

The neoclassical normative analysis of taxation, stemming from Ramsey [1927], soon found application to the theory of optimal fiscal policy.[1] A first wave, along the lines of Barro [1979], emphasized the welfare significance of smoothing tax distortions *across time*, while a second wave, along the lines of Lucas & Stokey [1983], added another dimension, that of smoothing tax distortions *across different states* in a stochastic world.[2] Public debt has thereby been identified as an instrument through which the government can acquire *insurance* from the private sector and stabilize the economy across *both* time and states.

For what we are concerned in this paper, the literature on optimal fiscal policy — including Lucas & Stokey [1983], Zhu [1992], Chari, Christiano & Kehoe [1991, 1994, 1996], and Chari & Kehoe [1999] — has heavily relied on the assumption that public debt is traded in state-contingent obligations, as if the government had access to a complete set of Arrow securities. In reality, however, debt is mostly non-contingent. There is hence a disturbing and uneasy gap between theory and reality, leaving one to question how relevant and applicable the complete-markets paradigm of optimal fiscal policy is to a world where the government can issue only non-contingent debt. This question gets indeed more pressing in the light of the recent work by Marcet, Sargent & Seppala [1999], who consider an economy where the government has access only to risk-free one-period debt, and find that optimal fiscal policy in such an environment may differ substantially from what is predicted by the complete-markets paradigm.

Lucas & Stokey [1983, p.88] themselves admit that "the option to issue state-contingent government debt is important: tax policies under uncertainty have an essential 'insurance' aspect to them." But the caveat is precisely that, while critical in the theory, in reality this option is not available.

So, *is the existing theory just as irrelevant as it is silent about optimal fiscal policy under uncertainty with non-contingent debt?*

The resolution we offer is quite reassuring: Strikingly enough, we show that a rich

[1] The pertinent literature is immense, including Barro [1979, 1995, 1997], Bohn [1990], Chari & Kehoe [1990, 1993, 1999], Chari, Christiano & Kehoe [1991, 1994, 1995, 1996], Hansen, Roberds & Sargent [1991], Judd [1985, 1987], Kydland & Prescott [1980], Lucas & Stokey [1983], Mankiw [1987], Marcet, Sargent & Seppala [1999], and Zhu [1992]. Barro [1989] offers an authoritative overview.

[2] Beyond Barro [1979], Pigou [1947] and Kydland & Prescott [1977, 1980] as well observed that Ramsey's approach could be applied to the study of optimal fiscal policy, by just reinterpreting the different goods in Ramsey's formulation as consumption at different dates. And similarly, Lucas & Stokey [1983] essentially reinterpreted Ramsey's economy in terms of an Arrow-Debreu economy, with different goods representing consumption in different dates and different states/events.

maturity structure can substitute, even perfectly, for state-contingency of debt. Hence, a world of non-contingent debt can be *isomorphic* to a world of contingent debt.

In particular, we consider a stochastic closed economy without capital like the one in Lucas & Stokey [1983],[3] but we allow for *incomplete markets* in the sense that public debt can *not* be state-contingent, and we essentially preclude any type of ex-post lump-sum transfers.[4] All debt has to be held in certain, *non-contingent* obligations, but these can be of various maturities. On the one hand, it is precisely the lack of contingent debt that differentiates our analysis from Lucas & Stokey [1983] and the pertinent literature. On the other hand, it is the possibility for long maturity that differentiates us from Marcet, Sargent & Seppala [1999], and this will turn out to be critical in restoring the validity of the complete-markets paradigm.

We first observe that any long maturity provides *some* state-contingency for the debt burden, thanks to the *endogenous* state-contingency of the *equilibrium* term structure of bond prices.[5] We then show that the government can manipulate the term structure so as to attain as much cross-state insurance as necessary to sustain with non-contingent debt essentially *any* tax policy that would be sustainable with state-contingent debt. More precisely, if the maturity structure is as rich as the number of possible continuation states, then non-contingent bonds complete the markets for any generic policy.

A simple example reveals the intuition: Consider an economy facing booms and recessions. Naturally, the Ramsey optimal policy dictates that the government runs a countercyclical budget deficit, coupled with a countercyclical debt burden and sustained by a procyclical present value of surpluses. This scheme serves as optimal insurance, transferring funds from booms to recessions. If the government had access to state-contingent debt, it would implement this scheme simply by borrowing in a debt contract that promises to pay a lot in a boom and little in a recession. But, what if instead debt is non-contingent? Allow the government at date t to borrow in long-term debt, and assume that short-term bond prices are procyclical.[6] As soon as date $t+1$ arrives and a boom or a recession is realized, the government can trade (buy) the outstanding past issues of long-term bonds at the contemporaneous short-term price. The procyclicality of the

[3]On the way we also get a reduced form to the social planner's problem which provides microfoundations to Barro's [1979] formulation.

[4]For example, we exclude default and ex-post state-contingent taxation of debt holdings or interest payments, like that in Zhu [1992] or Chari & Kehoe [1999].

[5]Bohn [1990] and Barro [1997] identified the risk-hedging possibilities in long-term debt, but their analyses were limited by the fact that they adopted a partial-equilibrium portfolio-management framework and treated interest rates as exogenous. Nobody whatsoever has examined the potential for *manipulating* equilibrium interest rates or the implications of the *endogeneity* of bond prices.

[6]The procyclicality assumption is not restrictive — see Section IV.D and Footnote 34.

latter then implies that the market value of the outstanding debt stock, or the effective debt burden, is also procyclical, just as the complete-markets Ramsey optimum dictates. So, the basic intuition is two-fold: First, via the design of the maturity structure of debt, the government can exploit the cyclical properties of the term structure of interest rates, and thereby condition its debt burden on the state of the economy as desired. Second, the government can induce and manipulate the endogenous term structure via tax policies. Thus, it is *as if* we had access to a complete set of Arrow securities.

Next, we know that introducing new assets in an incomplete-markets economy may *not* always increase welfare. With the government reoptimizing, however, monotonicity is ensured: Welfare at the third best always increases with market completeness.

Finally, a quite striking finding is that the optimal maturity structure, which implements the complete-markets Ramsey optimum with non-contingent debt, is *invariant over the business cycle*: Debt issues are the same in peaks and recessions. This result contrasts sharply both to common wisdom and to martingale models of taxation and debt à la Barro [1979].

Our result is good news for the existing theory on optimal fiscal policy, in that contingent debt is just a valid parable. Similarly, our finding could offer a resolution of the puzzle why we do not observe contingent debt while it appears to be so desirable for tax smoothing.[7] Apart from the moral-hazard issues involved in contingent debt, we may not have contingent debt simply because we do not need it. To put it differently, if we need more insurance along the business cycle, we can get it by expanding and appropriately managing the maturity structure.[8] And if, nonetheless, observed policy appears remote from the Ramsey optimal one, then at least we know that this is not due to the lack of contingent debt, but rather due to some other distortion.[9] Besides, contingent debt may face all the typical problems of complete contracts — e.g., difficulties in describing and verifying the state. On the contrary, bond prices are directly observable in the market, implying that the implementation of the endogenous contingency induced by the term structure is immediate.[10]

Further, advocates of the complete-markets paradigm have suggested that some of

[7] Lucas & Stokey [1983, p.77] prompt us to "wonder why governments forego gains in everyone's welfare by issuing only debt that purports to be a certain claim on future goods."

[8] This may relates to Shiller's [1993] proposals for new macro markets: We may not need new markets for contingent debt if the maturity structure is rich enough.

[9] For instance, the government may not know what the right policy is, or may not be benevolent, or may not be able to commit to the optimal but time-inconsistent policy plan.

[10] The last observation suggests a way moral-hazard and asymmetric-information problems may be mitigated when debt is held in long maturity rather than contracted as state-contingent obligations. This is an interesting open question.

the desired contingency for the debt burden can be induced by countercyclical inflation and/or procyclical taxation of debt holdings. None of this, however, seems empirically relevant, at least not to the extent that the optimal Ramsey policy would require,[11] so that the puzzle remained unanswered. Besides, raising asset taxes during a recession may exacerbate moral-hazard issues, involve political complications, or undermine government reputation/credibility — not to mention that countercyclical inflation would aggravate recessions and that monetary policy appears to aim at independent targets. However, provided a sufficiently rich maturity structure, monetary and tax policies can be disentangled from those considerations — and that's good news for both Alan Greenspan and Larry Summers!

The layout of the paper is as follows: Section II sets up our model economy and Section III proceeds to equilibrium analysis for any given tax policy. In Section IV we first determine the sets of sustainable tax policies *with* and *without* contingent debt; we next establish that the two sets coincide when the maturity structure is rich enough; and we reconsider in detail our business-cycle example. Section V turns to the implementation of the complete-markets optimal Ramsey policy with non-contingent debt and characterizes the optimal maturity structure. We conclude in Section VI with various remarks on robustness, scope, applicability, and empirical implications. The Appendix includes the proofs not appearing in the main text.

II. An Economy Without Capital

A. The Fundamentals of the Economy

We consider a standard neoclassical stochastic economy without capital. The fundamentals are identical to Lucas & Stokey [1983]. Labor is the only input in production, and the technology is subject to productivity shocks. Government spending is exogenous and stochastic, financed either by income taxes or by public debt. The government has a single tax instrument, a flat tax rate on total income/output, which distorts the labor-leisure choice.

Without loss of generality, we assume that the exogenous stochastic disturbances of the economy (government spending, productivity, endowments, etc.) are generated by a finite-support *stationary Markov process.* We let S be the number of possible

[11] In a portfolio-management framework, Bohn [1990, p.1226] crudely estimates for the US that historical inflation would induce the desired cross-state fluctuation in the real debt burden, if the level of nominal debt was as high as 2,680% of GDP. The actual figure of about 50% suggests the inflation-induced contingency is rather negligible.

states, $\mathcal{S} = \{\sigma^1, ..., \sigma^S\}$ the finite support of the Markov process (the state space),[12] and $\mu(.|.) : \mathcal{S} \times \mathcal{S} \to (0,1)$ the transition-probability function (the conditional p.d.f.). We let $s_t \in \mathcal{S}$ denote the state of the economy at date t; ${}_ts \equiv (s_0, ..., s_{t-1}, s_t) \in \mathcal{S}^t$ a typical history/path of states up to period t, or the event at t; and $\{{}_ts\} \times \mathcal{S}^j$ the set of all possible continuation events j periods ahead of node $(t,_t s)$. Finally, by the Markov property, $\Pr({}_ts, s_{t+1}|{}_ts) = \mu(s_{t+1}|s_t)$ and $\Pr({}_{t+j}s|{}_ts) = \mu(s_{t+j}|s_{t+j-1})...\mu(s_{t+1}|s_t)\forall j$.

Throughout we let $C_t({}_ts), L_t({}_ts), Y_t({}_ts), G(s_t)$, and $\tau_t({}_ts)$ denote aggregate consumption, the labor share of time, total output, government spending, and the tax rate, respectively, all as of date t and event ${}_ts = (s_0, ..., s_t)$.[13] The technology is given by a stationary function $F : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$, with $F_L > 0 \geq F_{LL}$,[14] so that the economy's resource constraint at date-event $(t,_t s)$ is:

$$C_t({}_ts) + G(s_t) = Y_t({}_ts) = F(L_t({}_ts), s_t) \tag{1}$$

We consider a labor-only economy first because that accords precisely with Lucas & Stokey [1983], secondly because our economy can be loosely reinterpreted as the limiting steady state of a growth economy, and finally because of space limitations and technical difficulties in incorporating capital.[15] Apart from some hints we give on the way, the extension to a capital economy remains an open project. Yet, by the end of our analysis we expect the reader to agree that leaving out capital is *not* critical for our main reasoning.

Finally, preferences are standard VonNeumann-Morgenstern, separable across time and states, given by $E_t\mathcal{U}_t = U(C_t, 1 - L_t) + \beta \cdot E_t\mathcal{U}_{t+1}$, where E_t is the expectation

[12] The assumption of a finite-support stationary Markov process is quite 'ergotic', but stronger than necessary. All we need for our main results is the number of possible continuation events, conditional on the current event, to be finite. Then our framework can allow, e.g., productivity and spending to be non-stationary and have a growth trend.

[13] The realization of the endogenous random variables may depend on the whole history of states, but the Markov assumption implies that the realization of government spending (like productivity, endowments, and any other exogenous disturbances) is determined by the contemporaneous state only — there is a fixed mapping $G : \mathcal{S} \to \mathbb{R}_+$ such that $G_t({}_ts) = G(s_t)\forall(t,_t s)$.

[14] An example is the linear specification $Y = F(L, s) = A(s)L + e(s)$, with A denoting labor productivity and e the endowment, both random.

[15] Consider the steady state of a neoclassical Solow-Cass-Koopmans economy, or a limiting Ak endogenous-growth economy: Following Chamley [1986], Judd [1987], Jones, Manuelli & Rossi [1995, 1997], Zhu [1993], and Chari & Kehoe [1999], tax rates on capital should be zero either asymptotically or even after finite time. Labor taxes are then the only taxes raised, so we conjecture the steady-state stochastic dynamics of such an economy to be quite close to the labor economy we consider here. This is not exactly true because of the persistence introduced by capital, and it is exactly there that the technical difficulty arises. See also Footnote 40 and the concluding Section.

operator conditional on $(t,_t s)$ and $U(C, 1-L)$ is the utility flow out of consumption C and leisure $(1-L)$. So, as of date 0:

$$E_0\mathcal{U}_0 = \sum_{t=0}^{\infty} \beta^t \sum_{_t s} \mu(_t s|s_0) \cdot U\left(C_t(_t s), 1 - L_t(_t s)\right) \tag{2}$$

where $\mu(_t s|s_0) = \Pr(_t s|s_0)$. The utility function $U : \mathbb{R}_{++} \times (0,1) \to \mathbb{R}$ is standard neoclassical: increasing, strictly concave, smooth, and assumed to satisfy the Inada conditions that preclude corner solutions. Finally, should we choose to incorporate preference shocks, U could be state-dependent.

B. The Household and the Government Budget, with Non-Contingent Debt

Let us index by $(t, j,_t s)$ a non-contingent bond of maturity j issued at date t and event $_t s$, and promising to pay one unit of consumable at date $t+j$, whatever the then state s_{t+j} and the then event $_{t+j}s$. Let $M \geq 1$ be the maximum maturity; let $j \in \{1, 2, ..., M\}$; and finally let $b_{t,j}(_t s)$ be the stock of maturity$-j$ bonds issued at date-event $(t,_t s)$, and $p_{t,j}(_t s)$ the price as of $(t,_t s)$ for any bond maturing j periods ahead.

As a matter of convention, the government refinances its debt every period,[16] so that the bonds market opens every period t with only issues dated $t-1$ and closes with only issues dated t. We can hence write the household budget at any date t and event $_t s = (_{t-1}s, s_t)$ as follows:

$$C_t(_t s) + \sum_{j=1}^{M} p_{t,j}(_t s) b_{t,j}(_t s) \leq [1 - \tau_t(_t s)] Y_t(_t s) + \sum_{j=0}^{M-1} p_{t,j}(_t s) b_{t-1,j+1}(_{t-1}s) \tag{3}$$

with the convention that $p_{t,0}(.) = 1$. The government budget, on the other hand, is:

$$G(s_t) + \sum_{j=0}^{M-1} p_{t,j}(_t s) b_{t-1,j+1}(_{t-1}s) \leq \tau_t(_t s) Y_t(_t s) + \sum_{j=1}^{M} p_{t,j}(_t s) b_{t,j}(_t s) \tag{4}$$

Notice that $\sum_{j=0}^{M-1} p_{t,j} b_{t-1,j+1}$ is the market value of old debt that has just matured or is bought out to be refinanced, while $\sum_{j=1}^{M} p_{t,j} b_{t,j}$ is the revenue from new debt issues. We finally denote by $P_t = (p_{t,1}, ..., p_{t,M})$ and $B_t = (b_{t,1}, ..., b_{t,M})$ the term structure of bond prices and the maturity structure of debt issues, respectively.

