Is There Any Escape To Equivalent Scales When Adjusting Incomes For Needs?♯

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Abstract

The aim of the paper is to provide guidelines in order to make meaningful comparisons of heterogeneous distributions when incomes are adjusted in order to accommodate differences in needs. We emphasize that the choice of the equivalent income function and the system of weights associated to the equivalent incomes affects significantly the conclusions to be drawn. Introducing simple but intuitively appealing conditions, we show that adjusting incomes by a scale factor and weighting the resulting equivalent incomes by the same factor – as was proposed by Pyatt (1985, 1990) and Ebert (1999) – does constitute the only consistent method of making comparisons of relative inequality and/or welfare across populations of heterogeneous households. When the focus is on comparisons in terms of absolute inequality, then lump-sum equivalent income functions and equal weights constitute the only admissible adjustment procedure. *Journal of Economic Literature* Classification Number: D31, D63, H24. *Keywords*: Heterogeneous households, Equivalence scales, Weighting schemes, Lorenz dominance.

1.Introduction and summary.

1.1.Motivation and relationship to the literature.

The ambition of the paper is to offer guidelines to the practioner in order to make meaningful comparisons of inequality and welfare when the population of households is heterogeneous. More precisely we propose a consistent approach for making inequality and welfare comparisons when households differ with respect to needs based on reasonable normative conditions.

The standard practice in empirical work when comparing distributions of incomes pertaining to households with differing needs typically involves a two-stage process[1]. Firstly, one derives the distribution of living standards by adjusting original households' incomes in order to accommodate differences in needs and weighting the resulting values in an appropriate way. Then, one applies to these virtual distributions the conventional criteria used for comparing homogeneous distributions in order to deduce the corresponding rankings of the actual heterogeneous distributions. This approach actually amounts to aggregate various dimensions into a single one by considering a fictitious and homogeneous population and to

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[1] It is fair to note that a different approach has been proposed by Atkinson and Bourguignon (1987) which compares directly the heterogeneous distributions and involves a test of Lorenz dominance sequentially. But it is equally true that at the moment very few empirical studies have been conducted that used the sequential Lorenz dominance approach.

apply the traditional techniques derived in the unidimensional case for assessing inequality and welfare. This adjustment process requires an agreement on the particular equivalence scale to be used, on the household type to be selected as the reference, and on the appropriate weights to be applied to the equivalent incomes. There is a wide agreement among practioners to apply a fixed scale factor to the household's income and to weight the resulting figure by the number of individuals in the household.

However, this general practice has been called in question by a number of scholars raising doubts on the theoretical justification of such a procedure [see e.g. Pollak and Wales (1979), Blundell and Lewbel (1991)]. The basic argument is that the welfare of a household cannot be derived from the observed behaviour so that extra and non-testable assumptions are needed in order to identify household welfare. A theoretical rationale for household equivalence scales and their use in normative economics is provided by Blackorby and Donaldson (1993). A second difficulty is that the adjustment procedure, which consists in choosing the income equivalence function in order to derive equivalent incomes and the weights to be associated to these equivalent incomes, is to a large extent arbitrary. This is particularly the case for the equivalence scale transformation which constitutes one equivalent income function among many other possible candidates. The equivalence scale transformation imposes a particular structure on the household welfare function which appears to be rejected by recent empirical tests [see e.g. Donaldson and Pendakur (1999)]. In most empirical studies, the equivalent income of the household is weighted by the number of persons in the household. This way of weighting equivalent incomes can be justified by the assumptions made regarding the decision process within the household as it is exemplified in Blackorby and Donaldson (1993). When one subscribes to equivalence scales, Ebert (1997b, 1999) and Pyatt (1985, 1990) have given arguments for employing the scale factors as the appropriate weights rather than the number of persons. The arbitrariness would not be too much a problem if the choice of the adjustment procedure were shown to have little impact on the normative conclusions to be drawn. However, this is far from being the case and a number of examples in the literature have made clear that the way incomes are transformed and weighted affects dramatically the results [see e.g. Coulter, Cowell and Jenkins (1992), Cowell and Mercader-Prats (1997), de Vos and Zaidi (1997), Figini (1998), Kakwani (1986)]. An even more serious problem has been pointed out by Glewwe (1991) who shows that a transfer of income between two households, which unambiguously reduces the inequality in terms of living standards between these two households, may imply a more unequal adjusted distribution. There is therefore a need for clarifying these issues and in this paper we will take a first step in this direction by providing arguments in order to resolve a substantial part of the arbitrariness regarding the choice of the adjustment procedure.

1.2.The theoretical approach developed in the paper.

Even if there is a substantial disagreement regarding the appropriate values of the scale factors and the weights, most of the profession would accept this general procedure for making inequality and welfare comparisons in a heterogeneous environment. Implicitly, this practice assumes that the choice of the adjustment process and the choice of the normative criteria one appeals to for comparing the distributions of equivalent incomes are distinct and independent issues. In this paper, we challenge this conventional view arguing that such an independence does not hold as long as one subscribes to some basic consistency requirements. If this is true, then it means that the choice of the equivalent income function and the way equivalent incomes have to be weighted is determined to some extent by the normative criteria one appeals to when comparing the adjusted distributions. To prove this conjecture we introduce two consistency requirements and we examine the restrictions these conditions impose on the adjustment process. Our first condition insists that the result of the comparisons is not affected by the choice of the reference type. Our second condition requires that a transfer of income that reduces the inequality of living standards between two households must improve welfare and/or inequality according to our heterogeneous quasi-orderings. Because we want our results to be valid for the largest possible classes of welfare and/or inequality measures, we will employ dominance criteria that are consistent with the basic Lorenz approach.

1.3.Organization of the paper.

Section 2 lays down the background considering the case of a homogeneous population and introduces the basic Lorenz quasi-orderings that will be used later on for comparing the adjusted distributions. Rather than focusing on particular inequality and welfare indices, we prefer to adopt a dominance approach in order to obtain unambiguous normative conclusions. As far as welfare judgements are involved, we refer to the generalized Lorenz quasi-ordering introduced by Shorrocks (1983). For the purpose of inequality measurement, we concentrate on the standard relative Lorenz quasi-ordering and on the absolute Lorenz quasi-ordering [Moyes (1987)] which is related to the class of absolute inequality indices introduced by Kolm (1976). We also assemble a number of technical results – concerning transformations that preserve the ranking of distributions – that will be needed in subsequent proofs.

The possibility that households differ in other respects than income is recognized in Section 3 where we formalize the current technique used when making welfare and inequality comparisons. We introduce the notion of an equivalent income function in order to compare the living standards of households with differing needs. Given any reference household-type, the equivalent income function specifies the transformations to be performed in order to convert households' incomes into equivalent incomes. These procedures are far less restrictive than the current equivalence scale method as they impose very little structure on the transformations to be used. The welfare and inequality quasi-orderings we propose for comparing heterogeneous distributions consist in applying the Lorenz quasi-orderings derived in the homogeneous case to the weighted distributions of equivalent incomes. These quasi-orderings are flexible enough, allowing the planner to choose the income equivalence function, the reference type, and the welfare or inequality quasi-ordering. Since there does not appear to be unanimous agreement regarding the appropriate way of weighting equivalent incomes, we also allow for possibly different systems of weights.

Section 4 investigates the implications for our heterogeneous quasi-orderings of the introduction of two basic consistency conditions and contains our main results. Our first condition requires that the result of the comparisons must not be affected by the choice of the reference type. This condition restricts considerably the class of admissible income equivalence functions which reduce to equivalence scales when the domain of household income is equal to the positive real line. When household income domain comprises in addition non-positive incomes – something which is permitted by the absolute Lorenz and the generalized Lorenz quasi-orderings – then the equivalent income function is of the lump-sum type. However the reference independence condition imposes no restriction on the way the equivalent incomes have to be weighted. The pictures changes when we impose our second consistency condition which requires that a transfer of income that reduces the inequality of living standards between two households must improve welfare and/or inequality according to our heterogeneous quasi-orderings. Then the income equivalence and the weighting function are no longer independent and are completely determined by the Lorenz criterion – or the corresponding admissible income domain – selected for comparing the adjusted distributions.

Finally, Section 5 concludes the paper summarizing our findings and discussing briefly related results in the literature while Appendices A and B contain the proofs of technical arguments used in Section 4.

2.Welfare and inequality in the case of a homogeneous population.

2.1.Notation and preliminary definitions.

We consider a fixed *population* or *society* $S := \{1, 2, \ldots, n\}$ consisting of n ($n \geq 3$) homogeneous households[2], and we assume that incomes are drawn from an interval D which, depending on the context, will be equal to $\mathbb{R}$ or $\mathbb{R}_{++}$. It is convenient to frame the analysis in terms of weighted samples of incomes as household weights will be extensively used in later developments where heterogeneous households will be considered. A typical *homogeneous income distribution* is a composite vector $(\mathbf{x} \mid \mathbf{w}) := (x_1, \ldots, x_n \mid w_1, \ldots, w_n)$, where $x_i \in D$ and $w_i > 0$ are respectively the income and the sample weight of household i, and we denote as $\mu(\mathbf{x} \mid \mathbf{w})$ the *weighted mean* of distribution $(\mathbf{x} \mid \mathbf{w})$. For notational convenience, we will call $\mathbf{x} := (x_1, \ldots, x_n)$ an *income profile* and $\mathbf{w} := (w_1, \ldots, w_n)$ a *weight profile*, and we represent the set of homogeneous income distributions by

$$(2.1) \qquad \mathcal{Y}(D) := \{(\mathbf{x} \mid \mathbf{w}) \mid x_i \in D \text{ and } w_i > 0, \text{ for all } i = 1, 2, \ldots, n\}.$$

Furthermore we let $\left(\mathbf{x}_{[\,]} \mid \mathbf{w}_{[\,]}\right) := \left(\mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[n]} \mid \mathbf{w}_{[1]}, \ldots, \mathbf{w}_{[n]}\right)$ stand for a non-decreasing rearrangement of $(\mathbf{x} \mid \mathbf{w})$ defined by $\mathbf{x}_{[\,]} = \Pi\mathbf{x}$ and $\mathbf{w}_{[\,]} = \Pi\mathbf{w}$ for some permutation matrix Π such that $\mathbf{x}_{[1]} \leq \mathbf{x}_{[2]} \leq \cdots \leq \mathbf{x}_{[n]}$. We are interested in defining rankings of distributions which have some ethical content and which used in a next step for constructing heterogeneous quasi-orderings.

2.2.Unidimensional inequality and welfare quasi-orderings.

Following Atkinson (1970), Kolm (1969) and Sen (1973), it is now a well-established practice to adopt a non-ambiguous approach and appeal to the Lorenz dominance criteria when making inequality and welfare judgements on the basis of the distribution of individual incomes. Before defining the unidimensional quasi-orderings we will exploit in the paper, it is convenient to introduce the basic conditions one is typically willing to impose on a welfare

[2]We will need this restriction when we will consider the general case of a population of heterogeneous households and in particular when we will investigate the effects on the ranking of the distributions of a reduction of the inequality in living standards.

and/or inequality quasi-ordering. To this end, we let $\geq_K$ represent an arbitrary reflexive and transitive relation on the set of distributions $\mathcal{Y}(D)$ and we denote respectively as $\sim_K$ and $>_K$ its symmetric and asymmetric components defined in the usual way.

We will say that distribution $(\mathbf{x} \mid \mathbf{w})$ is obtained from distribution $(\mathbf{y} \mid \mathbf{w})$ by an *increment* if there exists $\Delta > 0$ and a household $i \in S$ such that $x_i = y_i + \Delta$ and $x_k = y_k$, for all $k \neq i$. The following condition simply imposes that a distribution is ranked higher than another distribution when the income of at least one household in the former situation is larger than her income in the later situation.

