REDUCING OVERLAPPING GENERATIONS ECONOMIES TO FINITE ECONOMIES

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Abstract:

This paper establishes first an identification of stationary equilibria of an infinite-horizon economy and equilibria of a naturally related finite economy. Then it shows how to obtain new properties of each of the frameworks by means of importing known properties of the other framework. More specifically, the connection is established, firstly, between cycles of an overlapping generations economy and the equilibria of a finite economy with a cyclical structure. Such connection is then shown to hold also when extrinsic uncertainty is introduced in the models.

In order to obtain the adequate counterpart of the connection once the extrinsic uncertainty is introduced, we are lead to consider in the overlapping generations framework sunspot equilibria of a broad class, called here sunspot cycles. Their status with respect to the well-known k-SSE (stationary sunspot equilibria driven by a Markov chain fluctuating between k states) is quite subtle: it turns out to be the case that usual k-SSE are particularly simple instances of sunspot cycles of period 1, while it is also true that a sunspot cycle of the kind and any period is a k-SSE as well, but with a high number of states and strong symmetries in the matrix of probabilities of transition between them (including zero entries). The interest on these sunspot cycles comes from their ability to simplify considerably the beliefs on which the agents have to coordinate for one of these equilibria to be realized, while still being able to deliver dynamics which are complex enough to have a positive import. Thus, it is shown that this class of simple sunspot cycles of the overlapping generations economies, and the equilibria of the finite cyclical economies with asymmetric information on an extrinsic uncertainty, can actually be identified.

Finally, the previous connections between the equilibria of the two frameworks, with and without sunspots, are used to show that: a) overlapping generations economies allowing for heterogeneity across generations typically fluctuate aperiodically for arbitrarily long periods of time, while nevertheless being able to exhibit an approximate cyclical behavior during those periods (this result may account, in a stylized way, for the typical irregular recurrence of the actual business cycles); and b) the presence of asymmetric information on an extrinsic uncertainty may induce continua of equilibria in finite economies.

1. Introduction

Consider the simplest overlapping generations economy. Take two consecutive agents out of the never-ending chain of generations and, say, close a loop with them. The resulting economy is an Edgeworth box where each agent faces his mirror image. Any equilibrium in this Edgeworth box corresponds to a cycle of period 2 of the original overlapping generations economy and conversely, including the steady state as a "degenerate" cycle of period 2 (see the Figure 1 and Balasko and Ghiglino (1995)).

figure 1

This is the simplest example of the connections existing between stationary equilibria in a dynamic set-up and the equilibria of a symmetric static framework. Nevertheless, the previous thought experiment can actually be redone taking out any number n of consecutive generations and, again, closing a loop with them. The resulting finite economy has now a very peculiar cyclical structure, which makes of any of its equilibria a cycle of a period (divisor of) n (see Tuinstra and Weddepohl (1997) for a first approach to this idea, although not identical to the approach presented here).

The problem of clarifying the links between dynamic and static economies sharing some type of symmetry and between their equilibria underlies in previous attempts to establish a connection between sunspot equilibria of overlapping generations economies and correlated equilibria of strategic market games (see, for instance, Maskin and Tirole (1987), Forgès and Peck (1995), Dávila (1999)). Actually, the link conjectured by Maskin and Tirole (1987) between, on the one hand, the correlated equilibria of their 2-agents, 2-commodities exchange economy with asymmetric information on an extrinsic uncertainty and, on the other hand, the sunspot equilibria of an overlapping generations economy constructed after that finite economy, hints at a sort of extension to a framework with (extrinsic) uncertainty of the connection mentioned above between the cycles of the dynamic economy and the equilibria of the static one.

In effect, taking two consecutive agents out of the chain of overlapping generations and closing a loop with them makes of the publicly observed sunspot a distinct privately observed signal for each of the two agents[1]. Thus the possibility of a correlated equilibrium in which each agent uses his privately observed signal as randomizing device appears. Indeed, when considered in the Edgeworth box constructed with the two consecutive agents taken out of the chain, the sunspot equilibria of the original overlapping generations economy which fluctuate randomly between two states according to a Markov process become, quite naturally, correlated equilibria of the market game underlying the 2 agents economy with asymmetric information[2] (see Figure 2).

[1] This is the effect of collapsing the line of time in one instant as the loop is closed. Nevertheless, the "publicly observed" sunspot was already private information for each cohort of young agents actually, since it is disclosed sequentially.

[2] As it will be shown below (see Example 1 in section 4.1), the corners of the of the smaller box within the Edgeworth box in Figure 2 constitute the support of a sunspot equilibrium of the overlapping generations economy with consumer 1 as representative agent. They constitute as

figure 2

These sunspot equilibria are similar to cycles of period 2, but for the fact that the fluctuations between the two states are random instead of deterministic. Indeed, the cycles of period 2 can be considered as extreme (or degenerate) cases of stochastic equilibria[3] in which the probability of changing the state is actually 1 always. Nevertheless, as it will be argued below (see Example 2 in section 4.2), such sunspot equilibria may not be the most natural extension of cycles to a framework with extrinsic uncertainty. At any rate, they are definitely not the right extension if we want it to hold in general (see Dávila (1999)). Rather, the correct counterpart are the equilibria of a more general and, at the same time, somewhat special class that I shall call sunspot cycles (their seemingly contradictory character as simultaneously more general and particular than the usual sunspot equilibria is only apparent, since each statement correspond to different viewpoints; see the remarks in section 4.2 for clearer insights on this). Intuitively, these sunspot cycles consist of superimposing a sunspot signal to a cycle, instead of looking at the cycle as the sunspot signal itself. In such sunspot cycles, the periodicity of the underlying cycle and the number of values among which the sunspot fluctuates become thus completely unrelated.

Although the kind of sunspot cycles retained to establish the connection are, at any rate, usual k-SSE (sunspot equilibria driven by a Markov chain, see Azariadis and Guesnerie (1986), for instance), but with a very specific Markov process (with a matrix of probabilities of transitions full of symmetries and zero entries), they may deserve nevertheless some attention for their ability to deliver reasonably complex fluctuations, while keeping the sunspot process on which the agents are supposed to coordinate spontaneously much simpler than what an unstructured k-SSE would require. This may enhance the the likelihood of a positive import of the sunspot equilibrium concept as a rationale for the business cycle.

The ultimate goal of this investigation is to use the connections established as "bridges" joining the two frameworks and allowing to derive properties in each of them by means of "importing" known properties of the other. Some examples of this use are provided below. They essentially establish that: a) once the representative hypothesis is dropped, we should not expect to observe any pure cycle, although the equilibrium allocation can robustly exhibit irregular but recurrent fluctuations "looking like cycles"; and b) the use of private, irrelevant information by the agents of a finite economy may make it have continua of equilibria, worsening thus quite seriously the problem of spontaneous coordination on an equilibrium.

The rest of the paper is organized as follows. Section 2 introduces the two economies. In section 3, I present a straightforward connection between the cycles of the overlapping generations economy and the equilibria of its associated cyclical economies[4]. This is mainly done for the sake of completeness. Although strictly speaking section 3 is, so to speak, a special case of section 4, keeping the uncertainty

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well the allocation of resources of a correlated equilibrium of the economy formed by consumer 1 and his mirror image, consumer 2 (see the Example 2 in section 4.2).