C. Complete Markets and State-Contingent Debt

[16] This convention is without any loss of generality because bonds issued at any different dates but maturing at a common future date are perfect substitutes, and thus, by simple arbitrage, they have a common price.

Above we assumed the government to issue only non-contingent bonds. If instead markets are complete and debt is state-contingent, then the traded bonds take the form of Arrow securities that pay only in particular states/events. We index by $(t, j,_t s,_{t+j} s)$ securities of maturity j, issued at date-event $(t,_t s)$ and paying one unit of consumable at date-event $(t + j,_{t+j} s)$ and zero otherwise, and we let $q_{t,j}(_{t+1}s|_ts)$ be their price. We accordingly let $d_{t,j}(_{t+1}s|_ts)$ be the contingent debt raised at $(t,_t s)$ in securities paying at $(t + j,_{t+j} s)$. With state-contingent debt, the household and government budgets have to be adjusted appropriately, but we defer this to Section IV.B.

III. The Competitive Equilibrium

A. Definition and Characterization

We have fully determined the fundamentals of the economy — these are given as the collection $\mathcal{E} = \{\mathcal{S}, \mu, s_0, F, G, U, \beta, \bar{B}_{-1}, M\}$. A tax policy $\tau = \{\tau_t(.)\}_{t=0}^{\infty}$ is a sequence of mappings $\tau_t : \mathcal{S}^t \to [0, 1)$. The definition of a competitive equilibrium, for a given tax policy, is then as follows:

Definition 1 *An* ***Incomplete-Markets Competitive Equilibrium*** *for the economy $\mathcal{E}$ consists of a bounded sequence of tax rates and bond issues, $\{\tau_t(.), B_t(.)\}_{t=0}^{\infty}$, of consumption, labor and output allocations, $\{C_t(.), L_t(.), Y_t(.)\}_{t=0}^{\infty}$, and bond prices, $\{P_t(.)\}_{t=0}^{\infty}$, such that: (i) Given $\{\tau_t(.), P_t(.)\}_{t=0}^{\infty}$, $\{C_t(.), L_t(.), B_t(.), Y_t(.)\}_{t=0}^{\infty}$ maximizes the representative consumer's utility in (2) subject to her budgets (3) and the technology (1);*[17] *and (ii) given $\{P_t(.), Y_t(.)\}_{t=0}^{\infty}$, $\{\tau_t(.), B_t(.)\}_{t=0}^{\infty}$ satisfies the series of government budget constraints in (4), starting with given initial debt $\bar{B}_{-1}$. We then also say that the tax policy $\tau = \{\tau_t(.)\}_{t=0}^{\infty}$ is* ***sustainable.***

As usual, given any tax policy, competitive equilibrium allocations are characterized by the optimality conditions for the representative household's problem. Letting $U_c(t) \equiv U_c(C_t, 1-L_t)$, $U_l(t) \equiv U_l(C_t, 1-L_t)$, and $w_t = F_L(L_t, s_t)$, optimality requires that $U_l(t) = [1 - \tau_t]w_tU_c(t)$ and $U_c(t)p_{t,j} = \beta^j E_tU_c(t + j)\ \forall j$, at all dates and events — plus the transversality conditions, $\lim_{t\to\infty} \beta^t p_{t,j} b_{t,j} = 0\,\forall j$, or equivalently, $\lim_{t\to\infty} \beta^{t+j} E_tU_c(t + j)b_{t,j} = 0\,\forall j$. As a standard result, these conditions are both necessary and sufficient.

B. Equilibrium Allocations in an Economy without Capital

[17] To be precise, we should have added the standard no-Ponzi-game constraint.

In equilibrium, at any date and event, the MRS between consumption and leisure is equal to the net-of-tax wage rate:

$$\frac{U_l(C_t(_ts), 1 - L_t(_ts))}{U_c(C_t(_ts), 1 - L_t(_ts))} = [1 - \tau_t(_ts)] \cdot F_L(L_t(_ts), s_t) \tag{5}$$

If we combine (5) with the resource constraint (1), we see that the contemporaneous state s_t and the contemporaneous tax rate τ_t alone fully determine the static equilibrium allocation (C_t, L_t) at any date and event. The same holds true for the resulting utility flow U_t and the primary surplus $R_t \equiv \tau_t Y_t - G_t$. Moreover, an increase in τ_t reduces both contemporaneous C_t and L_t,[18] and reduces U_t as well. Letting $U_t(_ts) \equiv U\left(C_t(_ts), 1 - L_t(_ts)\right)$ and $R_t(_ts) \equiv \tau_t(_ts)Y_t(_ts) - G_t(_ts)$, we have:

Proposition 1 ***(Static Allocations)*** *For any generic*[19] *stationary economy $\mathcal{E}$ without capital, there are fixed mappings $C^*, L^*, u^*, R^* : [0,1] \times \mathcal{S} \to \mathbb{R}$ such that, in any competitive equilibrium and for any policy $\{\tau_t(.)\}_{t=0}^{\infty}$: $C_t(_ts) = C^*\left(\tau_t(_ts), s_t\right)$, $L_t(_ts) = L^*\left(\tau_t(_ts), s_t\right)$, $U_t(_ts) = u^*(\tau_t(_ts), s_t)$, and $R_t(_ts) = R^*(\tau_t(_ts), s_t)$, at all $(t,_t s)$. Further, $C^*_\tau(\tau, s) < 0$ and $L^*_\tau(\tau, s) < 0$ $\forall \tau$. Finally, $u^*_\tau(0, s) = 0 > u^*_\tau(\tau, s) > -\infty = u^*_\tau(1, s)$ $\forall \tau \in (0, 1)$.*

The result about the utility flow $U_t(_ts)$, simply translates the distortionary effects of taxation in terms of equilibrium welfare, with u^*_τ being the shadow cost or marginal disutility of taxation. In fact, $u^*(\tau, s)$ plays exactly the role of the welfare cost of taxation as in Barro's [1979] formulation: The period-t social welfare flow is decreasing in the contemporaneous tax rate ($u^*_\tau < 0$), and the marginal social cost of taxation may well be increasing in the rate itself ($u^*_{\tau\tau} < 0$).[20] We finally emphasize that the above are equally valid under either complete or incomplete markets.

C. Equilibrium Bond Pricing

As before, let $q_{t,j}(_{t+j}s|_ts)$ be the price of an Arrow security issued at date-event $(t,_t s)$ and paying at date-event $(t+j,_{t+j} s)$. Optimality on the household side implies that this

[18] The effect of the tax rate on leisure/labor is unambiguous because in general equilibrium there is no income effect counteracting the substitution effect.

[19] In non-generic cases, the mappings may simply fail to be single-valued, meaning multiple equilibria. From now on we ignore such multiplicity, without loss of generality.

[20] This differs from Barro [1979] is that the economy is expectably stochastic and the efficiency costs of taxation depend on the contemporaneous state of the economy. As seems intuitive, we expect the marginal cost, $u^*_\tau(.)$, to increase with government spending, to decrease with endowments, and to be ambiguous with respect to productivity.

is equal to the corresponding probability-weighted MRS in consumption: $q_{t,j}({}_{t+j}s|_t s) = \frac{\beta^j \mu({}_{t+j}s|_t s) U_c(C_{t+j}({}_{t+j}s), 1-L_{t+j}({}_{t+j}s))}{U_c(C_t({}_t s), 1-L_t({}_t s))}$. Letting $q_t({}_t s, s_{t+1}) \equiv q_{t,1}({}_t s, s_{t+1}|_t s)$ for the typical one-period security, simple arbitrage then dictates that $q_{t,j}({}_{t+j}s|_t s) = \prod_{n=0}^{j-1} q_{t+n}({}_{t+n}s, s_{t+n+1})$. Proposition 1 then provides us with the following:

Lemma 1 *$\exists\, q_j^* : [0,1)^2 \times \mathcal{S}^{j+1}\ \forall j \geq 1$ such that $q_{t,j}({}_{t+j}s|_t s) = q_j^*(\tau_t({}_t s), \tau({}_{t+j}s); s_t, ..., s_{t+j})$ $\forall_{t+j} s \in \{{}_t s\} \times \mathcal{S}^j$ and $q_{t,j}({}_{t+j}s|_t s) = 0$ otherwise. Also, $U_{cl} \geq 0$ or $U_{cl} > F_L U_{cc}$ implies $\partial q_j^*(\tau, \tau', .)/\partial \tau < 0 < \partial q_j^*(\tau, \tau', .)/\partial \tau'$. Finally, $\lim_{\tau \to 1} q_j^*(\tau, \tau', .) = 0$, $\lim_{\tau' \to 1} q_j^*(\tau, \tau', .) = +\infty$.*

We can now turn to the equilibrium pricing of non-contingent bonds. The household's optimality conditions dictate that $p_{t,j}({}_t s)$ equals the expected MRS in consumption between t and $t+j$. That is:

$$p_{t,j}({}_t s) = \frac{\sum_{t+j s} \beta^j \mu({}_{t+j}s|_t s) U_c\left(C_{t+j}({}_{t+j}s), 1 - L_{t+j}({}_{t+j}s)\right)}{U_c(C_t({}_t s), 1 - L_t({}_t s))} = \sum_{t+j s} q_{t,j}({}_{t+j}s|_t s) \tag{6}$$

Lemma 1 then implies:

Proposition 2 ***(Bond Pricing)*** *There are stationary pricing rules $p_j^* : [0,1) \times [0,1)^{jS} \times \mathcal{S} \to \mathbb{R}_+$ $(j = 1, ..., M)$, such that for all t,${}_t s$,j:*

$$p_{t,j}({}_t s) = p_j^*\left(\tau_t({}_t s), \tau_{t+j}(.|_t s), s_t\right) \equiv \sum_{t+j s} q_j^*(\tau_t({}_t s), \tau_{t+j}({}_{t+j}s), s_t, .., s_{t+j}) \tag{7}$$

where $\tau_{t+j}(.|_t s) \equiv \{\tau_t({}_{t+j}s) :\ {}_{t+j}s \in \{{}_t s\} \times \mathcal{S}^j\}$ denotes the continuation tax structure j periods ahead of $(t,{}_t s)$. Further, unless consumption and leisure are too strong substitutes, $\partial p_j^*(\tau, \tau', .)/\partial \tau < 0 < \partial p_j^*(\tau, \tau', .)/\partial \tau'$. That is, $\frac{\partial p_{t,j}({}_t s)}{\partial \tau_t({}_t s)} < 0 < \frac{\partial p_{t,j}({}_t s)}{\partial \tau_{t+j}({}_{t+j}s)}\ \forall_t s,_{t+j} s, t, j$, while $\frac{\partial p_{t,j}({}_t s)}{\partial \tau_n({}_n s)} = 0\ \forall_t s,_{t+n} s, t, j, n \notin \{t, t+j\}$.

So, the equilibrium price of a bond maturing j periods ahead depends only on (i) the contemporaneous state, (ii) the contemporaneous tax rate, and (iii) the expected tax structure at the maturity date. Further, the return typically increases with a rise in the contemporaneous tax rate, and decreases with an expected tax increase at the maturity date. The intuition is straightforward: An increase in the current tax rate reduces current consumption and thereby increases the contemporaneous marginal utility of consumption.[21] It follows that the intertemporal MRS between t and $t+j$ decreases with the tax rate in t and increases with any tax rate in $t+j$.

[21] The increase in the tax rate also increases leisure, but, unless consumption and leisure are too strong substitutes, the marginal utility of consumption still increases with the current tax rate.

Parenthetically, the term structure satisfies the following recursive pricing rule: $p_{t,1}({}_ts) = \sum_{s_{t+1}} q_t({}_ts, s_{t+1})$ and $p_{t,j}({}_ts) = \sum_{s_{t+1}} [q_t({}_ts, s_{t+1}) p_{t+1,j-1}({}_ts, s_{t+1})]$ $\forall j \geq 2$. This means $p_{t,1} = E_t[MRS_{t,t+1}]$ and $p_{t,j} = p_{t,1} E_t[p_{t+1,j-1}] + Cov_t(MRS_{t,t+1}, p_{t+1,j-1})$ $\forall j \geq 2$, so that the covariance term measures the risk premium on maturity j. From a portfolio-management perspective, these risk premia appear as a cost we have to incur in order to exploit the term structure. However, if exploiting and manipulating the term structure allows us to attain better tax smoothing, then consumption is stabilized across states, which mitigates uncertainty and may in turn reduce risk premia on all maturities — that's the beauty in general-equilibrium analysis!

D. Existence and Uniqueness of Equilibrium

Following Propositions 1 and 2, we have established that any given sustainable policy induces a unique competitive equilibrium:

Proposition 3 ***(Competitive Equilibrium)*** *For a generic economy $\mathcal{E}$, given any sustainable tax policy $\{\tau_t(.)\}_{t=0}^{\infty}$, a competitive equilibrium exists and is unique. The equilibrium allocations $\{C_t(.), L_t(.)\}_{t=0}^{\infty}$ and bond prices $\{P_t(.)\}_{t=0}^{\infty}$ are as in Propositions 1 and 2.*

We emphasize that uniqueness is meant in terms of allocations $\{C_t(.), L_t(.)\}_{t=0}^{\infty}$ and prices $\{P_t(.)\}_{t=0}^{\infty}$, but there might well be multiple debt issues $\{B_t(.)\}_{t=0}^{\infty}$ satisfying the series of government budgets in (4). That is, the maturity structure may be indeterminate. This is evident in the certainty case: What is then determinate is the value of net debt trade, not the exact maturity structure — the set of equilibrium-consistent B_t is indeed a continuum of dimension $M-1$. We conjecture that this indeterminacy is partly removed when we introduce uncertainty. The intuition is that different maturities are perfect substitutes in a deterministic economy, whereas they incorporate different risks and provide different hedging opportunities in a stochastic world. Proposition 6 later confirms this conjecture.

IV. Sustainable Policies and the Maturity Structure

We now seek to characterize the set of sustainable tax policies with non-contingent debt. To start with, consider the case of a deterministic economy with debt held in one-period bonds. The government budget constraint at t is $R_t \equiv \tau_t Y_t - G_t \geq p_t b_{t-1} - b_t$, and the series $\{R_t \geq p_t b_{t-1} - b_t\}_{t=0}^{\infty}$ is *equivalent* to the single intertemporal constraint at date 0: $\sum_{t=0}^{\infty} \left[\prod_{j=0}^{t} p_j\right] R_t \geq \bar{b}_{-1}$. Thus, in the certainty case, a policy $\{\tau_t\}_{t=0}^{\infty}$ is sustainable *if and only if* it satisfies this single constraint. But, what about a stochastic economy? Can we derive an analogous characterization of sustainable policies? And how then does the access to state-contingent debt or the maturity structure matter?

A. Sustainable Policies under Incomplete Markets

Fixing a date-event $(t,_t s)$ and summing up the temporal budgets (4) over all continuation date-events, we derive the intertemporal budget constraint:[22]

$$\sum_{n=0}^{\infty} \sum_{{}_{t+n}s \in \{{}_t s\} \times \mathcal{S}^n} q_{t,n}({}_{t+n}s|{}_t s) R_{t+n}({}_{t+n}s) \geq \sum_{j=0}^{M-1} p_{t,j}({}_{t-1}s, s_t) b_{t-1,j+1}({}_{t-1}s) \tag{8}$$

This simply requires that the expected present value of surpluses (across all future dates and events) covers the market value of the outstanding debt at that particular date-event.

