Local durability and long-run habit persistence : An evaluation of the U.S equity premium.

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Abstract

This paper studies the empirical properties of a dynamic representative-agent model which displays effects of substitution and complementarity of consumption over time. Specifically, I investigate whether the dynamic model can replicate the observed mean and the standard deviation of the U.S real returns in the 1889-1995 period. First, the intertemporal marginal rate of substitution of consumption implied by the model statistically fits the Hansen and Jagannathan bound. Secondly, combined effects of substituion and complementarity over consumption nearly solve the equity premium and the risk-free rate puzzles. Finally, the model does also resolve the Campbell's stock market volatility puzzle.

Keywords : Habit formation and durability, equity premia, volatility of the intertemporal marginal rate of substitution, Hansen and Jagannathan bound, projection method.

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1 Introduction

This paper is motivated by two empirical issues : the equity premium puzzle and the risk-free rate puzzle. Mehra and Prescott [1985] show that Von Neumann-Morgenstern preferences in the capital asset pricing model (CAPM) could not explain high equity premium, unless agents are extremely risk averse. One response to the equity premium puzzle is to accept these high values for the coefficients of relative risk aversion. However, Weil [1989] argues that this leads to a second puzzle. He shows that a low riskless interest rate is possible only if agents have negative rate of time preference, in a standard consumption-based asset pricing model. This is the risk-free rate puzzle. One of the reasons put forward for these puzzles has been the lack of variability of the intertemporal marginal rate of substitution (IMRS) of consumption, relatively to the high variability of asset returns, given plausible risk aversion and discount factor values. For number of authors[1], one way to generate variability in the IMRS is to relax the hypothesis of time-separability of the representative agent's utility, by introducing habit persistence. They show that the pure habit persistence model can explain the equity premium puzzle at a low level of curvature. However, this model also implies an highly volatile IMRS and so an highly volatile asset returns. Campbell [1999] calls this the stock market volatility puzzle. They also show that introducing substitution of consumption over time in the time additive model reduces the volatility of the IMRS and so the volatility of the assets returns. Therefore a natural way to solve the Campbell's stock market volatility seems to introduce substitution effects in the pure habit persistence model. Heaton [1995] has already introduced the both effects in the time additive model. He found using monthly data (1959,1-1980,12) that the model can explain either an high volatility in asset returns or the equity premium mean. Yet, the model reproduces relatively precisely the standard deviation of the bond return. He proceeded by assuming that dividend stream corresponds to an aggregate stock return. I choose to reexamine the three puzzles by using annual data (1889-1995) in the model and by reestablishing the usual link between consumption and dividend to be as close as possible to the hypotheses of Mehra and Prescott.

A projection method is implemented to solve the model. From these simulations two complementary experiments are carried out to see if the model could explain the U.S equity premium in the 1889-1995 period. First, I examine the implications of the substitution and habit persistence effects and the combination of the both effects over the volatility of the IMRS. The Hansen and Jagannathan [1991] bound and the Cecchetti, Lam, and Mark [1994] test are implemented to explain those effects. The expected effects on the IMRS volatility are found. In addition, I find that increasing the habit persistence

[1] We could find other analyses of models with internal or external habit persistence in Abel [1990], Campbell and Cochrane [1999], Cochrane and Hansen [1992], Constantinides [1990], Hansen and Jagannathan [1991], Sundaresan [1989], for example. See Kocherlakota [1996] for a survey

reduces the volatility of the IMRS, relatively to the volatility of the model where the habit effects die out after one period. Then, I test if the model's implications concerning the volatility of the IMRS are satisfied. I find that the IMRS implied by the model statistically can fit the Hansen and Jagannathan bound. Second, I study the capacities of the model of fitting the mean and the standard deviation of the asset returns. This study is carried out first by the simulations of the means and the standard deviations of the asset returns implied by the pure habit formation model. If the habit effects die out after one period, habits increase both the equity premium mean and the volatility of the asset returns. If an higher degree of habit persistence is considered, the equity premium mean and the volatility of the asset returns are slightly reduced relatively to the previous model. However, the volatilities are unreasonably high in the both cases. Consequently, the substitution effects are introduced in the pure habit formation model and the previous moments are simulated. I find that substitution effects reduce the means and the standard deviations of asset returns. Finally, I find that the model, which exhibits substitution and complementarity effects, can explain the equity premium and the volatility of the two returns at a low level of curvature.

The plan of the paper is as follows. In the next section, I outline the model and the asset pricing environment. In section 3, I present the methodology used to solve the model. Then, the results are presented in section 4, and section 5 concludes the paper with some remarks about potential extensions.

2 The model

I consider a single-agent economy with frictionless markets and no taxes. The assumption of a single-agent economy is standard and is made in the spirit of Lucas [1978] and Cox J and Ross [1985]. The representative agent has preferences over a good S_t, called service, which are represented by the constant relative risk aversion (CRRA) utility function

$$U(\mathcal{S}) = E_0 \sum_{t=0}^{\infty} \beta^t \frac{S_t^{1-\gamma} - 1}{1-\gamma}, \qquad \gamma > 0, \mathcal{S} \equiv \{S_t : t = 0, 1, 2, ...\} \tag{1}$$

where β is the agent's subjective time discount factor, $E_0(.)$ is the mathematical expectation operator conditional on information in period zero. S_t is modelled using the specification of Ferson and Constantinides [1991] and Heaton [1995] :

$$S_t = C_t^F - \alpha(1-\theta) \sum_{j=0}^{\infty} \theta^j C_{t-1-j}^F, \quad 0 \leq \alpha \leq 1, \quad 0 \leq \theta \leq 1, \text{ and } \sum_{j=0}^{\infty} \theta^j = 1 \tag{2}$$

where C_t produce a flow of consumptions C_t^F given by

$$C_t^F = \sum_{\tau=0}^{\infty} \delta^\tau C_{t-\tau}, \text{ where } 0 \leq \delta \leq 1 \text{ and } \sum_{\tau=0}^{\infty} \delta^\tau = 1 \tag{3}$$

The parameter δ measures the degree to which consumption is substitutable over time. For an high value of δ, the substitution effect is high. S_t is also a function of a weighted sum of lagged consumption flows, $(1-\theta)\sum_{j=0}^{\infty} \theta^j C_{t-1-j}^F$. It measures the habit stock and θ gives the degree of habit persistence. The parameter α gives the proportion of habit stock that enters into the preferences. Thus, time non-separability in preferences is introduced over the consumption goods C_t, such as the preferences display both :

i) substitution of consumption over time,

ii) habit persistence.