[3]This is not completely unrelated (or rather not unrelated at all) to the fact that they lay at the "extremes" of the 8-shaped figure containing the sunspot equilibria in Figure 2.

[4]Such a connection had been noticed for cycles of period 2 by Balasko and Ghiglino (1995), and then it was extended, in a different way than the one presented here, by Tuinstra and Weddepohl (1997) to cycles of any period.

out of the stage for a while eases the exposition a lot. Section 4 extends the connection between the equilibria of these economies to the case where there is (extrinsic) uncertainty too. Finally, section 5 shows how the connection can be used to obtain properties in these two frameworks.

2. A simple overlapping generations economy and its associated cyclical economies

Consider the simplest overlapping generations economy, i.e. an economy consisting of a never-ending sequence of generations dated by $t \in \mathbb{Z}$. All the members of a typical generation born at date t are identical to a representative agent who lives for two periods, is endowed with positive quantities[5] e_t^t and e_{t+1}^t of the single commodity of the economy at dates t and $t+1$, and has preferences over the consumption of this commodity along his lifetime which are represented by a utility function u^t depending on consumption when young c_t^t and consumption when old c_{t+1}^t, which is standard in the sense that it is continuous on $\mathbb{R}_+^2$, twice continuously differentiable on $\mathbb{R}_{++}^2$, strictly monotone[6], strictly quasi-concave[7] and well-behaved at the boundary[8]. Actually, throughout almost all the paper (except in section 5), all the generations are going to be identical, in such a way that (e_t^t, e_{t+1}^t) and u^t are all equal to a common endowments point $e = (e_1, e_2)$ and utility function u. This assumption is unimportant: the propositions in sections 3 and 4 hold for heterogeneous agents as well, while dropping their names by now helps in making the exposition lighter. Thus the entire overlapping generations economy is completely characterized by (u, e).

Now, to any given overlapping generations economy (u, e) and positive integer n, associate an economy with n commodities and n consumers defined as follows: consumer i's preferences and endowments are[9]

$$U^i(c_1^i, \ldots, c_n^i) = u(c_i^i, c_{i+1}^i)$$
$$e_h^i = \begin{cases} e_1 & \text{if } h = i \\ e_2 & \text{if } h = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $i = 1, \ldots, n$, and with the understanding from now on that $i+1$ stands for 1 when $i = n$, in such a way that the utility function and endowments of the n-th consumer are $u(c_n^n, c_1^n)$ and $(e_2, 0, \ldots, 0, e_1)$. This economy is a sort of closed loop of n consecutive agents of the overlapping generations economy (u, e). Let us refer to it as the n-cyclical economy (u, e, n) associated to the overlapping generations (u, e).

[5] More precisely, non-negative but not simultaneously equal to zero. In what follows, subscripts refer to dated commodities, superscripts to generations.

[6] In the sense that $Du(c_1, c_2)$ is always in the strictly positive orthant.

[7] In the sense that $D^2u(c_1, c_2)$ is always negative definite in the subspace orthogonal to $Du(c_1, c_2)$.

[8] In the sense that either there is no intersection between the axes and the indifference curves or any such intersection is tangent.

[9] Again, superscripts refer to consumers, subscripts to commodities.

3.1 The overlapping generations economy under certainty.

Consider first the problem of the generation t of the overlapping generations economy (u, e), i.e.

$$\begin{aligned} &\max_{0 \le c_t^t, c_{t+1}^t} \; u(c_t^t, c_{t+1}^t) \\ &p_t(c_t^t - e_1) + p_{t+1}(c_{t+1}^t - e_2) \le 0. \end{aligned} \tag{2}$$

Under the assumptions made on u and e, the unique solution of this problem is completely characterized by the corresponding first order conditions.

An equilibrium of the overlapping generations economy (u, e) consists of an allocation of resources $\{(c_t^t, c_{t+1}^t)\}_{t \in Z}$ and prices $\{p_t\}_{t \in Z}$ such that

(1) for all $t \in \mathbb{Z}$, (c_t^t, c_{t+1}^t) is the solution to (2), and
(2) the allocation of resources is feasible.

The next proposition gives a complete characterization of the equilibrium allocations of the overlapping generations economy (u, e).

Proposition 1.

(1) *If the allocation of resources* $\{(c_t^t, c_{t+1}^t)\}_{t \in Z}$ *and prices* $\{p_t\}_{t \in Z}$ *constitute an equilibrium of the overlapping generations economy* (u, e)*, then for all* $t \in \mathbb{Z}$

$$D_1 u(c_t^t, c_{t+1}^t)(c_t^t - e_1) + D_2 u(c_t^t, c_{t+1}^t)(c_{t+1}^t - e_2) = 0. \tag{3}$$

(2) *If the allocation of resources* $\{(c_t^t, c_{t+1}^t)\}_{t \in Z}$ *satisfies* (3) *and the feasibility condition*

$$c_{t+1}^t + c_{t+1}^{t+1} = e_1 + e_2, \tag{4}$$

for all $t \in \mathbb{Z}$*, then it is an equilibrium allocation of the overlapping generations economy* (u, e)*.*

Proof.

(1) If $\{(c_t^t, c_{t+1}^t)\}_{t \in Z}$ and $\{p_t\}_{t \in Z}$ is an equilibrium of the overlapping generations economy (u, e), then for every $t \in \mathbb{Z}$, (c_t^t, c_{t+1}^t) is a solution to (2) i.e. there exists a positive multiplier λ^t such that

$$\begin{aligned} D_1 u(c_t^t, c_{t+1}^t) - \lambda^t p_t &= 0 \\ D_2 u(c_t^t, c_{t+1}^t) - \lambda^t p_{t+1} &= 0 \\ p_t(c_t^t - e_1) + p_{t+1}(c_{t+1}^t - e_2) &= 0. \end{aligned} \tag{5}$$

Multiplying the first equation by $(c_t^t - e_1)$, the second by $(c_{t+1}^t - e_2)$ and adding them up taking into account the budget constraint in the third equation, the condition (3) follows.