Monotonicity (*MON*). *For all* $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$, *we have* $(\mathbf{x} \mid \mathbf{w}) \geq_K (\mathbf{y} \mid \mathbf{w})$, *whenever* $(\mathbf{x} \mid \mathbf{w})$ *is obtained from* $(\mathbf{y} \mid \mathbf{w})$ *by an increment.*

We will say that distribution $(\mathbf{x} \mid \mathbf{w})$ is obtained from distribution $(\mathbf{y} \mid \mathbf{w})$ by a *progressive transfer* if there exists $\Delta > 0$ and two households $i, j \in S$ such that

(2.2.a) $x_k = y_k$, for all $k \neq i, j$;

(2.2.b) $x_i = y_i + (\Delta / w_i)$ and $x_j = y_j - (\Delta / w_j)$;

(2.2.c) $y_i < x_i \leq y_j$ and $y_i \leq x_j < y_j$.

The next condition captures a concern for equality as it demands that a distribution be ranked above another distribution when the former is made more equal than the later as it is defined above.

Principle of Transfers (*PT*). *For all* $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$, *we have* $(\mathbf{x} \mid \mathbf{w}) \geq_K (\mathbf{y} \mid \mathbf{w})$, *whenever* $(\mathbf{x} \mid \mathbf{w})$ *is obtained from* $(\mathbf{y} \mid \mathbf{w})$ *by a progressive transfer*[3].

As we claimed in the Introduction, we are primarily interested in welfare and inequality measurement and we will now consider some basic quasi-orderings that have been proposed in the homogeneous case for making unambiguous comparisons of welfare or inequality. It is convenient to proceed in two steps and introduce first the following criterion proposed by Shorrocks (1983).

Definition 2.1: Given two income distributions $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$, we will say that $(\mathbf{x} \mid \mathbf{w})$ *generalized Lorenz dominates* $(\mathbf{y} \mid \mathbf{w})$, which we write $(\mathbf{x} \mid \mathbf{w}) \geq_{GL} (\mathbf{y} \mid \mathbf{w})$, if and only if

$$(2.3) \qquad \sum_{j=1}^{k} \left(\frac{w_{[j]}}{\sum_{i=1}^{n} w_{[i]}} \right) x_{[j]} \geq \sum_{j=1}^{k} \left(\frac{w_{[j]}}{\sum_{i=1}^{n} w_{[i]}} \right) y_{[j]}, \qquad \text{for all } k = 1, 2, \ldots, n.$$

The generalized Lorenz criterion may be considered a suitable welfare quasi-ordering as it satisfies conditions *MON* and *PT*[4]. The generalized Lorenz quasi-ordering can be given a

[3] This condition is a restatement of the well-known Pigou-Dalton condition [see e.g. Sen (1973), Shorrocks (1983)] in our more general setting where households are associated with possibly different weights.

[4] This is a consequence of the Hardy, Littlewood and Polya (1934) result [see e.g. Marshall and Olkin (1979), Shorrocks (1983)]. In the case of weighted distributions, one may refer to Cheng (1977) or Ebert and Moyes (2000).

nice graphical interpretation using the notion of the generalized Lorenz curve. Given the distribution $(\mathbf{x} \mid \mathbf{w}) \in \mathcal{Y}(D)$, let us define

$$(2.4) \qquad P(k;\mathbf{w}) := \sum_{j=1}^{k} \left(\frac{w_{[j]}}{\sum_{i=1}^{n} w_{[i]}} \right) \text{ and } L(k;(\mathbf{x} \mid \mathbf{w})) := \sum_{j=1}^{k} \left(\frac{w_{[j]}}{\sum_{i=1}^{n} w_{[i]}} \right) x_{[j]},$$

for $k = 1, 2, \ldots, n$. The generalized Lorenz curve of distribution $(\mathbf{x} \mid \mathbf{w})$ is then obtained by letting $L(0;(\mathbf{x} \mid \mathbf{w})) := 0$, $P(0;\mathbf{w}) := 0$, and by joining with straight segments the points with coordinates $(P(k;\mathbf{w}), L(k;(\mathbf{x} \mid \mathbf{w})))$, for $k = 0, 1, 2, \ldots, n$. Given two income distributions $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$, it is then a simple matter to verify that $(\mathbf{x} \mid \mathbf{w}) \geq_{GL} (\mathbf{y} \mid \mathbf{w})$ if and only if $L(k;(\mathbf{x} \mid \mathbf{w})) \geq L(k;(\mathbf{x} \mid \mathbf{w}))$, for all $k = 1, 2, \ldots, n$.

It is generally considered that inequality is concerned with the way individuals in the society share income. Such a view suggests that what is relevant for inequality appraisal is the income share rather than the amount of income received by every member of the society. To make this idea more precise, we will say that distribution $(\mathbf{x} \mid \mathbf{w})$ is obtained from distribution $(\mathbf{y} \mid \mathbf{w})$ by a *scale transformation* if there exists $\lambda > 0$ such that $x_i = \lambda y_i$, for all $i = 1, 2, \ldots, n$. Then, if we want inequality to depend only on the income shares pertaining to the different members of the society, then we will certainly require that inequality does not change when all incomes are modified in an equiproportionate way which secures that income shares are unchanged, hence the following condition:

Scale Invariance (*SI*). *For all* $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$ *with* $D = \mathbb{R}_{++}$, *we have* $(\mathbf{x} \mid \mathbf{w}) \sim_K (\mathbf{y} \mid \mathbf{w})$, *whenever* $(\mathbf{x} \mid \mathbf{w})$ *is obtained from* $(\mathbf{y} \mid \mathbf{w})$ *by a scale transformation.*

Given the distribution $(\mathbf{x} \mid \mathbf{w}) \in \mathcal{Y}(D)$ where $D = \mathbb{R}_{++}$, we will denote as $\hat{\mathbf{x}} := (\hat{x}_1, \ldots, \hat{x}_n)$ the reduced income profile obtained from $\mathbf{x}$ by letting $\hat{x}_i := x_i / \mu(\mathbf{x} \mid \mathbf{w})$, for all $i \in S$. We note that by definition, distributions $(\hat{\mathbf{x}} \mid \mathbf{w})$ and $(\hat{\mathbf{y}} \mid \mathbf{w})$ have the same mean: $\mu(\hat{\mathbf{x}} \mid \mathbf{w}) = \mu(\hat{\mathbf{y}} \mid \mathbf{w}) = 1$.

Definition 2.2: Given two income distributions $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$ with $D = \mathbb{R}_{++}$, we will say that $(\mathbf{x} \mid \mathbf{w})$ *relative Lorenz dominates* $(\mathbf{y} \mid \mathbf{w})$, which we write $(\mathbf{x} \mid \mathbf{w}) \geq_{RL} (\mathbf{y} \mid \mathbf{w})$, if and only if $(\hat{\mathbf{x}} \mid \mathbf{w}) \geq_{GL} (\hat{\mathbf{y}} \mid \mathbf{w})$.

Clearly, the relative Lorenz quasi-ordering satisfies conditions PT and *SI*, and may therefore be considered a suitable candidate for measuring inequality[5]. The preceding criterion is related to the relative Lorenz curve defined by $RL(k;(\mathbf{x} \mid \mathbf{w})) := L(k;(\hat{\mathbf{x}} \mid \mathbf{w}))$, for all $k = 0, 1, 2, \ldots, n$ and all $(\mathbf{x} \mid \mathbf{w}) \in \mathcal{Y}(D)$ where $D = \mathbb{R}_{++}$. Given the definition of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ above, one checks immediately that $(\mathbf{x} \mid \mathbf{w}) \geq_{RL} (\mathbf{y} \mid \mathbf{w})$ is equivalent to

$$(2.5) \qquad \sum_{j=1}^{k} \left(\frac{w_{[j]}}{\sum_{i=1}^{n} w_{[i]}} \right) \frac{x_{[i]}}{\mu(\mathbf{x} \mid \mathbf{w})} \geq \sum_{j=1}^{k} \left(\frac{w_{[j]}}{\sum_{i=1}^{n} w_{[i]}} \right) \frac{y_{[i]}}{\mu(\mathbf{y} \mid \mathbf{w})},$$

for all $k = 1, 2, \ldots, n-1$. A distribution is no less unequal than another distribution if its relative Lorenz curve is nowhere below the relative Lorenz curve of the other distribution.

[5] Actually, it can be shown that the relative Lorenz criterion is fully characterized by scale invariance and the within-type principle of transfers [see e.g. Foster (1985) in the case of discrete distributions, and Ebert and Moyes (2000) in the case of weighted distributions considered here].

In other words, the cumulated income share of every poorest fraction of the population of households in the former distribution is no less than its share in the latter distribution.

The relative inequality view has been criticized more than twenty years ago by Kolm (1976) who argued that one may take the view that equiproportionate additions to incomes actually increase inequality and that only equal additions to all incomes would leave inequality unchanged. As an alternative to the common relative inequality indices, he suggested the class of absolute inequality measures which are invariant with respect to equal additions to all incomes. Precisely, we will say that distribution $(\mathbf{x} \mid \mathbf{w})$ is obtained from distribution $(\mathbf{y} \mid \mathbf{w})$ by a *translation transformation* if there exists $\gamma \in \mathbb{R}$ such that $x_i = y_i + \gamma$, for all $i = 1, 2, \ldots, n$. The following condition states that inequality does not change when all incomes are increased or decreased by the same amount.

Translation Invariance (*TI*). *For all* $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$ *with* $D = \mathbb{R}$, *we have* $(\mathbf{x} \mid \mathbf{w}) \sim_K (\mathbf{y} \mid \mathbf{w})$, *whenever* $(\mathbf{x} \mid \mathbf{w})$ *is obtained from* $(\mathbf{y} \mid \mathbf{w})$ *by a translation transformation.*

Given the distribution $(\mathbf{x} \mid \mathbf{w}) \in \mathcal{Y}(D)$, let us denote as $\tilde{\mathbf{x}} := (\tilde{x}_1, \ldots, \tilde{x}_n)$ the centered income profile obtained from $\mathbf{x}$ by letting $\tilde{x}_i := x_i - \mu(\mathbf{x} \mid \mathbf{w})$, for all $i \in S$. We note that by definition, distributions $(\tilde{\mathbf{x}} \mid \mathbf{w})$ and $(\tilde{\mathbf{y}} \mid \mathbf{w})$ have equal means: $\mu(\tilde{\mathbf{x}} \mid \mathbf{w}) = \mu(\tilde{\mathbf{y}} \mid \mathbf{w}) = 0$.

Definition 2.3: Given two income distributions $(\mathbf{x} \mid \mathbf{w}), (\mathbf{y} \mid \mathbf{w}) \in \mathcal{Y}(D)$ with $D = \mathbb{R}$, we will say that $(\mathbf{x} \mid \mathbf{w})$ *absolute Lorenz dominates* $(\mathbf{y} \mid \mathbf{w})$, which we write $(\mathbf{x} \mid \mathbf{w}) \geq_{AL} (\mathbf{y} \mid \mathbf{w})$, if and only if $(\tilde{\mathbf{x}} \mid \mathbf{w}) \geq_L (\tilde{\mathbf{y}} \mid \mathbf{w})$.

Clearly, the absolute Lorenz quasi-ordering satisfies conditions *PT* and *TI*. The preceding criterion is related to the absolute Lorenz curve defined by $AL(k; (\mathbf{x} \mid \mathbf{w})) := L(k; (\tilde{\mathbf{x}} \mid \mathbf{w}))$, for all $k = 0, 1, 2, \ldots, n$ and all $(\mathbf{x} \mid \mathbf{w}) \in \mathcal{Y}(D)$. The absolute Lorenz curve of a distribution [Moyes (1987)] indicates for every fraction of the population the average short-fall from mean income i.e., the per capita amount of income needed in order to provide to these households the mean income. Given the definition of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ above, one notices that $(\mathbf{x} \mid \mathbf{w}) \geq_{AL} (\mathbf{y} \mid \mathbf{w})$ is equivalent to

$$(2.6) \qquad \sum_{j=1}^{k} \left(\frac{w_{[j]}}{\sum_{i=1}^{n} w_{[i]}} \right) \left[x_{[j]} - \mu(\mathbf{x} \mid \mathbf{w}) \right] \geq \sum_{j=1}^{k} \left(\frac{w_{[j]}}{\sum_{i=1}^{n} w_{[i]}} \right) \left[y_{[j]} - \mu(\mathbf{y} \mid \mathbf{w}) \right],$$

for all $k = 1, 2, \ldots, n-1$. In this paper we focus on the three Lorenz criteria introduced above: the generalized Lorenz quasi-ordering, the relative Lorenz quasi-ordering and the absolute Lorenz quasi-ordering. Much of the following conclusions could be easily extended to other related quasi-orderings such as the intermediate Lorenz criteria introduced by Pfingsten (1986) [see also Ebert and Moyes (2000)] or the poverty dominance criteria studied by Foster and Shorrocks (1988).