In the light of Proposition 2, we observe that, when $M > 1$, at t the government can affect the market value, $\sum_{j=0}^{M-1} p_{t,j} b_{t-1,j+1}$, of its inherited debt, $B_{t-1} = [b_{t-1,j}]$, by manipulating $P_t = [p_{t;j}]$, the contemporaneous term structure of bond prices. Further, different maturity structures B_{t-1} issued at $t-1$ will induce different cyclical behavior for the period-t debt burden, depending on the endogenous cyclical behavior of P_t. *Ex post*, this makes the inherited debt burden effectively contingent on the contemporaneous state — observe that $\sum_{j=0}^{M-1} p_{t,j}({}_{t-1}s, s_t) b_{t-1,j+1}({}_{t-1}s)$ depends on s_t via bond prices. *Ex ante*, this means that the government can induce the cyclical behavior in the term structure and manage the maturity structure in such a way that its debt obligations/claims become appropriately contingent on the state of the economy. This already sounds as if multiple maturity can substitute for state-contingent debt.

Now, define $PV_t({}_t s)$ as the left-hand side of (8), or, in recursive form, $PV_t({}_t s) = R_t({}_t s) + \sum_{s_{t+1}} q_t({}_t s, s_{t+1}) PV_{t+1}({}_t s, s_{t+1})$. That is, $PV_t({}_t s)$ is the present value of future surpluses expected at (t, s_t). Evaluated in equilibrium, $PV_t({}_t s) = \sum_{n=0}^{\infty} \sum_{{}_{t+n}s \in \{{}_t s\} \times \mathcal{S}^n} q_n^*(.) R^*(.) \equiv PV^*(s_t, \{\tau_{t+n}(.|{}_t s)\}_{n=0}^{\infty})$, so that it is a function of the current state and the continuation tax policy. Using (8), we can now characterize the set of sustainable policies with non-contingent debt as follows:

Proposition 4 *($\mathcal{SPI}$) The policy $\tau = \{\tau_t(.)\}_{t=0}^{\infty}$ is* ***sustainable*** *with non-contingent debt, given initial debt $\bar{B}_{-1} = [\bar{b}_{-1,j}] \in \mathbb{R}^M$,* ***if and only if****:*
(i) *The policy τ satisfies the equilibrium intertemporal budget at date* 0*:*

$$\underbrace{PV^*(s_0; \tau)}_{PV_0} \geq \sum_{j=0}^{M-1} \underbrace{p_j^*(\tau_0, \tau_j(.), s_0) \cdot \bar{b}_{-1,j+1}}_{p_{0,j} \cdot \bar{b}_{-1,j+1}} \tag{9}$$

[22] For the derivation see the proof of Proposition 4 below.

(ii) *At any date* $t \geq 1$ *and event* ${}_{t-1}s \in \{s_0\} \times \mathcal{S}^{t-1}$*, there is* ***some*** *vector* $B_{t-1}({}_{t-1}s) = [b_{t-1,j}({}_{t-1}s)] \in \mathbb{R}^M$ *such that the continuation sequence* $\{\tau_{t+n}(.|_{t-1}s)\}_{n=0}^{\infty}$ *satisfies*

$$\underbrace{PV^*\left(s_t, \{\tau_{t+n}(.|_{t-1}s, s_t)\}_0^{\infty}\right)}_{PV_t({}_{t-1}s, s_t)} \quad \geq \sum_{j=0}^{M-1} \underbrace{p_j^*\left(\tau_t({}_{t-1}s, s_t), \tau_{t+j}(.|_{t-1}s, s_t), s_t\right) \cdot b_{t-1,j+1}({}_{t-1}s)}_{p_{t,j}({}_{t-1}s, s_t) \cdot b_{t-1,j+1}({}_{t-1}s)} \tag{10}$$

for all $s_t \in \mathcal{S}$*. We then let* $\mathcal{SPI}$ *denote the set of policies satisfying both (i) and (ii).*

The intuition for (9) is clear: The present value of government surpluses at date 0 must finance the inherited debt burden. The intuition for (10) is analogous. But, while $\bar{B}_{-1}$ is historically given, $B_{t-1}({}_{t-1}s)$ is free to be chosen for every $t \geq 1$ and every ${}_{t-1}s$. To see how this relates to the 'degree' of market incompleteness, it helps first to derive the analogue of Proposition 4 for the case of contingent debt.

B. Sustainable Policies under Complete Markets

Suppose now that the government could issue bonds with face value contingent on all possible future states/events. Let $d_{t,j}({}_{t+1}s|_t s)$ be debt obligations issued at date-event $(t,_t s)$ and paying only at date-event $(t+j,_{t+j} s)$, and let $q_{t,j}({}_{t+1}s|_t s)$ be their price. The budget constraint for $(t,_t s)$ is:

$$G(s_t) + \sum_{j=0}^{M-1} \sum_{{}_{t+j}s \in \{_t s\} \times \mathcal{S}^j} q_{t,j}({}_{t+j}s|_t s) d_{t-1,j+1}({}_{t+j}s|_{t-1}s) \leq$$

$$\leq \tau_t({}_t s) Y_t({}_t s) + \sum_{j=1}^{M} \sum_{{}_{t+j}s \in \{_t s\} \times \mathcal{S}^j} q_{t,j}({}_{t+j}s|_t s) d_{t,j}({}_{t+j}s|_t s) \tag{11}$$

And the intertemporal budget under complete markets at $(t,_t s)$ is:

$$PV_t({}_t s) \equiv \sum_{n=0}^{\infty} \sum_{{}_{t+n}s} q_{t,n}({}_{t+n}s|_t s) R_t({}_t s) \geq \sum_{j=0}^{M-1} q_{t,j}({}_{t+j}s|_t s) d_{t-1,j+1}({}_{t-1+j}s|_{t-1}s) \tag{12}$$

In analogy to Proposition 4, we can thus characterize the set of sustainable policies under complete markets as follows:

Proposition 5 *($\mathcal{SPC}$) The tax policy* $\tau = \{\tau_t(.)\}_{t=0}^{\infty}$ *is* ***sustainable*** *with contingent debt, given initial debt*[23] $\bar{B}_{-1} = [\bar{b}_{-1,j}] \in \mathbb{R}^M$, ***if and only if*** τ *satisfies (9), the equilibrium intertemporal budget constraint at date* 0*. We then let* $\mathcal{SPC}$ *denote the set of policies satisfying (9) and hence being sustainable under complete markets.*

[23] Without loss of generality we set $d_{-1,j}({}_j s|_{-1}s) = \bar{b}_{-1,j}, \forall\, {}_j s \in \{_0 s\} \times \mathcal{S}^j, \forall j$. This makes the initial debt position the same under complete and incomplete markets.

We observe that the set of sustainable policies under complete markets is characterized by a single sustainability constraint, the initial intertemporal budget (**??**).[24] Given the initial debt burden, this constraint is independent of M. An immediate corollary is as Lucas & Stokey [1983] observed:

Corollary 1 *With state-contingent debt, the set of sustainable policies is independent of M, and the maturity structure is underdetermined iff $M > 1$.*

Next, compare the set of sustainable policies under contingent debt ($\mathcal{SPC}$) with that under non-contingent bonds ($\mathcal{SPI}$). The initial intertemporal budget constraint (9) is common to the two cases, but the lack of contingent debt imposes (10) for all $(t,_{t-1} s)$ as additional restrictions. The series of these constraints, or property (ii) of Proposition 4, is therefore what fundamentally distinguishes the incomplete-markets case. The maturity structure may now matter and Corollary 1 may now break down. But, how exactly does a richer maturity structure (a higher M) affect the set of sustainable policies when debt is non-contingent?

C. The Maturity Structure, Substituting for Contingent Debt

To get a first taste, consider the case that $M = 1$, meaning that debt is only in one-period risk-free bonds. (10) then becomes: $PV_t(_{t-}s, s_t) = b_{t-1,1}(_{t-1}s)\,\forall\, s_t \in \mathcal{S}$. That is, given history $_{t-1}s$, there must be *some* $b_{t-1,1}(.)$ that matches the expected present value of surpluses $PV_t(., s_t)$ for *all* current states $s_t \in \mathcal{S}$. This amounts to a set of S equations, one for each state $s_t \in \mathcal{S}$. But, $b_{t-1,1}(_{t-1}s)$ itself is free, meaning one degree of freedom, so that we are left with $S - 1$ independent constraints. Indeed, substituting away $b_{t-1,1}$, we restate (10) when $M = 1$ as $PV_t(., s_t) = PV_t(., s'_t)\,\forall\, s_t, s'_t \in \mathcal{S}$, meaning that present-value surpluses must be equated *across all current states.* The latter highlights two facts: First, the volume of debt works just as an auxiliary variable — at any $(t,_{t-1} s)$ what really matters is the cyclical behavior of P_t and PV_t, the term structure and the present-value surpluses, as induced in equilibrium by the underlying tax policy. Second, and related, what the lack of contingent debt fundamentally does is to impose restrictions on present-value surpluses *across different states*, but not across time. That is, *market incompleteness restricts possibilities for cross-state insurance.*

To further illuminate, let the state space be $\mathcal{S} = \left\{\sigma^1, \sigma^2, ..., \sigma^S\right\}$. Now, fix a date $t \geq 1$ and a past history $_{t-1}s$. Define then $Q_t(_{t-1}s) = [p_{t,j}(_{t-1}s, s_t)]_{s_t \in \mathcal{S}}^{j=0,...,M-1}$ as the $S \times M$

[24] When this constraint is expressed in terms of equilibrium allocations $\{C_t(.), L_t(.)\}_{t=0}^{\infty}$, it is commonly referred to as the "implementability constraint" — see, e.g., Chari & Kehoe (1999).

matrix formed by setting its $(i, j+1)$-th element to be the maturity-j price at date t and event ${}_t s = ({}_{t-1}s, \sigma^i)$:

$$Q_t({}_{t-1}s) = \begin{bmatrix} 1 & p_{t,1}({}_{t-1}s, \sigma^1) & \dots & p_{t,M-1}({}_{t-1}s, \sigma^1) \\ 1 & p_{t,1}({}_{t-1}s, \sigma^2) & \dots & p_{t,M-1}({}_{t-1}s, \sigma^2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_{t,1}({}_{t-1}s, \sigma^S) & \dots & p_{t,M-1}({}_{t-1}s, \sigma^S) \end{bmatrix} \downarrow s_t \tag{13}$$

Next, let $V_t({}_{t-1}s) = [PV_t({}_{t-1}s, s_t)]_{s_t \in \mathcal{S}}$ be the $S \times 1$ vector formed by the present-value surpluses across all states $s_t \in \mathcal{S}$:

$$V_t({}_{t-1}s) = \begin{bmatrix} PV_t({}_{t-1}s, \sigma^1) \\ \vdots \\ PV_t({}_{t-1}s, \sigma^S) \end{bmatrix} \downarrow s_t \tag{14}$$

The above are based on evaluating all $p_{t,j}(.)$ and $PV_t(.)$ at the equilibrium induced by the underlying tax policy. Last, observe that $B_{t-1}({}_{t-1}s) = [b_{t-1;j-1}({}_{t-1}s)]_{j=1,..,M}$, the maturity structure issued at date-event $(t-1, {}_{t-1}s)$, is an $M \times 1$ vector. With $Q = Q_t({}_{t-1}s)$ and $V = V_t({}_{t-1}s)$ so defined, we have:

Lemma 2 *Property (ii) in Proposition 4 is equivalent to the following:*
(ii) *For all $t \geq 1, {}_{t-1}s \in \{s_0\} \times \mathcal{S}^{t-1}$, $V = V_t({}_{t-1}s)$ is spanned by $Q = Q_t({}_{t-1}s)$:*

$$V_t({}_{t-1}s) \in Span\,[Q_t({}_{t-1}s)] \tag{15}$$

Proof: (10) yields $V \geq QB$, but we ignore slackness without loss of generality: Resources may not be wasted and $V = QB$ defines the efficient boundary of $\mathcal{SPI}$. Then, $V = QB$ for some B if and only if V is a linear combination of the columns of Q, meaning $V \in Span[Q]$. **QED**

Observe, in the light of Propositions 1 and 2, that any given $\{\tau_t(.)\}_{t=0}^{\infty}$ maps to a *unique* combination of $V = V_t({}_{t-1}s)$ and $Q = Q_t({}_{t-1}s)$ for every $(t, {}_{t-1}s)$. And according to Lemma 2, for the given $\{\tau_t(.)\}_{t=0}^{\infty}$ to be sustainable, it is necessary and sufficient that the induced V always be spanned by the contemporaneous Q. That is, the equilibrium term structure of interest rates should fluctuate over the business cycle in such a way that it can 'support' the variation in contemporaneous present-value surpluses.

But, what exactly is the nature of the restrictions embodied in (15), and when does the sustainability property (ii) bite? With S being the number of states, V is always a vector in $\mathbb{R}^S$. With M being the number of maturities, Q is an $S \times M$ matrix, and its span is necessarily a subspace of $\mathbb{R}^S$. So, pick a policy that is sustainable under complete markets, $\{\tau_t(.)\}_{t=0}^{\infty} \in \mathcal{SPC}$. If it turns out that $V_t(_{t-1}s) \notin Span[Q_t(_{t-1}s)]$ at some $(t,_{t-1}s)$, then $\{\tau_t(.)\}_{t=0}^{\infty} \notin \mathcal{SPI}$, meaning that this particular policy is not sustainable with non-contingent debt.

If $rank\,[Q] = S$, $Span\,[Q] = \mathbb{R}^S$ and then V necessarily falls in $Span[Q]$. But if $rank[Q] < S$, $Span[Q]$ is always a proper subset of $\mathbb{R}^S$ and we can then find a policy inducing $V \notin Span[Q]$. The latter means $\mathcal{SPI} \neq \mathcal{SPC}$ and is necessarily the case if the maturity structure is shorter than the number of continuation states:

Lemma 3 ***(a)*** *For any M and S, $\mathcal{SPC}$ includes $\mathcal{SPI}$.*
(b) *If $M < S$, then $\mathcal{SPI}$ is a proper subset of $\mathcal{SPC}$.*
(c) *If $M < S$, then $\mathcal{SPI}$ is not dense in $\mathcal{SPC}$:*

$$M < S \;\Rightarrow\; Closure[\mathcal{SPI}] \subsetneqq \mathcal{SPC} = Closure[\mathcal{SPC}]$$

(d) *For $M \leq S$, $\mathcal{SPI}$ is increasing in M, and:*

$$\tilde{M} < M \leq S \;\Rightarrow\; Closure[\mathcal{SPI}(\tilde{M})] \subsetneqq \mathcal{SPI}(M)$$

Notice that part (b) above establishes that there are tax policies that are sustainable with contingent debt but not with non-contingent debt whenever the maturity structure falls short of the number of possible states. However, part (c) is more important because it further excludes the possibility that the two sets $\mathcal{SPC}$ and $\mathcal{SPI}$ are 'almost' equal. If it were instead the case that $\mathcal{SPI} \neq \mathcal{SPC}$ but $Closure[\mathcal{SPI}] = \mathcal{SPC}$, then any policy in $\mathcal{SPC}$ not belonging to $\mathcal{SPI}$ could still be approximated by some other arbitrarily close policy in $\mathcal{SPI}$. Part (c) excludes exactly this possibility whenever $M < S$. Finally, part (d) means that a richer maturity structure expands the set of sustainable policies, in a non-trivial sense, which in turn implies that the government can do better with a richer maturity as long as $M < S$.