(2) In order to produce prices $\{p_t\}_{t \in Z}$ supporting $\{(c_t^t, c_{t+1}^t)\}_{t \in Z}$ as an equilibrium allocation, let p_1 be any positive price and define[10], for each $t \in \mathbb{Z}$,

[10] Notice that, since $Du(c) \in R_{++}^2$ guarantees that the terms of $\{(c_t^t, c_{t+1}^t)\}_{t \in Z}$ are all located either strictly northwest or strictly southeast of e (if not all equal to e), the feasibility condition implies that if $c_1^1 - e_1 < 0$ (> 0), then $c_t^t - e_1 < 0$ (> 0), i.e. $c_t^{t-1} - e_2 > 0$ (< 0), for all $t \in Z$ and thus p_t and λ^t are always positive. If the allocation of resources is the autarky, then the first order conditions will be satisfied by any positive price p_1, with $p_t = (D_2 u(e)/D_1 u(e))^{t-1} p_1$ and $\lambda^t = D_2 u(e)(D_1 u(e)/D_2 u(e))^t \cdot 1/p_1$, for each $t \in Z$.

the price

$$p_t = -\frac{c^1_1 - e_1}{c^{t-1}_t - e_2} p_1 \tag{6}$$

and the multiplier

$$\lambda^t = -D_2 u(c^t_t, c^t_{t+1}) \frac{c^t_{t+1} - e_2}{c^1_1 - e_1} \frac{1}{p_1}. \tag{7}$$

Then, for all $t \in \mathbb{Z}$, the allocation of resources is feasible by assumption and, moreover, the first order conditions (5) are satisfied since, firstly, the budget constraint holds

$$\frac{p_t}{p_{t+1}} = \frac{-\frac{c^1_1 - e_1}{c^{t-1}_t - e_2} p_1}{-\frac{c^1_1 - e_1}{c^t_{t+1} - e_2} p_1} = \frac{c^t_{t+1} - e_2}{c^{t-1}_t - e_2} = -\frac{c^t_{t+1} - e_2}{c^t_t - e_1}; \tag{8}$$

secondly, the partial derivative of the lagrangian with respect to c^t_{t+1} is zero because of the very definition of λ^t; and, finally, the partial derivative with respect to c^t_t is also zero since

$$\begin{aligned} D_1 u(c^t_t, c^t_{t+1}) - \lambda^t p_t &= \\ D_1 u(c^t_t, c^t_{t+1}) + D_2 u(c^t_t, c^t_{t+1}) \frac{c^t_{t+1} - e_2}{c^1_1 - e_1} \frac{1}{p_1} \cdot -\frac{c^1_1 - e_1}{c^{t-1}_t - e_2} p_1 &= \\ \frac{1}{c^t_t - e_1} \big(D_1 u(c^t_t, c^t_{t+1})(c^t_t - e_1) + D_2 u(c^t_t, c^t_{t+1})(c^t_{t+1} - e_2) \big) &= 0 \end{aligned} \tag{9}$$

by mere substitutions, recalling (4) and (3).

□

Some equilibria of the overlapping generations economy (u, e) among the uncountably many of them that typically exist, exhibit a regularity which lessens the heroic character of the assumption of a spontaneous coordination of every agent of the economy on one of them.[11] Specifically, a cycle of period n of the overlapping generations economy (u, e) is an equilibrium which treats equally any two generations n periods apart from each other. Notice that this condition defines a cycle of period n in a quite broad sense, since it allows to see any such cycle as a cycle of any other period n' multiple of n. In particular, it allows for a steady state (i.e. an equilibrium which treats equally all the generations) to be considered a cycle of any period. Thus $\{(c^t_t, c^t_{t+1})\}_{t \in Z}$ is the allocation of a cycle of period n if, and only if,

$$(c^t_t, c^t_{t+1}) = (c^{t'}_{t'}, c^{t'}_{t'+1}) \tag{10}$$

for any t and t' such that $t' = t + rn$ for some integer $r \in \mathbb{Z}$.

(Notice that the previous definition of cycles does not take into account the equilibrium prices. As a consequence, it encompasses as a cycle of any period the

[11] Although there may still be countably many of such recurrent equilibria: recall Grandmont's result (Grandmont(1985)) establishing that, should there be a cycle of period 3, then the economy would have cycles of any period.

autarky allocation $\{(e_1, e_2)\}_{t\in Z}$ too, which would not qualify typically as a cycle if the same kind of regularity was required for prices. In effect, for any cycle of period n distinct from the autarky, any p_t and $p_{t'}$ do coincide whenever they are n periods apart from each other as well: since, for all $\tau \in \mathbb{Z}$, it holds true that

$$p_\tau = -\frac{c^\tau_{\tau+1} - e_2}{c^\tau_\tau - e_1} p_{\tau+1} \tag{11}$$

because of the budget constraints, then, assuming $t < t'$ without loss of generality,

$$\begin{aligned} p_t &= \left(-\frac{c^t_{t+1} - e_2}{c^t_t - e_1}\right)\left(-\frac{c^{t+1}_{t+2} - e_2}{c^{t+1}_{t+1} - e_1}\right)\cdots\left(-\frac{c^{t'-1}_{t'} - e_2}{c^{t'-1}_{t'-1} - e_1}\right) p_{t'} \\ &= \left(-\frac{c^t_{t+1} - e_2}{c^{t+1}_{t+1} - e_1}\right)\left(-\frac{c^{t+1}_{t+2} - e_2}{c^{t+2}_{t+2} - e_1}\right)\cdots\left(-\frac{c^{t'-1}_{t'} - e_2}{c^t_t - e_1}\right) p_{t'} \\ &= \left(-\frac{c^{t'-1}_{t'} - e_2}{c^{t'}_{t'} - e_1}\right) p_{t'} \\ &= p_{t'}, \end{aligned} \tag{12}$$

after rearranging the denominators, substituting $c^{t'}_{t'}$ to c^t_t and noting that then each fraction becomes 1 because of the feasibility of the allocation of resources. As for the autarky allocation, any prices supporting it as an equilibrium must satisfy $p_t = \left(\frac{D_2u(e)}{D_1u(e)}\right)^{t-t'} \cdot p_{t'}$, which cannot show any cyclical regularity unless $\frac{D_2u(e)}{D_1u(e)} = 1$.)

Therefore a cycle of period n is completely characterized by at most n distinct consumption bundles, i.e. it consists of n points $(c^1_1, c^1_2), \ldots, (c^n_1, c^n_2)$ in $\mathbb{R}^2_{++}$ such that for all $i = 1, \ldots, n$

$$D_1u(c^i_1, c^i_2)(c^i_1 - e_1) + D_2u(c^i_1, c^i_2)(c^i_2 - e_2) = 0 \tag{13}$$

and

$$c^i_2 + c^{i+1}_1 = e_1 + e_2 \tag{14}$$

where, again, $i + 1$ stands for 1 if $i = n$.

3.2 The cyclical economy under certainty.

Considering now a cyclical economy (u, e, n'), its i-th consumer solves the problem[12]

$$\begin{gathered} \max_{0 \leq c^i_i, c^i_{i+1}} u(c^i_i, c^i_{i+1}) \\ p_i(c^i_i - e_1) + p_{i+1}(c^i_{i+1} - e_2) \leq 0 \end{gathered} \tag{15}$$

whose unique solution is again completely characterized by the first order conditions.

An equilibrium of the cyclical economy (u, e, n') consists of an allocation of resources $\{(c^i_i, c^i_{i+1})\}^{n'}_{i=1}$ and prices $\{p_i\}^{n'}_{i=1}$ such that

(1) for all $i = 1, \ldots, n'$, (c^i_i, c^i_{i+1}) is the solution to (15), and
(2) the allocation of resources is feasible.

[12]Strictly speaking, the problem should be posed in $R^{n'}_+$ instead of R^2_+, but the cyclical structure of the economy makes this absolutely unnecessary. Recall also that, in what follows, $i + 1$ stands for 1 whenever $i = n'$.