Up to now we have focused on distributions with the same weight profile, a restriction that might invalidate our subsequent results where households differ with respect to income and type. However, this assumption is not as restrictive as it might look at first sight as indicated in the following remark the proof of which is relegated to Appendix A.

Remark 2.1. *Let* $(\mathbf{x}\,|\,\mathbf{v}), (\mathbf{y}\,|\,\mathbf{u}) \in \mathcal{Y}(D)$ *be two arbitrary distributions such that* $\mathbf{v} \neq \mathbf{u}$. *Then, there exists two distributions* $(\mathbf{x}^*\,|\,\mathbf{w})$ *and* $(\mathbf{y}^*\,|\,\mathbf{w})$ *such that* $(\mathbf{x}^*\,|\,\mathbf{w})$ *and* $(\mathbf{x}\,|\,\mathbf{v})$ *(respectively* $(\mathbf{y}^*\,|\,\mathbf{w})$ *and* $(\mathbf{y}\,|\,\mathbf{u})$*) have the same quantile – equivalently the same inverse cumulative distribution – function.*

Because all the quasi-orderings we consider are based on the quantile function – or more precisely on the integral of the quantile function – this distributional equivalence result implies in particular that

$$(\mathbf{x}\,|\,\mathbf{v}) \geq_K (\mathbf{y}\,|\,\mathbf{u}) \Longleftrightarrow (\mathbf{x}^*\,|\,\mathbf{w}) \geq_K (\mathbf{y}^*\,|\,\mathbf{w})\,, \tag{2.7}$$

when $K = GL, RL, AL$. There is therefore no loss of generality restricting attention only to distributions with the same weight profile when the population is homogeneous.

2.3.Preservation of the welfare and inequality quasi-orderings.

As we will see later on, it is interesting to investigate the effect on the quasi-ordering of distributions of a given transformation of individual incomes. More precisely, we are concerned with the preservation of a given quasi-ordering $\geq_K$ $(K = GL, RL, AL)$ by a transformation which is captured by a function $f \in \mathcal{F}(D) := \{f : D \to D$ continuous and increasing$\}$. Given a transformation $f \in \mathcal{F}(D)$ and a distribution $(\mathbf{x}\,|\,\mathbf{w}) \in \mathcal{Y}(D)$, we denote as $(f(\mathbf{x})\,|\,\mathbf{w})$ the transformed distribution where $f(\mathbf{x}) := (f(x_1), \ldots, f(x_n))$. The question we are interested in is the following: given the quasi-ordering $\geq_K$, what are the properties of the transformation f that guarantee that $(f(\mathbf{x})\,|\,\mathbf{w}) \geq_K (f(\mathbf{y})\,|\,\mathbf{w})$ whenever $(\mathbf{x}\,|\,\mathbf{w}) \geq_K (\mathbf{y}\,|\,\mathbf{w})$? We consider successively the three quasi-orderings we have introduced above.

As far as we are concerned with the preservation of the generalized Lorenz ranking of distributions, we have the following proposition which extends previous results by Marshall and Olkin (1979), Moyes (1989) and Moyes and Shorrocks (1994).

Lemma 2.1. *Let* $D \subseteq \mathbb{R}$ *and* $f \in \mathcal{F}(D)$. *Then the two following statements are equivalent:*
(a) For all $(\mathbf{x}\,|\,\mathbf{w}), (\mathbf{y}\,|\,\mathbf{w}) \in \mathcal{Y}(D)$*:* $(\mathbf{x}\,|\,\mathbf{w}) \geq_{GL} (\mathbf{y}\,|\,\mathbf{w}) \Rightarrow (f(\mathbf{x})\,|\,\mathbf{w}) \geq_{GL} (f(\mathbf{y})\,|\,\mathbf{w})$.
(b) f *is concave.*

The class of order preserving transformations shrinks dramatically when attention is directed to our two inequality quasi-orderings. Regarding the preservation of the relative Lorenz quasi-ordering of distributions, we indeed obtain the following result.

Lemma 2.2. *Let* $D = \mathbb{R}_{++}$ *and* $f \in \mathcal{F}(D)$. *Then the two following statements are equivalent:*
(a) For all $(\mathbf{x}\,|\,\mathbf{w}), (\mathbf{y}\,|\,\mathbf{w}) \in \mathcal{Y}(D)$*:* $(\mathbf{x}\,|\,\mathbf{w}) \geq_{RL} (\mathbf{y}\,|\,\mathbf{w}) \Rightarrow (f(\mathbf{x})\,|\,\mathbf{w}) \geq_{RL} (f(\mathbf{y})\,|\,\mathbf{w})$.
(b) $f(y) := \beta y$, *for all* $y \in D$ $(\beta > 0)$.

Thus, only proportional transformations of income have the property to preserve the relative Lorenz quasi-ordering of distributions. When absolute Lorenz dominance is substituted for relative Lorenz dominance, the class of transformations enlarges slightly to comprise now increasing affine functions.

LEMMA 2.3. *Let $D \subseteq \mathbb{R}$ and $f \in \mathcal{F}(D)$. Then the two following statements are equivalent:*
(a) For all $(\mathbf{x} \,|\, \mathbf{w}), (\mathbf{y} \,|\, \mathbf{w}) \in \mathcal{Y}(D)$: $(\mathbf{x} \,|\, \mathbf{w}) \geq_{AL} (\mathbf{y} \,|\, \mathbf{w}) \Rightarrow (f(\mathbf{x}) \,|\, \mathbf{w}) \geq_{AL} (f(\mathbf{y}) \,|\, \mathbf{w})$.
(b) $f(y) := \alpha + \beta y$, for all $y \in D$ ($\alpha \in \mathbb{R}$, $\beta > 0$).

The proofs of these three lemmata are rather technical and lengthy, and the interested reader is referred to Ebert and Moyes (2000) for the details.

3.Inequality and welfare comparisons for heterogeneous populations.

3.1.Heterogeneous households and the distribution of income.

From now on we consider a population consisting of n households, where every household will be distinguished by two attributes: income and family size. We assume that there exists a given and finite number of types or family sizes H $(2 \leq H \leq n)$ and we let $\mathbb{H} := \{1, 2, \ldots, H\}$ represent the set of possible family sizes which is typically assumed to be a subset of the set of positive integers[6]. A *heterogeneous distribution* is a partitioned vector $(\mathbf{y}; \mathbf{m}) := (y_1, \ldots, y_n; m_1, \ldots, m_n)$, where $y_i \in D$ and $m_i \in \mathbb{H}$ are respectively the income and family size of household i, and we let

$$\mathcal{Z}(D) := \{(\mathbf{y}; \mathbf{m}) \mid y_i \in D \text{ and } m_i \in \mathbb{H}, \text{ for all } i = 1, 2, \ldots, n\} \tag{3.1}$$

represent the set of heterogeneous distributions. As we have stressed in the Introduction, welfare and inequality measurement in a heterogeneous environment involves a two-stage process: in a first step, the actual distributions are converted into virtual and equivalent distributions for a homogeneous population; in a second step, the quasi-ordering of the heterogeneous distributions is derived by applying a standard unidimensional criterion to the adjusted distributions.

3.2.Stage 1: Deriving the adjusted distributions for heterogeneous households.

In order to make meaningful comparisons of well-being across households of different types, we will suppose that one is able to adjust the income received by the different types of households in order to take needs into account. Formally, given a household type h we assume the existence of an *equivalent income function* $E : \mathbb{H} \times D \times \mathbb{H} \to D$ such that $E_h(y; m) := E(h; (y; m))$ represents the *equivalent income* of a type-m household with income y, i.e., the amount of income needed by a type-h household in order to achieve the same standard of living as a household of type m with income y. This definition necessitates that a particular reference household type – here type-h household – be specified. In most empirical work the reference type is either the single adult or the couple with no child. Since there is no particular reason for choosing one type rather than another, we will not fix the reference

[6]The framework might be extended by replacing m with a vector of household characteristics $\mathbf{a}$ which comprises in addition to household size the number of adults, the number and age of children, the health status of family members, the place where the household is living as well as any relevant information. Here we simplify things a little more paying only attention to the size of the household. However, all our results apply to the more general case provided that a general agreement can be reached regarding the ranking of needs or types on the basis of these vectors of characteristics. Formally this amounts to assume that there exists a one-to-one correspondance between the set of types $\mathbb{H}$ and the set of possible vectors of characteristics $\mathbf{A}$ so that $h < \ell$ signifies that a household with the characteristics vector $\mathbf{a}^h$ has less needs than a household with the characteristics vector $\mathbf{a}^\ell$.

type. One can think of an equivalent income function as a $H \times H$ matrix $E := [E_h(\,\cdot\,; m)]$, where row h indicates the transformations to be performed when the reference household is of size h and column m indicates the way the income of a type-m household must be converted depending on the chosen reference type. Specifying the equivalent income function E constitutes a value judgment in the same way as choosing a particular inequality or welfare measure. We will assume that the equivalent income function verifies the following properties:

(3.2.a) $E_h(y; h) = y,$ for all $h \in \mathbb{H}$ and all $y \in D$;

(3.2.b) $E_h(y; m)$ is continuous and increasing in $y,$ for all $h, m \in \mathbb{H}$;

(3.2.c) $E_h(y; m)$ is non-increasing in $m,$ for all $h \in \mathbb{H}$ and all $y \in D$;

(3.2.d) $E_h(y; m) = E_h(E_\ell(y; m); \ell),$ for all $h, \ell, m \in \mathbb{H}$ and all $y \in D$;

(3.2.e) $E_h(D; m) = D,$ for all $h, m \in \mathbb{H}$.

Property (3.2.a) simply states that the income of the chosen reference household type is not adjusted. Property (3.2.b) is rather natural and implies in particular that $E_h(y; m)$ is invertible in y. Since a larger family with the same household income is generally thought to be worse-off, it follows that $E_h(y; m)$ is non-increasing in m, hence condition (3.2.c). Property (3.2.d) is a path-independence condition which means that the sequence of transformations needed in order to convert the income of a type-m household into the equivalent income for the reference type h does not matter: one can first compute the equivalent income for type-ℓ using the transformation $E_\ell(\,\cdot\,; m)$ and then derive the equivalent income for type-h applying the transformation $E_h(\,\cdot\,; \ell)$ to the equivalent income $E_\ell(y; m)$. Taking $h = m$ and using (3.2.a), condition (3.2.d) reduces to $y = E_h(y; h) = E_h(E_\ell(y; h); \ell)$, which implies in turn that $E_\ell(y; h) = E_h^{-1}(E_h(y; h); \ell) = E_h^{-1}(y; \ell)$, for all $y \in D$. The four conditions above are rather innocuous and seem to put little structure on the equivalent income function. The remaining condition (3.2.e), which may seem more restrictive as it imposes that the income interval D is mapped into itself irrespective of the type of the household, is a direct consequence of the previous conditions. It follows that the equivalent income function defines an onto mapping on the interval D for all types. When $D = \mathbb{R}_{++}$, an implication of this condition is that, for all $h, \ell, m \in \mathbb{H}$, $E_h(y; \ell) \to E_h(y; m)$ whenever $y \to 0$, which means that differences in needs have little impact on the living standards when incomes are arbitrarily small and positive. A straightforward consequence, that will be useful later on, is that

$$\lim_{y \to 0} [E_h(y; \ell) - E_h(y; m)] = 0, \text{ for all } h, \ell, m \in \mathbb{H}, \tag{3.3}$$

when incomes are restricted to be positive. For further reference, we denote as $\mathbb{E} := \{E := [E_h(\,\cdot\,; m)] \mid \text{conditions (3.2.a) to (3.2.e) hold}\}$ the set of admissible equivalent income functions.