The introduction of habit persistence effects makes consumption complementary over time. This model is called the complete model. The complete model can be decomposed in four other models. Thus, if $\theta = 0$, then the model is just a one-period habit persistence model, as it was studied by Ferson and Constantinides [1991]. Further, if $\alpha = 0$, then utility function is time-separable in consumption flow, and the model reflects only substitution of consumption over time. It is the infinite durable model. If $\delta = 0$, then the model displays only habit effect. It is the infinite habit persistence model. Finally, if $\alpha = \delta = \theta = 0$, then the model reduces to the usual case of the time additive model. Therefore, five models can be studied.

Combining (3) and (2) implies that S_t is given by

$$S_t = \frac{1 - \phi L}{(1 - \delta L)(1 - \theta L)} C_t = H(L) C_t, \tag{4}$$

where $\phi = \theta + \alpha(1 - \theta)$, and L is the lag operator. $H(z)$ is a polynomial such as this model displays substitutability for low z and habit persistence for high z. Thus, if $\delta < \theta$, and if α is not too large, $\frac{1-\phi z}{(1-\delta z)(1-\theta z)}$ will be positive for low z and so the model will display durability. Then, as far as z will become higher, $H(z)$ will be positive. Thus, habit persistence will dominate durability. That is the reason why Heaton presents the model as a model which exhibits *local* substitution and *long-run* habit persistence.

The representative agent trades both in a one-period bond with a risk-free payoff and a one-period risky equity that delivers a random dividend D_t at each period. Let $P_{f,t}$, $P_{e,t}$ be respectively the bond and the equity prices at period t that yield respectively one unit of consumption good and $(D_{t+1} + P_{e,t+1})$ units of consumption. Denoting F_t and E_t be respectively the amount of bonds and equities that is purchased in period t and held until period $t + 1$ by the representative agent and C_t his consumption. The budget constraint is given by

$$P_{f,t+1} F_{t+1} + P_{e,t+1} E_{t+1} + C_{t+1} = (D_{t+1} + P_{a,t+1}) E_t + F_t \tag{5}$$

The representative agent maximizes his intertemporal utility function subject to (4) and (5). The agent solves the maximization problem by determining contingent plans for C_t, S_t, E_{t+1}, F_{t+1}. The Euler equation governing the equity price is given by

$$\Lambda_t = \beta E_t \left[\Lambda_{t+1} (\frac{P_{e,t+1} + D_{t+1}}{P_{e,t}}) \right]. \tag{6}$$

The left hand side of (6) is the marginal utility cost of consuming one unit of numeraire good less at time t ; the right-hand side is the expected marginal utility benefit from investing the unit in the risky asset at time t, selling it at time $t+1$ for $(\frac{P_{e,t+1}+D_{t+1}}{P_{e,t}})$ units, and consuming the proceeds. The agent equates marginal cost and marginal benefit, such as (6) describes the optimum. If we divide both the left and the right hand sides of (6) by Λ_t, we get the familiar form

$$1 = E_t \left[\beta \frac{\Lambda_{t+1}}{\Lambda_t}.(\frac{P_{e,t+1} + D_{t+1}}{P_{e,t}}) \right] = E_t \left[MRS_{t+1}.R_{e,t+1} \right], \tag{7}$$

where MRS_{t+1} is IMRS or the stochastic discount factor, and $R_{e,t+1}$ is the gross real rate of equity return. The first-order condition describing the bond price is

$$\Lambda_t = \beta E_t \left[\Lambda_{t+1} \frac{1}{P_{f,t}} \right], \tag{8}$$

and the interpretation is the same as the Euler equation of the equity.

In this model the marginal utility of consumption, Λ_t, is a function of the marginal utility and the expected marginal utility of S_t, M_t. We get the following Euler equation

$$\Lambda_t = M_t - \beta \phi E_t \left[M_{t+1} \right] \tag{9}$$

where M_t, the marginal utility of S_t, is a function of M_{t+1} and M_{t+2} such as

$$M_t = S_t^{-\gamma} + \beta \left(\delta + \theta \right) E_t \left[M_{t+1} \right] - \beta^2 \left(\delta.\theta \right) E_t \left[M_{t+2} \right]. \tag{10}$$

Market clearing further imposes $C_t = D_t$, $\forall t$.

Therefore, the economy is described by a vector of two state variables $Z_t = (S_t, S_{t-1})$, and a variable of an exogenously given shock D_t. The stochastic dividend evolves through time according to

$$D_{t+1} = x_{t+1} D_t$$

where x_t is the gross growth rate of dividend. It is governed by a first-order Markov process[2]. The representative agent receives either the high dividend, in which case x is

[2] This process allows the apparent non-stationarity observed in the per capita consumption stream over the sample period.

equal to x_h, or receives the low dividend in which case x is equal to x_l. Following Mehra and Prescott [1985], the Markov chain is given by

$$\begin{aligned} x_h &= 1+\nu+\sigma, \qquad x_l = 1+\nu-\sigma \\ \pi_{h,h} &= \pi_{l,l} = \pi, \qquad \pi_{l,h} = \pi_{h,l} = 1-\pi \end{aligned}$$

where $\pi_{a,a'}$ denotes the transition probability from state $a \in \{l, h\}$ toward $a' \in \{l, h\}$. The transition matrix is assumed to be symmetric. Denoting ν be the average real growth rate of per capita consumption, σ be the standard deviation of the real growth rate of per capita consumption and $\pi = \frac{(1-\rho)}{2}$, where ρ is the first-order serial correlation of this growth rate. As in Mehra and Prescott [1985], the dividend growth are chosen to match the mean, the standard deviation, and the first-order autocorrelation of the US real growth rate of per capita consumption, in the 1891-1995 period. Appendix B described the data. The following calibration is obtained : $\nu = 0.001842$, $\sigma = 0.033$, and $\pi = 0.43$.

To empirically investigate the properties of this model, I must first assess the marginal utility of C_t. The solution to the equation (11), along with (12), define Δ_t which then can be used to solve (6) and (8) for asset prices. Unfortunately, I cannot analytically solve these functional equations. In the next section, the numerical method applied to solve the model is presented.