The equilibrium allocations of the cyclical economy (u, e, n') are completely characterized by the following proposition.

Proposition 2.

(1) *If the allocation of resources* $\{(c_i^i, c_{i+1}^i)\}_{i=1}^{n'}$ *and prices* $\{p_i\}_{i=1}^{n'}$ *constitute an equilibrium of the cyclical economy* (u, e, n')*, then for all* $i = 1, \ldots, n'$

$$D_1 u(c_i^i, c_{i+1}^i)(c_i^i - e_1) + D_2 u(c_i^i, c_{i+1}^i)(c_{i+1}^i - e_2) = 0. \tag{16}$$

(2) *If the allocation of resources* $\{(c_i^i, c_{i+1}^i)\}_{i=1}^{n'}$ *satisfies* (16) *and the feasibility condition*

$$c_{i+1}^i + c_{i+1}^{i+1} = e_1 + e_2, \tag{17}$$

for all $i = 1, \ldots, n'$*, then it is an equilibrium allocation of the cyclical economy* (u, e, n').

Proof. It is identical to that of Proposition 1 characterizing the equilibrium allocations of the overlapping generations (u, e), substituting i to t wherever the latter appears and modifying the range of the indices from $t \in \mathbb{Z}$ to $i = 1, \ldots, n'$ accordingly. □

3.3 The connection under certainty.

The next proposition establishes the connection between the cycles of the overlapping generations economy (u, e) and the equilibria of its cyclical economies (u, e, n).

Proposition 3. *Any cycle of period* n *of an overlapping generations economy* (u, e) *can be translated into an equilibrium of any of its cyclical economies* (u, e, n') *with* n' *multiple of* n*, and conversely*

Proof. The statement above is a straightforward consequence of comparing the equations characterizing the equilibrium allocations in each of the two frameworks, which happen to be the same up to a change of notation.

In effect, eliminating c_2^i in (13,14) and c_{i+1}^i in (16,17) and letting[13] $c_1^i = c_i^i$, the two sets of equations turn out to be the same but for the number of equations: n in (13) while n' in (16). Since n' is a multiple of n, a solution to (13) becomes a solution to (16) by replication and, the other way round, a solution to (16) is at least a solution to (13) for the case $n = n'$, while it may well be also a replication of a solution to (13) for some n divisor of n'.[14] □

Proposition 3 applies obviously to any steady state also, considered as a degenerate cycle of period 1.

[13] Two minor points about notation are in order here. Firstly, it would have been preferable to use different letters to denote consumption in each of the economies, should we have wanted to avoid now the somewhat confusing condition $c_1^n = c_n^n$ which appears when $i = n$. Such a condition does not refer by any means to a constraint on the consumption bundle of the n-th consumer of the cyclical economy, but to the identification of his consumption of commodity n (the right-hand side) to the n-th consumption when young in the cycle of the overlapping generations economy (the left-hand side). A more rigorous notation has been discarded for the sake of the readability of the paper. Secondly, another source of confusion may originate in the fact that the left-hand side index i runs from 1 through n, while the right-hand side index i runs from 1 through n', multiple of n. Whenever $n < n'$ it goes without saying that, for any right-hand side i exceeding n, c_i^i is to be identified to $c_1^{i \bmod n}$.

[14] If it is invariant to the n-th power of the $n' \times n'$ matrix ρ whose typical entry ρ_{ij} equals 1 whenever $j = i + 1$ (recall that $n' + 1$ stands for 1) and is 0 otherwise.

Corollary 1. *If* $(\bar{c}_1, \bar{c}_2)$ *is a steady state of the overlapping generations economy* (u, e)*, then any* n*-replica of it* $\{(\bar{c}_1, \bar{c}_2), \ldots, (\bar{c}_1, \bar{c}_2)\}$ *constitutes an equilibrium allocation of the cyclical economy* (u, e, n) *and conversely, any symmetric equilibrium of* (u, e, n) *is an* n*-replica of a steady state of* (u, e)*.*

The next corollary follows also straightforwardly from a mere re-labeling of any equilibrium allocation of a cyclical economy.

Corollary 2. *Any cyclical permutation*[15] *of an equilibrium allocation of a cyclical economy* (u, e, n) *is still an equilibrium allocation of this economy.*

4. The connection under uncertainty

4.1 The overlapping generations with sunspots.

Assume now that in the overlapping generations economy (u, e) there is a sunspot signal[16] σ_t which is publicly observed at every period[17] and takes one of $k > 1$ values at random[18]. Before getting into further details, the fact that the extrinsic uncertainty represented by this sunspot signal may end up having an influence on the outcome of the economy can be given the following rationale.

The agent born at t cares, in order to make his decision, about prices at t and $t+1$. In the presence of the (extrinsic) uncertainty introduced by the sunspot signal, he cannot exclude a priori a possible dependence of the prices on the values taken by the sunspot and therefore he is uncertain about the price he may face at $t+1$ (at date t the price is observed as well as the sunspot). Thus, the agent t has to choose at t his current consumption and a plan of consumption at $t+1$ contingent to the price actually realized at $t+1$, bearing in mind some expectations about that price. These expectations may take, for instance, the form of a dependence of the price at $t+1$ on the value of the sunspot at that date. Thus the agent t will rather use, in the making of his decision, the information conveyed by the value of the sunspot at t about the probability distribution of the sunspot at $t+1$, i.e. about the probability distribution which applies to the prices at $t+1$ as well actually, according to his beliefs. The consumption choice at t will thus show a dependence on the sunspot observed at t, and it will moreover determine the price

[15] That is to say, the permutation of the indices of the allocation and prices by any power of ρ.

[16] A signal with no influence on the fundamentals of the economy, i.e. a signal representing states of the world with respect to which the fundamentals (preferences, endowments, technology also if there were production) remain unchanged. It was first noticed in Shell (1977) that such signals could nevertheless have an influence on the outcome of the economy, leading to a so-called sunspot equilibrium. Later on Cass and Shell (1983) provided a characterization of the circumstances in which no sunspot equilibrium can exist (in few words, the Arrow-Debreu world), which amounts to a characterization by negation of the set-ups where they are likely to emerge. See Chiappori and Guesnerie (1991) and Guesnerie and Woodford (1996) for surveys on the subsequent literature that followed.

[17] Notice that, because of the demographic structure of the model, the signal publicly observed at each date, is actually private information of the generation currently making its consumption choices.

[18] As a matter of fact, if we let $k = 1$ in what follows in such a way that the stochastic process driving σ_t is a trivial one giving to the signal the same constant value at every period with probability 1, then all the claims and proofs still go through. Thus the results shown in section 3 for the case under certainty are a particular case of those proved in this section. They have nevertheless been presented separately in order to ease the exposition.

at t, resulting so in a dependence of the price at t on the current sunspot too. If this dependence ends up being the dependence believed to hold between the price $t+1$ and the sunspot at $t+1$, then such beliefs turn out to be self-fulfilling[19].