But, what if $M \geq S$, meaning that the maturity structure is as rich as the state space? In this case we establish the converse to (c) above:

Lemma 4 *If $M \geq S$, then $\mathcal{SPI}$ is dense in $\mathcal{SPC}$:*

$$M \geq S \;\Rightarrow\; Closure[\mathcal{SPI}] = \mathcal{SPC} = Closure[\mathcal{SPC}]$$

Proof: Assume $M \geq S \geq 2$ ($S = 1$ is the trivial deterministic case) and take any $\tau = \{\tau_t(.)\}_{t=0}^{\infty} \in \mathcal{SPC}$. Form $\{V_t(.), Q_t(.)\}_{t=0}^{\infty}$ as in (14) and (13), evaluated in the equilibrium induced by τ. Recall that $Q_t(_{t-1}s)$ is an $S \times M$ matrix. With $M \geq S$, we can have either (a) $rank[Q_t(_{t-1}s)] = S$ for all $t \geq 1$, $_{t-1}s \in \{s_0\} \times \mathcal{S}^{t-1}$, or (b) $rank[Q_t(_{t-1}s)] < S$ for some $t \geq 1$, $_{t-1}s \in \{s_0\} \times \mathcal{S}^{t-1}$. Consider first case (a): □ In lieu of Lemma 2, $\tau \in \mathcal{SPC}$ satisfies both sustainability properties (i) and (ii) of Proposition 4 and thus belongs to $\mathcal{SPI}$. ■ Next, consider case (b). □ Let $Q = Q_t(_{t-1}s), V = V_t(_{t-1}s)$, and $Span[Q] \subsetneqq \mathbb{R}^S$ at some $(t,_{t-1}s)$. If still $V \in Span[Q]$, we are just fine, and $\tau \in \mathcal{SPI}$. But, if $V \notin Span[Q]$, then $\tau \notin \mathcal{SPI}$. Yet, in that case, for any small $\varepsilon > 0$, we can find $\hat{\tau} = \{\hat{\tau}_t(.)\}_{t=0}^{\infty} \in \mathcal{SPI}$ such that $||\hat{\tau} - \tau||^* < \varepsilon$. Here we choose the distance induced by the norm $||x||^* \equiv \sup_t\{\frac{1}{S^{t/2}}||x_t||\}$, with which we endow the space of sequences $x = \{x_t\}_{t=0}^{\infty}, x_t \in \mathbb{R}^{S^t}$, for $||x_t|| \equiv \sqrt{x_t' x_t}$ being the usual Euclidean norm. The way to perturb τ and construct $\hat{\tau}$ is as follows: For the given $\tau = \{\tau_t(.)\}_{t=0}^{\infty} \in \mathcal{SPC}$ and the corresponding $\{V_t(.), Q_t(.)\}_{t=0}^{\infty}$, we start to ascend the date-event tree until we find a $(t,_{t-1} s)$ such that $rank[Q_t(_{t-1}s)] < S$. For $1 \leq i \leq S$ and $1 \leq j \leq M-1$, consider the $(i, j+1)$ element of $Q = Q_t(_{t-1}s)$. By (7), this is $p_{t,j}(_{t-1}s, \sigma^i) = p_j^*\left(\tau_t(_{t-1}s, \sigma^i), \tau_{t+j}(.|_{t-1}s, \sigma^i), \sigma^i\right)$. Consider now perturbing $\tau_{t+j}(.|_{t-1}s, \sigma^i)$ slightly to some $\hat{\tau}_{t+j}(.|_{t-1}s, \sigma^i)$ for only the particular (i, j, σ^i) and the given $(t,_{t-1}s)$ — let in particular $|\tau_{t+j}(_{t+j}s) - \hat{\tau}_{t+j}(_{t+j}s)| < \varepsilon \;\forall_{t+j}s \in \{_{t-1}s, \sigma^i\} \times \mathcal{S}^j$. This perturbation affects as of date t **only** the particular price $p_{t,j}(_{t-1}s, \sigma^i)$, or the particular $(i, j+1)$ element of the matrix $Q = Q_t(_{t-1}s)$ for the particular $(t,_{t-1} s)$, and no other element of this Q. It affects prices at $t+j$ as well, but not any prices before the particular t. Thus, $\tau_{t+j}(.|_{t-1}s, \sigma^i)$ is an instrument with which we can control the corresponding $p_{t,j}(_{t-1}s, \sigma^i)$, or the $(i, j+1)$ element in matrix $Q = Q_t(_{t-1}s)$, for the particular $t,_{t-1}s, \sigma^i, i, j$, with no other contemporaneous or past effect. But then, we can easily break any linear dependence in Q. Having ensured full rank for $Q = Q_t(_{t-1}s)$ up to the particular t, we next proceed ascending the date-event tree until we hit another situation where $V \notin Span[Q]$, and we then make an analogous perturbation to restore full rank. Proceeding this way for $t \to \infty$, we ensure that the perturbed policy $\hat{\tau} = \{\hat{\tau}_t(.)\}_{t=0}^{\infty} \in \mathcal{SPC}$ has $rank[\hat{Q}_t(_{t-1}s)] = S$, implying $\hat{V}_t(_{t-1}s) \in Span[\hat{Q}_t(_{t-1}s)]$, at all $(t,_{t-1} s)$. And that means $\hat{\tau} \in \mathcal{SPI}$. What is more, by construction, $|\tau_t(_ts) - \hat{\tau}_t(_ts)| < \varepsilon$ at all $(t,_t s)$, implying $||\tau_t(.) - \hat{\tau}_t(.)|| = \sqrt{\sum_{_ts}[\tau_t(_ts) - \hat{\tau}_t(_ts)]^2} < \sqrt{S^t}\varepsilon = S^{t/2}\varepsilon$ at all t, and thus $||\tau - \hat{\tau}||^* \equiv \sup_t\{\frac{1}{S^{t/2}}||\tau_t(.) - \hat{\tau}_t(.)||\} < \sup_t\left\{\frac{1}{S^{t/2}}S^{t/2}\varepsilon\right\} = \varepsilon$. That is, the perturbation may be arbitrarily small. ■ **QED**

The last lemma is quite strong: It tells us that, if the maturity structure is sufficiently rich, in that $M \geq S$, then the set of sustainable policies with non-contingent debt is *essentially* identical to that under complete markets! It means that any generic policy from $\mathcal{SPC}$ falls into $\mathcal{SPI}$ as well, and any non-generic policy from $\mathcal{SPC}$ can be approximated arbitrarily well by some policy in $\mathcal{SPI}$.

This is so because, when $M \geq S$, non-contingent bonds generically complete the markets and the lack of contingent debt then imposes no constraint whatsoever. And in non-generic situations, the government can do with non-contingent debt almost as well as with contingent debt, because it can manipulate the term structure of bond prices and break the linear dependence across different states by a small perturbation in the underlying tax policy.[25] In manipulating $p_{t,j}({}_ts)$, the price of a bond of maturity j at any date-event $(t, {}_ts)$, we have to use as an instrument not the contemporaneous tax rate, but rather $\tau_{t+j}(.|{}_ts) \equiv \{\tau_{t+j}({}_{t+j}s)\}_{{}_{t+j}s \in \{{}_ts\} \times \mathcal{S}^j}$, the structure of taxes expected to prevail at the maturity date.

In conclusion, combining Lemmas 3 and 4:

Theorem 1 *($\mathcal{SPI}$ **and** $\mathcal{SPC}$) Consider an economy $\mathcal{E}$ without capital, and let $S \geq 1$ be the number of possible states and $M \geq 1$ the length of the maturity structure. Let $\mathcal{SPC}$ denote the set of policies sustainable with contingent debt and $\mathcal{SPI}$ that with non-contingent debt. Then:*

◦ If $M \geq S$, then and only then $\mathcal{SPI}$ is equal to $\mathcal{SPC}$ up to a set of measure zero. That is, any policy that is sustainable with contingent debt, either is sustainable itself with non-contingent debt, or can be approximated arbitrarily well.

◦ If instead $M < S$, then and only then $\mathcal{SPI}$ is not dense in $\mathcal{SPC}$. That is, there are policies that are sustainable under complete markets but are remote from any policy that is sustainable under incomplete markets. Further, $\mathcal{SPI}$ is then increasing in M.

The result may appear odd because it relies on comparing M, which is in time units, with S, which is in state units: How can the state and the time domain be comparable? A reflection on the Arrow-Debreu complete-markets world provides the resolution, because there time and states are utterly indistinguishable. All we have is an abstract economy with many different goods, one for every date and every event — in that context, dates and events make no sense other than providing an arbitrary indexing of goods. The ability then to make debt obligations contingent on *both* time and states allows the government to 'move' and transfer budget funds across *both* the time and the state domain under no constraint other than the single initial intertemporal budget constraint (9). So, the question is what happens when we deviate from the Arrow-Debreu world to an incomplete-markets situation. What the lack of contingent debt does is to restrict the ways the government can transfer funds across the state domain — these cross-state constraints are embodied in sustainability property (ii) of Proposition 4 or Lemma 2. We then showed that introducing a richer maturity structure relaxes the restrictions over the state domain, and established that all the cross-state constraints are generically redundant when $M \geq S$.[26]

[25] It is precisely the latter point, showing a generic perturbation in the equilibrium allocation to make Q non-singular, which establishes the former point, that any generic equilibrium has a full-rank Q.

[26] As regards dynamic security trading, our result may relate to Harrison & Kreps [1979], Duffie & Huang [1985], etc. But, to quote the latter [p.1339], "in all [that] literature, the takeoff point is a given

When the maturity structure is rather short ($M < S$), the gap between S and M is a rough index for market incompleteness: $S - M$ are the non-redundant constraints. However, how important *economically* the difference $S - M$ can be, this is an empirical question. The example in the next section suggests that, while no cross-state insurance is possible with only risk-free short-term debt ($M = 1$), quite a lot can be attained by just adding long-term debt ($M = 2$). We thus conjecture that even a short maturity structure (small M) may do pretty well. In the same spirit, our *finite* state space should be viewed as a reasonable approximation.[27] In other words, a finite maturity structure can do just as well as a finite set of state-contingent debt instruments. What is a 'good' approximation is an empirical question — an interesting one but beyond the focus of this paper.[28]

Finally, recall our discussion in Section III.D about the indeterminacy of the maturity structure.[29] We then conjectured that part of the indeterminacy is removed in a stochastic economy because different maturities provide different hedging opportunities. Here is the confirmation:

Proposition 6 ***(The Maturity Structure)*** *The maturity structure is uniquely determined for a generic policy in $\mathcal{SPI}$, if and only if $M \leq S$. If instead $M > S$, it is underdetermined, with generically $M - S$ redundancy degrees. If $M = S$ and the policy is generic, the corresponding maturity structure is:*

$$B_t({}_ts) = Q_{t+1}({}_ts)^{-1} V_{t+1}({}_ts) \qquad \forall\, t \geq 0,\ {}_ts \in \{s_0\} \times \mathcal{S}^t \tag{16}$$

D. An Example for the Business Cycle

We now treat in detail the example we first encountered in the Introduction. We consider an economy that faces two states only — this is intended to capture the basic intuition about taxation and debt management over the business cycle. Let the state space be $\mathcal{S} = \{\sigma^g, \sigma^b\}$. The 'good' state, σ^g, or a peak, is identified as the one where productivity,

set of security price processes," which are presumed to span the continuation state space. Our result instead exploits the very endogeneity of bond returns, and may be of independent interest to the finance literature, in that a given set of assets, the set of non-contingent bonds, completes the markets for any generic equilibrium.

[27]Technically, any continuum-state economy can be approximated arbitrarily well by a discrete-state economy.

[28]Simulations could help further evaluate the last points, but that is beyond our scope here.

[29]The indeterminacy discussed here should not be confused with that discussed in, say, Zhu [1992] and Chari & Kehoe [1999]. There, debt is contingent and an indeterminacy arises because of the ex-post lump-sum nature of state-dependent capital taxation.

wages and profits are all relatively high, and government expenditure relatively low. We thus specify: $F(., \sigma^g) > F(., \sigma^b)$, $F_L(., \sigma^g) > F_L(., \sigma^b)$, and $G(\sigma^g) < G(\sigma^b)$. To simplify, the Markov state process is assumed symmetric and the shocks i.i.d., implying no persistence — this means $\mu(s'|s) = \frac{1}{2} \forall s', s \in \mathcal{S}$. Preferences are separable and isoelastic in C, so that $U_c(C, 1-L) = C^{-\theta}$ for $\theta > 0$.

We have not thus far defined or characterized an 'optimal' tax policy, but it is most natural to assume that it smoothes tax distortions and consumption both across time and across states.[30] It follows that output and consumption are relatively high in good times, coupled with high tax revenues. With spending lower in peaks as well, the government runs a *countercyclical deficit:*[31] $R_t({}_{t-1}s, \sigma^g) > R_t({}_{t-1}s, \sigma^b) \;\forall\, t, {}_{t-1}s$. This in turn implies a *procyclical present value of surpluses*:

$$PV_t({}_{t-1}s, \sigma^g) > PV_t({}_{t-1}s, \sigma^b) \quad \forall\, t, {}_{t-1}s \tag{17}$$

Finally, given that consumption is relatively high in peaks, expected consumption growth and hence interest rates are relatively low, meaning *procyclical bond prices*:[32]

$$p_{t,1}({}_{t-1}s, \sigma^g) > p_{t,1}({}_{t-1}s, \sigma^b) \quad \forall\, t, {}_{t-1}s \tag{18}$$

The rationale for (17) should be clear: The government would like to insure across states by transferring funds from peaks to recessions. This smoothing may or may not be feasible — it depends on the debt instruments the government has access to. If contingent debt were available, it would be easy: Just sell at $t-1$ contingent obligations $d_{t-1,1}({}_{t-1}s, \sigma^g) = PV_t({}_{t-1}s, \sigma^g)$ and $d_{t-1,1}({}_{t-1}s, \sigma^b) = PV_t({}_{t-1}s, \sigma^b)$. If this were done, as soon as period t arrives, the government would have to pay just $d_{t-1,1}(., \sigma^b)$ if it is a boom, and as much as $d_{1,t-1}(., \sigma^g)$ if it is a recession. But, what if contingent debt is not available?

For whatever M, we have $V_t({}_{t-1}s) = \begin{bmatrix} PV_t({}_{t-1}s, \sigma^g) \\ PV_t({}_{t-1}s, \sigma^b) \end{bmatrix}$. If there were only one-period risk-free debt ($M = 1$), then $Q_t({}_{t-1}s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $B = b_{t-1,1}({}_{t-1}s) \in \mathbb{R}$. The sustainability constraint, $V = QB$, would then require $PV_t(., \sigma^g) = PV_t(., \sigma^b) = b_{t-1,1}(.)$. Just as we had discussed earlier, $M = 1$ imposes a rigid constraint on fiscal management and tax smoothing: *In each period*, the present value of surpluses has to be equated *across all states*. But, this contradicts a procyclical present value of surpluses, as in (17), which characterizes the Ramsey optimum or any policy with enough smoothing. Thus, the

[30] In fact, whatever we assume here is consistent with the complete-markets Ramsey optimal policy. This draws on Lucas & Stokey [1983], Chari & Kehoe [1999], and our Theorem 2.

[31] Besides, depending on the elasticity of labor supply, the tax rate may be countercyclical, so as to encourage employment in recessions.

[32] For the proof of both (18) and (17) above, see Appendix. We may also show $p_{t,2}(., \sigma^g) > p_{t,2}(., \sigma^b)$.

desirable policy would not be sustainable with non-contingent debt of only one-period maturity.