3 Solving method

The model presented above is a function of a non-stationary variable D_t. The model can easily be transformed to a system of equations that contains only stationary variables. To see this, define all non-stationary variables relatively to dividend[3]. The first order conditions of the problem can then be written as

$$\lambda_t = \mu_t - \beta\phi E_t \left[x_{t+1}^{-\gamma} . \mu_{t+1} \right], \tag{11}$$

$$\mu_t = s_t^{-\gamma} + \beta(\delta+\theta) E_t \left[x_{t+1}^{-\gamma} \mu_{t+1} \right] - \beta^2 (\delta.\theta) E_t \left[x_{t+2}^{-\gamma} \mu_{t+2} \right], \tag{12}$$

$$p_{e,t} = \beta E_t \left[\left(\frac{\lambda_{t+1}}{\lambda_t} \right) x_{t+1}^{-\gamma+1} (p_{e,t+1}+1) \right] \tag{13}$$

$$P_{f,t} = \beta E_t \left[\left(\frac{\lambda_{t+1}}{\lambda_t} \right) x_{t+1}^{-\gamma} \right] \tag{14}$$

[3]Analytically, the deflations are such as $s_t = \frac{S_t}{D_t}$, $y_t = \frac{Y_t}{D_{t-1}}$, $\mu_t = \frac{M_t}{D_t^{-\gamma}}$, $\lambda_t = \frac{\Lambda_t}{D_t^{-\gamma}}$, and $p_{e,t} = \frac{P_{e,t}}{D_t}$. From now on, lower case variables are used to denote variables that are growth rates or variables that are relative to dividend.

where λ_t and μ_t represent the deflated policy variables and $p_{e,t}$ the deflated equity price. The recursive equation of S_t is also transformed such as there are only stationary variables in its expression. Applying the market clearing restriction, the recursive properties of the good service are governed by the following functions

$$\begin{aligned} s_{t+1} &= (\delta+\theta)x_{t+1}^{-1}s_t - (\delta.\theta)y_t(x_{t+1}x_t)^{-1} - \phi x_{t+1}^{-1} + 1 \\ y_{t+1} &= s_t \end{aligned} \tag{15}$$

In this economy, the state variables are s_t, s_{t-1},and x_t. Solving the model involves characterizing the marginal utility functions of s and c, and the asset price functions, $p_{e,t}$ and $P_{f,t}$. These variables have to satisfy the Euler equations (11), (12), (13), (14), and the recursive properties of s. The model is solved relying on the projection method (see Judd [1998] and Christiano and Fisher [2000]). Appendix A described the algorithm. To approximate μ_t, I use the function $\hat{\mu}\left(s, y, x; \boldsymbol{\varphi}_x\right)$, where $\boldsymbol{\varphi}_x$ is an unknown vector of parameters and $\hat{\mu}\left(.\right)$ is a polynomial chosen from the class of Chebyshev polynomials. Similarly, I use the function $\hat{p}_e\left(s, y, x; \boldsymbol{\eta}_x\right)$ to approximate the deflated equity price.

In the model, the marginal utility of service μ_t depends on current expectations of marginal utility of s more than one period ahead, which do make the computational resolution harder. Given this computational difficulty, the following iteration scheme is used to find $\boldsymbol{\varphi}_x$ and $\boldsymbol{\eta}_x$:

Step 1. *I approximate the marginal utility of s. That is, obtain parameter values for* φ_x.

Step 2 *Given parameter values of* φ_x *obtained in the first step, I solve the whole problem. That is, I both approximate the marginal utilities of s and c, and the asset prices. So I find parameter values for* φ_x *and* η_x.

Given the asset prices, the asset returns can easily be calculated using these equations

$$R_{e,x_i} = E_{t+1}\left[\left(x_{t+1}.\left(\frac{\hat{p}_e\left(s_{t+1}, s_t, x_t; \hat{\boldsymbol{\eta}}_x\right)}{\hat{p}_e\left(s_t, s_{t-1}, x_t; \hat{\boldsymbol{\eta}}_x\right)}\right)\right)\right] - 1,\ i = l, h \tag{16}$$

$$R_{f,x_i} = E_{t+1}\left[\left(\frac{1}{\hat{P}_f\left(s_{t+1}, s_t, x_t; \hat{\varphi}_x\right)}\right)\right] - 1,\ i = l, h \tag{17}$$

and the expected returns are

$$R_i = 0.5.R_{i,x_l} + 0.5.R_{i,x_h}, \quad \text{for } i = e,\ f.$$

4 The results

In this section, I carry out two complementary experiments to see wether the previous model can explain the U.S risk premia in the period 1889-1995. First, I test the model's implications in term of volatility of the IMRS. Second, I analyze the time-series properties of the complete model by decomposing the habit persistence and the substitution effects. Specifically, I compute the first and the second empirical moments for a particular set of parameter values, and I compare these moments with their empirical counterpart. Then, I choose the set of parameter values that better replicate the two observed moments. I compute the moments with a simulated draw of 10000 observations, discarding the first 500 simulations. In the first part of this section, I present the results concerning the volatility of the IMRS. In the second part of the section, the time-series properties of the complete models are reported.

4.1 The Hansen and Jagannathan bound.

According to Constantinides [1990], the equity and the risk-free puzzles occur in the time additive model because of the lack of variability of the IMRS relatively to the strong volatility of the asset returns. In this section, I study the effects of durability and habit on the volatility of the IMRS. The Hansen and Jagannathan [1991] method is carried out to study these effects. Hansen and Jagannathan [1991] derived a lower bound on the volatility of the IMRS that correctly prices the assets[4]. The bound is denoted by HJ. One advantage of their procedure is that the bound they construct makes no reference to a particular model. It is solely calculated from asset returns data. To estimate the bound, equity and bond returns are considered. I use the $R_{f,t}$, and the $R_{e,t}$ described in the data appendix and two additional artificial returns, $R_{e,t-1}.R_{f,t}$, $R_{f,t-1}.R_{e,t}$, which prices are respectively $R_{e,t-1}$, $R_{f,t-1}$. $x_t = (R_{e,t}, R_{f,t}\ , R_{e,t-1}.r_{f,t}\ , R_{f,t-1}.R_{e,t}\ )$ is the vector of asset returns and $q_t = (1, 1,\ R_{e,t-1},\ R_{f,t-1})$ is the vector of asset prices. I extend this visual method by implementing a statistical procedure for judging whether the complete model of the previous section is able to statistically fit this lower bound. I use here the methodology of Cecchetti, Lam, and Mark [1994] to perform the statistical inference. Their statistic measures the vertical distance, labeled Δ, between a sample pair (μ_v, σ_v) and the lower bound HJ σ_x, where μ_v and σ_v respectively represent the empirical mean and standard error of a particular $IMRS$. The candidate IMRS is rejected if its sample pair significantly lies below the bound. In order, to evaluate whether the difference is