In effect, consider the problem of a member of a generation t of the overlapping generations economy (u, e) with a sunspot signal $\{\sigma_t\}_{t\in Z}$. What he knows about the prevailing state of the world $s = (\ldots, \sigma_{t-1}, \sigma_t, \sigma_{t+1}, \ldots)$, from the continuum $S = \{1, \ldots, k\}^Z$ of possible states, is at most the history of sunspot values up to t, $s_t = (\ldots, \sigma_{t-1}, \sigma_t)$, as well as the price[20] p_{ts_t} for consumption $c^t_{ts_t}$ at t, and expects a price $p_{t+1s'_{t+1}}$ for consumption $c^t_{t+1s'_{t+1}}$ at $t+1$ if s'_{t+1} happens (where s'_{t+1} is necessarily such that[21] $s'_t = s_t$) with probability $P(s'_{t+1}|s'_t = s_t)$. Then the member of a generation t born in a state of the world s faces the problem

$$\max_{\substack{0\le c^t_{ts_t}, c^t_{t+1s'_{t+1}} \\ s'_{t+1}|s'_t=s_t}} \sum_{s'_{t+1}|s'_t=s_t} P(s'_{t+1}|s'_t = s_t)\ u(c^t_{ts_t}, c^t_{t+1s'_{t+1}}) \tag{18}$$
$$p_{ts_t}(c^t_{ts_t} - e_1) + p_{t+1s'_{t+1}}(c^t_{t+1s'_{t+1}} - e_2) \le 0\ ,\ s'_{t+1}|s'_t = s_t.$$

The unique solution to this problem is completely characterized by its first order conditions.

An equilibrium of the overlapping generations economy (u, e) with a sunspot signal $\{\sigma_t\}_{t\in Z}$ consists of an allocation $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})\}_{t\in Z, s\in S}$ and prices $\{p_{ts_t}\}_{t\in Z, s\in S}$ such that

(1) for all $s \in S$ and all $t \in \mathbb{Z}$, $(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})$ is the solution to (18), and
(2) the allocation of resources is feasible.

The following proposition provides a complete characterization of the equilibrium allocations of the overlapping generations economy (u, e) with a sunspot signal $\{\sigma_t\}_{t\in Z}$.

Proposition 4.

(1) *If the allocation* $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})_{s_t}\}_{t\in Z, s\in S}$ *and prices* $\{p_{ts_t}\}_{t\in Z, s\in S}$ *constitute an equilibrium of the overlapping generations economy* (u, e) *with a sunspot signal* $\{\sigma_t\}_{t\in Z}$*, then for all* $s \in S$ *and all* $t \in \mathbb{Z}$

$$\sum_{s'_{t+1}|s'_t=s_t} P(s'_{t+1}|s'_t = s_t)\big(D_1u(c^t_{ts_t}, c^t_{t+1s'_{t+1}})(c^t_{ts_t} - e_1) + D_2u(c^t_{ts_t}, c^t_{t+1s'_{t+1}})(c^t_{t+1s'_{t+1}} - e_2)\big) = 0. \tag{19}$$

[19]More exactly, the expectations held after such beliefs turn out to be rational expectations actually.

[20]In what follows superscripts refer again to generations, while subscripts to dated, sunspot history-contingent commodities. The sequential unfolding of the allocation of resources of an overlapping generations economy prevents any generation to make consumption decisions contingent to information which will be disclosed after the consumption takes place. This is a distinctive feature of an overlapping generations economy under uncertainty which does not appear in a one-shot economy (like, for instance, an Arrow-Debreu economy) where consumption takes place after every uncertainty, if any, is resolved, even if the decisions may be made ex ante contingent to the realization of any uncertainty.

[21]Here s'_t denotes the truncation of the history s'_{t+1} up to t.

(2) *If the allocation of resources* $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})\}_{t\in Z, s\in S}$ *satisfies* (19) *and the feasibility condition*

$$c^{t-1}_{ts_t} + c^t_{ts_t} = e_1 + e_2 \tag{20}$$

for all $s \in S$ *and all* $t \in \mathbb{Z}$, *then it is an equilibrium allocation of the overlapping generations economy* (u, e) *with the sunspot signal* $\{\sigma_t\}_{t\in Z}$.

Proof.

(1) If $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})\}_{t\in Z, s\in S}$ and prices $\{p_{ts_t}\}_{t\in Z, s\in S}$ constitute an equilibrium, then, for any $s \in S$ and any $t \in \mathbb{Z}$, $(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})$ is the solution to (18), i.e. there exist positive multipliers $\lambda^t_{s'_{t+1}}$ for all s'_{t+1} such that $s'_t = s_t$, for which

$$\begin{gathered}\sum_{s'_{t+1}|s'_t=s_t} P(s'_{t+1}|s'_t = s_t) D_1 u(c^t_{ts_t}, c^t_{t+1s'_{t+1}}) - \sum_{s'_{t+1}|s'_t=s_t} \lambda^t_{s'_{t+1}} p_{ts_t} = 0 \\ P(s'_{t+1}|s'_t = s_t) D_2 u(c^t_{ts_t}, c^t_{t+1s'_{t+1}}) - \lambda^t_{s'_{t+1}} p_{t+1s'_{t+1}} = 0 \ , \ s'_{t+1}|s'_t = s_t \\ p_{ts_t}(c^t_{ts_t} - e_1) + p_{t+1s'_{t+1}}(c^t_{t+1s'_{t+1}} - e_2) = 0 \ , \ s'_{t+1}|s'_t = s_t.\end{gathered} \tag{21}$$

Multiplying the first equation by $(c^t_{ts_t} - e_1)$, each of the equations in the second line by the corresponding $(c^t_{t+1s'_{t+1}} - e_2)$ and adding all them up taking into account the budget constraints in the third line, then the condition (19) follows.

(2) In order to produce prices supporting $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})\}_{t\in Z, s\in S}$ as an equilibrium allocation, let p_{1s_1}, for every[22] $s \in S$, be any positive price such that, for any other $s' \in S$,

$$\frac{p_{1s_1}}{p_{1s'_1}} = \frac{c^1_{1s'_1} - e_1}{c^1_{1s_1} - e_1}. \tag{22}$$

Then define, for all $s \in S$ and all $t \in \mathbb{Z}$, the prices

$$p_{ts_t} = -\frac{c^1_{1s_1} - e_1}{c^{t-1}_{ts_t} - e_2} p_{1s_1}, \tag{23}$$

and for all $s' \in S$ such that $s'_t = s_t$, the multipliers

$$\lambda^t_{s'_{t+1}} = -P(s'_{t+1}|s'_t = s_t) D_2 u(c^t_{ts_t}, c^t_{t+1s'_{t+1}}) \frac{c^t_{t+1s'_{t+1}} - e_2}{c^1_{1s_1} - e_1} \frac{1}{p_{1s_1}}. \tag{24}$$

Then, for all $t \in \mathbb{Z}$ and all $s \in S$, the feasibility constraint is satisfied by assumption and the first order conditions (21) are satisfied too: firstly, the budget constraints are satisfied since, for all s'_{t+1} such that $s'_t = s_t$,

$$\frac{p_{ts_t}}{p_{t+1s'_{t+1}}} = \frac{-\frac{c^1_{1s_1} - e_1}{c^{t-1}_{ts_t} - e_2} p_{1s_1}}{-\frac{c^1_{1s'_1} - e_1}{c^t_{t+1s'_{t+1}} - e_2} p_{1s'_1}} = \frac{c^t_{t+1s'_{t+1}} - e_2}{c^{t-1}_{ts_t} - e_2} = \frac{c^t_{t+1s'_{t+1}} - e_2}{c^t_{ts_t} - e_1}, \tag{25}$$

[22] Actually, just one p_{1s_1} needs to be fixed arbitrarily, all the other prices at date 1 and at every state of the world s being then determined by the normalization (22).

where the second equality results from the normalization (22) adopted; secondly, the partial derivatives of the lagrangian with respect to $c^t_{t+1s'_{t+1}}$ are satisfied by the very definition of the multipliers $\lambda^t_{s'_{t+1}}$, and finally, the partial derivative with respect to $c^t_{ts_t}$ is again satisfied by mere substitutions, recalling (20) and (19).