This procedure is sufficiently general and flexible to encompass most of the possibilities. For instance, taking singles as the reference type ($h = 1$), the conventional *equivalence scale* approach assumes a proportional transformation such that[7]

$$E_1(y;m) := y/b(1,m), \qquad \text{for all } y \in D = \mathbb{R}_{++} \text{ and all } m \in \mathbb{H}, \tag{3.4}$$

where the equivalence scale factors $b(1,m)$ are independent of income, non-decreasing in m, and $b(1,1) = 1$. Ebert (1995) has shown that such equivalent income functions are consistent with the Atkinson-Kolm-Sen family of inequality indices in a heterogeneous framework. Other specifications are also plausible, such as the one for instance where an additional person in the household implies a fixed extra expense

$$E_1(y;m) = y - a(1,m), \qquad \text{for all } y \in D = \mathbb{R} \text{ and all } m \in \mathbb{H}, \tag{3.5}$$

where the amount deducted $a(1,m)$ is non-decreasing with family size, and $a(1,1) = 0$, as it is suggested by Ebert (1997a). One may conceive of more general transformations combining allowances and scale factors or even think of more complicated functional forms such as for instance:

$$E_1(y;m) = \frac{1}{\eta} \ln\left(1 + \frac{\exp(\eta y) - 1}{b(1,m)}\right), \qquad \text{for all } y \in D = \mathbb{R} \text{ and all } m \in \mathbb{H}, \tag{3.6}$$

where $\eta > 0$ and $1 = b(1,1) \leq b(1,2) \leq \cdots \leq b(1,H)$ [Ebert (2000)].

Given the heterogeneous distribution $(\mathbf{x};\mathbf{m}) \in \mathcal{Z}(D)$, the equivalent income function $E \in \mathbb{E}$ and the reference type $h \in \mathbb{H}$, we will let $E_h(\mathbf{x};\mathbf{m}) := (E_h(x_1;m_1), \ldots, E_h(x_n;m_n))$ represent the equivalent income profile. The adjustment process is not yet complete since one has to specify the way equivalent incomes have to be weighted. There is much disagreement in the literature about the appropriate way equivalent incomes have to be weighted and therefore the right procedure to derive the adjusted distributions. Most practitioners agree on weighting the equivalent income by the number of persons in the household. On the other hand, Ebert (1999) and Pyatt (1985, 1990) have provided arguments for weighting the equivalent incomes by the equivalence scale factors rather than by the number of persons constituting the household. It is important to note that, even in the case where there is a general agreement regarding the appropriate equivalence scale, the choice of the system of weights may affect the result of the comparisons. Here we will assume that the weight attributed to every household's equivalent income depends only on family size. We therefore define a weighting function as a function $w : \mathbb{H} \times \mathbb{H} \to \mathbb{R}_{++}$ such that $w_h(m) := w(h;m)$ is the weight attributed to a type-m household when the reference type is h and we denote as $\mathbf{W}$ the set of weighting functions. Let us further introduce the following subset of weighting functions:

$$\overline{\mathbf{W}} := \{w \in \mathbf{W} \mid \exists \lambda(h,\ell) > 0 : w_h(m) = \lambda(h,\ell) w_\ell(m), \forall h, \ell, m \in \mathbb{H}\}. \tag{3.7}$$

All the elements in $\overline{\mathbf{W}}$ have the property that the weights vary proportionally when the reference type changes. Given the equivalent income function $E \in \mathbb{E}$, the reference type $h \in \mathbb{H}$,

[7] Here and in the following examples, we assume that the set D is appropriately chosen so that the equivalent income $E_1(y;m)$ is well-defined.

and the weighting function $w \in \mathbf{W}$, we therefore associate to every heterogeneous distribution $(\mathbf{x};\mathbf{m}) \in \mathcal{Z}(D)$ the adjusted weighted homogeneous distribution $(E_h(\mathbf{x};\mathbf{m}) \,|\, w_h(\mathbf{m}))$, where $w_h(\mathbf{m}) := (w_h(m_1), \ldots, w_h(m_n))$. This construction makes clear that, given the reference type, the adjustment process involves two distinct – and not necessarily independent – stages: (i) the choice of the equivalent income function, and (ii) the choice of the weighting function. Ebert (1997b) reviews four different adjustment procedures depending on the choices made regarding the equivalent income function and the weighting function and provides a justification in each case.

3.3.Stage 2: Inequality and welfare quasi-orderings of heterogeneous distributions.

The current practice when making comparisons for heterogeneous distributions is to compare the distributions of equivalent incomes on the basis of a particular unidimensional quasi-ordering or measure. This approach actually amounts to aggregate various dimensions into a single one by considering a fictitious and homogeneous population and to apply the traditional techniques derived in the homogeneous case for assessing inequality and welfare. Summing up, the heterogeneous quasi-orderings one refers to for making welfare and inequality comparisons in this multidimensional setting involve four parameters: (i) an equivalent income function; (ii) a reference household type; (iii) a weighting function which depend on household composition but not on household income; and (iv) a particular quasi-ordering for homogeneous distributions. Formally, we will define a multi-dimensional quasi-ordering in the following way.

Definition 3.1: Given the equivalent income function $E \in \mathbb{E}$, the reference type $h \in \mathbb{H}$, the weighting function $w \in \mathbf{W}$ and the basic quasi-ordering $\geq_K$ $(K = GL, RL, AL)$, we will say that *distribution* $(\mathbf{x}^*;\mathbf{m}^*)$ *weakly dominates distribution* $(\mathbf{x}^\circ;\mathbf{m}^\circ)$ *for* E*,* h*,* w *and* $\geq_K$, which we write $(\mathbf{x}^*;\mathbf{m}^*) \geq [E,h,w,K]\ (\mathbf{x}^\circ;\mathbf{m}^\circ)$ if and only if

$$(3.7)\quad (E_h(x_1^*;m_1^*), \ldots, E_h(x_n^*;m_n^*)\,|w_h(\mathbf{m}^*)) \geq_K (E_h(x_1^\circ;m_1^\circ), \ldots, E_h(x_n^\circ;m_n^\circ)\,|w_h(\mathbf{m}^\circ)).$$

We denote as $>[E,h,w,K]$ and $\sim[E,h,w,K]$ the asymmetric and symmetric components of $\geq[E,h,w,K]$ obtained by substituting respectively $>_K$ and $\sim_K$ for $\geq_K$ in (3.7)[8]. This procedure consists in using the transformation $E_h(\,\cdot\,;m)$ for determining the equivalent incomes for the reference type h, and then applying the quasi-ordering $\geq_K$ to the distributions obtained by weighting the equivalent incomes by means of the weight profile $w_h(\mathbf{m})$. One could in principle use an analogous procedure to define a multidimensional inequality or welfare index starting with some unidimensional index. Definition 3.1 makes clear that the heterogeneous quasi-ordering involves two successive stages – the derivation of the adjusted distribution and the selection of a criterion for making comparisons in a homogeneous environment – which are typically taken to be independent. The quasi-ordering $\geq[E,h,w,K]$ is flexible enough to

[8]Rigorously, the quasi-orderings $\geq[E,h,w,K]$ are not correctly defined because the vectors of weights $w_h(\mathbf{m}^*)$ and $w_h(\mathbf{m}^\circ)$ in (3.7) are not equal, so that definitions (2.3), (2.5) and (2.6) do not apply. However, as it is hinted in Section 2, it is always possible to frame the analysis in terms of distributions of equivalent incomes with the same vector of weights.

encompass most of the different approaches taken in the literature. For instance, the usual strategy in most empirical studies consists in taking (i) $h = 1$ (single as the reference type), (ii) $E_h(y; m) = y/b(h; m)$ (equivalence scales), (iii) $w_h(m) = m$ (weights equal to the number of persons in the household), and then in applying any unidimensional quasi-ordering or index to the adjusted distributions for obtaining a verdict. There are however other possibilities, in particular regarding the choice of the equivalent income function and/or the weighting function, and there does not appear to be at the moment any convincing reason – either theoretically or empirically – for choosing one option rather than the other.

4.Consistent procedures for making inequality and welfare comparisons.

Up to now we have introduced a flexible method for comparing heterogeneous distributions which, assuming that a given unidimensional criterion has been agreed upon, typically depends on: the equivalent income function and the weighting function. In practice, the selection of equivalent income function and the weighting function are considered as independent choices. We will argue that, contrary to what is currently and implicitly assumed, one cannot choose independently the weight function and the equivalent income function. Even more, the adjustment process is determined by the basic quasi-ordering one takes for comparing the adjusted distributions. In order to reach these conclusions, one obviously needs impose some conditions on the heterogeneous quasi-orderings defined by (3.7). We examine successively the implications for the adjustment process of two types of consistency requirements. Our first consistency condition simply requires that the ranking of the heterogeneous distributions does not depend on the household type chosen as the reference type. Our second condition is intended to prevent the kind of situation depicted by Glewwe (1991) where an inequality reducing transfer of income between two households of different types results in a non-conclusive verdict.

4.1.The implications of the condition of reference independence.

We will restrict attention for a while to the case where the weighting function changes proportionally with the reference type. This actually amounts to consider adjustment methods $(E, w) \in \mathbb{E} \times \overline{\mathbf{W}}$ which are consistent with the current practice where the weights are fixed and equal to the number of persons in the households.

Example 4.1: Consider a population $S := \{1, 2, 3\}$ consisting of two singles ($m_1 = m_2 = 1$) and one couple ($m_3 = 2$). The weighting function is given by $w_h(m) := 1$, for all $h, m \in \mathbb{H}$, and the equivalent income function E is defined by $E_1(y; 1) = E_2(y; 2) = y$, $E_1(y; 2) := \sqrt{y+1} - 1$ and $E_2(y; 1) = E_1^{-1}(y; 2) = (y + 1)^2 - 1$, for all $y \in D = \mathbb{R}_{++}$, which verifies conditions (3.2.a) to (3.2.e). Choose distributions $(\mathbf{x}; \mathbf{m}) := (2, 2, 23.01; 1, 1, 2)$ and $(\mathbf{y}; \mathbf{m}) := (1, 3, 24.00; 1, 1, 2)$. Taking singles as the reference type and letting $y_i^* := E_1(y_i; m_i)$ and $x_i^* := E_1(x_i; m_i)$, the adjusted distributions are given by $(\mathbf{y}^* \mid \mathbf{w}) = (1, 3, 4.0 \mid 1, 1, 1)$ and $(\mathbf{x}^* \mid \mathbf{w}) = (2, 2, 3.9 \mid 1, 1, 1)$ respectively, and one checks that $(\mathbf{x}^* \mid \mathbf{w}) >_{RL} (\mathbf{y}^* \mid \mathbf{w})$. If we choose now the couple as the reference type, then we obtain $(\mathbf{x}^* \mid \mathbf{w}) = (8, 8, 23.01 \mid 1, 1, 1)$ and $(\mathbf{y}^* \mid \mathbf{w}) = (3, 15, 24.00 \mid 1, 1, 1)$, and the relative Lorenz curves of $(\mathbf{x}^* \mid \mathbf{w})$ and $(\mathbf{y}^* \mid \mathbf{w})$ intersect.

The example above shows that, by choosing suitably the reference type, one can turn domination into non-comparability by applying the relative Lorenz criterion to the distributions of equivalent incomes. Therefore the choice of the reference household type plays a crucial role in the appraisal of relative inequality in a heterogeneous context. One can easily provide similar examples in the case of absolute inequality or welfare comparisons by adapting the above example. The following condition precisely aims at avoiding the situation illustrated in Example 4.1 by requiring that other things equal the ranking of heterogeneous distributions does not depend on the particular chosen reference type when adjusting incomes.