[4]In this paper, the positivity restriction on the IMRS is not imposed to compute the lower bound.

large, they compute the following statistic :

$$\begin{aligned} H_0 &: \quad \Delta \leq 0 \\ \frac{\Delta}{\hat{\sigma}_\Delta} &= \left(\frac{\hat{\sigma}_v - \hat{\sigma}_x}{\hat{\sigma}_\Delta}\right) \\ \hat{\sigma}_\Delta &= \left(\frac{\partial \Delta}{\partial \theta'}\right)_{\hat{\theta}} \widehat{\Sigma}_\theta \left(\frac{\partial \Delta}{\partial \theta}\right)_{\hat{\theta}} \end{aligned}$$

where Δ has asymptotically gaussian distribution with mean 0 and variance σ_Δ^2, and $\widehat{\Sigma}_\theta$ is the estimated covariance matrix of the parameter θ, such as $\theta = \left(\mu_q, \mu_x, \Sigma_x\right)'$. Here, μ_q is the mean vector of the four asset prices, and μ_x, Σ_x are respectively the mean vector and covariance matrix of the 2×2 assets payoffs. In practice, I compute $\hat{\theta}$, and $\hat{\Sigma}_\theta$ by generalized method of moments using the first two moments of asset returns and the first moment of asset prices[5]. The covariance matrix $\hat{\Sigma}_\theta$ is the Newey and West [1987] covariance matrix estimator. These complementary methods are implemented for the complete model and its special cases.

As noted by Hansen and Jagannathan [1991], the time additive model is not able to generate enough volatility in the IMRS for low level of γ. This also the case here. The test of the volatility bound restrictions reports that a γ as large as 20 cannot fit statistically the Hansen and Jagannathan bound. When I consider a model in which only the durability effect is taking into account ($\alpha = \theta = 0$), the volatility of the IMRS goes significantly down. For example, when $\delta = 0.5$, and $\gamma = 20$, the statistic of the test of the volatility bound restrictions is equal to -3.943, against only -3.35 for the same value of γ and $\delta = 0$. Further, as δ goes up, the volatility fall. These results are consistent with Cecchetti, Lam, and Mark [1994], Ferson and Constantinides [1991], and Heaton [1995]. Yet, here the drop of volatility of the IMRS is strengthened because the durability effects do not die out after one period.

An opposite effect can be obtained by considering a model which exhibits only habit effects ($\delta = 0$). First, suppose that habit effects die out after one period ($\theta = 0$). For this kind of model, a $\gamma = 2$, and $\alpha = 0.1$ generate an IMRS sufficiently high to statistically fit the Hansen and Jagannathan bound. Further, increases in α do significantly goes up the volatility of the IMRS. For example, when $\gamma = 2$, and $\alpha = 0.1$, the volatility of the IMRS is equal to 2.03%, for $\alpha = 0.5$, the volatility is equal to 18.28%. The table 1 reports values of the statistics of the test of the volatility bound restrictions and the P-values of the tests of the hypothesis that the distance between the second moment of the IMRS and the Hansen and Jagannathan bound is less than or equal to zero against the alternative

[5] The moment conditions used in estimation are $E\left[x_t - \mu_x\right] = 0$,

$$E\left[q_t - \mu_q\right] = 0,$$
$$E\left[vec\left(x_t x_t'\right) - vec\left(\Sigma_x\right) + vec\left(\mu_x \mu_x'\right)\right] = 0.$$

that it is positive. The tests are implemented for α ranges from 0.05 to 0.4 in increments of 0.05. For $\beta = 1$, $\delta = 0$, $\gamma = 3$, a value of α higher than 0.25 is needed to statistically fit the bound. In addition, as α becomes larger, the volatility is stronger. So imposing habit formation increases significantly the volatility of the IMRS. This is consistent with the results of Cecchetti, Lam, and Mark [1994], Cochrane and Hansen [1992], Ferson and Constantinides [1991], Gallant, Hansen, and Tauchen [1990], and Heaton [1995]. So, a sufficiently high proportion of habit stock is needed to fit the HJ bound.

When θ is different from zero (so the habits effects do not die out after one period), the equations (11), (12), and the dynamic of s_t have the following form

$$\begin{aligned}
\lambda_t &= s_t^{-\gamma} - \beta\alpha(1-\theta)E_t\left[x_{t+1}^{-\gamma}\mu_{t+1}\right], && (18)\\
\mu_t &= s_t^{-\gamma} + \beta.\theta E_t\left[x_{t+1}^{-\gamma}\mu_{t+1}\right], && (19)\\
s_t &= 1-\alpha(1-\theta)(x_t^{-1} + \theta x_t^{-1}x_{t-1}^{-1} + \theta^2 x_t^{-1}x_{t-1}^{-1}x_{t-2}^{-1} + ...) && (20)
\end{aligned}$$

The table 1 reports the statistics of the test of the volatility bound restrictions, and its corresponding P-value for infinite period habit persistence model. In this table, θ can also take two values : 0.1, and 0.7. As for the one-period habit model and for each values of θ, increasing α tends to increase the volatility of the IMRS. Yet for α constant, as θ becomes larger, the volatility of the IMRS goes down. For example, when $\theta = 0.7$, values of α lower than 0.4 are not able to yield an IMRS that fit the HJ bound. Whereas, when $\theta = 0.1$, the nul hypothesis is accepted for a value of $\alpha = 0.3$.

Table 1 : Results of tests of the volatility bound restrictions, β=1, δ=0, γ=3.

α	*Test statistic* $(P-value)$ $\theta = 0$		*Test statistic* $(P-value)$ $\theta = 0.1$		*Test statistic* $(P-value)$ $\theta = 0.7$	
0.05	-2.074	(0.01907)	-2.074	(0.01906)	-2.058	(0.01979)
0.1	-2.017	(0.02186)	-2.080	(0.01879)	-2.028	(0.02130)
0.15	-1.937	(0.02638)	-2.009	(0.02227)	-2.026	(0.02138)
0.2	-1.821	(0.03432)	-1.828	(0.03377)	-1.854	(0.03191)
0.25	-1.644	(0.05008)	-1.743	(0.04068)	-1.842	(0.03277)
0.3	-1.360	(0.08687)	-1.495	(0.06747)	-1.882	(0.02995)
0.35	-0.878	(0.1900)	-1.208	(0.1136)	-1.724	(0.04237)
0.4	-0.178	(0.4292)	-0.7079	(0.2395)	-1.171	(0.1209)

Notes : 9500 observations of the simulated series are used to calculate the moments of IMRS.