□

Any equilibrium of the overlapping generations economy (u,e) with a sunspot signal $\{\sigma_t\}_{t\in Z}$ whose allocation of resources does actually depend (in a non trivial way) on the state s realized is known as a sunspot equilibrium.

Notice that the allocation of an equilibrium of the overlapping generations economy (u,e) with a sunspot signal $\{\sigma_t\}_{t\in Z}$ must specify in general not only how the resources are allocated at every date $t \in \mathbb{Z}$ (as in the case with certainty) but for every possible state $s \in S$ also. An equilibrium allocation must thus specify for every generation t what its typical member will receive for any possible history[23] s_t in which he may be born and any continuation σ_{t+1} of it in $t+1$. Nonetheless, such an allocation may exhibit some regularities, as in the case without uncertainty, which make it simpler and, hence, likelier to emerge from spontaneous coordination of the agents. Notice, however, that the regularities may arise now not only with respect to the dates $t \in \mathbb{Z}$, as in the cycles, but with respect to the histories s_t as well.

Concerning the regularities with respect to t, let us call a sunspot cycle of period n (and of order k, following the literature on k-SSE, for the number of values taken by the sunspot) any equilibrium whose allocation treats equally any two generations n periods apart from each other, i.e. such that

$$(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t}) = (c^{t'}_{t'\tilde{s}_{t'}}, (c^{t'}_{t'+1\tilde{s}'_{t'+1}})_{\tilde{s}'_{t'+1}|\tilde{s}'_{t'}=\tilde{s}_{t'}}) \tag{26}$$

for every $t, t' \in \mathbb{Z}$ such that $t' = t + rn$ for some $r \in \mathbb{Z}$, and every $s, \tilde{s} \in S$ such that $s_t = \tilde{s}_{t'}$. Thus the equilibrium allocation of a sunspot cycle of period n is completely characterized by at most n distinct sunspot-history contingent plans[24] $(c^i_{1s_1}, (c^i_{2s_2})_{s_2|s_2^{-1}=s_1})_{s_1\in\{1,\dots,k\}^{-N}}$, for all $i = 1, \dots, n$, such that for all $s_1 \in \{1,\dots,k\}^{-N}$

$$\begin{aligned}\sum_{s_2|s_2^{-1}=s_1} P(s_2|s_2^{-1}=s_1)\big(D_1u(c^i_{1s_1},c^i_{2s_2})(c^i_{1s_1}-e_1)\\ + D_2u(c^i_{1s_1},c^i_{2s_2})(c^i_{2s_2}-e_2)\big) = 0\end{aligned} \tag{27}$$

and for all $s_2 \in \{1,\dots,k\}^{-N}$,

$$c^i_{2s_2} + c^{i+1}_{1s_2} = e_1 + e_2. \tag{28}$$

The simplest of such equilibria are obviously those for which $n = 1$.

[23]Notice that there are uncountably many possible histories $(\dots,\sigma_{t-1},\sigma_t) \in \{1,\dots,k\}^{-N}$

[24]Here s_2^{-1} stands for the one-step forward shift of the history $s_2 = (\dots, s_{23}, s_{22}, s_{21})$, i.e. the history $(\dots, s_{24}, s_{23}, s_{22})$. With this notation the usual condition $s'_{t+1}|s'_t = s_t$ would have become $s'_{t+1}|{s'}_{t+1}^{-1} = s_t$. The first one seems to be eloquent enough, while being less cumbersome, to justify this minor redundancy in notation.

As for regularities with respect to sunspot histories, an equilibrium may be such that, for instance, it allocates the resources in the same way at any two histories up to any given date $t \in \mathbb{Z}$ which coincide in the m last periods besides the current one, i.e. such that

$$(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t}) = (c^t_{t\tilde{s}_t}, (c^t_{t+1\tilde{s}'_{t+1}})_{\tilde{s}'_{t+1}|\tilde{s}'_t=\tilde{s}_t}) \tag{29}$$

for all $t \in \mathbb{Z}$ and any $s, \tilde{s} \in S$ such that $\sigma_{t-i} = \tilde{\sigma}_{t-i}$, for all $i = 0, \ldots, m$. The simplest of such equilibria are clearly those with $m = 0$, in which case, for any $s, \tilde{s} \in S$ and $t \in \mathbb{Z}$ such that $\sigma_t = \tilde{\sigma}_t$, it holds $c^t_{ts_t} = c^t_{t\tilde{s}_t}$. For such equilibria, the feasibility condition implies necessarily $c^t_{t+1s_{t+1}} = c^t_{t+1\tilde{s}_{t+1}}$ as well, for any $s, \tilde{s} \in S$ and $t \in \mathbb{Z}$ such that $\sigma_{t+1} = \tilde{\sigma}_{t+1}$. Therefore, these are allocations in which the consumption decisions depend on the current sunspot only and, hence, the history s_t of sunspot values up to t can be identified to the current sunspot value σ_t. Thus the conditions characterizing an equilibrium with this sort of regularity with respect to sunspot histories, can be written as

$$\begin{aligned}\sum_{\sigma_{t+1}=1}^{k} m^{\sigma_t\sigma_{t+1}}_{s_{t-1}} \big(D_1 u(c^t_{t\sigma_t}, c^t_{t+1\sigma_{t+1}})(c^t_{t\sigma_t} - e_1) \\ + D_2 u(c^t_{t\sigma_t}, c^t_{t+1\sigma_{t+1}})(c^t_{t+1\sigma_{t+1}} - e_2)\big) = 0\end{aligned} \tag{30}$$

for all $t \in \mathbb{Z}$ and all $\sigma_t = 1, \ldots, k$, and

$$c^t_{t+1\sigma_{t+1}} + c^{t+1}_{t+1\sigma_{t+1}} = e_1 + e_2 \tag{31}$$

for all $t \in \mathbb{Z}$ and all $\sigma_{t+1} = 1, \ldots, k$, where $m^{\sigma_t\sigma_{t+1}}_{s_{t-1}}$ in (30) stands for $P(s'_{t+1}|s'_t = s_t)$, i.e. the probability of transition from σ_t to each σ_{t+1} given the history s_{t-1} up to $t-1$.[25]