Reference Independence (*RI*). *Let the equivalent income function* $E \in \mathbb{E}$*, the reference type* $h \in \mathbb{H}$*, the weighting function* $w \in \overline{\mathbf{W}}$ *and the quasi-ordering* $\geq_K$ $(K = GL, RL, AL)$ *be given and consider any* $(\mathbf{x};\mathbf{m}),(\mathbf{y};\mathbf{m}) \in \mathcal{Z}(D)$ *such that* $E_h(\mathbf{x};\mathbf{m})$ *and* $E_h(\mathbf{y};\mathbf{m})$ *are non-decreasingly arranged. Then:*

$$\text{(4.1)} \qquad \textit{For all } h \neq \ell \in \mathbb{H} : (\mathbf{x};\mathbf{m}) \geq [E,h,w,K]\,(\mathbf{y};\mathbf{m}) \Rightarrow (\mathbf{x};\mathbf{m}) \geq [E,\ell,w,K]\,(\mathbf{y};\mathbf{m}).$$

Actually, *RI* is a condition imposed on the sequence $\{\geq [E,h,w,K], h = 1,2,\ldots,H\}$ of heterogeneous quasi-orderings. On the other hand, it is a rather weak condition since we impose independence in very specific cases: the distributions of equivalent incomes are similarly arranged and the distributions of household characteristics are identical in both situations. The following result indicates the implications of the condition of reference independence for the equivalent income function.

Proposition 4.1. *Let the equivalent income function* $E \in \mathbb{E}$*, the reference type* $h \in \mathbb{H}$*, the weighting function* $w \in \overline{\mathbf{W}}$ *and the quasi-ordering* $\geq_K$ $(K = GL, RL, AL)$ *be given. Then,* $\geq [E,h,w,K]$ *verifies condition RI if and only if, for all* $y \in D$ *and all* $h, m \in \mathbb{H}$*:*

(4.2.a) $E_h(y;m) = \beta(h,m)y$*, when* $K = RL, GL$ $(D = \mathbb{R}_{++})$*;*

(4.2.b) $E_h(y;m) = \alpha(h,m) + y$*, when* $K = AL, GL$ $(D = \mathbb{R})$*,*

for some $\alpha(h,m)$ *and* $\beta(h,m)$ *non-increasing in* m *and such that* $\alpha(h,\ell) = \alpha(h,k) + \alpha(k,\ell)$ *and* $\beta(h,\ell) = \beta(h,k)\beta(k,\ell) > 0$*, for all* $h,k,\ell \in \mathbb{H}$*, and* $\alpha(h,h) = 0$ *and* $\beta(h,h) = 1$*, for all* $h \in \mathbb{H}$*.*

Proof: Because sufficiency is obvious, we only have to prove that condition *RI* implies (4.2.a) and (4.2.b). To simplify notation, let us define $\mathbf{w}_h := (w_h(m_1),\ldots,w_h(m_n))$ and $\mathbf{w}_\ell := (w_\ell(m_1),\ldots,w_\ell(m_n))$. Letting $\mathbf{h} := (h,\ldots,h) \in \mathbb{R}^n$ and using (3.2.d), (3.7) and (4.1), we show that, if we have

$$\text{(4.3)} \quad (E_h(\mathbf{x};\mathbf{m})|\mathbf{w}_h) \geq_K (E_h(\mathbf{y};\mathbf{m})|\mathbf{w}_h) \Rightarrow (E_\ell(E_h(\mathbf{x};\mathbf{m});\mathbf{h})\,|\mathbf{w}_\ell) \geq_K (E_\ell(E_h(\mathbf{y};\mathbf{m});\mathbf{h})\,|\mathbf{w}_\ell)\,,$$

for all $h,\ell \in \mathbb{H}$, then $E_h(\,\cdot\,;m)$ must verify (4.2.a) and (4.2.b), where $K = GL, RL, AL$. Because $w_\ell(m) = \lambda(h,\ell)w_h(m)$, for all $h,\ell,m \in \mathbb{H}$, and given definitions (2.3), (2.5) and (2.6), condition (4.3) reduces to

$$\text{(4.4)} \qquad (\mathbf{s}^* \mid \mathbf{v}) \geq_K (\mathbf{s}^\circ \mid \mathbf{v}) \Rightarrow (E_\ell(\mathbf{s}^*;\mathbf{h}) \mid \mathbf{v}) \geq_K (E_\ell(\mathbf{s}^\circ;\mathbf{h}) \mid \mathbf{v})\,,$$

where $\mathbf{v} = (w_h(m_i), \ldots, w_h(m_n))$, $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$, $\mathbf{s}^\circ = (s_1^\circ, \ldots, s_n^\circ)$, and $s_i^* = E_h(x_i; m_i)$ and $s_i^\circ = E_h(y_i; m_i)$, for all $i = 1, 2, \ldots, n$.

Case 1: $K = GL$. Appealing to Lemma 2.1, we know that (4.3) will hold if and only if $E_\ell(\,\cdot\,; h)$ is concave. Interchanging the indices h and ℓ in (4.3) and using again Lemma 2.1, we deduce that $E_h(\,\cdot\,;\ell) = E_\ell^{-1}(\,\cdot\,; h)$ must be concave. Therefore, we conclude that $E_\ell(\,\cdot\,; h)$ is affine i.e.,

$$E_\ell(y; h) = \alpha(\ell, h) + \beta(\ell, h)y,\ (\beta(\ell, h) > 0),\ \text{for all } y \in D \text{ and all } h, \ell \in \mathbb{H}. \tag{4.5}$$

Appealing to condition (3.2.c), we know that $E_\ell(y; h) \leq E_\ell(y; h+1)$, for all $\ell \in \mathbb{H}$ and all $h \in \mathbb{H}$ $(h \neq H)$. This is secured by setting $\beta(\ell, h) = 1$ when $D = \mathbb{R}$ and $\alpha(\ell, h) = 0$ when $D = \mathbb{R}_{++}$.

Case 2: $K = RL$ $(D = \mathbb{R}_{++})$. We deduce from Lemma 2.2 that (4.3) will hold if and only if $E_\ell(\,\cdot\,; h)$ is proportional so that its inverse $E_\ell^{-1}(\,\cdot\,; h)$ is also proportional. Thus

$$E_\ell(y; h) = \beta(\ell, h)y,\ (\beta(\ell, h) > 0),\ \text{for all } y \in D \text{ and all } h, \ell \in \mathbb{H}, \tag{4.6}$$

and, given the domain restriction, condition (3.2.c) is automatically satisfied.

Case 3: $K = AL$ $(D = \mathbb{R})$. Using a similar reasoning, we deduce from Lemma 2.3 that $E_\ell(\,\cdot\,; h)$ is defined by (4.5). Invoking condition (3.2.c), we finally obtain $\beta(\ell, h) = 1$ so that $E_\ell(y; h) = \alpha(\ell, h) + y$, for all $y \in D$ and all $h, \ell \in \mathbb{H}$.

Finally, appealing to condition (3.2.d), we obtain $\alpha(h, h) = 0$, $\alpha(h, \ell) = \alpha(h, k) + \alpha(k, \ell)$, $\beta(h, h) = 1$, and $\beta(h, \ell) = \beta(h, k)\beta(k, \ell) > 0$, for all $h, k, \ell \in \mathbb{H}$. □

It is surprising how strong the implications for the adjustment process of such an apparently weak condition are. Proposition 4.1 also makes clear that the transformation performed in order to accommodate differences in needs depends on the basic quasi-ordering used in the second stage when making comparisons of homogeneous distributions. Therefore, contrary to what practice suggests, the value judgements made at each stage are not independent as the way incomes have to be adjusted in order to take needs into account is determined by the basic quasi-ordering one will employ when comparing the distributions of equivalent incomes. As far as generalized Lorenz and relative Lorenz dominance are concerned, and under suitable domain restrictions, adjusting incomes by means of scale factors is the appropriate technique when making comparisons of income distributions across heterogeneous populations. However, any verdict about inequality as measured by the absolute Lorenz criterion may be highly misleading as the conclusion depends on the chosen reference type when equivalence scales are employed. It is a simple matter to verify that, if the heterogeneous quasi-ordering satisfies condition *RI*, then changing the reference type does not modify the ranking of any arbitrary distributions, as is pointed out below.

Corollary 4.1. *Let the equivalent income function $E \in \mathbb{E}$, the reference type $h \in \mathbb{H}$, the weighting function $w \in \overline{\mathbf{W}}$ and the quasi-ordering $\geq_K$ $(K = GL, RL, AL)$ be given. If $\geq[E, h, w, K]$ verifies condition RI, then, for all $(\mathbf{x}^*; \mathbf{m}^*), (\mathbf{x}^\circ; \mathbf{m}^\circ) \in \mathcal{Z}(D)$ and all $\ell \neq h$, we have:*

$$(\mathbf{x}^*; \mathbf{m}^*) \geq[E, h, w, K] (\mathbf{x}^\circ; \mathbf{m}^\circ) \text{ if and only if } (\mathbf{x}^*; \mathbf{m}^*) \geq[E, \ell, w, K] (\mathbf{x}^\circ; \mathbf{m}^\circ). \tag{4.7}$$

4.2.The implications of the between-type principle of transfers.

On the other hand, Proposition 4.1 tells nothing about the weights to be used and the practitioner will be reassured by learning that she is allowed to choose any system of weights, provided that these weights depend only on family size. The practitioner may thus find some comfort knowing that weighting the equivalent household income by the number of persons is consistent with condition *RI* as long as she is concerned with welfare and/or inequality assessment. However there is still another problem left which is best illustrated by the counter-intuitive example provided by Glewwe (1991).

Before considering a variant of this example, it is convenient to introduce a generalization of the notion of a progressive transfer in a heterogeneous environment. Given the equivalent income function $E \in \mathbb{E}$, the reference type $h \in \mathbb{H}$, and two heterogeneous distributions $(\mathbf{x};\mathbf{m}),(\mathbf{y};\mathbf{m}) \in \mathcal{Z}(D)$, we will say that distribution $(\mathbf{x};\mathbf{m})$ is obtained from distribution $(\mathbf{y};\mathbf{m})$ by a *[rank-preserving] E_h-progressive transfer* if there exists $\Delta > 0$ and two households $i, j \in S$ $(m_i \neq m_j)$ such that

(4.8.a) $$x_k = y_k, \text{ for all } k \neq i, j;$$

(4.8.b) $$x_i = y_i + \Delta \text{ and } x_j = y_j - \Delta;$$

(4.8.c) $$E_h(y_i; m_i) < E_h(x_i; m_i) \leq E_h(x_j; m_j) < E_h(y_j; m_j);$$

and the positions of all households on the equivalent income scale are not affected i.e.,

(4.9) $$E_h(y_1; m_1) \leq \cdots \leq E_h(y_n; m_n) \text{ and } E_h(x_1; m_1) \leq \cdots \leq E_h(x_n; m_n),$$

assuming that households are labelled in such a way that equivalent incomes are non-decreasing with $i = 1, 2, \ldots, n$.

Intuitively, one expects that a E_h-progressive transfer has an inequality reducing – or a welfare improving – impact on the distribution of living standards.