So, increase the habit persistence effect reduce the volatility of the IMRS. This result is different from the result of Heaton [1995]. Heaton found that larger values of θ tend to increase the volatility of the IMRS. To understand how does it occur, considering first the model where $\alpha = 0.5$, $\theta = 0.01$, and β sets to unity. Notice that for this parameter setting, the previous equations are approximately equal to

$$\lambda_t \simeq s_t^{-\gamma} - 0.495 E_t \left[x_{t+1}^{-\gamma} s_{t+1}^{-\gamma} \right], \tag{21}$$

$$\mu_t \simeq s_t^{-\gamma} \tag{22}$$

$$s_t \simeq 1 - 0.495 x_t^{-1} \tag{23}$$

For this parameter setting, these three equations are very closed to those of the one-period habit persistence model. In this kind of model, Gallant, Hansen, and Tauchen [1990] showed that the larger value of α, the higher volatility of IMRS. Here, $\alpha(1-\theta)$ is always lower than α, $(0.495 < 0.5)$. Therefore, introducing a low degree of habit persistence upper than one period slightly reduces the volatility of the IMRS relatively to the one period habit persistence model. Notice also that for a high value of θ, for example $\theta = 0.99$, the equation (11), and the dynamic of s_t are such as

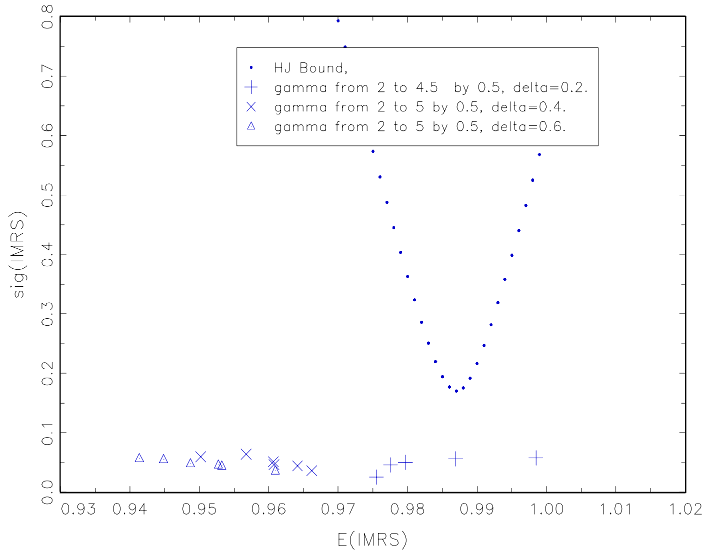
$$\lambda_t = s_t^{-\gamma} - 0.005 E_t \left[x_{t+1}^{-\gamma} \mu_{t+1} \right] \simeq s_t^{-\gamma}, \tag{24}$$

$$s_t = 1 - 0.005(x_t^{-1} + 0.99 x_t^{-1} x_{t-1}^{-1} + 0.99^2 x_t^{-1} x_{t-1}^{-1} x_{t-2}^{-1} + ...) \simeq 1 - 0.005 x_t^{-1} \tag{25}$$

Note also that when α is low, the marginal utility of consumption of the one-period habit persistence model is approximately equals to $s_t^{-\gamma}$, where $s_t = 1 - \alpha x_t^{-1}$. Therefore, the two previous equations are very closed to the corresponding equations of the one-period habit formation model where $\alpha = 0.005$. So, the higher θ, the lower volatility of IMRS.

Given these previous effects, I ask wether the complete model can yield an IMRS sufficiently high to fit the HJ bound. Figure 1 plots the Hansen and Jagannathan bound of the complete model. It is represented by the U-shaped region. The figure represents the standard deviation bound, $std(IMRS)$, as a function of the mean of the IMRS, $E(IMRS)$. The figure 1 also plots the simulated mean-standard deviation pairs of the $IMRS$, for different values of β, γ, δ, θ, α. These moments are computed with a simulated draw of 10000 observations, discarding the first 500 simulations and with $\beta = 1$, $\theta = 0.8$, $\alpha = 0.6$. The parameter δ is allowed to take three values : 0.2, 0.4, 0.6. Further, in the plot the parameter γ ranges from 2 to 4.5 in increments of 0.5. The figure 1 shows that the volatility of the $IMRS$ increases, but the mean decreases as γ increases, for δ equals to 0.4, 0.6. Therefore, the triangles ($\delta = 0.4$) and the stars ($\delta = 0.6$) move away from the admissible region. Whereas, for δ equals to 0.2, the plus get nearer to the HJ bound. Nevertheless, the model for this specification do not generate enough volatility in the $IMRS$, to fit the HJ bound.

Figure 1: HJ bound and the simulated mean-standard deviation pairs



Note : These moments are computed with a simulated draw of 10000 observations, discarding the first 500 simulations and with β, θ , α respectively equal to 1, 0.8, and 0.6, for δ equals to 0.2, 0.4, 0.6 and ranging from 2 to 4.5 in increments of 0.5.

For the same parameter settings, the volatility bound test is implemented. For δ sets to 0.4, and 0.6, the model is not able to generate a sufficiently high volatility to fit the HJ bound. Whereas, when $\delta = 0.2$, and $\gamma = 4$, the test statistic is equal to -0.4818 and its corresponding P-value is 0.3150. Therefore, although the simulated mean-standard deviation pairs of the $IMRS$ for this latter parameter setting do lie below the HJ bound in the figure 1, the distance is not significant. So even with substitution effect, the complete model can fit the HJ bound and so the model can correctly price the assets. Nevertheless, the substitution effect should be low.

4.2 The equity premium and the risk-free rate puzzles

In this section, I study the capacities of the complete model of fitting the mean and the standard deviation of the observed assets returns. To understand how the complete model can explain the equity premium puzzle, I decompose the effects of substitution and habit persistence. Thus, the infinite durable model, the one-period habit persistence model, and the infinite period habit persistence model are studied. For each possible set of parameter values, the two moments of each model are simulated. Then, the set of parameters values which generates a risk-free return mean lower than 3%, and an equity return mean larger than 5% are put aside. Furthermore, the parameter values that generates a negative simulated marginal utility of consumption are rule out.