Considering the two types of regularity together, the simplest sunspot cycle of period n and order k of the overlapping generations economy (u, e) with a general sunspot signal $\{\sigma_t\}_{t\in Z}$ that can be thought of, is an equilibrium which, besides treating equally any two generations n periods apart from each other, allocates consumption depending on the current sunspot only, i.e. such that

$$(c^t_{t\sigma_t}, (c^t_{t+1\sigma_{t+1}})^k_{\sigma_{t+1}=1})^k_{\sigma_t=1} = (c^{t'}_{t'\sigma_{t'}}, (c^{t'}_{t'+1\sigma_{t'+1}})^k_{\sigma_{t'+1}=1})^k_{\sigma_{t'}=1} \tag{32}$$

whenever $t' = t + rn$ for some $r \in \mathbb{Z}$. Thus, a sunspot cycle of period n and order k, dependent on the current sunspot only, is completely characterized by at most n distinct sunspot-contingent consumption plans $(c^1_{1\sigma_1}, (c^1_{2\sigma_2})^k_{\sigma_2=1})^k_{\sigma_1=1}$, $\ldots$,$(c^n_{1\sigma_1}, (c^n_{2\sigma_2})^k_{\sigma_2=1})^k_{\sigma_1=1}$, such that, for all $i = 1, \ldots, n$, all $\sigma_1 = \{1, \ldots, k\}$ and all $s \in \{1, ..., k\}^{-N}$

$$\sum_{\sigma_2=1}^{k} m^{\sigma_1\sigma_2}_{s} \big(D_1 u(c^i_{1\sigma_1}, c^i_{2\sigma_2})(c^i_{1\sigma_1} - e_1) + D_2 u(c^i_{1\sigma_1}, c^i_{2\sigma_2})(c^i_{2\sigma_2} - e_2)\big) = 0 \tag{33}$$

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[25] Notice that, as long as $k \geq 3$, the continuum of possible histories up to $t-1$ is not a problem for this dependence of the probabilities of transition on s_{t-1} because of the positive number of degrees of freedom for determining such probabilities. On the contrary, if $k = 2$, the condition (30) and $\Sigma m = 1$ determine unambiguously each row of the matrix of probabilities of transition and, hence, any such equilibrium has to be necessarily markovian.

and for all $i = 1, \ldots, n$ and all $\sigma_2 = 1, \ldots, k$,

$$c^i_{2\sigma_2} + c^{i+1}_{1\sigma_2} = e_1 + e_2, \tag{34}$$

where, as usual, $i+1$ stands for 1 when $i = n$.

Yet, a particularly interesting class of equilibria is that of those equilibria for which the process followed by the sunspot signal is especially simple, i.e. such that the probabilities of transition $m_s^{\sigma_1\sigma_2}$ do not depend on the entire history of σ_t, but only on that of the most recent periods up to a finite number of them, maybe none, in which case the sunspot signal (and, we will say, the sunspot cycle too) is markovian. Such equilibria are specially appealing because, unlike the equilibria where $m_s^{\sigma_1\sigma_2}$ depends on the entire history s, they do not require the agents to be able to handle infinite sets of information, but rather a finite number of past values of σ_t along with a matrix of probabilities of transition.

Thus, a markovian sunspot cycle of period n and order k, dependent on the current sunspot only, is completely characterized by at most n distinct sunspot-contingent consumption plans $(c^1_{1\sigma_1}, (c^1_{2\sigma_2})^k_{\sigma_2=1})^k_{\sigma_1=1}, \ldots, (c^n_{1\sigma_1}, (c^n_{2\sigma_2})^k_{\sigma_2=1})^k_{\sigma_1=1}$, such that, for all $i = 1, \ldots, n$, all $\sigma_1 = \{1, \ldots, k\}$

$$\sum_{\sigma_2=1}^{k} m^{\sigma_1\sigma_2}\big(D_1u(c^i_{1\sigma_1}, c^i_{2\sigma_2})(c^i_{1\sigma_1} - e_1) + D_2u(c^i_{1\sigma_1}, c^i_{2\sigma_2})(c^i_{2\sigma_2} - e_2)\big) = 0 \tag{35}$$

and for all $i = 1, \ldots, n$ and all $\sigma_2 = 1, \ldots, k$,

$$c^i_{2\sigma_2} + c^{i+1}_{1\sigma_2} = e_1 + e_2. \tag{36}$$

Once arrived to this point, some remarks are in order[26]. Firstly, notice that the role of steady state in the deterministic framework (a cycle of period 1, indeed), is played now by the sunspot cycles of period 1. Secondly, notice too that a markovian sunspot cycle of period 1 and order k, whose allocation of consumption to each agent depends on the current sunspot only[27], turns out to be a sunspot equilibrium of the class k-SSE (for Stationary Sunspot Equilibrium of order k) studied in, for instance, Azariadis and Guesnerie (1986), Guesnerie (1986), Chiappori and Guesnerie (1989), Chiappori, Geoffard and Guesnerie (1992). As a matter of fact, the k-SSE are the simplest sunspot equilibria that can be produced in an overlapping generations economy (u, e). The Example 1 below shows how the characterization of equilibrium allocations provided by Proposition 4 works for them.

Finally, notice that a markovian sunspot cycle dependent only on the current sunspot, whose period is n and driven by a sunspot signal taking k values, fluctuates typically between kn states. For instance, in the Figure 3 the support of a markovian sunspot cycle of period 2 and order 2 as well is depicted[28]. At such an equilibrium, the partition at each date of the economy resources between the

[26] Besides those already made on the notion of cycle, which apply straightforwardly to that of sunspot cycle, namely that any sunspot cycle of period n is also a sunspot cycle of any other period n' multiple of n.

[27] Note that there is no relation at all between the, so to speak, "memory" of an equilibrium allocation and the "memory" of the sunspot process driving it.

[28] The gradients at the supporting points have to be such that the first order conditions are satisfied.

contemporary young and old agents, fluctuates between the four points on the diagonal, although alternating (deterministically) the inner and the outer points for the (random) choice of the partition. Thus sunspot cycles are able to deliver considerably finer grids for fluctuations than the k-SSE, for a sunspot signal of a given cardinality k, i.e. assuming the same level of sophistication of the sunspot theory held by the agents or, to put it in other words, for the same cost in terms of their assumed ability to attain spontaneous coordination. Nonetheless, every markovian sunspot cycle of this kind is actually a "high" order k-SSE with a Markov matrix with special symmetries . For instance, that of the sunspot cycle in Figure 3 may be of the form

$$\begin{pmatrix} 0 & 1-m^{12} & m^{12} & 0 \\ 1-m^{12} & 0 & 0 & m^{12} \\ m^{21} & 0 & 0 & 1-m^{21} \\ 0 & m^{21} & 1-m^{21} & 0 \end{pmatrix}. \tag{37}$$

Figure 3

Example 1. A k-SSE of the overlapping generations economy (u, e) with a sunspot signal driven by a $k \times k$ Markov matrix (m^{ij}) is an equilibrium whose allocation of resources treats equally all the generations of the economy and the consumption when young depends only on the current sunspot signal and not on its entire history (the feasibility condition on the allocation of resources imposes then that the consumption when old depends on the current sunspot only as well). Let c_1^i denote in this example[29] the consumption when young at any t such that $\sigma_t = i$, and c_2^j the consumption when old at any t such that $\sigma_t = j$. Then the allocation of resources of a k-SSE has to satisfy

(1) for all $i = 1, \ldots, k$,

$$\sum_{j=1}^{k} m^{ij} \big(D_1 u(c_1^i, c_2^j)(c_1^i - e_1) + D_2 u(c_1^i, c_2^j)(c_2^j - e_2)\big) = 0 \tag{38}$$

(2) and for all $j = 1, \ldots, k$

$$c_2^j + c_1^j = e_1 + e_2 \tag{39}$$

(the prices supporting such an allocation as an equilibrium are also time-independent and contingent only on the current sunspot only because of the normalization (22) used).

