Convergence of Adaptive Learning in Models of Pure Exchange

By Jan Wenzelburger *

Fakultät für Wirtschaftswissenschaften
Universität Bielefeld
Postfach 100 131
D-33501 Bielefeld
Germany

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Abstract

This paper develops an adaptive learning scheme for a standard version of the overlapping generations model with pure exchange using the notion of an error function. Trajectories generated by this scheme converge globally to the monetary steady state for arbitrary consumers' savings behavior. The resulting learning dynamics is therefore globally asymptotically stable. This shows that with the efficient use of structural knowledge on the market mechanism, learning schemes which generate complex dynamics with non-vanishing forecast errors such as ordinary least squares can be avoided. This finding holds for all possible parameterizations guaranteeing the existence of a monetary steady state and generalizes to all one-dimensional models of the Cobweb type.

Key words: Learning Dynamics, rational expectations, bounded rationality.
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1 Introduction

The recent economic literature has used the notion of an adaptive learning scheme to justify the rational expectations hypothesis as a long-run concept where agents have eliminated all systematic forecast errors. This idea was first supported by authors, e.g. see Bray (1982), Fourgeaud, Gourieroux & Pradel (1986) or Marcet & Sargent (1989a), who succeeded in providing conditions under which adaptive learning schemes converge to an equilibrium under rational expectations. Unfortunately, however, in economies whose evolution is driven by non-linear maps, the question what learnings schemes generate forecasts which converge to such rational expectations equilibria remains to a large extent unresolved. This seems to hold equally true for all adaptive schemes such as Bayesian learning, where agents update beliefs according to Bayes's rule (cf. the recent survey article by Blume & Easley (1998) and references therein) or the ordinary least squares scheme (OLS) used by many theorists in the field (cf. Evans & Honkapohja (1998)).

For the standard OLG model of pure exchange, Bullard (1994), and Schönhofer (1999a, b) showed that ordinary least squares learning may generate forecasts which do not converge to perfect-foresight orbits of the system. Moreover, the long-run behavior of the system may be irregular and chaotic and for this reason may differ substantially from the dynamics under perfect foresight. However, as shown in Schönhofer (1999a), the forecast errors associated with such attractors, also called *learning equilibria*, may have vanishing sample mean and vanishing autocorrelation coefficients. Hommes & Sorger (1998) and Hommes (1998) call such learning schemes *consistent*. They argue that agents with limited statistical tools do not identify systematic forecast errors in these equilibria. Hence, agents would see no reason to revise their learning scheme.

This paper argues that the conclusions from these results are misleading. The criticism is twofold. Apart from the problem of existence of perfect foresight in a particular environment, it is commonplace that the quality of forecasts hinges essentially on the 'rationality' of an economic agent, that is, on both his or her theoretical knowledge of possible economic scenarios and on available statistical tools. On the one hand, it seems not to be surprising that economic agents in deterministic non-linear models may fail in finding orbits with perfect foresight when relying on techniques originally designed for linear stochastic models such as OLS. They simply do not use the appropriate statistical tools. On the other hand, the notion of consistency is rather coarse, because the structural information which is available from observing a particular market is used inefficiently. For example, it is well-known that the market mechanism determining the actual inflation factors in a stationary OLG model under market clearing is of the Cobweb type, that is, actual interest factors of the economy depend on expected inflation factors only and not on previous interest factors, cf. Böhm & Wenzelburger (1997). Hence, in the absence of exogenous noise, the dynamics in an OLG exchange economy stems exclusively from the expectations formation. As a consequence, an agent who knows or suspects that he or she lives in a Cobweb environment should replace any scheme which generates complex dynamics with non-vanishing forecast errors.

In contrast to the findings of Bullard (1994), Schönhofer (1999b) or Marcet & Sargent (1989b), the present paper supports the original intention of the learning literature (see e.g. Blume & Easley (1982)) in showing that a forecasting agency in a stationary OLG exchange economy may indeed find the perfect foresight steady state of the economy. This implies in particular that an agent endowed with appropriate tools can generate forecasts which are more precise than consistent expectations equilibria, as forecast errors become pointwise arbitrarily small. The forecasting agency modeled in the present paper is boundedly rational in the usual sense. It uses well-known theoretical results and, in addition, is aware of the basic market mechanism underlying an OLG exchange economy without knowing consumers' savings behavior. It is shown that this structural knowledge, comprised in the notion of an *error function*, is enough to generate forecasts which converge to the monetary steady state for all initial conditions. This concept arises naturally when carefully distinguishing between an *economic law* describing the basic market mechanism of an economy and a *forecasting rule*, Böhm & Wenzelburger (1997, 1999). The result holds for all possible parameterizations guaranteeing the existence of a monetary steady state and hence for all monetary policies. It can be generalized to all one-dimensional models of the Cobweb type. Contrary to earlier results, the induced learning dynamics is therefore globally stable with forecast errors which vanish in the long run.

The paper is organized as follows. Section 2 briefly reviews the stationary OLG exchange model. The notion of an error function is introduced in Section 3. Section 4 provides a discussion of the learning dynamics with a particular emphasis of the ordinary least squares scheme. In Sections 5 and 6 we introduce and investigate our adaptive learning scheme. Conclusions are given in Section 7. An appendix includes extensions and refinements of the approach presented in the main text.

2 The Model

Consider a standard version of the overlapping generations model with one non-storable commodity per period and fiat money as the only store of value between periods. There will be neither growth of the population nor production. Given the usual assumption of price taking behavior of all generations, young agents need to transfer purchasing power from the first to the second period of their lifes. To avoid the problem of heterogenous beliefs, let θ^e_{t+1} denote the common expected gross inflation rate for period $t+1$ on which all members of the young generation in t base their decisions. We assume that a forecasting agency is in charge of issuing these forecasts and that this agency knows the basic market mechanism of the economy but not the savings behavior of consumers. Under the standard two-period optimizing behavior of a consumer h born in period t, her optimal consumption plan given an initial endowment of goods w_1^h and w_2^h, respectively is defined by a savings function $s^h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. $s^h(\theta_{t+1}^e)$ is the amount saved (and supplied to the market) in period t and $w_2^h + s^h(\theta_{t+1}^e)/\theta_{t+1}^e$ is the amount expected to be consumed in period $t+1$.

A government is infinitely lived and consumes g_t at time t. At time $t = 1$ the government owns $M_1 > 0$ units of currency, and agents save by holding currency, since fiat money is the only store of value. Given a nonnegative price level p_t, the government finances its consumption by creating additional currency given by the process $M_t = \gamma M_{t-1}$, where $\gamma > 1$ is the gross rate of currency growth. The government's budget constraint is given by $M_t - M_{t-1} = g_t p_t$. The policy rule γ is chosen by the monetary authorities such that the government expenditures g_t become endogenous.