The infinite durable model ($\alpha = \theta = 0$),
The model cannot fit the observed mean and standard deviation. For example, for δ sets to 0.1 and a value of γ lower than 8.1, the model is unable to generate an equity premium mean upper than 1%. Whereas, the time additive model yields an equity premium of 1.94%, for γ sets to 8.1. So, durability substantially reduces the equity premium mean. For larger values of curvature parameter and δ still sets to 0.1, the equity premium goes up, but it is still low. For example, a $\gamma = 18$ cannot generate an equity premium higher than 2%. And as δ becomes larger, an higher γ is needed to generate an equity premium above 1%. For δ sets to 0.15, the model generate an equity premium above 1%, if γ is above 14.6. Therefore, the positive effect of γ on the equity premium is outclassed by the negative effect of the substitution effect. Similar results are found in the one-period durability model (see Cecchetti, Lam, and Mark [1994], Ferson and Constantinides [1991]). Here, the negative effect on the equity premium is strengthened by the infinite substitution effect of consumption over time. Moreover, the introduction of substitution reduces the volatility of the asset returns. Note that in the time additive model, increasing the relative risk aversion coefficient makes asset returns more volatile. Here, the effect is weakened by the substitution effect. For example, for δ still sets to 0.1, and γ ranging from 8.1 to 18.1, the volatility of the equity and bond returns only increases respectively by 0.83 and 0.78 percentage point, against respectively 5.56 and 4.35 percentage point in the time additive model. So, as far as γ goes up, the standard deviation of the asset returns stay relatively constant. So, δ slow down the volatility of the asset returns. This is consistent with the results of Cecchetti, Lam, and Mark [1994], Ferson and Constantinides [1991], and Heaton [1995].

The one-period habit persistence model ($\delta = \theta = 0$)
The model can fit the observed means of the asset returns. However, the one-period habit persistence model yields an unreasonably high volatility for asset returns. The table 2 reports these two effects. The introduction of the habit formation in the time-separable model has two effects. The first one is to increase the simulated mean of the equity premium. For example, a γ as little as 0.2 and a $\alpha = 0.8$ generate a mean of equity premium above 2%. Further, as the habit parameter becomes larger, the equity premium increases. For example, when the curvature parameter is set to 3.3, and α equal to 0.1, 0.2, 0.3, 0.4, 0.5, the model respectively yields equity premium of 0.86%, 1.42%, 2.53%, 5.04%, and 11.78%. The second effect is to dramatically increases the volatility of the asset returns. Further, increasing the habit effect makes asset returns more volatile. Cecchetti, Lam, and Mark [1994], Ferson and Constantinides [1991], and Heaton [1995] found the same result. Campbell [2000] calls this the stock market volatility puzzle. For example, when $\gamma = 3.3$, $\alpha = 0.4$, the model exhibits an equity and a bond returns of 9.06% and 4.01%, but also standard deviations of 38.09% and 24.67% respectively. For the same curvature level and $\alpha = 0.5$, the standard deviations of each asset returns

increases respectively by 10.7 and 5.66 percentage point.

Table 2 : The one-period habit persistence model δ=θ=0, β=1, γ=3.3.

α	0.1	0.2	0.3	0.4	0.5
mean $R^e - R^f$	0.86	1.42	2.53	5.04	11.78
std R^e	7.44	11.21	17.31	38.09	48.02
std R^f	3.62	6.48	10.92	24.67	30.34

The infinite period habit persistence model ($\delta = 0$)
This model is also a pure habit formation model which displays infinite habit persistence. The introduction of an higher persistence effect has two effects. The first one is to reduce the asset returns and so the equity premium. For example, when γ, α equal respectively to 3.3, 0.5, and θ equals 0.1, the model generates an equity premium of 10.65%, against 11.78 in the one-period habit persistence model. In addition, as θ becomes higher, the equity premium slows down. When the persistence parameter equals to 0.4, 0.5, 0.6, 0.7, the model respectively generates equity premium of 8.26%, 7.58%, 6.89%, and 6.11%. All these results are reported in the table 3. The second effect is to reduce the volatility of the asset returns. In addition as θ becomes higher, the volatility becomes lower. However, the volatility still remains high. For example, when γ, α equal respectively to 3.3, 0.5, the volatilities of the equity and the bond returns are respectively 48.02%, and 30.34% in the one-period habit model. In the infinite habit formation model, θ equals 0.4, the volatilities are 43.10%, and 26.39%. For higher habit persistence effect, θ equals 0.7, the volatilities are 24.57%, and 10.50%. Therefore as in the previous subsection, increasing the habit persistence effect reduces the volatilities of the asset returns relatively to those of one-period habit persistence model.

Table 3 : The infinite period habit persistence model δ=0, β=1, γ=3.3, α=0.5.

θ	0.4	0.5	0.6	0.7
mean $R^e - R^f$	8.26	7.58	6.89	6.11
std R^e	43.10	30.05	27.39	24.57
std R^f	26.39	15.01	12.72	10.50

The complete model
The intuition behind this model is to introduce durability in the infinite habit formation model, in order to reduce the volatilities of the asset returns. However, introducing durability will also reduces the mean of the equity premium. For example, when $\gamma = 3.3$, $\alpha = 0.5$ and $\theta = 0.6$, and δ sets to 0.1, the standard deviations of the equity and the bond returns are respectively 19.64% and 7.8%, against respectively 27.37%, 12.73% in the infinite period habit persistence model. Yet, the equity premium is 4.09%, against

6.8% in the infinite period habit persistence model . So to generate an higher equity premium, a larger α should be considered and to reduce the rise in volatility implied by the increase of α, a higher δ can be considered. So, for $\delta = 0.2$, $\alpha = 0.6$ and the other parameters remaining constant, the equity premium is 5.58% and the standard deviations of the equity and the bond returns are respectively 22.74% and 7.81%.

Finally, to increase the equity premium of asset returns, I set θ to 0.5. So the parameter setting is such as γ, δ, α and θ are respectively equal to 3.3, 0.2, 0.6 and 0.5, for a discount factor sets to unity. First, for these parameter values the vertical distance between the sample pair $(E[IMRS], V[IMRS])$ and the lower bound is not significant. I obtain a P-value equals to 0.3488. So, the admissible region is fitted for these parameter values. Therefore, this consumption-based asset pricing model generates enough volatility in the IMRS to correctly price the assets. As it was shown in the previous section, a high proportion of habit stock and a low substitution effect fit the HJ bound. The first two moments implied by these parameter values are reported in the table 4. In this table, I respectively denote by r_i^{obs} and r_i^{sim} the observed and the simulated asset i return. The means and standard deviations of the real returns are reported on annualized basis. The simulated mean of real returns on equity and bond are coherent compared to those of the sample means. The model still undervalues the mean of the equity return. But the simulated mean of the equity return is now comparable to the observed equity return. In addition, the simulated mean risk-free rate is very close to observed mean bond return. The third column of table 4 displays the average equity premium, denoted by $(r_e - r_f)^{obs}$ for the observed and $(r_e - r_f)^{sim}$ for the simulated equity premium. Simulated equity premium is 6.11% per annum in the model, against 0.39% in the time additive model. Therefore, combined effects of substitution and complementarity over consumption can fit the equity premium mean at a low level of curvature. Though, Heaton [1995] finds that the complete model does not fit the equity premium well.