Let us now enrich the maturity structure to $M = S = 2$, allowing for both short-term ($j = 1$) and long-term ($j = 2$) bonds. Then, the matrix of bond prices becomes $Q = Q_t({}_{t-1}s) = \left[\begin{smallmatrix} 1 & p_{t,1}({}_{t-1}s,\sigma^g) \\ 1 & p_{t,1}({}_{t-1}s,\sigma^b) \end{smallmatrix}\right]$. For this, (18) implies $\det(Q) = p_{t,1}(.,\sigma^b) - p_{t,1}(.,\sigma^g) < 0\,(\neq 0)$, so that it has full rank: $rank[Q] = S = 2$. The latter ensures that any vector of present-value surpluses $V = V_t({}_{t-1}s) \in \mathbb{R}^2$ falls into the span of $Q = Q_t({}_{t-1}s)$, and Theorem 1 applies in all its beauty: The complete-markets Ramsey optimal policy is generic and sustainable with non-contingent debt.

We can then apply Proposition 6 and compute the optimal maturity structure as:

$$\begin{bmatrix} b_{t-1,1}(.) \\ b_{t-1,2}(.) \end{bmatrix} = \frac{1}{p_{t,1}(.,\sigma^b) - p_{t,1}(.,\sigma^g)} \begin{bmatrix} p_{t,1}(.,\sigma^b)PV_t(.,\sigma^g) - p_{t,1}(.,\sigma^g)PV_t(.,\sigma^b) \\ PV_t(.,\sigma^b) - PV_t(.,\sigma^g) \end{bmatrix} \tag{19}$$

Given that prices and surpluses are procyclical, as in (18) and (17), (19) implies:

$$b_{t-1,1}({}_{t-1}s) < 0 < b_{t-1,2}({}_{t-1}s) \quad \forall\, t, {}_t s \tag{20}$$

In words, the government *lends* in risk-free short-term bonds and *borrows* in risky long-term bonds. To examine the rational behind this structure, consider what happens in the period following the issue of this debt. At date t, short-term bonds ($j = 1$) issued at $t-1$ are expiring and the government is receiving a certain revenue $-b_{t-1,1}({}_{t-1}s) > 0$, an amount independent of the date-t state s_t. This is essentially a countercyclical budget revenue, because, compared to tax revenues and government spending, it is relatively higher in recessions. On the other hand, long-term bonds ($j = 2$) issued at $t - 1$ have one period to maturity as of date t, and may be traded at price $p_{t,1}(., s_t)$. So, the government budget incurs an outlay $-p_{t,1}({}_{t-1}s, s_t)b_{t-1,2}({}_{t-1}s) < 0$, an amount *endogenously* contingent on the date-t state, s_t, via short-term bond prices, $p_{t,1}$. (18) and (20) imply in particular $-p_{t,1}(.,\sigma^g)b_{t-1,2}(.) < -p_{t,1}(.,\sigma^b)b_{t-1,2}(.) < 0$; that is, a procyclical budget outlay.[33,34] In total, the proposed maturity structure induces a procyclical debt burden.

[33] An alternative rational for lending in short-term bonds, $b_{t-1,1}(.) < 0$, is that it allows us to take an even shorter position in the long-term market, inducing an even higher variation in the outlay $p_t(., s_t)b_{t-1,2}(.)$ along the cycle.

[34] Procyclical short-term bond prices, meaning countercyclical short-term interest rates, might be contradicted empirically. But we are comfortable with that, for two reasons: First, the example predicts countercyclical interest rates because the state space is very coarse, so that on average there is reversion rather than persistency in the state. If we had more than two states, or introduced capital, and imposed some persistency over the business cycle, we could generate procyclical interest rates, and even calibrate the cyclical behavior of the term structure. Second, if short-term bond prices were instead countercyclical, we would simply reverse the optimal maturity structure, and lend rather than borrow in long-term debt.

And that is exactly the nature of desirable insurance along the business cycle.

V. The Optimal Policy with Non-Contingent Debt

So far we have examined the sets of sustainable policies, with contingent or non-contingent debt, but we made no reference whatsoever to the objectives of the government. We now turn to the optimal design of tax policy and debt management, assuming that the government is benevolent and can credibly commit to its optimal plan.

A. Optimal Fiscal Policy: The Second and the Third Best

The social planner's problem consists of choosing, among the set of tax policies that are sustainable under the particular market structure, the one that maximizes social welfare. Thus, evaluating the equilibrium welfare induced by a policy $\tau = \{\tau_t(.)\}_{t=0}^{\infty}$ as $E_0\mathcal{U}(\tau) = \sum_{t=0}^{\infty}\sum_{s_t \in S}\mu(s_t|s_{t-1})u^*\left(\tau_t({}_ts), s_t\right)$, the **second-best** or **Ramsey optimal policy**, under complete markets, is the $\arg\max_\tau \left\{E_0\mathcal{U}(\tau) \mid \tau \in \mathcal{SPC}\right\}$. We similarly define the **third-best policy**, under incomplete markets and non-contingent debt, as the $\arg\max_\tau \left\{E_0\mathcal{U}(\tau) \mid \tau \in \mathcal{SPC}\right\}$.[35] Observe that, while the second-best or Ramsey optimal policy is subject only to the initial intertemporal budget constraint (9), the third best faces in addition the spanning constraints (15) at all dates and events.

For the complete-markets case, Lucas & Stokey [1983] did *not* provide an existence result. Nonetheless:

Proposition 7 ***(Ramsey Policy with Contingent Debt)*** *Consider an economy $\mathcal{E}$ without capital, under complete markets. Provided that the initial debt $\bar{B}_{-1}$ is not too high, a sufficient condition for the Ramsey optimal policy to exist is that the tax rate is bounded above by some $\tilde{\tau} < 1$, or that $\lim_{\tau\to 1}\frac{\partial}{\partial\tau}\left[U_c^*(\tau,.)R^*(\tau,.)\right] < \infty$.*

What about the incomplete-markets case? Unfortunately, even with the tax rate bounded below 1, the set $\mathcal{SPI}$ of sustainable policies with non-contingent debt may fail to be compact.[36] And if $\mathcal{SPI}$ is not compact, a fixed-point existence argument does not apply. Nonetheless, if the second best both exists itself and is generic, and if the

[35] The qualification "second best" is due to the presence of distortionary (non-lump-sum) taxation. The qualification "third best" reflects the additional distortion introduced by market incompleteness (the luck of contingent debt).

[36] In particular, if we take a convergent sequence of policies in $\mathcal{SPI}$, their limit may fall out of $\mathcal{SPI}$, meaning that $\mathcal{SPI}$ need not be closed. Technically, this might be the case because the rank of the matrix Q of bond prices may collapse as we take the limit of a sequence of policies in $\mathcal{SPI}$.

maturity structure is long enough ($M \geq S$), existence of the third best is a fortiori ensured by Theorem 1, for then the second and the third best simply coincide:[37]

Corollary 2 *Consider an economy $\mathcal{E}$ without capital, and suppose the second-best Ramsey policy exists. If the maturity structure is sufficiently rich, then with non-contingent we can do almost equally well:*

$$M \geq S \quad \Rightarrow \quad \sup_\tau \{E_0\mathcal{U} \,|\, \tau \in \mathcal{SPI}\} \;=\; \max_\tau \{E_0\mathcal{U} \,|\, \tau \in \mathcal{SPC}\}$$

If further the Ramsey policy itself is generic, then the third best exists and coincides with the second best.

So, either the complete-markets Ramsey policy is itself sustainable with non-contingent debt, or it can be approximated arbitrarily well by some other policy in $\mathcal{SPI}$. The latter is indeed ensured by the government's ability to manipulate equilibrium allocations, thereby MRS's, and thereby equilibrium interest rates.

The importance of this result should be quite obvious: The literature on optimal fiscal policy has relied heavily on the assumption of contingent debt, an assumption leaving an uneasy gap between theory and reality. The resolution we provide, however, is quite reassuring. All the same can be achieved with non-contingent debt by appropriately managing the maturity structure and if necessary manipulating the term structure. What is critical about the alternative situation examined by Marcet, Sargent & Seppala [1999] is that they do *not* allow for long-term debt ($M = 1$). In that case, there is no possibility whatsoever for cross-state insurance. But, as soon as we introduce long-term debt ($M \geq 2$), insurance opportunities expand, and this can help the government attain the Ramsey outcome.

An interesting result is then the monotonicity between third-best welfare and market completeness. It is well known that introducing new assets in an incomplete-markets economy does not always increase equilibrium welfare — it depends critically on the equilibrium response of prices. For a fixed fiscal policy, the ambiguity still prevails. If, however, the government optimally adjusts its policy to the new asset structure, the prices respond in such a way that monotonicity is ensured:

Corollary 3 *The third-best welfare,* $\sup_\tau \{E_0\mathcal{U} \,|\, \tau \in \mathcal{SPI}\}$, *is non-decreasing in M, the richness of the maturity structure.*

B. The Optimal Maturity Structure: Time and State Invariance

[37]On the other hand, the characterization of the third-best when $M < S$ is an open question. Marcet, Sargent & Seppala [1999] made the first step examining the case of $M = 1$, and our Propositions 1 & 4 and Lemma 2 may help formulate the problem for the more general case of $1 \leq M < S$.

The question that arises naturally is what is the maturity structure that implements the Ramsey optimum with non-contingent debt, and how it should be managed over the business cycle. A strong property of the Ramsey optimum is that the optimal tax rate and the corresponding equilibrium allocation depend *only* on the contemporaneous state and *not* on past history. So do the equilibrium interest rates and government surpluses, implying in turn that the optimal maturity structure is *invariant across all states and dates*:

Theorem 2 ***(Optimal Maturity Structure)*** *Let* $\{\tau_t(.)\}_{t=0}^{\infty}$ *be the Ramsey optimal policy. Then, there are stationary mappings* $\bar{\tau}, \bar{C}, \bar{L}, \bar{p}_j, \bar{R}, \overline{PV} : \mathcal{S} \rightarrow \mathbb{R}$ *such that, for all* $t \geq 1$, ${}_{t-1}s \in \{s_0\} \times \mathcal{S}^{t-1}$, *and* ${}_ts \in \mathcal{S}$, *the equilibrium has:* $\tau_t({}_{t-1}s, s_t) = \bar{\tau}(s_t)$, $C_t({}_{t-1}s, s_t) = \bar{C}(s_t)$, $L_t({}_{t-1}s, s_t) = \bar{L}(s_t)$, $R_t({}_{t-1}s, s_t) = \bar{R}(s_t)$, $PV_t({}_{t-1}s, s_t) = \overline{PV}(s_t)$, *and* $p_{t,j}({}_{t-1}s, s_t) = \bar{p}_j(s_t)\, \forall j \geq 1$. *Further,* $V_t({}_{t-1}s) = \bar{V}$ *and* $Q_t({}_{t-1}s) = \bar{Q}$ *for some fixed* $M \times 1$ *vector* $\bar{V}$ *and* $M \times S$ *matrix* $\bar{Q}$. *Finally, if* $M = S$ *and if the Ramsey optimal policy is generic, then the optimal maturity structure is time- and state-invariant, given by:*[38]

$$B_t({}_ts) = \bar{B} \equiv \bar{Q}^{-1}\bar{V} \qquad \forall\ t \geq 0, {}_ts \in \{s_0\} \times \mathcal{S}^t$$

Martingale models of taxation and debt à la Barro [1979], Bohn [1990], Hansen, Roberds & Sargent [1991], etc., predict that the level of debt should be both persistent and countercyclical — and common wisdom holds it so. But, this conviction is sharply contradicted by our result above. It is indeed striking, and counterintuitive at first glance, that the level of debt issued in any period should not correlate per se with the level of debt inherited, and that the optimal maturity structure should be invariant over the business cycle.[39] But, what is the underlying rational?

In both peaks and recessions, independently of the current state or past history, we issue the same amount of short-term and long-term debt. All the required cross-state insurance is then attained via the cyclical variation in the effective debt burden, thanks to the equilibrium fluctuation of the term structure. There is thus no need to introduce any persistence in tax policy, nor in debt management. The optimal maturity structure in a given date-event is designed with the focus *not* on contemporaneous budgetary needs, but rather on the next-period cyclical variation in interest rates and present-

[38]This result requires that the Ramsey policy be generic, so that $\bar{Q}$ has full rank. We conjecture the Ramsey policy to be generic for generic economies.

[39]Observe, however, that primary deficits and net borrowing, $-R_t({}_{t-1}s, s_t) = -\bar{R}(s_t)$, may well be countercyclical.

value surpluses — that is, B_t is designed to match Q_{t+1} and V_{t+1}. With the latter being stationary, the maturity structure is invariant along the business cycle.[40]

On the other hand, as regards comparative statics, the optimal structure $\bar{B} \equiv \bar{Q}^{-1}\bar{V}$ is dependent on s_0 and B_{-1}, the state prevailing and the debt inherited at the time that the optimal plan is first designed, as well as on the underlying stochastic process for government spending and productivity. For instance, it is a safe conjecture that a higher initial debt, B_{-1}, or an increase in the expected stream of government expenditure, captured by an upward shift in $G(.)$, implies a higher level of optimal debt, $B_t({}_ts) = \bar{B}$, along all the optimal plan. Similarly, the optimal tax rule, $\bar{\tau}(.)$, may shift upwards with an increase in B_{-1} or $G(.)$.

The last observation relates to what we said before about martingale models of taxation and debt: When we set the optimal policy plan, we anticipate that government expenditures and tax revenues will fluctuate over the business cycle, and we design the optimal maturity structure so as to smooth out these fluctuations. That is, any variation in revenues and expenditures is *automatically* absorbed by the counterbalancing variation in the effective debt burden. Taxes and debt thus do not follow a martingale, but rather inherit the persistence properties of the exogenous state process of the economy.[41] It is only a change in the fundamentals, a *sudden* shift in the underlying shock-generating processes, that brings a permanent shift in policy — and the latter would be reminiscent of the "random-walk" result. Thus, if a war is an unanticipated event, it should be financed with unusually large deficits and should induce a shift in the policy plan, but an ordinary recession may not justify higher debt issues.

[40] If we introduce capital, Theorem 2 has to be modified as follows: At the Ramsey optimum, tax rates and equilibrium allocations and prices are contingent on (s, k), the contemporaneous state and the capital stock. And then V and Q depend on k but not on s: $V = \bar{V}(k)$ and $Q = \bar{Q}(k)$, implying an optimal maturity structure $\bar{B}(k) = \bar{Q}(k)^{-1}\bar{V}(k)$. Conditional on investment k, this is again invariant over the cycle. A immediate question is then whether $\bar{B}(k)$ is monotonic: Bond prices should be increasing as an economy grows, so that $\bar{Q}(k)^{-1}$ "falls" with k. Because MRS's fall due to diminishing returns, for given budget surpluses, the present value $\bar{V}(k)$ should also fall, making $\bar{B}(k)$ to fall with k. On the other hand, a richer economy is expected to raise higher surpluses in the future, which may drive up $\bar{V}(k)$ and hence $\bar{B}(k)$. Therefore, it is not clear how debt issues should behave over the cycle, or as the economy grows. We conjecture that, if diminishing returns are weak, then $\bar{B}(k)$ is increasing in k and thus debt issues are procyclical — in an Ak economy, e.g., $\bar{B}(k)$ has to be homogenous in k. In any case, even in a capital economy, the basic rationale is the same: The current state *does not matter per se* — if it matters, it matters only through expectations about future interest rates and future present-value surpluses.

[41] This last point has been emphasized by Chari, Christiano, and Kehoe.

Finally, as regards the nature of the optimal maturity structure, a taste was given by the example of Section IV.D: It pays to borrow in the maturities with countercyclical returns and lend in the ones with procyclical returns. How big a position should be taken in each maturity will critically depend on the nature of uncertainty. To give an example, suppose that 'good times' are more probable than 'bad times'. It then pays to take a big position in both the short-term and the long-term market. This way, the government essentially rolls over the same debt burden as long as the times are good. When times turn bad, however, bond prices fall, and the large inherited stock of long-term obligations implies a large drop in the debt burden.