Market clearing on the goods market in any period t requires that the real savings of young consumers, which defines the amount supplied, has to be equal to the demand of the old generation and the government, which is equal to the real purchasing power of the money acquired in the previous period. In other words, aggregate savings S has to be equal to real money balances

$$\frac{M_t}{p_t} = S(\theta_{t+1}^e) := \sum_{h=1}^{H} s^h(\theta_{t+1}^e)$$

for all times t. The actual inflation rate $\theta_t = p_t/p_{t-1}$ is then determined by

$$\theta_t = F(\theta_t^e, \theta_{t+1}^e) := \frac{\gamma S(\theta_t^e)}{S(\theta_{t+1}^e)}, \tag{1}$$

as long as the expected inflation factors are such that aggregate savings is different from zero. Let $0 < \gamma < \theta_{aut}$, where we assume that the autarky factor θ_{aut}, given by $S(\theta_{aut}) = 0$, is uniquely determined. Thus the monetary steady state γ exists. Observe that from a sequential view point, the forecast θ_{t+1}^e for the inflation rate θ_{t+1} has to be picked before the actual trading takes place and thus prior to the realization of θ_t. Moreover, M_t denotes the final money balance of generation t after trading. The map $F : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ given in (1) includes a two-step-ahead forecast and for this reason defines an economic law with an expectational lead, see Böhm & Wenzelburger (1997). Since only expected inflation factors enter F as an argument, the economic law (1) has the structure of a cobweb model. The goods price of period t generated by (1) is given by

$$p_t = \frac{\gamma S(\theta_t^e)}{S(\theta_{t+1}^e)} p_{t-1}, \quad t \in \mathbb{N}. \tag{2}$$

The market mechanism (1) or equivalently (2) has been used previously to demonstrate that the least squares learning scheme may generate chaotic trajectories, see Schönhofer (1999b), Bullard (1994), and Marcet & Sargent (1989b). The purpose of this paper is to devise an adaptive learning scheme which converges to the monetary steady state for all reasonable parameterizations of (1).

3 Perfect Prediction

In order to address the question which forecasting rules generate perfect foresight along orbits of the economy, we first consider the error function associated with the economic law (1). For an arbitrary period let θ^e_{old} and θ^e_{you} denote the forecasts of the old and the young generation, respectively. The forecast error for the old generation is then given by the error function

$$e_F : \mathbb{R}^2_+ \longrightarrow \mathbb{R}, \quad (\theta^e_{old}, \theta^e_{you}) \longmapsto \frac{\gamma S(\theta^e_{old})}{S(\theta^e_{you})} - \theta^e_{old}. \tag{3}$$

The function e_F describes all possible forecast errors, independently of which forecasting rule or learning scheme has been used to obtain the forecasts. Geometrically, the graph of e_F is a surface over the θ^e_{old}–θ^e_{you} plane. The zero-contour of e_F describes the loci of all forecasts with vanishing forecast errors. It follows from $S(\gamma) > 0$ that $e_F(\gamma, \gamma) = 0$ which means that the monetary steady state (γ, γ) belongs to the zero contour of e_F. In order to obtain perfect foresight for any previously determined forecast θ^e_{old}, a new forecast θ^e_{you} has to be chosen such that the forecast error $e_F(\theta^e_{old}, \theta^e_{you})$ vanishes. This implies in particular that only forecasting rules, also called predictors, of the form

$$\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \quad \theta^e_{you} = \psi(\theta^e_{old}) \tag{4}$$

have a chance of generating perfect foresight along orbits of the system. According to Böhm & Wenzelburger (1997), predictors of the form (4) which satisfy $e_F(\theta^e, \psi(\theta^e)) = 0$ for all θ^e in an open subset U of $\mathbb{R}_+$ are called *locally perfect*. If $U = \mathbb{R}_+$, then ψ is called *globally perfect*. The following proposition shows that locally perfect predictors for the OLG exchange model always exist, whereas globally perfect predictors exist only under very restrictive assumptions.

Proposition 3.1 *Let $0 < S(\gamma) < w_1$, where $w_1 := \sum_{h=1}^{H} w_1^h$ is aggregate endowment of the young generation, and $S'(\gamma) \neq 0$. Then there exists an open neighborhood U of γ and a predictor $\psi_\star$ given by*

$$\psi_\star(\theta^e) = S^{-1}(\gamma S(\theta^e)/\theta^e), \quad \theta^e \in U \tag{5}$$

which is locally perfect on U. If in addition, S is monotonically increasing in θ^e and $\gamma S(\theta^e)/\theta^e < w_1$ for all $\theta^e \in \mathbb{R}_+$, then $\psi_\star$ is globally perfect on $\mathbb{R}_+$.

Proof. As noticed above $e_F(\gamma, \gamma) = 0$. Since $S'(\gamma) \neq 0$, we have $\partial_2 e_F(\gamma, \gamma) \neq 0$ and the equation $e_F(\theta^e_{old}, \theta^e_{you}) = 0$ satisfies the conditions of the implicit function theorem. This shows that $\psi_\star$ given in (5) is well-defined in a neighborhood U of γ. If, in addition, S is monotonically increasing and $\gamma S(\theta^e)/\theta^e < w_1$ for all $\theta^e \in \mathbb{R}_+$, then $\psi_\star$ is globally defined on $\mathbb{R}_+$.

Q.E.D.

Corollary 3.2 $\psi_\star' > 0$ $(= 0)$ *on* U *if and only if* $S' < 0$ $(= 0)$ *on* U.

If youthful and old-age consumption are normal goods or if the aggregate savings function is non-increasing, then the term $S(\theta^e)/\theta^e$ will become unbounded for sufficiently small θ^e. In these cases globally perfect predictors will not exist. Most preferences with intertemporal substitution properties usually assumed in the OLG literature will therefore not allow for global perfect foresight, see Böhm & Wenzelburger (1997).

Proposition 3.1 states that perfect predictors depend exclusively on previous forecast and not on observed states of the economy. Geometrically, the graph of a locally perfect predictor $\psi_\star$ is contained in the zero contour of the error function. The error function of an OLG exchange model has some general features which are stated in the next proposition. For doing so, let $E_S(\theta) := S'(\theta)\theta/S(\theta)$ denote the elasticity of the savings function.

Proposition 3.3 *The error function* e_F *of an OLG exchange model has the following properties:*

(i) e_F *is linear along the* $45°$*-degree line.*

$$e_F(\theta^e, \theta^e) = \gamma - \theta^e \quad \textit{for all} \quad \theta^e \in \mathbb{R}_+ \quad \textit{with} \quad S(\theta^e) > 0.$$

(ii) *Its gradient along the zero contour satisfies*

$$\operatorname{grad} e_F(\theta^e_{old}, \theta^e_{you})_{|e_F = 0} = \left(E_S(\theta^e_{old}) - 1, \frac{-S'(\theta^e_{you})(\theta^e_{old})^2}{\gamma S(\theta^e_{old})} \right),$$

where $E_S(\theta^e_{old}) < 1$ *for* $\theta^e_{old} \neq \theta_{aut}$.