Table 4 : The moments implied by the following parameter setting : β=1, γ=3.3, δ=0.2, α=0.6, θ=0.5.

	r_e^{obs}	r_e^{sim}	r_f^{obs}	r_f^{sim}	$(r_e - r_f)^{obs}$	$(r_e - r_f)^{sim}$
mean	0.0880	0.0824	0.0238	0.021	0.0642	0.061
std	0.2045	0.245	0.0910	0.102	0.1990	0.280

Note : This table displays the sample and the simulated means and standard deviations of the asset returns and the equity premium.

The simulated moments are obtained for γ, δ, α and θ respectively set to 3.3, 0.2, 0.6 and 0.5, and for a discount factor sets to unity. 10000 observations of the simulated series are used to calculate the simulated moments of asset returns.

The observed asset returns moments are calculated for the period 1889-1995.

However, despite the negative effect of the substitution effect on the volatilities of the asset returns, the simulated second order moments are above the observed second order moments in the both cases. Nevertheless, the standard deviation of the risk-free rate seems in line with the data. It is a good result compared to the one-period habit persistence model. The latter model actually implies extremely volatile stochastic discount factor to explain the equity premium puzzle. Therefore, it generates a very volatile risk-free rate (see above). For example, when $\gamma = 3.3$, $\alpha = 0.5$, $\theta = 0.6$, and $\delta = 0$, the standard deviation of the simulated risk-free rate is 30.33%. Increasing δ yields a much lower standard deviation. Yet, it is still above the observed standard deviation. Heaton [1995] also estimated relatively precisely the second moment of the bond return. Therefore, the introduction of local substitution substantially improves the model's ability to fit the risk-free rate volatility. Introducing substitution effects also reduces the volatilities of the equity returns. For example, the complete model generates a simulated equity volatility 4.05 percentage point higher than the observed equity volatility, against 9.6 percentage point in the pure habit formation model. Heaton [1995] is capable of explaining the highly volatile stock return or the equity premium. This occurs according to Heaton because the complete model is required to fit the extremely low volatility of the monthly bond return. Here with this annual data set, the standard deviation of bond is much more higher. So, the constrain to have a little volatility in the bond does not work here. In addition, the negative effect of an habit persistence upper than one period on the volatility of the asset returns, relatively to the one-period habit persistence model, strengthens the negative effect of the substitution effect. Therefore, these two effects help to fit the standard deviation of the asset returns. Taking into account the annual data set and these double negative effect, the complete model can both fit the second moment of the asset returns and the equity premium.

5 Conclusion

The purpose of this paper was to examine the empirical properties of a non-linear stochastic dynamic model with rational expectations, in which the representative agent is assumed to display time non separable preferences. Specifically, I carried out two complementary studies to check the empirical relevance of the model. First, I used the Hansen and Jagannathan bound to check if the consumption based asset-pricing model with local substitution and long run habit persistence over consumption correctly prices the assets. I found that introducing a low degree of habit persistence upper than one period slightly reduces the volatility of the IMRS relatively to the one period habit persistence model. In addition, despite the negative effects of the latter effect and the substitution effect, the IMRS implied by the model statistically can fit the Hansen and Jagannathan bound, if the degree of substitutability is relatively low and the proportion of habit stock is high. Secondly, I compared the simulated two first order moments with those observed. I concluded that combined effects of substitution and complementarity over consumption can explain the equity premium and the risk free rate puzzles at a low level of curvature. In addition, I found that the introduction of local substitution substantially improves the model's ability to fit the volatility of the asset returns, compared to the pure habit persistence model. Finally, the model does resolve the Campbell's stock market volatility puzzle.

Nonetheless, these results may be improved in three ways. First, I studied a partial equilibrium representative-agent model. It would be interesting to consider the same preferences in a general equilibrium model. Secondly, I also maintain the assumption of homogeneous agents. One other possibility would be to investigate a non-linear stochastic dynamic model with heterogeneous agents. Thirdly, I suppose a complete-market economy. However, the implications of equilibrium incomplete-market economy deserve to be studied, because agents will be limited in their ability to smooth consumption.

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A Approximating the policy rules with the projection method

The model is solved relying on the method of weighted residuals (see Judd [1998], and Christiano and Fisher [2000]), and the approximating functions are chosen from the class of Chebyshev polynomials. Thus, if we consider the marginal utility of service, the approximating function is such as

$$\hat{\mu}(s, y, x; \boldsymbol{\varphi}_x) = \hat{\mu}_{\varphi_x} = \sum_{i=1}^{n_s} \sum_{j=1}^{n_y} \varphi_{ij,x} \psi_{ij}(s, y), \quad \text{for } x = x_l, x_h \tag{26}$$

where $\psi_{ij}(s, y) \equiv T_{i-1}(2((s - s_l)/(s_h - s_l)) - 1) T_{j-1}(2((y - s_l)/(s_h - s_l)) - 1)$, $T_i(.)$ and $T_j(.) : [-1, 1] \to [-1, 1]$, are Chebyshev polynomials, and $\boldsymbol{\varphi}_x$ is a $n_s \times n_y$ vector of parameters. I use the linear transformation $2((s - s_l)/(s_h - s_l)) - 1$ in order to take into account that Chebyshev polynomials are defined in $[-1, 1]$. The approximating function must satisfy the following residual function :

$$R\left(s, y, x; \hat{\mu}_{\varphi_x}\right) = 0, \text{ for all } s, y, x \in [s_l, s_h] \times [s_l, s_h] \times [x_l, x_h]$$

where the residual function is just defined by the residuals of the Euler equation (12), such as

$$\begin{aligned} R\left(s, y, x; \hat{\mu}_{\varphi_x}\right) &= \hat{\mu}(s, y, x; \boldsymbol{\varphi}_x) - s^{-\gamma} - \beta(\delta + \theta) E\left[x'^{-\gamma}.\hat{\mu}(f(s, y, x, x'), s, x'; \boldsymbol{\varphi}_x) \mid x\right] \\ &+ \beta^2 (\delta.\theta) E\left[E\left[(x'')^{-\gamma}.\hat{\mu}(f(s', y', x', x''), f(s, y, x, x'), x''; \boldsymbol{\varphi}_x) \mid x'\right] \mid x\right], \quad \text{for } x = x_l, x_h \end{aligned}$$

where I denote the current value of x_t by x, the next period's value by x' and x_{t+2} by x'', and the recursive properties of s are summarized by the function f. The problem is then to identify the set of parameters $\boldsymbol{\varphi}_x$, for each state of the economy, defining the approximation. This is undertaken using the projection algorithm