At a more sophisticated level, it is quite interesting to run calibration exercises: The complete-markets Ramsey equilibrium could first be simulated along the lines of Chari & Kehoe [1999], and then our formula (16) could be used to 'translate' the outcome to a non-contingent-debt world and simulate the optimal maturity structure. This is left as a future project.

C. Optimality and Time Consistency

So far we have assumed that the date-0 government can credibly commit to the optimal policy plan. If, however, the government can set and reset tax policies sequentially, we run into the standard time-inconsistency problems raised by Kydland & Prescott [1977, 1980].[42,43]

Reputation offers the typical resolution both for default or capital levies, and for tax policies.[44] We point out that, as in Theorem 2, the optimal tax scheme is stationary: $\bar{\tau}(.)$ is a simple time-invariant function of the contemporaneous state. In particular, a fixed progressive tax system may be close to the optimal scheme. This means that a

[42] Lucas & Stokey [1983] *partly* resolved this problem through the maturity structure but thanks to state-contingent consoles. The latter are not available in our (theoretical or real) world, and thus their argument does not apply. Besides, neither Lucas & Stokey offered a *complete* resolution to the time-inconsistency issues that are intrinsic to fiscal policy. After all, they also had to preclude debt default or capital levies a priori — thanks to some unmodeled commitment mechanism.

[43] The relevant externality (incentive incompatibility) works precisely through the term structure: The government at t issues debt in bonds whose prices, P_t, depend on both the current tax rate τ_t and the next-period tax rate, τ_{t+1}. In assessing the optimal τ_{t+1}, the period-t government would internalize the effect of τ_{t+1} on P_t. The government at $t+1$, however, may ignore the effect of τ_{t+1} on past realized P_t. Nonetheless, a possibility of mitigating the problem arises when $M > S$: We then have $M - S$ degrees of indeterminacy in debt issues which can help realign incentives across different dates and states.

[44] The relevant literature includes Stokey [1989, 1991], Chari, Kehoe & Prescott [1989], Chari & Kehoe [1990, 1993], Marcet & Marimon [1993], and Benhabib, Rustichini & Velasco [1999].

deviation from the optimal rule may be easily detectable, the implicit 'social contract' is simple to 'write', and thus reputation enforcement may work pretty well. Other explicit commitment devices (like constitutional constraints, time delays in implementing tax changes, etc.) may also help.

Besides, our main result (Theorem 1) is about the set of feasible policies: *Any* policy that is sustainable with state-contingent debt *can* be sustained as well with non-contingent debt, provided an appropriate maturity structure — in this, commitment is irrelevant. Whether and how a society resolves time-inconsistency problems is a compelling question, but it is clearly beyond the scope of this paper.

VI. Concluding Remarks – Discussion

A. Empirical Implications

As regards the empirical implications of our analysis, we distinguish two relevant questions: The first refers to testing our main result (Theorem 1), namely the possibility of substituting for contingent debt through the maturity structure and *potentially* attaining the Ramsey optimum. The second is to test whether the observed policy is the Ramsey optimal one.

In answering the first question, we need to explore the cyclical properties of the term structure of bond prices and test how they are affected by government policies. If we get empirical counterparts for S and $rank(Q)$, and observed time series suggest $rank(Q) = S$, then historically there has been a potential for substantial cross-state smoothing. But even if this test fails, nothing rules out that there was room for manipulating the term structure so as to attain the desired cyclical behavior in bond prices.

The second question asks whether the theoretical optimal policy accords with the observed one, assuming that the maturity structure has been rich enough. In the light of Theorem 2, our analysis provides a *new* test for that *old* question: Test whether debt issues at any given maturity have been acyclical and uncorrelated. If we generalize to allow for capital, the relevant implication is that all variation in debt should be explained by the variation in capital investment. Controlling for the latter, debt issues should be invariant.

Another study could assess the relation between efficiency in smoothing tax distortions and richness of maturity structure in a panel of countries: Economies with a richer maturity structure should ceteris paribus be more stable.

Testing these empirical implications is left for future research.[45] It was not the point of our paper, after all, to test the complete-markets paradigm empirically.

B. Robustness, Scope, and Applicability

Our model economy is a standard closed neoclassical economy without capital, and the set-up of optimal taxation is fairly standard as well. There is a single infinite-horizon representative private agent, so that we can abstract from both intra- and inter-generation redistribution issues. The government is benevolent and all politicoeconomic considerations are set aside. In this, we follow Lucas & Stokey [1983], with the fundamental modification that we do *not* allow for state-contingent debt.

Our theoretical economy was closed. Consider instead the other extreme, of a small open economy with free access to an international asset market and perfect capital mobility: Such an economy faces completely exogenous interest rates, and the term structure is independent of domestic fiscal policies.[46] Many open economies, however, are sufficiently near the closed-economy paradigm, because the government can affect domestic interest rates via, say, tax policies or capital controls. In addition, big economies like the US or the EU could possibly manipulate even international prices.[47] Either way, openness can help smooth consumption through external aggregate insurance, but our own argument will be valid to the extent that the government can manipulate domestic interest rates.

Government expenditure was taken as exogenous, but our analysis can be readily extended to incorporate endogenous government spending over the business cycle.

Our economy, just like that of Lucas & Stokey [1983], had no capital. The extension of Lucas & Stokey's analysis to a complete-markets economy with capital has been well explored.[48] Regarding now our own argument, what we need is that the government can

[45] Another related test is whether the term structure of bond prices can reveal, or predict, the contemporaneous government spending. However, a failure of this test does not necessarily contradict our argument: The term structure depends on the contemporaneous state, but, as long as the state combines government expenditure with other (observed or non-measurable) disturbances, bond prices will be a noisy only statistic for government expenditure. It is of independent interest, nonetheless, to investigate how well the term structure predicts government spending and productivity in the economy.

[46] In terms of our model, the matrix $Q = Q_{t-1}({}_{t-1}s)$ is exogenous, and will fail to have full rank if the country faces idiosyncratic risk.

[47] Work in progress explores the idea that a *big* country could get rid of its idiosyncratic risk by manipulating international bond prices, with no need for contingent bonds.

[48] See Zhu [1992], Chari, Christiano & Kehoe [1991, 1994, 1995], and Chari & Kehoe [1999]. The results about the taxation of capital are reminiscent of Chamley [1986], Judd [1987], and Jones, Manuelli & Rossi [1993, 1997].

manipulate the allocation of resources and thereby the intertemporal MRS's — through tax or other policies. The rationale seems to apply even when there is capital, and we thus expect our main results to be robust to such an extension.

The equilibrium pricing of bonds is along the lines of standard general-equilibrium asset pricing and the CAPM literature that built on Lucas [1978]. To the extent, then, that the government can manipulate competitive allocations by use of some policy instruments, it can also manipulate the term structure of interest rates.

Finally, as regards time-consistency considerations, we repeat that Theorem 1 is *not* dependent on the availability of a commitment technology. The same reasoning suggests that our result is equally robust to redistribution or politicoeconomic concerns: The latter refer mostly to the objectives of the government, not to the set of feasible policies.

In conclusion, the two *critical elements* in our argument are, first, that interests rates are endogenous in a CAPM-like way and, second, that the government has some instruments with which it can manipulate marginal rates of intertemporal substitution. Provided so, the government can attain through an appropriate maturity structure as much insurance as that offered by a complete set of Arrow securities and thereby sustain the complete-markets Ramsey policy with only non-contingent debt.

So, a main lesson from this paper — and this is quite reassuring for the pertinent literature — is that the complete-markets paradigm of optimal fiscal policy is quite relevant for a world with non-contingent debt. The other striking result — and this contradicts common wisdom and martingale models of debt — is the optimality of a state-invariant maturity structure, meaning acyclical and uncorrelated debt issues.

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Appendix: Proofs

Proof of Proposition 1: Consider the non-linear system $\frac{U_l}{U_c}(C, 1-L) = [1-\tau]F_L(L)$ and $C + G = F(L)$ in $x = (C, L) \in \mathbb{R}_+ \times (0,1)$, and let $x(\tau)$ be the set of solutions, for given $\tau \in [0,1)$. The first best would be $x(0)$, the point(s) where an indifference curve is tangent to the production frontier, meaning $\frac{U_l}{U_c} = F_L$. For $\tau > 0$, $x(\tau)$ is simply the point(s) on the trasformation frontier whereby the indifference curve crosses with $\frac{U_l}{U_c} = [1-\tau]F_L$. With preferences being convex and smooth and satisfying the Inada conditions, $x(.)$ is non-empty-valued, convex-valued, and upper hemicontinuous. It may yet fail to be single-valued. However, for any generic U or F, the points in $x(\tau)$ have to be isolated. Together with convexity, this implies that $x(\tau)$ is a singleton. Thus, for any generic economy, we solve (5) and (1) for $C_t({}_ts)$ and $L_t({}_ts)$ as single-valued functions of s_t and $\tau_t({}_ts)$ alone: Define $C^*, L^* : [0,1) \times \mathcal{S} \to \mathbb{R}$ such that: $\frac{U_l}{U_c}(C^*(\tau, s), 1 - L^*(\tau, s)) = [1-\tau]F_L(L^*(\tau, s), s)$ and $C^*(\tau, s) + G(s) = F(L^*(\tau, s), s)\ \forall(\tau, s)$. By the IFT (implicit function theorem), assuming U and F smooth, it follows that $C^*(.)$ and $L^*(.)$ are continuously differentiable in τ, with:

$$\begin{bmatrix} \partial C^*/\partial\tau \\ \partial L^*/\partial\tau \end{bmatrix} = \begin{bmatrix} \frac{U_cU_{cl} - U_lU_{cc}}{U_c^2} & \frac{-U_cU_{ll} + U_lU_{cl} - (1-\tau)F_{LL}U_c^2}{U_c^2} \\ 1 & -F_L \end{bmatrix}^{-1} \begin{bmatrix} -F_L \\ 0 \end{bmatrix} = \ldots = D \cdot \begin{bmatrix} F_L^2 \\ F_L \end{bmatrix}$$

where $1/D \equiv \det \begin{bmatrix} \frac{U_cU_{cl} - U_lU_{cc}}{U_c^2} & \frac{-U_cU_{ll} + U_lU_{cl} - (1-\tau)F_{LL}U_c^2}{U_c^2} \\ 1 & -F_L \end{bmatrix}$.

Next, notice that:

$$
\begin{aligned}
U_c^3/D &= U_c[U_cU_{ll} - U_lU_{cl} + (1-\tau)F_{LL}U_c^2] - U_cF_L[U_cU_{cl} - U_lU_{cc}] \\
&\leq U_c[U_cU_{ll} - U_lU_{cl}] - U_cF_L[U_cU_{cl} - U_lU_{cc}] \qquad \text{since } F_{LL} \leq 0 < U_c \\
&\leq U_c[U_cU_{ll} - U_lU_{cl}] - (1-\tau)F_LU_c[U_cU_{cl} - U_lU_{cc}] \qquad \text{since } 1-\tau \leq 1 \\
&= U_c[U_cU_{ll} - U_lU_{cl}] - U_l[U_cU_{cl} - U_lU_{cc}] \qquad \text{since } U_l/U_c = (1-\tau)F_L \\
&= U_c^2U_{ll} + U_l^2U_{cc} - 2U_cU_lU_{cl} < 0 \qquad \text{by strict quasiconcavity of } U(.)
\end{aligned}
$$

Hence, $D < 0$, implying $\begin{bmatrix} \partial C^*/\partial\tau \\ \partial L^*/\partial\tau \end{bmatrix} = DF_L\begin{bmatrix} F_L \\ 1 \end{bmatrix} < 0$. Also, $L^*(1,s) = C^*(1,s) = 0\,\forall s$. We next define u^* by $u^*(\tau,s) \equiv U\left(C^*(\tau,s), 1-L^*(\tau,s)\right)$. It follows that u^* is continuously differentiable in τ, with $\frac{\partial u^*}{\partial\tau} = \begin{bmatrix} U_c \\ -U_l \end{bmatrix}'\begin{bmatrix} \partial C^*/\partial\tau \\ \partial L^*/\partial\tau \end{bmatrix} = ... = (-DF_L)(U_l - U_cF_L)$. Since $D < 0$ as shown above, and since $U_l - (1-\tau)F_LU_c = 0$ by (5), we have $\tau > 0 \Rightarrow U_l - U_cF_L < U_l - (1-\tau)F_LU_c = 0 \Rightarrow \frac{\partial u^*}{\partial\tau} < 0$ while $\tau = 0 \Rightarrow \frac{\partial u^*}{\partial\tau} = 0$. Further, $\lim_{\tau\to1} c(\tau,s) = 0$, and then $u_\tau^*(1,s) = -\infty$ follows from the Inada condition $\lim_{C\to0} U_c(C, 1-L) = \infty$. Next, for the surplus $R({}_ts) \equiv \tau_t({}_ts)Y_t({}_ts) - G(s_t)$ we define R^* as $R^*(\tau,s) \equiv \tau F(L^*(\tau,s),s) - G(s)$. **QED**

Proof of Lemma 1: In equilibrium, $C_t({}_ts) = C^*(\tau_t({}_ts), s_t)$ and $L_t({}_ts) = L^*(\tau_t({}_ts), s_t)$, as in Proposition (1). Letting $U_c^*(\tau,s) \equiv U_c\left(C^*(\tau,s), 1-L^*(\tau,s)\right)$ and ${}_js = (s_0,...,s_j)$, define $q_j^*(\tau,\tau';{}_j\, s) \equiv$ $\equiv \frac{\beta^j\mu(s_j|s_{j-1})...\mu(s_1|s_0)U_c^*(\tau',s_j)}{U_c^*(\tau,s_0)}$. As in the main text, $q_{t,j}({}_{t+j}s|{}_ts) = \frac{\beta^j\mu({}_{t+j}s|{}_ts)U_c(C_{t+j}({}_{t+j}s),1-L_{t+j}({}_{t+j}s))}{U_c(C_t({}_ts),1-L_t({}_ts))}$. Using $\mu({}_{t+j}s|{}_ts) = \mu(s_{t+j}|s_{t+j-1})...\mu(s_{t+1}|s_t)$ for all ${}_{t+j}s \in \{{}_ts\}\times\mathcal{S}^j$ and $\mu({}_{t+j}s|{}_ts) = 0$ otherwise, we hence get $q_{t,j}({}_{t+j}s|{}_ts) = q_j^*(\tau({}_ts), \tau({}_{t+j}s); s_t,...,s_{t+j})$ for all ${}_{t+j}s \in \{{}_ts\}\times\mathcal{S}^j$ and $q_{t,j}({}_{t+j}s|{}_ts) = 0$ otherwise. Next, $\frac{\partial q_j^*(\tau,\tau';s_0,...,s_j)}{\partial\tau} = -\frac{q_j^*(\tau,\tau';s_0,...,s_j)}{U_c^*(\tau,s_0)}\cdot\frac{\partial U_c^*(\tau,s_0)}{\partial\tau}$ and $\frac{\partial q_j^*(\tau,\tau';s_0,...,s_j)}{\partial\tau'} = +\frac{q_j^*(\tau,\tau';s_0,...,s_j)}{U_c^*(\tau',s_j)}\cdot\frac{\partial U_c^*(\tau',s_j)}{\partial\tau'}$. Bby definition of $U_c^*(\tau,s)$, $\frac{\partial U_c^*(\tau,s)}{\partial\tau} = \begin{bmatrix} U_{cc} \\ -U_{cl} \end{bmatrix}'\begin{bmatrix} \partial C^*/\partial\tau \\ \partial L^*/\partial\tau \end{bmatrix} = DF_L\begin{bmatrix} U_{cc} \\ -U_{cl} \end{bmatrix}'\begin{bmatrix} F_L \\ 1 \end{bmatrix} = DF_L(F_LU_{cc} - U_{cl})$. Since $D < 0$ (see the proof of Proposition 1) and $U_{cc} < 0$, a sufficient condition for $\partial U_c^*(\tau,s)/\partial\tau > 0$ is that $U_{cl} > F_LU_{cc}$, meaning that consumption and leisure are either complements or not too-strong substitutes. And then, $\frac{\partial q_j^*(\tau,\tau',.)}{\partial\tau} < 0$ and $\frac{\partial q_j^*(\tau,\tau',.)}{\partial\tau'} > 0$. That is, $\frac{\partial U_c(t)}{\partial\tau_t} > 0$ implying $\frac{\partial q_{t,j}({}_{t+j}s|{}_ts)}{\partial\tau_t({}_ts)} < 0$; and $\frac{\partial U_c(t+j)}{\partial\tau_{t+j}} > 0$ implying $\frac{\partial q_{t,j}({}_{t+j}s|{}_ts)}{\partial\tau_{t+j}({}_{t+j}s)} > 0$. Finally, $\lim_{\tau\to1} q_j^*(\tau,\tau',.) = 0$ and $\lim_{\tau'\to1} q_j^*(\tau,\tau',.) = +\infty$ follow from the fact that $\lim_{\tau\to1} C^*(\tau,s) = 0$ and the Inada condition $\lim_{C\to0} U_c(C,1-L) = 0$. **QED**