(iii) *The zero contour intersects the* $45°$*-degree line of* $\mathbb{R}^2_+$ *transversally or, equivalently,*

$$\operatorname{grad} e_F(\gamma, \gamma) \neq \lambda(1, -1) \quad \textit{for all} \quad \lambda \in \mathbb{R}.$$

Proof. The first statement is obvious from the definition of e_F. The second statement follows from the definition of a gradient and the fact that $\gamma S(\theta^e_{old}) = \theta^e_{old} S(\theta^e_{you})$ along the zero contour. The Slutsky matrix for a individual savings function s^h is negative semi-definite if and only if

$$s^h(\theta^e) - s^{h\prime}(\theta^e)\theta^e \geq s^h(\theta^e)^2.$$

Therefore

$$S(\theta^e) - S'(\theta^e)\theta^e \geq \sum_{h=1}^{H} s^h(\theta^e)^2$$

and the first component of the gradient in **(ii)** is negative as long as $\theta^e \neq \theta_{aut}$. The third statement follows directly from **(ii)**. *Q.E.D.*

It follows from Proposition 3.3 that the zero contour of the error function has a unique intersection point with the diagonal of the $\theta^e_{old} - \theta^e_{you}$ plane which is the monetary steady state. The surface has a singularity at a possible autarky inflation factor where aggregate savings is zero. The gradient of the zero contour shows in the direction of increasingly positive forecast errors. Vice versa, the negative gradient shows in the direction of increasingly negative forecast errors. The second statement in Proposition 3.3 states that the gradient of the zero contour always points in the direction of the θ^e_{you}–axis. To illustrate the concept of an error function, we discuss three examples which are slight generalizations of examples taken from Bullard (1994) and Schönhofer (1999b).

Example 3.1 *The Cobb-Douglas case. Let each consumer be characterized by the same Cobb-Douglas utility function*

$$u(c_1, c_2) = \ln c_1 + \delta \ln c_2, \quad 0 < \delta \leq 1,$$

and the same endowments $w_i^h = w_i^{h'}$, $i = 1, 2$ *for all* $h, h' \in H$. *The savings function is*

$$s^h(\theta^e) = \min\left\{0, \tfrac{1}{1+\delta}[\delta w_1^h - \theta^e w_2^h]\right\}, \quad \theta^e \in \mathbb{R}_+.$$

The common autarky inflation factor of all households is $\theta_{aut} = \frac{\delta w_1^h}{w_2^h}$, *such that aggregate savings becomes*

$$S(\theta^e) = \begin{cases} \frac{1}{\delta}[w_1 - \theta^e w_2] & \text{if } \theta^e \in [0, \theta_{aut}) \\ 0 & \text{else} \end{cases}$$

with w_1 *and* w_2 *denoting aggregate endowments when young and when old, respectively. Let* $\gamma < \theta_{aut}$ *as before. In this case there exists a locally perfect predictor* $\psi_\star$, *defined by*

$$\psi_\star(\theta^e) = \theta_{aut} - \gamma\left[\frac{\theta_{aut}}{\theta^e} - 1\right] \quad \text{for} \quad \theta^e \in \left[\frac{\gamma}{1+\gamma\theta_{aut}}, \theta_{aut}\right]. \tag{6}$$

$\psi_\star$ *is uniquely determined and the domain of definition in (6) is maximal. This case is illustrated in Fig. 1.*

Example 3.2 *The CES case. Let each consumer be characterized by a CES utility function of the form*

$$u(c_1, c_2) = [c_1^\rho + (\delta\, c_2)^\rho]^{\frac{1}{\rho}}, \quad 0 < \delta \leq 1, \ \rho < 1, \ \rho \neq 0.$$

Then aggregate savings becomes

$$S(\theta^e) = w_1 - \frac{w_1 + w_2\,\theta^e}{1 + (\delta^{-1}\theta^e)^{\frac{\rho}{\rho-1}}} \quad \theta^e \in \mathbb{R}_+,$$

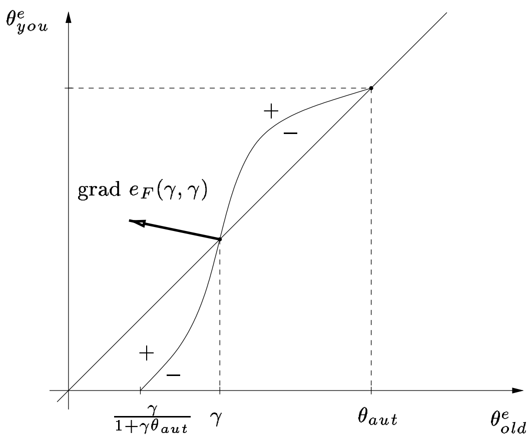

Figure 1: Zero contour for the Cobb-Douglas case.

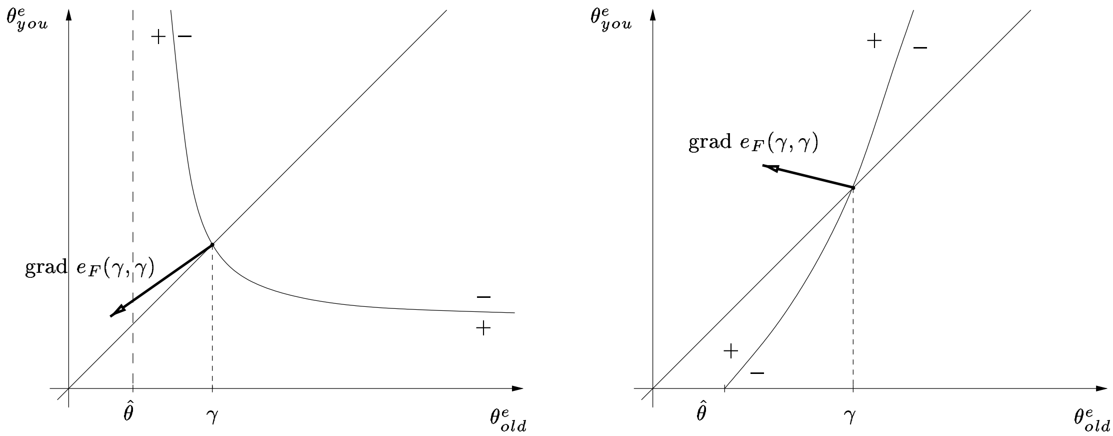

Figure 2: Zero contour for the CES case, $\rho < 0$ (left) and $\rho > 0$ (right).

where w_1 and w_2 denotes aggregate endowment when young and when old, respectively. Suppose $w_2 = 0$. Then there exists a locally perfect predictor $\psi_\star$, defined by

$$\psi_\star(\theta) = \left(\frac{\gamma \theta^{\frac{\rho}{\rho-1}}}{\delta^{\frac{\rho}{\rho-1}}\theta + \theta^{\frac{\rho}{\rho-1}}[\theta - \gamma]} \right)^{\frac{\rho-1}{\rho}} \qquad \theta^e \in (\hat{\theta}, \infty), \tag{7}$$

where $\hat{\theta}$ is defined by $\delta^{\frac{\rho}{\rho-1}}\hat{\theta}^{\frac{1}{1-\rho}} + \hat{\theta} - \gamma = 0$. $\psi_\star$ is uniquely determined and the interval $(\hat{\theta}, \infty)$ is its maximal domain. Therefore, a global perfect predictor does not exist. This case is illustrated in Fig. 2 (left) for $\rho < 0$ and in Fig. 2 (right) for $\rho > 0$.