Example 4.2: We consider a population $S := \{1, 2, 3, 4\}$ consisting of three singles ($m_1 = m_2 = m_4 = 1$) and one couple ($m_3 = 2$). The system of weights $\mathbf{w} := (w_1, w_2, w_3, w_4)$ is given by $w_i = 1$, for all $i = 1, 2, 4$, and $w_3 = \vartheta$. The equivalent income function E is defined by $E_1(y;1) = E_2(y;2) = y$, $E_1(y;2) := y/b$, $E_2(y;1) := by$, for all $y \in D$. Now choose distributions $(\mathbf{y};\mathbf{m}) := (1, 2, 6, 4; 1, 1, 2, 1)$ and $(\mathbf{x};\mathbf{m}) := (1, 2+\Delta, 6-\Delta, 4; 1, 1, 2, 1)$, where $2 \leq 6/b \leq 4$ and $2 + \Delta \leq (6-\Delta)/b \leq 4$. The two preceding inequalities are verified if $b = 2$ and $0 < \Delta \leq 2/3$. Taking singles as the reference type and letting $y_i^* := E_1(y_i; m_i)$ and $x_i^* := E_1(x_i; m_i)$, the adjusted distributions are given by $(\mathbf{y}^* \mid \mathbf{w}) = (1, 2, 6/b, 4 \mid 1, 1, \vartheta, 1)$ and $(\mathbf{x}^* \mid \mathbf{w}) = (1, 2+\Delta, (6-\Delta)/b, 4 \mid 1, 1, \vartheta, 1)$. Therefore distribution $(\mathbf{x};\mathbf{m})$ is obtained from distribution $(\mathbf{y};\mathbf{m})$ by means of a E_1-progressive transfer. Define $\Delta(k;\vartheta) := RL(k; (\mathbf{x}^* \mid \mathbf{w})) - RL(k; (\mathbf{y}^* \mid \mathbf{w}))$, for $k = 1, 2, 3, 4$, so that $(\mathbf{x}^* \mid \mathbf{w}) \geq_{RL} (\mathbf{y}^* \mid \mathbf{w})$ if and only if $\Delta(k;\vartheta) \geq 0$, for all $k = 1, 2, 3$. Then it is a matter of simple computations to show that:

k	$\vartheta < b$	$\vartheta = b$	$\vartheta > b$
1	$\Delta(1;\vartheta) < 0$	$\Delta(1;\vartheta) = 0$	$\Delta(1;\vartheta) > 0$
2	$\Delta(2;\vartheta) > 0$	$\Delta(2;\vartheta) > 0$	$\Delta(2;\vartheta) > 0$
3	$\Delta(3;\vartheta) > 0$	$\Delta(3;\vartheta) = 0$	$\Delta(3;\vartheta) < 0$

For any value of the weight attributed to the couple greater than the scale factor, the relative Lorenz curve of $(\mathbf{x}^* \mid \mathbf{w})$ crosses the relative Lorenz curve of $(\mathbf{y}^* \mid \mathbf{w})$ once from above, though the opposite situation occurs for any value of the weight smaller than the scale factor. This implies in particular that it is always possible to find two relative inequality indices I° and I^* such that $I^\circ(\mathbf{x}^* \mid \mathbf{w}) < I^\circ(\mathbf{y}^* \mid \mathbf{w})$ and $I^*(\mathbf{x}^* \mid \mathbf{w}) > I^*(\mathbf{y}^* \mid \mathbf{w})$[9]. Therefore, assuming that there exists an agreement regarding the equivalent income function, the picture changes completely as one chooses different systems of weights.

Similar examples can be constructed to show that a E_h-progressive transfer may result in intersecting absolute Lorenz curves and/or generalized Lorenz curves. In the above example, it is interesting to note that the E_1-progressive transfer has the expected impact on the ranking of distributions in the special case where the weight given to the couple is set equal to the scale factor employed for computing her equivalent income. Actually this observation is robust to manipulations of the values of the different parameters in the example. Although our example is admittedly a particular case, it might suggest that the weighting function and the equivalent income function cannot be chosen independently if one wants that a E_h-progressive transfer reduces inequality in a heterogeneous environment. In order to avoid the kind of situation depicted in the example above, one may wish to impose the following condition on any of our heterogeneous quasi-orderings.

Between-Type Principle of Transfers (*BTPT*). *Let the equivalent income function $E \in \mathbb{E}$, the reference type $h \in \mathbb{H}$, the weighting function $w \in \mathbf{W}$ and the quasi-ordering $\geq_K$ $(K = GL, RL, AL)$ be given and consider any $(\mathbf{x};\mathbf{m}), (\mathbf{y};\mathbf{m}) \in \mathcal{Z}(D)$. Then, $(\mathbf{x};\mathbf{m}) \geq [E,h,w,K]\,(\mathbf{y};\mathbf{m})$, whenever $(\mathbf{x};\mathbf{m})$ is obtained from $(\mathbf{y};\mathbf{m})$ by a rank-preserving E_h-progressive transfer.*

Condition *BTPT* is reminiscent of the *Equity Axiom* introduced by Hammond (1976) in his characterization of the Maximin rule. Before investigating the implications of *BTPT* for the comparison of heterogeneous distributions, we find convenient to introduce the following technical result the proof of which is relegated to the Appendix.

Lemma 4.1. *Let the equivalent income function $E \in \mathbb{E}$, the types $h, k, \ell \in \mathbb{H}$, and the weighting function $w \in \mathbf{W}$ be given. Then, the solution to*

$$(4.10) \qquad w_h(k)E_h(u+\Delta;k) + w_h(\ell)E_h(v-\Delta;\ell) = w_h(k)E_h(u;k) + w_h(\ell)E_h(v;\ell),$$

[9]Glewwe (1991) provides an example with the Theil index of inequality to show that an E_h-regressive transfer may imply an increase in inequality in terms of equivalent incomes.

for all $u, v \in D$ and all $\Delta > 0$ such that $v - \Delta \in D$, is given by

$$(4.11) \qquad w_h(m)E_h(y;m) = \chi(h,m) + \lambda(h)y, \text{ for all } y \in D \text{ and all } m \in \mathbb{H}.$$

Here again the implications of the apparently harmless condition $BTPT$ imposes quite a lot of structure on the adjustment process. Indeed, as the following result demonstrates, the income equivalence function and the weighting function can no longer be chosen independently from each other. Precisely, we have:

PROPOSITION 4.2. *Let the equivalent income function $E \in \mathbb{E}$, the reference type $h \in \mathbb{H}$, the weighting function $w \in \mathbf{W}$ and the quasi-ordering $\geq_K$ ($K = GL, RL, AL$) be given. Then, $\geq [E, h, w, \geq_K]$ verifies condition $BTPT$ if and only if, for all $y \in D$ and all $h \in \mathbb{H}$:*

(4.12.a) *$E_h(y;m) = \beta(h,m)y$ and $w_h(m) = \lambda(h)/\beta(h,m)$, for all $m \in \mathbb{H}$, when $K = RL, GL$ ($D = \mathbb{R}_{++}$);*

(4.12.b) *$E_h(y;m) = \alpha(h,m) + y$ and $w_h(m) = \lambda(h)$, for all $m \in \mathbb{H}$, when $K = AL, GL$ ($D = \mathbb{R}$),*

for some $\lambda(h) > 0$, $\alpha(h,m) \in \mathbb{R}$ and $\beta(h,m) > 0$ such that $E_h(y;m)$ verifies conditions (3.2.a) to (3.2.e).

PROOF: Because sufficiency is obvious, we only have to prove that condition $BTPT$ implies (4.12.a) and (4.12.b) which is done in a number of steps.

STEP 1: Given the equivalent income function $E \in \mathbb{E}$ and the reference type $h \in \mathbb{H}$, we indicate the way of constructing two distributions $(\mathbf{x}; \mathbf{m})$ and $(\mathbf{y}; \mathbf{m})$ such that the former results from the later by means of a rank-preserving E_h-progressive transfer. To this aim, we let $m \in \mathbb{H}$ be an arbitrary type and we consider the two following cases.

Case 1: $m < h$. Then, we have $E_h(y;h) = y \leq E_h(y;m)$, for all $y \in D$, by condition (3.2.c). We claim that, given any $v_m \in D$, it is always possible to find $u_m, t_m \in D$ ($u_m < v_m < t_m$) and $\Delta > 0$ such that:

$$(4.13.a) \qquad E_h(v_m;m) < E_h(v_m + \Delta;m) \leq E_h(t_m - \Delta;h) < E_h(t_m;h); \text{ and}$$

$$(4.13.b) \qquad E_h(u_m - \Delta;h) < E_h(u_m;h) \leq E_h(v_m;m) < E_h(v_m + \Delta;m).$$

Condition (4.13.a) follows from the fact that the income domain D is unbounded from above. Since the income domain D is open, given any $v \in D$, it is always possible to find $0 < u < v$ such that $E_h(u;h) \leq E_h(v;m)$, from which condition (4.13.b) follows. We then define distributions $(\mathbf{x}^1; \mathbf{m}^1)$, $(\mathbf{y}^1; \mathbf{m}^1)$, $(\mathbf{x}^2; \mathbf{m}^2)$ and $(\mathbf{y}^2; \mathbf{m}^2)$ as shown below:

i	1	$\cdots$	$n-3$	$n-2$	$n-1$	n
m_i^1	m	$\cdots$	m	m	h	h
y_i^1	v_m	$\cdots$	v_m	v_m	t_m	t_m
x_i^1	v_m	$\cdots$	v_m	$v_m+\Delta$	$t_m-\Delta$	$t_m-\Delta$
m_i^2	h	$\cdots$	h	h	m	m
y_i^2	$u_m-\Delta$	$\cdots$	$u_m-\Delta$	$u_m-\Delta$	$v_m+\Delta$	$v_m+\Delta$
x_i^2	$u_m-\Delta$	$\cdots$	$u_m-\Delta$	u_m	v_m	v_m

Case 2: $h \leq m$. Then, we have $E_h(y;h) = y \geq E_h(y;m)$, for all $y \in D$, by condition 3.2.c. We claim that, given any $v_m \in D$, it is always possible to find $u_m, t_m \in D$ $(u_m < v_m < t_m)$ and $\Delta > 0$ such that:

(4.14.a) $\quad E_h(v_m - \Delta; m) < E_h(v_m; m) \leq E_h(t_m; h) < E_h(t_m + \Delta; h)\,;$ and

(4.14.b) $\quad E_h(u_m; h) < E_h(u_m + \Delta; h) \leq E_h(v_m - \Delta; m) < E_h(v_m; m)\,.$

Condition (4.14.a) follows from the openess of the income domain D. The fact that $D = \mathbb{R}$ is open gurantees that it is always possible to find incomes such that (4.14.b) holds. When $D = \mathbb{R}_{++}$, we claim that, for all $v \in D$, there exists u $(0 < u < v)$ such that $E_h(u;h) < E_h(v;m)$. Indeed, suppose that, there exists $v \in D$ such that $E_h(u;h) \geq E_h(v;m)$, for all u $(0 < u < v)$. But this implies that $\lim_{y\to 0} E_h(y;h) \geq E_h(v;m) > \lim_{y\to 0} E_h(y;m)$, which is impossible since $\lim_{y\to 0}\left[E_h(y;h) - E_h(y;m)\right] = 0$, for all $E \in \mathbb{E}$ and all h, m. We introduce the distributions $(\mathbf{x}^3;\mathbf{m}^3)$, $(\mathbf{y}^3;\mathbf{m}^3)$, $(\mathbf{x}^4;\mathbf{m}^4)$ and $(\mathbf{y}^4;\mathbf{m}^4)$ defined below:

i	1	$\cdots$	$n-3$	$n-2$	$n-1$	n
m_i^3	h	$\cdots$	h	h	m	m
y_i^3	u_m	$\cdots$	u_m	u_m	v_m	v_m
x_i^3	u_m	$\cdots$	u_m	$u_m+\Delta$	$v_m-\Delta$	$v_m-\Delta$
m_i^4	m	$\cdots$	m	m	h	h
y_i^4	$v_m-\Delta$	$\cdots$	$v_m-\Delta$	$v_m-\Delta$	$t_m+\Delta$	$t_m+\Delta$
x_i^4	$v_m-\Delta$	$\cdots$	$v_m-\Delta$	v_m	t_m	t_m

We note that by construction, distribution $(\mathbf{x}^s;\mathbf{m}^s)$ is obtained from distribution $(\mathbf{y}^s;\mathbf{m}^s)$ by means of a rank-preserving E_h-progressive transfer, for $s = 1,2,3,4$, and that this holds for all $v_m \in D$ and all $\Delta > 0$ sufficiently small.