Proof of Proposition 2: It follows from Lemma 1 and (6), by defining p_j^* as $p_j^*(\tau(s), \tau'(.|s), s)) \equiv \sum_{s'\in\mathcal{S}^j} q_j(\tau(s), \tau'(s,s'), s')$. Recall that $\frac{\partial U_c^*(\tau,s)}{\partial\tau} > 0$ and thus $\frac{\partial p_{t,j}({}_ts)}{\partial\tau_t({}_ts)} < 0 < \frac{\partial p_{t,j}({}_ts)}{\partial\tau_{t+j}({}_{t+j}s)}$ hold if $U_{cl} > 0$; or iff $U_{cl} > AU_{cc}$. **QED**

Proof of Proposition 3: Follows immediately from Propositions 1 and 2. **QED**

Proof of Proposition 4: Letting $P_t = (p_{t,1},...,p_{t,M})$ and $B_t = (b_{t,1},...,b_{t,M})$, we can rewrite the temporal budget (4) more compactly as $G_t + \begin{bmatrix} 1 \\ P_t \end{bmatrix}'\begin{bmatrix} I \\ o' \end{bmatrix}B_{t-1} \leq \tau_tY_t + P_t'B_t$, or $R_t \equiv \tau_tY_t - G_t \geq P_t'B_t - \begin{bmatrix} 1 \\ P_t \end{bmatrix}'\begin{bmatrix} I \\ o' \end{bmatrix}B_{t-1}$ where a prime ($'$) denotes vector transpose, I is the $M\times M$ identity matrix, and o' a raw of M zeros. Also, given $\{\tau_t(.)\}_{t=0}^{\infty}$, throughout this proof we let $R_t({}_ts) = R^*(\tau_t({}_ts), s_t)$, $U_c(t,{}_t\, s) = U_c^*(\tau_t({}_ts), s_t)$, $q_t({}_ts, s_{t+1}) = q_{t,1}({}_ts, s_{t+1}|{}_ts)$ and $\forall j$ $q_{t,j}({}_{t+j}s|{}_ts) = \frac{\beta^j\mu({}_{t+j}s|{}_js)U_c(t+j,{}_{t+j}s)}{U_c(t,{}_ts)} = q_j^*\left(\tau_t({}_ts), \tau_{t+j}({}_{t+j}s), s_t,...,s_{t+j}\right)$, at all $t,{}_t\, s$. First suppose that $\{\tau_t(.)\}_{t=0}^{\infty}$ is sustainable (in the sense of Definition 1) and let us prove it has to satisfy both (9) and (10): $\square$ For any $t,n \geq 0$, fix a node $(t+n,{}_{t+n}\, s)$ and consider the temporal budget; from (4), $R_{t+n}({}_ns) \geq \begin{bmatrix} 1 \\ P_{t+n}({}_{t+n}s) \end{bmatrix}'\begin{bmatrix} I \\ o' \end{bmatrix}B_{t+n-1}({}_{t+n-1}s) - P_{t+n}({}_ts)'B_{t+n}({}_{t+n}s)$. With $q_{t,0}(.) \equiv 1$, multiply both sides with $q_{t,n}({}_{t+n}s|{}_ts) = \frac{\beta^n\mu({}_{t+n}s|{}_ts)U_c(t+n,{}_{t+n}s)}{U_c(t,{}_ts)} =$

$\prod_{j=t}^{t+n-1} q_j({}_js, s_{j+1})$; then for given ${}_ts = ({}_{t-1}s, s_t) \in \{{}_{t-1}s\} \times \mathcal{S}$ and for fixed $t+n$ sum up over all ${}_{t+n}s \in \{{}_ts\} \times \mathcal{S}^n$, to get:

$$\sum_{{}_{t+n}s} q_{t,n}({}_{t+n}s|{}_ts) R_{t+n}({}_{t+n}s) \geq$$
$$\geq \sum_{{}_{t+n}s} q_{t,n}({}_{t+n}s|{}_ts) \left\{ -P_{t+n}({}_{t+n}s)' B_{t+n}({}_{t+n}s) + \left[\begin{smallmatrix} 1 \\ P_{t+n}({}_{t+n}s) \end{smallmatrix}\right]' \left[\begin{smallmatrix} I \\ o' \end{smallmatrix}\right] B_{t+n-1}({}_{t+n-1}s) \right\}$$

for all $(t+n, {}_{t+n}s)$. Next sum up over all $n \in \{0, 1, ...\}$, rearrange the right-hand side, and use the arbitrage condition $q_{t,n+1}({}_{t+n}s, s_{t+n+1}|{}_ts) = q_{t,n}({}_{t+n}s|{}_ts) q_{t+n}(s_{t+n}, s_{t+n+1})$, to get:

$$\sum_{n=0}^{\infty} \sum_{{}_{t+n}s} q_{t,n}({}_{t+n}s|{}_ts) R_{t+n}({}_{t+n}s) \geq$$
$$\geq \sum_{n=0}^{\infty} \sum_{{}_{t+n}s} q_{t,n}({}_{t+n}s|{}_ts) \left\{ \left[\begin{smallmatrix} 1 \\ P_{t+n}({}_{t+n}s) \end{smallmatrix}\right]' \left[\begin{smallmatrix} I \\ o' \end{smallmatrix}\right] B_{t+n-1}({}_{t+n-1}s) - P_{t+n}({}_{t+n}s)' B_{t+n}({}_{t+n}s) \right\} =$$
$$= \left[\begin{smallmatrix} 1 \\ P_t({}_ts) \end{smallmatrix}\right]' \left[\begin{smallmatrix} I \\ o' \end{smallmatrix}\right] B_{t-1}({}_{t-1}s) + \sum_{n=0}^{\infty} \sum_{{}_{t+n}s} \{ q_{t,n}({}_{t+n}s|{}_ts) [-P_{t+n}({}_{t+n}s)' B_{t+n}({}_{t+n}s) +$$
$$+ \sum_{s_{t+n+1}} q_{t+n}(s_{t+n}, s_{t+n+1}) \left[\begin{smallmatrix} 1 \\ P_{t+n+1}({}_{t+n+1}s) \end{smallmatrix}\right]' \left[\begin{smallmatrix} I \\ o' \end{smallmatrix}\right] B_{t+n}({}_{t+n}s)] \} =$$
$$= \left[\begin{smallmatrix} 1 \\ P_t({}_ts) \end{smallmatrix}\right]' \left[\begin{smallmatrix} I \\ o' \end{smallmatrix}\right] B_{t-1}({}_{t-1}s) + \sum_{n=0}^{\infty} \sum_{{}_{t+n}s} q_{t,n}({}_{t+n}s|{}_ts) \times$$
$$\times \left\{ -P_t({}_ts)' + \sum_{s_{t+1}} q_t(s_t, s_{t+1}) \left[\begin{smallmatrix} 1 \\ P_{t+1}({}_{t+1}s) \end{smallmatrix}\right]' \left[\begin{smallmatrix} I \\ o' \end{smallmatrix}\right] \right\} B_{t+n}({}_{t+n}s)$$

Above we used $\lim_{t\to\infty} \sum_{{}_ts} q_{0;t}({}_ts|{}_0s) P_t({}_ts)' B_t({}_ts) = 0$, which is ensured by the no-Ponzi-game and transversality conditions. Next, $p_{t,j}({}_ts) = \sum_{s_{t+1}} [q_t(s_t, s_{t+1}) p_{t+1,j-1}({}_ts, s_{t+1})]$, $\forall j$, can be written more compactly as $P_t({}_ts)' = \sum_{s_{t+1}} q_t(s_t, s_{t+1}) \left[\begin{smallmatrix} 1 \\ P_{t+1}({}_{t+1}s) \end{smallmatrix}\right]' \left[\begin{smallmatrix} I \\ o' \end{smallmatrix}\right]$. So, the last term above vanishes. Therefore:

$$\sum_{n=0}^{\infty} \sum_{{}_{t+n}s} q_{t,n}({}_{t+n}s|{}_ts) R_{t+n}({}_{t+n}s) \geq \left[\begin{smallmatrix} 1 \\ P_t({}_ts) \end{smallmatrix}\right]' \left[\begin{smallmatrix} I \\ o' \end{smallmatrix}\right] B_{t-1}({}_{t-1}s) = \sum_{j=0}^{M-1} p_{t,j}({}_ts) b_{t-1,j+1}({}_{t-1}s)$$

Given any t and ${}_{t-1}s$, and thus given $B_{t-1}({}_{t-1}s)$, the last condition should hold at all ${}_ts = ({}_{t-1}s, s_t) \in \{{}_{t-1}s\} \times \mathcal{S}$; this proves that any sustainable policy must satisfy (10) for all $(t, {}_{t-1}s)$. Evaluating at $t=0$ and $B_{-1}({}_{-1}s) = \bar{B}_{-1}$, we get (9) as well. ■ We hence proved that the series of temporal budgets (4) implies the sustainability constraints (10) and (9). Now, the converse: □ Given $\{\tau_t(.)\}_{t=0}^{\infty}$ and thus $\{R_t(.)\}_{t=0}^{\infty}$ and $\{q_{t,j}(.)\}_{t,j=0}^{\infty}$ satisfying (10) and (9), we just let the sequence $\{B_t(.)\}_{t=-1}^{\infty}$ with initial condition $B_{-1}({}_{-1}s) = \bar{B}_{-1}$ form the debt structures supporting the given tax policy. It is immediate then that $\{R_t(.), B_t(.)\}_{t=0}^{\infty}$ satisfies (4) for all $(t, {}_ts)$. Hence, that policy is sustainable. ■ **QED**

Proof of Proposition 5: This is a well-established result. For the comparison of the two cases, however, it helps to provide a proof that relates to Proposition 4. The analogue of (10) and (ii) here would state: "At any $t \geq 1$ and any ${}_{t-1}s \in \{s_0\} \times \mathcal{S}^{t-1}$, there are *some* vectors $D_{t-1}({}_{t-1}s, s_t) = \left[[d_{t-1,j+1}({}_{t+j}s|{}_{t-1}s)]_{{}_{t+j}s \in \{{}_{t-1}s, s_t\} \times \mathcal{S}^j} \right]_{j=0,\dots,M-1}$ for $s_t \in \mathcal{S}$, such that the continuation sequence $\{\tau_{t+n}(.|{}_{t-1}s)\}_{n=0}^{\infty}$ satisfies

$$\underbrace{PV^*\left(s_t; \{\tau_{t+n}(.)\}_{n=0}^{\infty}\right)}_{PV_t({}_{t-1}s, s_t)} \geq \sum_{j=0}^{M-1} \sum_{{}_{t+j}s} \underbrace{q_j^*\left(\tau_t(.), \tau_{t+j}(.), .\right)}_{q_{t,j}({}_{t+j}s|{}_{t-1}s, s_t)} d_{t-1,j+1}({}_{t+j}s|{}_{t-1}s) \qquad \text{(10.b)}$$

for all $s_t \in \mathcal{S}$." If we add this, then the proof here would be just a replica of that of Proposition 4. However, imposing (10.b) puts no constraints whatsoever, because for every $(t, {}_{t-1}s)$ and every s_t we can use a different $D_{t-1}({}_{t-1}s, s_t)$, contingent on the particular s_t, to support $PV_t({}_{t-1}s, s_t)$ above. For instance, set $d_{t-1,1}({}_{t-1}s, s_t|{}_{-1}s) = PV_t({}_{t-1}s, s_t)$ for all s_t and all ${}_{t-1}s, t$, and $d_{t,j}(.) = 0$ for $j > 1$. Thus, (10.b) is redundant. **QED**

Proof of Lemma 2: In main text.