Example 3.3 *An 'irregular' savings function, see Bullard (1994, p. 480). Consider*

$$S(\theta^e) = \exp\left\{ \cos\left[\frac{10}{1+(\theta^e)^{\frac{\rho}{\rho-1}}} \right] \right\}^{-1}, \quad \theta^e \in \mathbb{R}_+.$$

In this case multiple locally perfect predictors exist, as the zero contour of the corresponding error function folds back. In Fig. 4 points in the θ^e_{old}-θ^e_{you} plane which lie in the same error range are colored according to the adjacent color code. The contour lines of the corresponding error function are given by the boundaries between to colored regions. The contour lines could, in principle, be computed analytically.

4 Learning dynamics

To discuss the dynamics of the inflation factors under a particular learning scheme, consider for a moment predictors ψ of the simple form (4). The evolution of the economy is then governed by the two-dimensional map

$$F_\psi : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+, \quad (\theta, \theta^e_{old}) \longmapsto (F(\theta^e_{old}, \psi(\theta^e_{old})), \psi(\theta^e_{old})).$$

Notice that the evolution of the system is exclusively driven by the predictor ψ and that the dynamics of the economy are one-dimensional. More precisely, the dynamics governed by F_ψ is clearly topological conjugate to the dynamics of the predictor alone which is given by

$$\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \quad \theta^e_{old} \longmapsto \psi(\theta^e_{old}).$$

The (global) perfect foresight dynamics, if it exists, is given by

$$F_{\psi_\star} : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+, \quad (\theta, \theta^e) \longmapsto (\theta^e, \psi_\star(\theta^e)). \tag{8}$$

It is straightforward to see that the unique positive fixed point of $F_{\psi_\star}$ with the perfect foresight property is the monetary steady state $(\theta_\star, \theta^e_\star) = (\gamma, \gamma)$. Clearly, under perfect foresight $\theta^e_{t+1} = \theta_{t+1}$ for all times t and the dynamics of (8) is equivalent to the dynamics generated by the perfect predictor $\psi_\star$ alone. Since perfect predictors are completely determined by the economic fundamentals, so is the prefect foresight dynamics.

Similar observations remain valid for any more complex forecasting rule and any learning scheme which may depend on whatever variable the agent (or the modeler) may believe in. Bullard (1994) and Schönhofer (1999b) have made use of the fact that the recursive formula of the least squares learning scheme (OLS) can be viewed as a *predictor* which depends in a well-specified manner on current states and certain other auxiliary variables, see e.g. Chen & Guo (1991). In the case of an OLG exchange economy, their OLS-based forecasts are defined by the maps

$$\begin{aligned} \psi &: \mathbb{R}^2_+ \times [0,1] \longrightarrow \mathbb{R}_+, & \psi(\theta_{t-1}, \theta^e_t, g_{t-1}) &:= \theta^e_t + g_{t-1}[\theta_{t-1} - \theta^e_t] \\ \varphi &: \mathbb{R}_+ \times [0,1] \longrightarrow [0,1], & \varphi(\theta_{t-1}, g_{t-1}) &:= \frac{g_{t-1}\,\theta^2_{t-1}}{1 + g_{t-1}\,\theta^2_{t-1}}, \end{aligned} \tag{9}$$

such that $\theta^e_{t+1} = \psi(\theta_{t-1}, \theta^e_t, g_{t-1})$ is the forecast for θ_{t+1} and $g_t = \varphi(\theta_{t-1}, g_{t-1})$ the auxiliary variable[1]. The evolution of the inflation factors is then governed by the map $F_{\psi,\varphi} : \mathbb{R}^2_+ \times [0,1] \longrightarrow \mathbb{R}^2_+ \times [0,1]$, given by

$$(\theta_{t-1}, \theta^e_t, g_{t-1}) \longmapsto (F(\theta^e_t, \psi(\theta_{t-1}, \theta^e_t, g_{t-1})), \psi(\theta_{t-1}, \theta^e_t, g_{t-1}), \varphi(\theta_{t-1}, g_{t-1})).$$

Replacing θ_{t-1} by $F(\theta^e_{t-1}, \theta^e_t)$, the dynamics of $F_{\psi,\varphi}$ is essentially induced by the pair of maps (ψ, φ) and hence by the OLS scheme alone. The dynamics can be analyzed in the $\theta^e_{old} - \theta^e_{you}$ plane as has been done in Bullard (1994) and Schönhofer (1999a). Figs. 5 and 6 show two complex attractors[2] in the $\theta^e_{old} - \theta^e_{you}$ plane generated by the OLS scheme (9) in the CES case and the case of an irregular savings function, respectively. Points $(\theta^e_t, \theta^e_{t+1})$ on the two attractors which lie in the same error range are colored according to the adjacent color code. As is apparent from the figures, the contour lines and hence the shape of the error function becomes to a large extent revealed through the course of time, if the learning dynamics is complex[3]. In particular, the location of the zero contour in these figures is disclosed, giving a clear indication of the location of the monetary steady state. In other words, Figs. 5 and 6 provide a clear incentive to abandon the employed OLS scheme. It therefore seems to be worthwhile to construct learning schemes which make a more efficient use of information on the market mechanism of the particular economy.

5 An adaptive learning scheme

In this section we introduce an adaptive learning scheme for stationary OLG exchange economies. Since the perfect foresight dynamics may most likely fail to exist globally and may have unfavorable properties such as convergence to the non-monetary steady state, we concentrate on schemes which find the perfect foresight monetary steady state.

Consider a forecasting agency which is in charge of forecasting the future evolution of the economy and hence has to decide on θ^e. We assume here that *all* young consumers in period t share the belief in θ^e_{t+1} issued by the forecasting agency prior to their savings decisions. This excludes heterogenous beliefs and strategic behavior of consumers. If the number of households is large, the effect of strategic behavior of a single consumer can be neglected. Furthermore, we assume that the forecasting agency itself has no strategic interest other than issuing good forecasts in order to be credible for households.

[1]The forecast θ^e_{t+1} given by (9) seems to be counterintuitive from the econometric point of view. In the expression for ψ there seem to appear the wrong differences, namely $\theta_{t-1} - \theta^e_t$ instead of $\theta_t - \theta^e_t$. This is so because the regression is done on prices rather than on inflation factors. We argue in Appendix A that OLS-based forecasts for inflation factors should differ from (9).

[2]Observe that the system is three-dimensional and that Figs. 5 and 6 actually show projections of attractors onto the $\theta^e_{old} - \theta^e_{you}$ plane.

[3]All simulations were carried out using the program package MACRODYN , cf. Böhm & Schenk-Hoppé (1998). We used similar parameterizations to those in Bullard (1994) and Schönhofer (1999a).