Step 2: We show that, if $\geq[E,h,w,K]$ satisfies condition $BTPT$, then, given any $k, \ell \in \mathbb{H}$, it must be the case that

(4.15) $\quad w_h(k)E_h(v_k;k) + w_h(\ell)E_h(v_\ell;\ell) = w_h(k)E_h(v_k - \Delta;k) + w_h(\ell)E_h(v_\ell - \Delta;\ell)\,,$

for all $v_k, v_\ell \in D$ and all $\Delta > 0$ sufficiently small. We consider successively three cases and examine in each case the implication of condition $BTPT$ when $K = GL, RL, AL$. To simplify notation, given any $(\mathbf{x};\mathbf{m})$ and $(\mathbf{y};\mathbf{m})$, we find convenient to let $\mathbf{x}^* := E_h(\mathbf{x};\mathbf{m})$, $\mathbf{y}^* := E_h(\mathbf{y};\mathbf{m})$ and $\mathbf{w}^* := (w_1^*, \ldots, w_n^*)$ where $w_i^* = w_h(m_i)$, for all $i = 1, 2, \ldots, n$.

Case 1: $h \leq k < \ell$. Let $m \in \{k, \ell\}$ and choose first $(\mathbf{x};\mathbf{m}) = (\mathbf{x}^3;\mathbf{m}^3)$ and $(\mathbf{y};\mathbf{m}) = (\mathbf{y}^3;\mathbf{m}^3)$. A necessary condition for $(\mathbf{x}^* \mid \mathbf{w}) \geq_{GL} (\mathbf{y}^* \mid \mathbf{w})$ is that

$$\sum_{j=1}^{q}\left(\frac{w_j^*}{\sum_{i=1}^{n} w_i^*}\right) x_j^* \geq \sum_{j=1}^{q}\left(\frac{w_j^*}{\sum_{i=1}^{n} w_i^*}\right) y_j^*, \tag{4.16}$$

for all $q = 1, 2, \ldots, n-2$, which implies that

$$w_h(h)E_h(u_m + \Delta; h) + w_h(m)E_h(v_m - \Delta; m) \geq w_h(h)E_h(u_m; h) + w_h(m)E_h(v_m; m)\,. \tag{4.17}$$

Now for $(\mathbf{x}^* \mid \mathbf{w}^*) \geq_{RL} (\mathbf{y}^* \mid \mathbf{w}^*)$ it is necessary that

$$\sum_{j=1}^{q}\left(\frac{w_j^*}{\sum_{i=1}^{n} w_i^*}\right) \frac{x_j^*}{\mu(\mathbf{x}^*;\mathbf{w}^*)} \geq \sum_{j=1}^{q}\left(\frac{w_j^*}{\sum_{i=1}^{n} w_i^*}\right) \frac{y_j^*}{\mu(\mathbf{y}^*;\mathbf{w}^*)}, \tag{4.18}$$

for all $q = 1, 2, \ldots, n-2$, which actually reduces to (4.17). Similarly a necessary condition for $(\mathbf{x}^* \mid \mathbf{w}^*) \geq_{AL} (\mathbf{y}^* \mid \mathbf{w}^*)$ is that

$$\sum_{j=1}^{q}\left(\frac{w_j^*}{\sum_{i=1}^{n} w_i^*}\right) \left[x_j^* - \mu(\mathbf{x}^*;\mathbf{w}^*)\right] \geq \sum_{j=1}^{q}\left(\frac{w_j^*}{\sum_{i=1}^{n} w_i^*}\right) \left[y_j^* - \mu(\mathbf{y}^*;\mathbf{w}^*)\right], \tag{4.19}$$

for all $q = 1, 2, \ldots, n-2$, which is again equivalent to (4.17). Choosing next $(\mathbf{x};\mathbf{m}) = (\mathbf{x}^4;\mathbf{m}^4)$ and $(\mathbf{y};\mathbf{m}) = (\mathbf{y}^4;\mathbf{m}^4)$, we establish along a similar reasoning that

$$w_h(m)E_h(t_m - \Delta; m) + w_h(h)E_h(v_m + \Delta; h) \leq w_h(m)E_h(t_m; m) + w_h(h)E_h(v_m; h)\,. \tag{4.20}$$

Using condition (3.2.a), and combining inequalities (4.17) and (4.20), we obtain

$$w_h(h)\Delta = w_h(m)\left[E_h(v_m; m) - E_h(v_m - \Delta; m)\right], \tag{4.21}$$

for $m \in \{k, \ell\}$, and we conclude that (4.15) holds, for all $v_k, v_\ell \in D$ and all $\Delta > 0$ sufficiently small.

Case 2: $k < \ell \leq h$. One proves along a similar argument that (4.15) holds, for all $v_k, v_\ell \in D$ and all $\Delta > 0$ sufficiently small, choosing successively $(\mathbf{x};\mathbf{m}) = (\mathbf{x}^s;\mathbf{m}^s)$ and $(\mathbf{y};\mathbf{m}) = (\mathbf{y}^s;\mathbf{m}^s)$ with $s = 1, 2$, for $m \in \{k, \ell\}$.

Case 3: $k < h \leq \ell$. Repeated application of the above argument gives (4.15) choosing (i) $(\mathbf{x};\mathbf{m}) = (\mathbf{x}^s;\mathbf{m}^s)$ and $(\mathbf{y};\mathbf{m}) = (\mathbf{y}^s;\mathbf{m}^s)$ with $s = 1, 2$, when $m = k$, and (ii) $(\mathbf{x};\mathbf{m}) = (\mathbf{x}^s;\mathbf{m}^s)$ and $(\mathbf{y};\mathbf{m}) = (\mathbf{y}^s;\mathbf{m}^s)$ with $s = 3, 4$, when $m := \ell$.

STEP 3: We have shown that, if $\geq [E, h, w, K]$ satisfies condition $BTPT$, then condition (4.15) must hold. Invoking Lemma 4.1, we conclude that

$$w_h(m)E_h(y;m) = \chi(h,m) + \lambda(h)y, \text{ for all } y \in D \text{ and all } h, m \in \mathbb{H}. \tag{4.22}$$

Letting $\alpha(h,m) := \chi(h,m)/w_h(m)$ and $\beta(h,m) := \lambda(h)/w_h(m)$, (4.22) can be equivalently rewritten as

$$E_h(y;m) = \alpha(h,m) + \beta(h,m)y, \text{ for all } y \in D \text{ and all } h, m \in \mathbb{H}. \tag{4.23}$$

Case 1: $D = \mathbb{R}$ ($K = GL, AL$). Then, condition (3.2.c) implies that $\beta(h,m) = 1$, for all $h, m \in \mathbb{H}$ ($h \neq m$). Indeed, if for instance $\beta(h,m) > 1$ for some $h \neq m$, then it is possible to find $y^\circ, y^* \in D$ such that

$$E_h(y^\circ;m) = \alpha(h,m) + \beta(h,m)y^\circ < y^\circ = E_h(y^\circ;h), \text{ and} \tag{4.24.a}$$

$$E_h(y^*;m) = \alpha(h,m) + \beta(h,m)y^* > y^* = E_h(y^*;h), \tag{4.24.b}$$

which contradicts condition (3.2.c). Therefore, we conclude that, for all $h \in \mathbb{H}$:

$$E_h(y;m) = \alpha(h,m) + y, \text{ for all } y \in D \text{ and all } m \in \mathbb{H}, \text{ and} \tag{4.25.a}$$

$$w_h(m) = \lambda(h), \text{ for all } h, m \in \mathbb{H}. \tag{4.25.b}$$

Case 2: $D = \mathbb{R}_{++}$ ($K = GL, RL$). Then, condition (3.2.c) implies that $\alpha(h,m) = 0$, for all $h, m \in \mathbb{H}$ ($h \neq m$). Consider first the case where $m < h$ and suppose for the contrary that $\alpha(h,m) > 0$. Then

$$\lim_{y \to 0} [E_h(y;m) - E_h(y;m)] = \alpha(h,m) > 0, \tag{4.26}$$

which is impossible. Similarly, supposing that $\alpha(h,m) > 0$ when $h < m$, we obtain

$$\lim_{y \to 0} [E_h(y;m) - E_h(y;m) - E_h(y;m)] = -\alpha(h,m) > 0, \tag{4.27}$$

and we therefore conclude that

$$E_h(y;m) = \beta(h,m)y, \text{ for all } y \in D \text{ and all } m \in \mathbb{H}, \text{ and} \tag{4.28.a}$$

$$w_h(m) = \lambda(h)/\beta(h,m), \text{ for all } h, m \in \mathbb{H}, \tag{4.28.b}$$

for all $h \in \mathbb{H}$, which makes the proof complete. □

Proposition 4.2 makes clear that the equivalent income function and the weighting function cannot be chosen independently from each other if we want that the resulting heterogeneous quasi-orderings obey the between-type principle of transfers. Actually Proposition 4.2 tells even more than that since it fully determines the structure of the equivalent income function which appears to be identical to the structure indicated by Proposition 4.1. We also would like to stress that the weighting function we get is an element of $\overline{\mathbf{W}}$: the resulting weights vary proportionally with the reference type, a restriction that was assumed in condition *RI*. It follows that, given the definition of an equivalent income function and the domain restrictions, condition *BTPT* is much stronger than condition *RI*.

It is important to note that the definition of the income equivalence function combined with the domain restrictions play a crucial role in the derivation of the results. From this point of view, condition (3.2.c), which aims at preventing that the functions $E_h(\,\cdot\,;k)$ and $E_h(\,\cdot\,;\ell)$ intersect, has a strong impact on the results by considerably narrowing down the class of admissible equivalent income functions.

We summarize our findings by means of the following table where we have set: $b(h,m) := 1/\beta(h,m)$ and $a(h,m) := -\alpha(h,m)$. It makes clear that: (i) the adjustment process is determined by the chosen basic quasi-ordering one employs when comparing the adjusted distributions, and (ii) the equivalent income function and the weighting function cannot be chosen independently.

Unidimensional Ordering K	**Income Domain** D	**Equivalence Function** $E_h(y;m)$	**Weighting Function** $w_h(m)$
RL GL	$\mathbb{R}_{++}$ $\mathbb{R}_{++}$	$y/b(h,m)$	$w_h(m) = \lambda(h)b(h,m)$
AL GL	$\mathbb{R}$ $\mathbb{R}$	$y - a(h,m)$	$w_h(m) = \lambda(h)$

In establishing Propositions 4.1 and 4.2, we have implicitly restricted our attention to heterogeneous distributions that differ with respect to households' incomes but not to households' composition. In other words, we have focused on the comparisons of distributions for the same population of households which was legitimate because the consistency conditions we have imposed precisely involved such a restriction. As we already insisted on, imposing such constraints results in particularly weak consistency conditions as they are only required to hold in very specific cases. Needless to say, the adjustement procedures characterized in Propositions 4.1 and 4.2 apply to all heterogeneous distributions and not only to situations where the distributions of household sizes are identical.

5.Summary and concluding remarks.

5.1.Summary of results.

The current practice when making comparisons of heterogeneous distributions consists of a two stage process. In the first stage the distributions are adjusted by deflating the income of every household by the equivalence scale corresponding to her type and weighting the obtained figure by the number of persons in the household. In the second stage one applies the conventional tools designed in the homogeneous case to the resulting distributions. It is typically assumed that these two steps are independent so that one does not have to care about the criterion that will be used for comparing the adjusted distributions when deciding which equivalent income function and weighting function to choose.

Contrary to the dominant practice in the empirical literature, we have argued in this paper that the adjustment procedure is to a large extent determined by the normative criteria one appeals to when comparing the adjusted distributions. This issue is particularly crucial as there is strong evidence that the choice of the adjustment procedure – in particular the way the equivalent income function and the weighting function are combined – affects significantly the conclusions drawn. Because we want our results to be valid for the largest possible value judgements available regarding inequality and welfare, we have adopted a dominance approach in terms of Lorenz consistent quasi-orderings. This will ensure that the conclusions obtained will hold for the largest variety of inequality indices and social welfare functions, as well as most poverty measures. In order to prove our claim, we introduced two alternative consistency requirements that the procedure for making comparisons of heterogeneous distributions sketched above should satisfy.