Proof of Lemma 3: Part (a): $\square$ Obviously, $\mathcal{SPI}$ is always a weak subset of $\mathcal{SPC}$, because any element of $\mathcal{SPC}$ has to satisfy only sustainability property (i), while any element of $\mathcal{SPI}$ has to satisfy both properties (i) and (ii). $\blacksquare$ Part (b): $\square$ That $M < S \Rightarrow \mathcal{SPI} \neq \mathcal{SPC}$ follows from our discussion in the main text: Pick a $V \in \mathbb{R}^S$ and, with $M < S$, pick an $M \times S$ matrix Q such that $V \notin Span[Q]$. We can always find some $\{\tau_t(.)\}_{t=0}^{\infty} \in \mathcal{SPC}$ such that the induced $\{V_t(.), Q_t(.)\}_{t=0}^{\infty}$ have $V_t({}_{t-1}s) = V$ and $Q_t({}_{t-1}s) = Q$ at some t and some ${}_{t-1}s$. But then $V_t({}_{t-1}s) \notin Span[Q_t({}_{t-1}s)]$ and thus $\{\tau_t(.)\}_{t=0}^{\infty} \notin \mathcal{SPI}$. $\blacksquare$ Part (c): $\square$ Observe that the typical element of $\mathcal{SPC}$ or $\mathcal{SPI}$ is a sequence $x = \{x_t\}_{t=0}^{\infty}$ where $x_t \in \mathbb{R}^{S^t}$. Let us endow the space of such sequences with the norm $||x||^* = \sup_t\left\{\frac{1}{S^{t/2}}\sqrt{x_t' x_t}\right\}$ and the induced distance — the choice of the particular norm is immaterial for our results. Now, pick some policy τ, $\tau \in \mathcal{SPC}$ but $\tau \notin \mathcal{SPC}$. This means that, at some $(t,_{t-1} s)$, we have $V(\tau) \notin Span[Q(\tau)]$, where $V(\tau)$ and $Q(\tau)$ denote the $V_t({}_{t-1}s)$ and $Q_t({}_{t-1}s)$ induced in equilibrium by policy τ at the particular $(t,_{t-1} s)$. Now, for small $\varepsilon > 0$, let $N_\varepsilon(\tau)$ be a radius-ε ball around τ and let then $\mathcal{X} = N_\varepsilon(\tau) \cap \mathcal{SPC} \neq \emptyset$. Next, consider all $\hat{\tau} \in \mathcal{X}$ and form the corresponding $V(\hat{\tau})$ and $Q(\hat{\tau})$. Let $\mathbb{W} \equiv \cup_{\hat{\tau}\in\mathcal{X}} Span[Q(\hat{\tau})]$, and observe that each $Span[Q(\hat{\tau})]$ is a subspace of dimension at most M, with $M < S$. Let then $\delta = distance(V(\tau), \mathbb{W}) > 0$. The latter is ensured for $\varepsilon > 0$ small enough by the continuity of bond prices, and thus of $Q(.)$, and by $V(\tau) \notin Span[Q(\tau)]$. We can further pick $\varepsilon > 0$ small enough so that we also have $distance(V(\hat{\tau}), V(\tau)) < \delta/2$ for all $\hat{\tau} \in \mathcal{X}$. This is now ensured by the continuity of present-value surpluses, and thus of $V(.)$. But then $distance(V(\hat{\tau}), \mathbb{W}) \geq \delta/2$, implying $V(\hat{\tau}) \notin \mathbb{W}$. Since $Span[Q(\hat{\tau})] \subseteq \mathbb{W}$ by construction of $\mathbb{W}$, we get $V(\hat{\tau}) \notin Span[Q(\hat{\tau})]$, meaning $\hat{\tau} \notin \mathcal{SPI}$. We therefore conclude $\mathcal{X} \cap \mathcal{SPI} = \emptyset$, which means $Closure[\mathcal{SPI}] \subsetneqq \mathcal{SPC}$. $\blacksquare$ Next, that $\mathcal{SPC} = Closure[\mathcal{SPC}]$ is rather trivial: $\square$ Take any convergent sequence $\{\tau^m\}_{m=1}^{\infty}$ with $\tau^m = \{\tau_t^m(.)\}_{t=0}^{\infty} \in \mathcal{SPC}$ for all m, and let $\tau = \lim_{m\to\infty} \tau^m$. We have that $\tau^m \in \mathcal{SPC}$ if and only if τ^m satisfies the initial intertemporal budget (9). But if τ^m satisfies (9) for all m, and $\tau = \lim_{m\to\infty} \tau^m$, then, by continuity, τ satisfies (9) as well, which means $\tau \in \mathcal{SPC}$. That is, $\mathcal{SPC}$ includes its limit points, and $\mathcal{SPC} = Closure[\mathcal{SPC}]$. $\blacksquare$ Part (d): $\square$ Following our previous reasoning, if $\tilde{M} < M \leq S$, then we can find a (small) open set $\mathcal{X} \subset \mathcal{SPI}(M)$ such that $\mathcal{X} \cap \mathcal{SPI}(\tilde{M}) = \emptyset$. In particular, $\mathcal{X} \subset \mathcal{SPI}(\tilde{M})$ is picked so that $\tau \in \mathcal{X}$ implies $V \in Span[Q]$ but $V \notin Span[\tilde{Q}]$, where $V = V_t({}_{t-1}s)$ and $Q = Q_t({}_{t-1}s)$ are the induced ones by the particular τ and where $\tilde{Q}$ is the $S \times \tilde{M}$ matrix formed by the first $\tilde{M}$ out of the M columns of Q. (Notice that $V \in Span[Q]$ and $V \notin Span[\tilde{Q}]$ can hold generically because we generically have $rank(\tilde{Q}) = \tilde{M} < M = rank(Q)$; the latter is established in Lemma 4.) In other words, there are as many as $S - M$ non-redundant independent constraints in sustainability property (ii), and these are decreasing in M. $\blacksquare$ **QED**

Proof of Lemma 4: In main text.

Proof of Theorem 1: Just combining Lemmas 3 and 4. **QED**

Proof of Proposition 6: This is a straightforward linear-algebra exercise: $V_t(.) = Q_t(.)B_{t-1}(.)$, or $V = QB$, is a system of S linear equations in M unknowns. Assume first $M \leq S$: $\square$ For any generic policy, Q has always full rank, $rank(Q) = M \leq S$. Thus, if the $M \times S$ system $V = QB$ has a solution, this has to be unique. Existence of some solution is ensured by $\tau \in \mathcal{SPI}$, as in sustainability property (ii). Therefore, $V = QB$ generically has a unique solution when $M \leq S$, meaning that B is uniquely determined. ■ Now assume $M > S$: □ In this case $V = QB$ has more unknowns than equations, so that B is necessarily underdetermined. The degrees of indeterminacy are just $M - rank[Q_t({}_{t-1}s)]$; generically these are $M - S$. ■ Finally, for $M = S$: □ Q_t is now square. Generically it is non-singular, and thus $V_t = Q_t B_{t-1} \Rightarrow B_{t-1} = Q_t^{-1} V_t$. Re-dating gives (16). ■ **QED**

Proof of Condition (18): Consider first the shadow prices of Arrow securities. With CRRA preferences, $U_c(.) = C^{-\theta}$, and $\mu(.|.) = \frac{1}{2}$, the equilibrium pricing for $q_t(.)$ gives $q_t({}_{t-1}s, s_t, s_{t+1}) = \frac{1}{2}\beta \left[\frac{C_t({}_{t-1}s, s_t)}{C_{t+1}({}_{t-1}s, s_t, s_{t+1})}\right]^\theta \ \forall\, t, {}_ts, s_{t+1}$. Since consumption is procyclical, $C_t({}_{t-1}s, \sigma^g) > C_t({}_{t-1}s, \sigma^b)$, we get $q_t({}_{t-1}s, \sigma^g, s_{t+1}) > q_t({}_{t-1}s, \sigma^b, s_{t+1}) \ \forall\, t, {}_{t-1}s, s_{t+1}$. It follows that, for all $(t, {}_{t-1}s)$: $q_t({}_{t-1}s, \sigma^g, \sigma^g) + q_t({}_{t-1}s, \sigma^g, \sigma^b) > q_t({}_{t-1}s, \sigma^b, \sigma^g) + q_t({}_{t-1}s, \sigma^b, \sigma^b)$. Next, from bond-pricing rule (6): $p_{t,1}({}_{t-1}s, s_t) = q_t({}_{t-1}s, s_t, \sigma^g) + q_t({}_{t-1}s, s_t, \sigma^b)$. Combining gives $p_{t;1}({}_{t-1}s, \sigma^g) > p_{t;1}({}_{t-1}s, \sigma^b)$. **QED**

Proof of Condition (17): From Chari & Kehoe [1990] and our Theorem 2, there are stationary functions $\bar{q}, \bar{R}, \overline{PV}$ such that, at the Ramsey optimum, $q_t({}_{t-1}s, s_t, s_{t+1}) = \bar{q}(s_t, s_{t+1})$, $R_t({}_{t-1}s, s_t) = \bar{R}(s_t)$, and $PV_t({}_{t-1}s, s_t) = \bar{R}(s_t)$, for all $t, {}_{t-1}\, s$. Using this and the recursive form for PV's, we get:

$PV(\sigma^g) = R(\sigma^g) + q(\sigma^g, \sigma^g)PV(\sigma^g) + q(\sigma^g, \sigma^b)PV(\sigma^b)$

$PV(\sigma^b) = R(\sigma^b) + q(\sigma^b, \sigma^g)PV(\sigma^g) + q(\sigma^b, \sigma^b)PV(\sigma^b)$

As before, $R(\sigma^g) > R(\sigma^b)$, $q(\sigma^g, \sigma^g) > q(\sigma^b, \sigma^g)$ and $q(\sigma^g, \sigma^b) > q(\sigma^b, \sigma^b)$. Combining we conclude $PV(\sigma^g) > PV(\sigma^b)$. Then, by continuity, for any policy that is sufficiently close to the Ramsey optimal we have $PV_t({}_{t-1}s, \sigma^g) > PV_t({}_{t-1}s, \sigma^b)$ at all $t, {}_{t-1}s$. **QED**

Proof of Proposition 7: W.l.o.g., assume the initial debt to be only in one-period bonds ($d_{-1;j} = 0$ for $j \geq 2$). Then, the sustainability/implementability constraint (9) writes $d_{-1;1} \leq PV_0$, or:

$d_{-1;1} \leq PV^*(\tau, s_0) \equiv \sum_{t=0}^{\infty} \sum_{{}_ts \in \{s_0\} \times \mathcal{S}^t} q_t^* \left(\tau_0(s_0), \tau_t({}_ts), s_0, ..., s_t\right) R^* \left(\tau_t({}_ts), s_t\right)$

$\mathcal{SPC}$ is just the set of all $\tau = \{\tau_t(.)\}_{t=0}^{\infty}$ with $\tau_t(.) \in [0, 1)$. that satisfy the above. Since both $q_t^*(\tau, \tau', .)$ and $R^*(\tau, .)$ are continuous in $\tau, \tau' \in [0, 1)$, the functional $PV_0 = PV^*(\tau, s_0)$ is continuous $\tau = \{\tau_t(.)\}_{t=0}^{\infty}$. Also, since $u^*(\tau, .)$ is both continuous in $\tau \in [0, 1)$ and bounded, the functional $E_0\mathcal{U} = \sum_{t=0}^{\infty} \sum_{{}_ts \in \{s_0\} \times \mathcal{S}^t} \beta^t \mu({}_ts|s_0) u^*(\tau_t({}_ts), s_t)$ is continuous in $\tau = \{\tau_t(.)\}_{t=0}^{\infty}$. Now, consider first the case that we impose on the tax rate an exogenous upper bound $\hat{\tau} = \{\hat{\tau}_t(.)\}_{t=0}^{\infty}$ s.t. $\hat{\tau}_t(.) \in (0, 1)$ $\forall t$. □ If so, define $\mathcal{SPC}^* \equiv \{\tau = \{\tau_t(.)\}_{t=0}^{\infty} \in \mathcal{SPC} \,|\, 0 \leq \tau_t(.) \leq \hat{\tau}_t(.) \forall t \geq 1\}$. Then, $\mathcal{SPC}^*$ is compact; it is also non-empty, provided $d_{-1;1}$ small enough, or $d_{-1;1} \leq \sup_\tau PV_0$. And $E_0\mathcal{U}$ is continuous over all $\mathcal{SPC}^*$. It then follows by the Maximum Theorem that $\arg\max_\tau \{E_0\mathcal{U} | \tau \in \mathcal{SPC}^*\} \neq \emptyset$. ■ Now, suppose there is no exogenous bound, but $\lim_{\tau' \to 1} \frac{\partial}{\partial \tau'}[U_c^*(\tau', .)R^*(\tau', .)] < +\infty$. □ From Proposition 1, $\lim_{\tau' \to 1} \frac{\partial}{\partial \tau'} u^*(\tau', .) = -\infty$. Assuming that $\lim_{\tau' \to 1} \frac{\partial}{\partial \tau'}[U_c^*(\tau', .)R^*(\tau', .)] < +\infty$, we also get $\lim_{\tau' \to 1} \frac{\partial}{\partial \tau'}[q_t^*(\tau, \tau', .)R^*(\tau', .)] < +\infty$. Combining, and letting $\lambda \in [0, +\infty)$ be the LaGrange multiplier for (9) and $\mathcal{L} = \mathcal{L}\left(\tau_0, \{\tau_t(.)\}_{t=1}^{\infty}, \lambda\right) = E_0\mathcal{U} + \lambda\left[PV_0 - d_{-1;1}\right]$ the Lagrangian, we get $\lim_{\tau' \to 1} \frac{\partial}{\partial \tau'}[\beta^t \mu(.) u^*(\tau', .) + \lambda q_t^*(\tau, \tau', .)R^*(\tau', .)] = -\infty$ and thus $\lim_{\tau_t({}_ts) \to 1} \frac{\partial \mathcal{L}}{\partial \tau_t({}_ts)} = -\infty$ for all

$t \geq 1$,${}_t s$. It follows that it never pays to let $\tau_t({}_t s) \to 1$ for any $t \geq 1$ or any ${}_t s$. Intuitively, as $\tau_t({}_t s) \to 1$, the welfare cost of taxation explodes, in that $u_\tau(\tau_t({}_t s), s_t) \to -\infty$. Letting $\tau_t({}_t s) \to 1$ may increase the contemporaneous surplus, but $\lim_{\tau_t({}_t s) \to 1} \frac{\partial}{\partial \tau'}[U_c(.)R_t({}_t s)] < +\infty$ ensures that this effect is bounded and dominated by the welfare loss. Thus, $\tau_t({}_t s)$ may be bounded away from 1, and $\arg\max_\tau \{E_0\mathcal{U}|\tau \in \mathcal{SPC}\} \neq \emptyset$. ■ **QED**

Proof of Corollary 2: From Theorem 1, $\mathcal{SPI} \subseteq \mathcal{SPC} \Rightarrow \sup_{\tau \in \mathcal{SPI}} E_0\mathcal{U} \leq \max_{\tau \in \mathcal{SPC}} E_0\mathcal{U}$, $\forall M$,S. Next, $M \geq S \Rightarrow Closure\mathcal{SPI} = \mathcal{SPC} \Rightarrow \sup_{\tau \in \mathcal{SPI}} E_0\mathcal{U} = \max_{\tau \in \mathcal{SPC}} E_0\mathcal{U}$. Finally, if the second best is generic and if $M \geq S$, then $\arg\max_{\tau \in \mathcal{SPC}} E_0\mathcal{U} \in \mathcal{SPI}$ implying $\arg\max_{\tau \in \mathcal{SPI}} E_0\mathcal{U} = \arg\max_{\tau \in \mathcal{SPC}} E_0\mathcal{U} \neq \emptyset$. **QED**

Proof of Corollary 3: From part (d) of Lemma 3, in particular, $M' < M \Rightarrow SPI(M') \subset SPI(M) \Rightarrow \sup_{\tau \in \mathcal{SPI}(M')} E_0\mathcal{U} \leq \sup_{\tau \in \mathcal{SPI}(M)} E_0\mathcal{U}$. **QED**

Proof of Theorem 2: That the Ramsey policy for an economy with a Markov state process is characterized by $\tau_t({}_t s) = \bar{\tau}(s_t)$ for all $t \geq 1$ and all ${}_t s$, was first observed by Lucas & Stokey [1983]. See Chari & Kehoe [1999] for a more detailed treatment. Given this property, the rest characterization follows immediately. We just have to define:

$\bar{C}(s) \equiv C^*(\bar{\tau}(s), s)$, $\bar{L}(s) \equiv L^*(\bar{\tau}(s), s)$, $\bar{U}_c(s) \equiv U_c\left(\bar{C}(s), 1 - \bar{L}(s)\right)$,

$\bar{R}(s) \equiv R^*(\bar{\tau}(s), s) \equiv \{\bar{\tau}(s)[A(s)\bar{L}(s) + e(s)] - G(s)\}$, $\overline{PV}(s) \equiv \sum_{j=0}^{\infty} \sum_{{}_j s \in \{s\} \times \mathcal{S}^j} \bar{q}_j({}_j s)\bar{R}(s_j)$,

$\bar{q}_1(s, s') \equiv \beta\mu(s'|s)\bar{U}_c(s')/\bar{U}_c(s) \equiv q_1^*\left(\bar{\tau}(s), \bar{\tau}(s'), s, s'\right)$,

$\bar{q}_j(s_0, s_1, ..., s_j) \equiv \bar{q}_1(s_0, s_1)...\bar{q}_1(s_{j-1}, s_j) \equiv q_j^*\left(\bar{\tau}(s_0), \bar{\tau}(s_1), s_0, ..., s_j\right)$, $\forall j \geq 1$, and

$\bar{p}_j(s) \equiv \sum_{{}_j s \in \{s\} \times \mathcal{S}^j} \bar{q}_j({}_j s) \equiv p_j^*\left(\bar{\tau}(s), \bar{\tau}(.), s\right)$, $\forall j \geq 1$.

And then: $\bar{V} \equiv \left[\overline{PV}(s)\right]_{s \in \mathcal{S}}$ and $\bar{Q} \equiv [\bar{p}_j(s)]_{s \in \mathcal{S}}^{j=0,...,M-1}$.

Next, provided $M = S$ and assuming that the Ramsey optimal is generic so that $\bar{Q}$ is non-singular, we let $\bar{B} \equiv \bar{Q}^{-1}\bar{V}$ and apply Proposition 6 to complete the proof. **QED**