In period τ the agency has observed past prices and therefore knows all past real rates of return $\{\theta_t\}_{t=0}^{\tau-1}$ and all corresponding forecasts $\{\theta_t^e\}_{t=0}^{\tau}$ including the forecast for the current period. Recall that θ_τ is not available prior to the decision on $\theta_{\tau+1}^e$. Let the agency be aware of the basic market mechanism of the economy, that is, the basic structure of the economic law given by (1) but not the specific parameterization of (1). Hence, neither the preferences nor the savings behavior of young consumers are known to the agency. In other words, the forecasting agency has the concept of the error function without a concrete functional specification and for this reason is bounded rational in the sense of Sargent (1993).

The key idea now is that the geometric shape of the error function becomes disclosed through the course of time. This allows an agency, in principle, to improve estimates of the zero contour and hence to improve the quality of the forecasts. If $\epsilon_t = \theta_t - \theta_t^e$ denotes the forecast error made in period t, then the point $(\theta_t^e, \theta_{t+1}^e, \epsilon_t)$ belongs to the graph of the error function

$$\operatorname{graph} e_F := \{(\theta_{old}^e, \theta_{you}^e, \epsilon) \in \mathbb{R}_+^2 \times \mathbb{R} \mid e_F(\theta_{old}^e, \theta_{you}^e) = \epsilon\}.$$

At the beginning of an arbitrary period, say τ, the sequence of points $\{(\theta_t^e, \theta_{t+1}^e, \epsilon_t)\}_{t=0}^{\tau-1}$ have been observed and may be visualized as points on the unknown graph of the error function. These points reveal information on the shape of the error function and thus on the location of its zero contour. In the case of the OLS scheme this was visualized in Figs. 5 and 6.

To construct an adaptive learning scheme which makes use of the notion of an error function, let us summarize and fix the assumptions on the structural knowledge of the forecasting agency.

Assumption 5.1 *The information of the forecasting agency comprises the following:*

(i) *the relevant variables of the economy are $(\theta_{old}^e, \theta_{you}^e)$ and the market mechanism is of the form $(\theta_{old}^e, \theta_{you}^e) \mapsto \tilde{F}(\theta_{old}^e, \theta_{you}^e)$,*

(ii) *there exists a monetary steady state which lies in some* uncertainty interval $[\underline{\theta}, \overline{\theta}]$, *and*

(iii) *the errors at the boundaries $\underline{\theta}, \overline{\theta}$ satisfy $e_F(\underline{\theta}, \underline{\theta}) > 0$ and $e_F(\overline{\theta}, \overline{\theta}) < 0$.*

Proposition 3.3 **(i)** guarantees the existence of $\underline{\theta}$ and $\overline{\theta}$ with the given properties. One may think of the errors $e_F(\underline{\theta}, \underline{\theta})$ and $e_F(\overline{\theta}, \overline{\theta})$ at the boundaries of the uncertainty interval as known or being previously observed. The lower bound $\underline{\theta}$ could be equal or close to zero and $\overline{\theta}$ may, for instance, be assumed to be less than the autarky inflation factor θ_{aut} a known quantity to the agency. A learning scheme which searches for the monetary steady state will have to search close to the diagonal of the $\theta_{old}^e - \theta_{you}^e$ plane. The following algorithm implements this idea using Assumption 5.1. It can be started at any time $t_0 > 0$, so we might as well assume $t_0 = 0$. Then θ^e_{-1} and θ^e_0 denote the expected inflation factors of the last and the second last generation, respectively which are assumed to lie in the uncertainty interval.

Algorithm 5.1 *Let* $\theta^e_{-1}, \theta^e_0 \in [\underline{\theta}, \overline{\theta}]$, $\theta^e_{-1} \neq \theta^e_0$ *be arbitrary.*

1. *If* $e_F(\theta^e_{-1}, \theta^e_0) \geq 0$, *then define* $\epsilon = \frac{\overline{\theta} - \theta^e_0}{k}$ *with some positive integer* $k \geq 2$.
   (a) *Set* $\theta^e_t = \theta^e_0 + t\epsilon$, $t > 0$ *until* $e_F(\theta^e_{t-1}, \theta^e_t) \leq 0$ *or* $t = k - 1$. *Let* τ *denote the first time for which this condition is satisfied.*
   (b) *Set* $\theta^e_{\tau+1} = \theta^e_\tau$. *If* $e_F(\theta^e_\tau, \theta^e_{\tau+1}) < 0$, *then set* $\underline{\theta}' = \underline{\theta}$ *and* $\overline{\theta}' = \theta^e_{\tau+1}$. *If* $e_F(\theta^e_\tau, \theta^e_{\tau+1}) > 0$, *then set* $\underline{\theta}' = \theta^e_{\tau+1}$ *and* $\overline{\theta}' = \overline{\theta}$. *If* $e_F(\theta^e_\tau, \theta^e_{\tau+1}) = 0$, *then* $\theta^e_t = \theta^e_{\tau+1}$ *for all* $t > \tau + 1$.

2. *If* $e_F(\theta^e_{-1}, \theta^e_0) < 0$, *then let* $\epsilon = \frac{\theta^e_0 - \underline{\theta}}{k}$ *with some positive integer* $k \geq 2$.
   (a) *Set* $\theta^e_t = \theta^e_0 - t\epsilon$, $t > 0$ *until* $e_F(\theta^e_{t-1}, \theta^e_t) \geq 0$ *or* $t = k - 1$. *Let* τ *denote the first time for which this condition is satisfied.*
   (b) *Set* $\theta^e_{\tau+1} = \theta^e_\tau$. *If* $e_F(\theta^e_\tau, \theta^e_{\tau+1}) < 0$, *then set* $\underline{\theta}' = \underline{\theta}$ *and* $\overline{\theta}' = \theta^e_{\tau+1}$. *If* $e_F(\theta^e_\tau, \theta^e_{\tau+1}) > 0$, *then set* $\underline{\theta}' = \theta^e_{\tau+1}$ *and* $\overline{\theta}' = \overline{\theta}$. *If* $e_F(\theta^e_\tau, \theta^e_{\tau+1}) = 0$, *then* $\theta^e_t = \theta^e_{\tau+1}$ *for all* $t > \tau + 1$.

Fig. 3 conveys the economic intuition for this learning scheme. As long as there are positive (negative) forecast errors, increase (reduce) the current forecast by some quantity ϵ. A positive (negative) forecast error means that the expected inflation factor of the old generation was too low (high). As soon as a negative (positive) forecast error is obtained, check how close the current forecast is to the monetary steady state. It is clear that any pair of forecasts $(\theta^e_{old}, \theta^e_{you})$ lying on the diagonal reduces the uncertainty interval. Clearly $e_F(\theta^e, \theta^e) = 0$ if and only if θ^e is the monetary steady state γ. Geometrically, the resulting learning dynamics takes place along straight lines which are parallel to the 45°-degree line.