The first condition simply requires that the result of the comparisons must not be affected by the choice of the reference type. For instance, if a distribution is ranked above another distribution when the single is taken as the reference type, then this should still be the case when the couple is substituted for the single. This condition restricts considerably the class of admissible equivalent income functions which reduces to equivalence scales when the domain of household income is an interval of the positive real line. When household income domain comprises in addition non-positive incomes – something which is permitted by the absolute Lorenz and the generalized Lorenz quasi-orderings – then the equivalent income function is of the lump-sum type. The reference independence condition imposes no restriction on the way the equivalent incomes have to be weighted. Taking the number of persons in the household or the scale factor will have no incidence on the ranking of the heterogeneous distributions when one is concerned with relative inequality.

The pictures changes totally when we impose our second consistency condition which requires that a transfer of income that reduces the inequality of living standards between two households must improve welfare and/or inequality according to our heterogeneous quasi-orderings. Then the income equivalence and the weighting function are no longer independent and are completely determined by the Lorenz criterion – or the corresponding admissible income domain – selected for comparing the adjusted distributions. The relationship between the equivalent income function and the weighting function which is needed in order to meet our second consistency condition actually amounts to require that the mean of the adjusted distribution is not modified by the transfer. It is fair to recognize that this equal-mean condition has been touched upon – although its implications have not been fully comprehended – in the stimulating article of Glewwe (1991).

5.2.Comparison with related results in the literature.

The results above have to some extent been exploited in a former paper by the authors dealing with the consistent design of tax systems for heterogeneous populations. Part of the results presented in here can be anticipated from results in Ebert and Moyes (1999). However the approach taken here is far more general as we put little structure on the distributions under comparison contrary to Ebert and Moyes (1999) where one distribution is obtained from the other one by means of a function.

Although equivalence techniques are routinely used by the practioner, theoretical investigation has raised doubts about its relevance for making normative comparisons. Most of this literature has focused on the identification issue which results in imposing severe restrictions on the household welfare function which do not appear to be validated by the econometric studies [see e.g. Blackorby and Donaldson (1993), Blundell and Lewbel (1991)]. In all this theoretical work, the structure of the equivalent income function – and more generally the adjustment method – is chosen *a priori* and then its implication for the household utility function is derived. In this paper, we have taken a completely different route putting forward arguments for choosing one or another adjustment method.

Appendix A

Proof of Remark 2.1: Consider two distributions $(\mathbf{x} \mid \mathbf{v}) := (x_1, \ldots, x_n \mid v_1, \ldots, v_n)$ and $(\mathbf{y} \mid \mathbf{u}) := (y_1, \ldots, y_n \mid u_1, \ldots, u_n)$ and define:

$$\text{(A.1.a)} \qquad \bar{u}_i = u_1 + u_2 + \cdots + u_i; \quad \bar{u}'_i = \frac{u_1 + u_2 + \cdots + u_i}{u_1 + u_2 + \cdots + u_n}; \text{ and}$$

$$\text{(A.1.b)} \qquad \bar{v}_i = v_1 + v_2 + \cdots + v_i; \quad \bar{v}'_i = \frac{v_1 + v_2 + \cdots + v_i}{v_1 + v_2 + \cdots + v_n};$$

for all $i = 1, 2, \ldots, n$, which implies that

$$\text{(A.2.a)} \qquad 0 =: \bar{u}'_0 < \bar{u}'_1 < \bar{u}'_2 < \cdots < \bar{u}'_{n-1} < \bar{u}'_n = 1; \text{ and}$$

$$\text{(A.2.b)} \qquad 0 =: \bar{v}'_0 < \bar{v}'_1 < \bar{v}'_2 < \cdots < \bar{v}'_{n-1} < \bar{v}'_n = 1.$$

Now consider the set

$$\text{(A.3)} \qquad Q := \{\vartheta'_h \in (0,1] \mid \exists i \in S \, : \, \vartheta'_h = \bar{u}'_i \text{ or } \vartheta'_h = \bar{v}'_i\},$$

and arrange the elements of Q in ascending order so that

$$\text{(A.4)} \qquad 0 =: \vartheta'_0 < \vartheta'_1 < \vartheta'_2 < \cdots < \vartheta'_{q-1} < \vartheta'_q = 1,$$

where $q = \#Q$ and $n \leq q \leq 2n$. Letting $\vartheta_h = \vartheta'_h - \vartheta'_{h-1}$, for $h = 1, 2, \ldots, q$, we obtain a normalized vector $(\vartheta_1, \ldots, \vartheta_q)$ with $\vartheta_h > 0$, for all $h = 1, 2, \ldots, q$, and $\sum_{h=1}^{q} \vartheta_h = 1$. Define next

$$\text{(A.5)} \qquad k_i = max\{h \mid \vartheta'_h \leq \bar{v}'_i\} \text{ and } \ell_i = max\{h \mid \vartheta'_h \leq \bar{u}'_i\}, \text{ for } i = 1, 2, \ldots, n,$$

and consider the income profiles $\mathbf{x}^*$ and $\mathbf{y}^*$ given by

$$\text{(A.6.a)} \qquad x^*_h = x_i, \text{ for } k_{i-1} < h \leq k_i, \text{ and}$$

$$\text{(A.6.b)} \qquad y^*_h = y_i, \text{ for } \ell_{i-1} < h \leq \ell_i,$$

for all $i = 1, 2, \ldots, n$. Consider the weight profile $\mathbf{w} = (w_1, \ldots, w_q)$ such that $w_i = \lambda\vartheta_i$ for some $\lambda > 0$ and all $i = 1, 2, \ldots, n$. The distributions $(\mathbf{x}^* \,|\, \mathbf{w})$ and $(\mathbf{y}^* \,|\, \mathbf{w})$ are elements of $\mathcal{Y}(D)$. Furthermore, it is a simple matter to check that $(\mathbf{x}^* \,|\, \mathbf{w})$ and $(\mathbf{x} \,|\, \mathbf{v})$ (respectively $(\mathbf{y}^* \,|\, \mathbf{w})$ and $(\mathbf{y} \,|\, \mathbf{u})$) have the same quantile function. It is worth mentioning that the preceding argument can be generalized to an arbitrary number of distributions and applies also to distributions $(\mathbf{x} \,|\, \mathbf{v}) := (x_1, \ldots, x_n \,|\, v_1, \ldots, v_n)$ and $(\mathbf{y} \,|\, \mathbf{u}) := (y_1, \ldots, y_r \,|\, u_1, \ldots, u_r)$ such that $n \neq r$. □

Appendix B

Proof of Lemma 4.1: Let us rewrite (4.10) as follows

$$w_h(m)E_h(u+\Delta; m) - w_h(q)E_h(v; q) = w_h(m)E_h(u; m) - w_h(q)E_h(v-\Delta; q). \tag{B.1}$$

Fix $v = \tilde{v}$ and consider the following functions:

$$\tilde{f}(y) := w_h(m)E_h(y; m) - w_h(q)E_h(\tilde{v}; q), \tag{B.2.a}$$

$$\tilde{g}(y) := w_h(m)E_h(y; m), \text{ and} \tag{B.2.b}$$

$$\tilde{\phi}(y) := -w_h(q)E_h(\tilde{v} - y; q). \tag{B.2.c}$$

Then we can rewrite (B.1) equivalently as

$$\tilde{f}(u+\Delta) = \tilde{g}(u) + \tilde{\phi}(\Delta), \tag{B.3}$$

for all $u \in D$ and all $\Delta > 0$ such that $u + \Delta \in D$. This is a Pexider equation the solution of which [Aczel (1966, Theorem 1, p. 142)] is

$$\tilde{f}(y) = \tilde{\alpha} + \tilde{\gamma} + \tilde{\beta}y; \quad \tilde{g}(y) = \tilde{\alpha} + \tilde{\beta}y; \quad \tilde{\phi}(y) = \tilde{\gamma} + \tilde{\beta}y, \tag{B.4}$$

where $\tilde{\alpha} = \alpha(\tilde{v}; (h, m))$, $\tilde{\gamma} = \gamma(\tilde{v}; (h, m))$ and $\tilde{\beta} = \beta(\tilde{v}; h)$. Substituting into (B.2.a), (B.2.b) and (B.2.c), we get

$$\tilde{f}(y) = \tilde{\alpha} + \tilde{\gamma} + \tilde{\beta}y, \tag{B.5.a}$$

$$\tilde{g}(y) = \tilde{\alpha} + \tilde{\beta}y, \tag{B.5.b}$$

$$\tilde{\phi}(y) = \tilde{\gamma} + \tilde{\beta}y. \tag{B.5.c}$$

Subtracting (B.5.b) from (B.5.a), we obtain

$$\tilde{\gamma} = -w_h(q)E_h(\tilde{v}; q). \tag{B.6}$$

Allow now v to vary and let $v = \hat{v} \neq \tilde{v}$. Using a similar reasoning as above, we get

$$\hat{f}(y) = \hat{\alpha} + \hat{\gamma} + \hat{\beta}y, \tag{B.7.a}$$

$$\hat{g}(y) = \hat{\alpha} + \hat{\beta}y, \tag{B.7.b}$$

$$\hat{\phi}(y) = \hat{\gamma} + \hat{\beta}y, \tag{B.7.c}$$

where $\hat{\alpha} = \alpha(\hat{v};(h,m))$, $\hat{\gamma} = \gamma(\hat{v};(h,q))$ and $\hat{\beta} = \beta(\hat{v};h)$. Subtracting (B.7.b) from (B.5.b), we have

$$\text{(B.8)} \qquad \big[\tilde{\alpha} - \hat{\alpha}\big] + \big[\tilde{\beta} - \hat{\beta}\big]y = 0, \text{ for all } y \in D,$$

which implies that $\hat{\alpha} = \tilde{\alpha} = \alpha(h,m) =: \alpha$ and $\hat{\beta} = \tilde{\beta} = \beta(h) =: \beta$. Substituting the value of $\tilde{\gamma}$ given by (B.6) into (B.5.c), we obtain

$$\text{(B.9)} \qquad w_h(q)E_h(\tilde{v} - y; q) = w_h(q)E_h(\tilde{v}; q) - \beta y.$$

Letting $s = \tilde{v} - y$, this can be rewritten as

$$\text{(B.10)} \qquad w_h(q)E_h(s + y; q) = w_h(q)E_h(s; q) + \beta y, \text{ for all } s, y \in D \text{ such that } s + y \in D.$$

Defining

$$\text{(B.11.a)} \qquad f^*(z) := w_h(q)E_h(z; q),$$

$$\text{(B.11.b)} \qquad g^*(z) := w_h(q)E_h(z; q), \text{ and}$$

$$\text{(B.11.c)} \qquad \phi^*(z) = \beta z,$$

condition (B.10) can be rewriten as

$$\text{(B.12)} \qquad f^*(s + y) = g^*(s) + \phi^*(y), \text{ for all } s, y \in D \text{ such that } s + y \in D.$$

Appealing to Aczel (1966, Theorem 1, p. 142) again, we obtain

$$\text{(B.13.a)} \qquad f^*(z) = \chi + c + bz,$$

$$\text{(B.13.b)} \qquad g^*(z) = \chi + bz,$$

$$\text{(B.13.c)} \qquad \phi^*(z) = c + bz,$$

where $\chi = \chi(h,q)$, $c = c(h)$ and $b = \lambda(h)$. Comparing (B.13.a) and (B.13.b), we conclude that $c = 0$, which upon substituting into (B.11.c) implies that $\beta(h) = \lambda(h)$. Therefore, we finally get

$$\text{(B.14)} \qquad f^*(z) := w_h(q)E_h(z; q) = \chi(h,q) + \lambda(h)z, \text{ for all } z \in D,$$

which makes the proof complete. □

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