Step 1. *Compute $m_s \times m_y$ nodes at which the residual function will be evaluated. These nodes correspond to the roots, z^s, z^y ,of the Chebyshev polynomials of order $m_s \times m_y$. Then form the $(m_s \times m_y, n_s \times n_y)$ matrix given by*

$$X = \begin{pmatrix} \psi_{11}(z_1^s, z_1^y) & \psi_{12}(z_1^s, z_1^y) & \cdots & \psi_{(n_s+1)(n_y+1)}(z_1^s, z_1^y) \\ \psi_{11}(z_1^s, z_2^y) & \psi_{12}(z_1^s, z_2^y) & \cdots & \psi_{(n_s+1)(n_y+1)}(z_1^s, z_2^y) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{11}\left(z_{m_s}^s, z_{m_y}^y\right) & \psi_{12}\left(z_{m_s}^s, z_{m_y}^y\right) & \cdots & \psi_{(n_s+1)(n_y+1)}\left(z_{m_s}^s, z_{m_y}^y\right) \end{pmatrix}$$

Step 2. *Given an initial guess of φ_x, compute at each node the residual function. The following $(m_s \times m_y, 1)$ vector of residual function is obtained for the state x*

$$R\left(s, y, x; \hat{\mu}_{\varphi_x}\right) = \left[R\left(s_1, y_1, x; \hat{\mu}_{\varphi_x}\right), ..., R\left(s_1, y_{m_y}, x; \hat{\mu}_{\varphi_x}\right), ..., R\left(s_{m_s}, y_{m_y}, x; \hat{\mu}_{\varphi_x}\right)\right]'$$

Step 3. Then form the following projections

$$X'.R\left(s,y,x;\hat{\mu}_{\varphi_x}\right)=0,\quad \text{for } x=x_m,x_M. \tag{27}$$

(27) represents a nonlinear system of $2.n_s.n_y$ *equations in the* $2.n_s.n_y$ *unknown* $\varphi_x=\left[\boldsymbol{\varphi}_{x_l},\boldsymbol{\varphi}_{x_h}\right]'$.

Step 4. By iterating over step 2 and 3, find φ_x *which sets the* $2.n_s.n_y$ *projections equal to zeros.*

This system can be solved using the version of Newton-Raphson method implemented in the GAUSS routine, $NLSYS$.

I use the same algorithm to approximate the equity price dividend ratio, $p_{e,t}$. As previously, the approximation of $p_{e,t}$ is given by

$$\hat{p}_e\left(s,y,x;\boldsymbol{\eta}_x\right)=\sum_{i=1}^{n_s}\sum_{j=1}^{n_y}\eta_{ij,x}\psi_{ij}\left(s,y\right),\quad \text{for } x=x_l,x_h \tag{28}$$

$\hat{p}_e\left(s,y,x;\boldsymbol{\eta}_x\right)$ has to satisfy the residual function $\Re$, defined by the residuals of (13), such as

$$\Re\left(s,y,x;\hat{\mu}_{\varphi_x},\hat{p}_{e,\eta_x}\right)=\hat{p}_e\left(s,y,x;\boldsymbol{\eta}_x\right)-$$
$$\beta E\left[\left(\frac{\hat{\lambda}\left(f\left(s,y,x,x'\right),s,x';\boldsymbol{\varphi}_x\right)}{\hat{\lambda}\left(s,y,x;\boldsymbol{\varphi}_x\right)}\right)\left(x'\right)^{-\gamma+1}\left(\hat{p}_e\left(f\left(s,y,x,x'\right),s,x';\boldsymbol{\eta}_x\right)+1\right)\mid x\right]$$

where the approximation of the marginal utility of consumption is given by

$$\hat{\lambda}\left(s,y,x;\boldsymbol{\varphi}_x\right)=\hat{\mu}\left(s,y,x;\boldsymbol{\varphi}_x\right)-\beta\phi E\left[\left(x'\right)^{-\gamma}.\hat{\mu}\left(f\left(s,y,x,x'\right),s,x';\boldsymbol{\varphi}_x\right)\mid x\right] \tag{29}$$

Then, I solve the system of $4.n_s.n_y$ equations in the $4.n_s.n_y$ unknowns φ_x and $\boldsymbol{\eta}_x$

$$\left\{\begin{array}{c} X'.R\left(s,y,x;\hat{\mu}_{\varphi_x}\right)=0\\ X'.\Re\left(s,y,x;\hat{\mu}_{\varphi_x},\hat{p}_{e,\eta_x}\right)=0\end{array}\right.,\quad \text{for } x=x_l,x_h$$

I denote $\hat{\varphi}_x$ and $\hat{\boldsymbol{\eta}}_x$ the solutions of the previous system. The resolution is made according to the iteration scheme described in section 3. Then, the bond price approximation for the state x is given by

$$\hat{P}_f\left(s,y,x;\hat{\varphi}_x\right)=\beta E\left[\left(\frac{\hat{\lambda}\left(f\left(s,y,x,x'\right),s,x';\hat{\varphi}_x\right)}{\hat{\lambda}\left(s,y,x;\hat{\varphi}_x\right)}\right)\left(x'\right)^{-\gamma}\mid x\right] \tag{30}$$

B Data Appendix

The US annual data, for the period 1891-1995, on equity markets and macroeconomics variables are the Campbell [1999]'s updated version of the data in Grossman and Shiller [1982]. The series are described below :

i) Prices and real dividends refer to annual Standard and Poor's composite Stock. The price index is divided by the Consumption deflator.

ii) The aggregate nominal per capita consumption of non durables and services and the consumption deflator refer to the series used by Grossman and Shiller [1982], and updated using the National Account from CITIBASE.

iii) The annual interest rate series is the return on 6-month commercial paper bought in January and rolled over July. The Grossman and Shiller [1982] series was updated using the commercial paper series in CITIBASE (FYCP series).