Assumption 5.1 can easily be relaxed to the case where no initial uncertainty interval is known. A forecasting agency needs to know only the qualitative properties of the error surface graph e_F given in Proposition 3.3. Roughly speaking, once it is known that the left hand side of the zero-contour line is on a positive level and the right hand side on a negative level, it is clear in which direction to search for the monetary steady state. We note in passing that Algorithm 5.1 can be rephrased as a (time-independent) predictor in the above sense. Moreover, it is applicable for any one-dimensional economic law of the Cobweb type with an expectational lead.

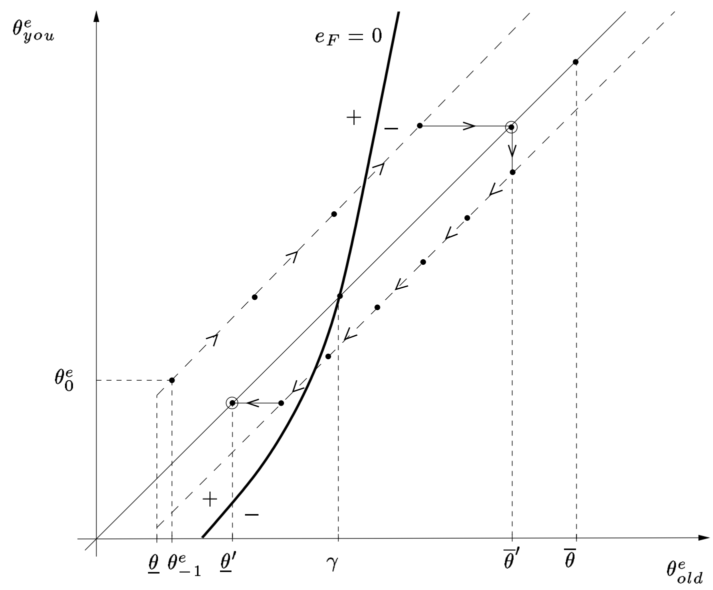

Figure 3: Adaptive learning scheme.

6 Global Convergence of adaptive learning schemes

We show first that the adaptive learning scheme introduced in the previous section converges in the sense that it generates forecasts which converge to the monetary steady state when repeatedly applied.

Proposition 6.1 *For arbitrary initial forecasts $\theta^e_{-1}, \theta^e_0 \in [\underline{\theta}, \overline{\theta}]$, Algorithm 5.1 ends in at most k time steps and reduces the initial uncertainty interval $[\underline{\theta}, \overline{\theta}]$ to a smaller uncertainty interval $[\underline{\theta}', \overline{\theta}']$ whose length $\overline{\theta}' - \underline{\theta}'$ is less than $\frac{k-1}{k}(\overline{\theta} - \underline{\theta})$.*

Proof. By construction, the algorithm 5.1 picks a least one additional point on the diagonal whose coordinates lie in $(\underline{\theta}, \overline{\theta})$. This yields a new uncertainty interval $[\underline{\theta}', \overline{\theta}']$ which is contained in $[\underline{\theta}, \overline{\theta}]$. Now clearly $\overline{\theta}' - \underline{\theta}' \leq \frac{k-1}{k}(\overline{\theta} - \underline{\theta})$ which yields the proposition. *Q.E.D.*

Proposition 6.1 shows that Algorithm 5.1 reduces the uncertainty interval by a factor which is at most $(k-1)/k$. Since this factor is less than unity, the following theorem is obvious.

Theorem 6.2 *Under Assumption 5.1, repeated application of Algorithm 5.1 yields a sequence of forecasts $\{\theta^e_t\}_{t \in \mathbb{N}}$ which converges to the monetary steady state γ.*

Observe that by Theorem 6.2 Algorithm 5.1 converges for all initial conditions and hence globally on $[\underline{\theta}, \overline{\theta}]$. It stabilizes the monetary steady state in particular for cases, where the perfect foresight dynamics is complicated as in Grandmont (1985). Algorithm 5.1 is inspired by a minimization scheme introduced by Berman (1966) and applies for practically all one-dimensional models of the Cobweb type (see Appendix C for the case of one-step-ahead forecasts). In principle, any numerical method to compute zeros of a function such as Newton's method (see e.g. Ortega & Rheinboldt (1970)) can be employed to find the monetary steady state. In Appendix B we show that the monetary steady state of an OLG exchange economy can be reached in 5 time steps using Newton's method. This is possible because the error function is linear along the 45°-degree line. In the general case, however, it is well known that Newton's method may not converge and even lead to chaotic behavior if the initial guesses are not sufficiently close to the solution of the problem. For this reason methods which work globally are needed. In the case of the standard OLG model, one simple global method is Algorithm 5.1. More advanced numerical continuation methods such as presented in Allgower & Georg (1990) seem to be applicable as well.

In the case of the exchange economy described above, one feature of a learning scheme seems to be sufficient to guarantee convergence to the monetary steady state. Consider a scheme which generates a Cauchy sequence of forecasts $\{\theta_t^e\}_{t\in\mathbb{N}}$ such that there exists some $\theta_\star^e$ with $\theta_t^e \to \theta_\star^e$ as t tends to infinity. The following proposition gives an easy criterion which ensures that the learning scheme converges to the monetary steady state γ.

Proposition 6.3 *Let $\{\theta_t^e\}_{t\in\mathbb{N}}$ be a Cauchy sequence of forecasts generated by some learning scheme. Assume that there exists a strictly monotonic sequence of positive integers $\{t_j\}_{j\in\mathbb{N}}$ with $e_F(\theta^e_{t_{2j}}, \theta^e_{t_{2j}+1}) > 0$ and $e_F(\theta^e_{t_{2j+1}}, \theta^e_{t_{2j+1}+1}) < 0$. If $\{\theta_t^e\}_{t\in\mathbb{N}}$ does not converge to θ_{aut}, then $\theta_t^e \to \gamma$ as t tends to infinity.*

Proof. Since $\{\theta_t^e\}_{t\in\mathbb{N}}$ is a Cauchy sequence there exists a limit point $\theta_\star^e$. Let $\theta_\star^e \neq \theta_{aut}$ and suppose $\theta_\star^e \neq \gamma$. Then either $e_F(\theta_\star^e, \theta_\star^e) < 0$ or $e_F(\theta_\star^e, \theta_\star^e) > 0$. Consider the first case. Since the forecasts θ_t^e converge to $\theta_\star^e$ and e_F is continuous, there exists $t_\star$ such that

$$e_F(\theta_t^e, \theta_{t+1}^e) < 0 \quad \text{for all} \quad t > t_\star.$$

This contradicts the assumption on $\{t_j\}_{j\in\mathbb{N}}$. An analogous argument applies for the second case. It follows that $e_F(\theta_\star^e, \theta_\star^e) = 0$ and hence $\theta_\star^e = \gamma$. *Q.E.D.*

7 Conclusions

The analysis of adaptive learning in a standard OLG economy showed that convergence of an employed scheme required a forecasting agency to use the correct structural information about the market mechanism. For the OLG economy it was the fact that only the previous and the current expected inflation factors matter. This structural information is encoded in the error function associated with the economy. By definition, an error function gives all forecast errors as a function of all possible states of the economy and all possible forecasts, independently of the employed learning scheme. It depends exclusively on the economic fundamentals encoded in the economic law and captures the nature of the true expectations feedback. The graph of the error function is the essential time-invariant object about which information is to be obtained along the evolution of the system. A successful learning scheme must therefore locate the zero contour of the error function by successive approximations.

It was shown that along an orbit of the system, a forecasting agency receives more and more information about the shape of the error function and the location of its zero contour. In the case of the OLG economy, this was exploited to generate forecasts which converge to the monetary steady state. The learning scheme proposed in the present paper was based on a simple geometric intuition and converges globally for all initial conditions and all parameterizations which guarantee the existence of a monetary steady state. It is equally well applicable and successful for all one-dimensional models of the Cobweb type. Thus, learning schemes which induce diverging or complex behavior in these types of models ignore their correct structural features. This may lead to their ultimate failure to predict perfectly.

A The OLS scheme and expectational leads

In this section we comment on the recursive OLS scheme for prices in relation to the economic law (1) which has been employed by Bullard (1994) and Schönhofer (1999b). Assume that a forecasting agency makes forecasts based on a 'perceived law of motion' given by an AR(1) process

$$p_t = \beta p_{t-1} + \xi_t, \tag{10}$$

where the parameter β is unknown and $\{\xi_t\}_{t\in\mathbb{N}}$ is a sequence of unobservable i.i.d. random variables with zero mean. The least squares estimator for β based on the times series $\{p_i\}_{i=0}^{t-1}$ is given by

$$\hat{\beta}_{t-1} = \left(\sum_{i=1}^{t-1} p_{i-1}^2\right)^{-1} \left(\sum_{i=1}^{t-1} p_{i-1} p_i\right). \tag{11}$$

According to Chen & Guo (1991, p. 91), the recursion formula for $\hat{\beta}_{t-1}$ is

$$\begin{array}{rcl} \hat{\beta}_{t-1} & = & \hat{\beta}_{t-2} + \dfrac{C_{t-2}p_{t-2}}{1+C_{t-2}p_{t-2}^2}[p_{t-1} - \hat{\beta}_{t-2}p_{t-2}] \\ C_{t-1} & = & \dfrac{C_{t-2}}{1+C_{t-2}p_{t-2}^2} \end{array}.$$

Setting $g_t := C_t p_{t-1}^2$, this recursion formula in terms of inflation factors $\theta_{t-1} = p_{t-1}/p_{t-2}$ reads

$$\begin{array}{rcl} \hat{\beta}_{t-1} & = & \hat{\beta}_{t-2} + g_{t-1}[\theta_{t-1} - \hat{\beta}_{t-2}] \\ g_t & = & \dfrac{g_{t-1}\theta_{t-1}^2}{1+g_{t-1}\theta_{t-1}^2} \end{array}. \tag{12}$$

The forecasts p_t^e and p_{t+1}^e for prices p_t and p_{t+1} on the basis of the estimator $\hat{\beta}_{t-1}$ are

$$\begin{array}{rcl} p_t^e & = & \mathbb{E}_{t-1}(\hat{\beta}_{t-1}p_{t-1} + \xi_t) = \hat{\beta}_{t-1}p_{t-1}, \\ p_{t+1}^e & = & \mathbb{E}_{t-1}(\hat{\beta}_{t-1}[\hat{\beta}_{t-1}p_{t-1} + \xi_t] + \xi_{t+1}) = \hat{\beta}_{t-1}^2 p_{t-1}. \end{array}$$

Therefore, $\theta_t^e := \hat{\beta}_{t-1}$ and $\theta_{t+1}^e := \hat{\beta}_{t-1}^2$ are the forecasts for inflation factors θ_t and θ_{t+1} based on the same estimator $\hat{\beta}_{t-1}$ and which are consistent with the perceived law (10).

Now, according to the actual price process (2) associated with the economic law (1), θ_{t+1}^e has to be determined prior to θ_t and hence prior to p_t. As pointed out in Sec. 2, this is due to the fact that (1) contains an expectational lead. It implies that the information set at time t cannot contain p_t. An OLS-based forecast for θ_{t+1} at time t as required by (1) should therefore be $\theta_{t+1}^e = \hat{\beta}_{t-1}^2$ instead of $\theta_{t+1}^e = \hat{\beta}_{t-1}$ as done by Bullard (1994) and Schönhofer (1999b). However, their recursion scheme (9) does not differ from (12).

B Newton's method as an adaptive learning scheme

This appendix is to show that the so-called Newton's discrete method can be incorporated in an adaptive learning scheme for the case of an OLG exchange economy described in Sec. 1. In this case it is shown that the monetary steady state (γ, γ) is in fact reached within 5 steps. We make explicit use of the fact that the monetary steady state lies on the 45°–degree line of the θ^e_{old}–θ^e_{you} plane.

Consider a forecasting agency which applies Newton's method to the map $\theta^e \mapsto E(\theta^e) := e_F(\theta^e, \theta^e)$, where as before the error function e_F is unknown. Observe that the derivative E' of E is also unknown such that E' has to be replaced by a suitable approximation. This is done by Newton's discrete method which formally is given by the successive iterates

$$x_{t+1} = \mathrm{NEW}(x_t, x_{t-1}) := x_t - \left[\frac{E(x_{t-1}) - E(x_t)}{x_{t-1} - x_t}\right]^{-1} E(x_t), \quad t \in \mathbb{N}$$

(see Ortega & Rheinboldt (1970)). As it is assumed that all forecasts are known and forecast errors are observable, this formula can be integrated into the following adaptive learning scheme. The only remaining difficulty is that no scheme can move on the 45°–degree line directly.

Algorithm B.1 *Let* $(\theta^e_{-1}, \theta^e_0) \in \mathbb{R}^2_+$ *be arbitrary.*

1. *Set* $\theta^e_1 = \theta^e_0$.
2. *Choose some small number* ν *and such that* $\theta^e_2 = \theta^e_1 + \nu > 0$.
3. *Set* $\theta^e_3 = \theta^e_2$.
4. *Apply a Newton step by setting* $\theta^e_4 = \mathrm{NEW}(\theta^e_3, \theta^e_1)$.
5. *Set* $\theta^e_5 = \theta^e_4$.

Now since E is linear by Proposition 3.3, a routine calculation shows that Step 4 already yields the monetary steady state, i.e., $\theta^e_4 = \gamma$ which finally is reached in Step 5. Notice that the initial state $(\theta^e_{-1}, \theta^e_0)$ was arbitrary.

C The Cobweb case without leads

We show that Algorithm 5.1 has a simple modification which is applicable to one-dimensional models of the Cobweb type without leads, that is, when only one-step-ahead forecast matter. The algorithm presented in this appendix will converge to the perfect foresight steady state, if it exists. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ and consider an economic law of the Cobweb type, given by

$$\theta_t = F(\theta_t^e),$$

where θ_t^e is the forecast for θ_t based on information available at date $t-1$. The corresponding error function is defined by $e_F(\theta^e) := F(\theta^e) - \theta^e$. Assume that there exists a perfect foresight steady state $\gamma = F(\gamma) > 0$ of F known to lie in some interval $[\underline{\theta}, \overline{\theta}]$ with $e_F(\underline{\theta}) > 0$ and $e_F(\overline{\theta}) < 0$. Then the following algorithm applies.

Algorithm C.1 *Let $\theta_0^e \in [\underline{\theta}, \overline{\theta}]$ be arbitrary.*

1. *If $e_F(\theta_0^e) > 0$, then define $\epsilon = \frac{\overline{\theta} - \theta_0^e}{k}$ with some positive integer $k \geq 2$.*
   (a) *Set $\theta_t^e = \theta_0^e + t\epsilon$, $t > 0$ until $e_F(\theta_t^e) \leq 0$. Let τ denote the first time for which this condition is satisfied.*
   (b) *If $e_F(\theta_\tau^e) = 0$, then $\theta_t^e = \theta_\tau^e$ for all $t > \tau$; otherwise set set $\underline{\theta}' = \underline{\theta}$ and $\overline{\theta}' = \theta_\tau^e$.*
2. *If $e_F(\theta_0^e) < 0$, then let $\epsilon = \frac{\theta_0^e - \underline{\theta}}{k}$ with some positive integer $k \geq 2$.*
   (a) *Set $\theta_t^e = \theta_0^e - t\epsilon$, $t > 0$ until $e_F(\theta_t^e) \geq 0$. Let τ denote the first time for which this condition is satisfied.*
   (b) *If $e_F(\theta_\tau^e) = 0$, then $\theta_t^e = \theta_\tau^e$ for all $t > \tau$; otherwise set set $\underline{\theta}' = \theta_\tau^e$ and $\overline{\theta}' = \overline{\theta}$.*

By a reasoning similar to that of Sec. 6, Algorithm C.1 reduces the initial uncertainty interval $[\underline{\theta}, \overline{\theta}]$ and repeated application yields a sequence of forecasts which converge to the perfect foresight steady state.

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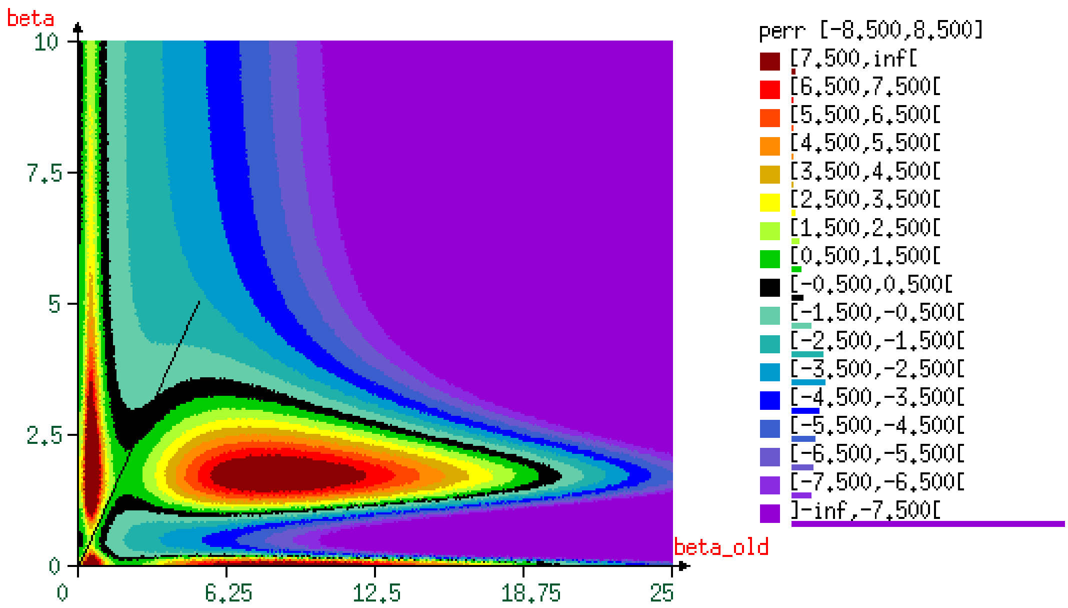

Figure 4: Error levels of an 'irregular' error function with $\gamma = 2.7,\ \rho = 0.5$.

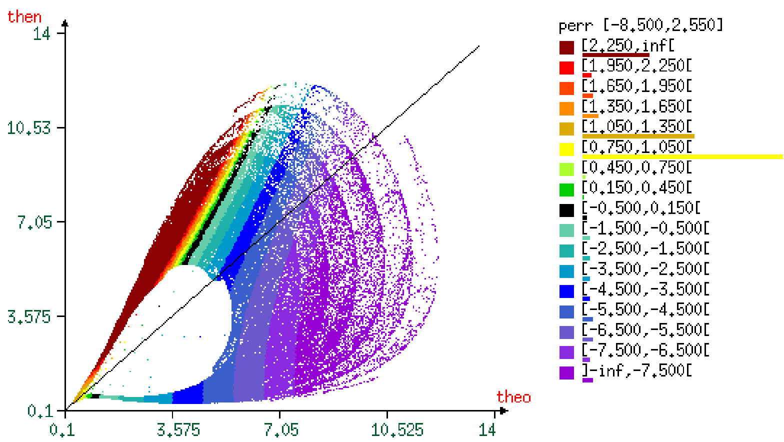

Figure 5: 500000 points of an attractor with corresponding error levels in the CES case; parameters $\gamma = 1.5$, $\rho = 0.735$, $w_1 = 2$, $w_2 = 0$, $\delta = 1$, initial conditions $\theta_0^e = 1.2$, $\theta_{-1}^e = 0.75$, $g_0 = 0.5$.

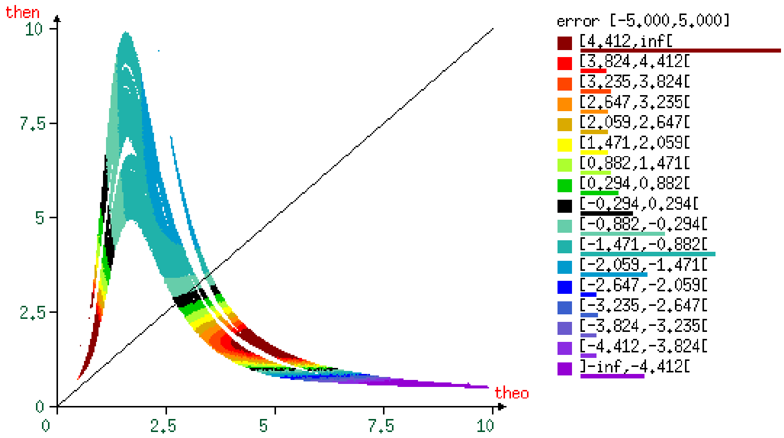

Figure 6: 500000 points of an attractor with corresponding error levels in the 'irregular' case; parameters $\gamma = 2.7$, $\rho = 0.5$, initial conditions $\theta_0^e = 0.5$, $\theta_{-1}^e = 1.0$, $g_0 = 0.5$.