Notice that the left-hand side of equation (38) is a convex linear combination of the inner products of the gradients of the utility function at each point (c_1^i, c_2^j) with its corresponding excess demand. A quite natural interpretation of equation (38) is therefore that, for $(c_1^i, (c_2^j)_{j=1}^{k})$ to be the optimal current consumption and contingent plan of future consumption of a generation observing the i-th value of the sunspot, the orthogonality of the gradient with the corresponding excess demand that characterizes the optimal choices in the case with certainty, has to be satisfied in mean. As a consequence, not all the inner products in (38) can have the same

[29]Instead of c_{1i} (resp. c_{2j}), as we should write according to the previous notation.

sign and there must be at least two points (c_1^i, c_2^j) and $(c_1^i, c_2^{j'})$ which are separated by the offer curve. Assuming $c_1^1 < c_1^2 < \cdots < c_1^k$, without loss of generality, then $c_2^k < c_2^{k-1} < \cdots < c_2^1$ and therefore, at a k-SSE the offer curve must necessarily separate (c_1^i, c_2^k) and (c_1^i, c_2^1) for every $i = 1, \ldots, k$ (see Figure 4).

Figure 4

Thus, in order to exhibit such a k-SSE we only need to be able to produce a box with its top-left and bottom-right corners on the line going through the endowments e with slope -1,[30] and such that the top corners and bottom corners are separated by the offer curve. It may be clear now why the indeterminacy of the steady state in the perfect foresight dynamics (i.e. that the slope of the offer curve at the non autarkic steady state is smaller than 1 in absolute value) is a sufficient condition for the existence of sunspot equilibria of this class (a continuum of them indeed), although it is by no means necessary. It is clearly nonetheless a necessary condition to be possible to produce such sunspot equilibria arbitrarily close to the steady state, i.e. local sunspot equilibria.[31]

4.2 The cyclical economy with private sunspots.

Consider now the cyclical economy (u, e, n). Assume that each consumer i observes privately a different sunspot signal s_i which can take k values at random each. Thus the state of the world is an extrinsic random variable $s = (s_1, \ldots, s_n)$ taking values in $S = \{1, \ldots, k\}^n$. Let π^s denote the probability of s being the prevailing state and $\{(p_{is})_{i=1}^n\}_{s \in S}$ be the prices of each commodity i contingent to the state of the world s. Actually, as it will be seen below, for any equilibrium allocation the prices supporting it will be such that each p_{is} depends only on s_i.

[30] This condition takes care of the feasibility of the allocation of resources.

[31] See Chiappori, Geoffard and Guesnerie (1992) for a characterization of the existence of local k-SSE around the steady state of an abstract one-step forward looking dynamical system, Dávila (1997) for the existence of the same kind of local sunspot equilibria in similar systems but with predetermined variables or memory, and Woodford (1984) for a result showing that the indeterminacy of the steady state in the perfect foresight dynamics is a necessary and sufficient condition for the existence of more general local sunspot equilibria. Dávila, Gottardi and Kajii (1999) provide an argument showing that, however, any economy with a determinate steady state can be approximated arbitrarily closely by another economy with k-SSE in an arbitrarily small neighborhood of its determinate steady state.

The problem that the i-th consumer observing s_i faces is[32],[33]

$$\max_{\substack{0\leq c^i_{is_i}, c^i_{i+1s'}\\ \forall s'|s'_i=s_i}} \sum_{\forall s'|s'_i=s_i} \pi^{s'} u(c^i_{is_i}, c^i_{i+1s'}) \tag{40}$$
$$p_{is'}(c^i_{is_i} - e_1) + p_{i+1s'}(c^i_{i+1s'} - e_2)\ ,\ \forall s'|s'_i = s_i$$

where, as usual, $i+1$ stands for 1 when $i = n$, and the denominator has been dropped from the conditional probability for the sake of readability. An equilibrium of the cyclical economy (u, e, n) with private sunspot signals consists of an allocation of resources $\{(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ and prices $\{(p_{is})^n_{i=1}\}_{s\in S}$ such that

(1) for all $i = 1, \ldots, n$ and all $s_i = 1, \ldots, k$, $(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})$ is the solution to the problem (40) above, and
(2) the allocation of resources is feasible.

The next proposition gives a complete characterization of the equilibrium allocation of the cyclical economy (u, e, n) with asymmetric information on the extrinsic uncertainty.

Proposition 5.

(1) *If the allocation of resources $\{(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ and the prices $\{(p_{is})^n_{i=1}\}_{s\in S}$ constitute an equilibrium of the cyclical economy (u, e, n) with private sunspot signals, then for all $i = 1, \ldots, n$ and all $s_i = 1, \ldots, k$*

$$\sum_{s'|s'_i=s_i} \pi^{s'} \big(D_1 u(c^i_{is_i}, c^i_{i+1s'})(c^i_{is_i} - e_1) + D_2 u(c^i_{is_i}, c^i_{i+1s'})(c^i_{i+1s'} - e_1)\big) = 0. \tag{41}$$

(2) *If the allocation of resources $\{(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ satisfies (41) and the feasibility condition*

$$c^i_{i+1s'} + c^{i+1}_{i+1s_{i+1}} = e_1 + e_2 \tag{42}$$

for all $i = 1, \ldots, k$, all $s_{i+1} = 1, \ldots, k$ and all s' such that $s'_{i+1} = s_{i+1}$, then it is an equilibrium allocation of the cyclical economy (u, e, n).

[32] As previously, superscripts refer to consumers, while subscripts to sunspot contingent commodities. Notice that consumption by each consumer i of the "same-label" good i depends only on his private signal, while his consumption of commodity $i + 1$ depends on the whole arrays of sunspot signals, i.e. on the state of the world. This amounts to say that consumption of the "same-label" commodity is decided as if it took place ex ante to the realization of the uncertainty while consumption of any other commodity takes place ex post, and therefore can be made contingent to it. Thus, this is not a one-shot economy, but it rather shares somehow the sequential character of the overlapping generations economy, while staying a finite horizon economy. This pattern of dependence of the consumption of different commodities on different information sets is imported here from Maskin and Tirole (1987), and is reminiscent of the interpretation of the "same-label" commodity as leisure which can be either consumed or used to produce a commodity to be traded with other consumers, very much as in the usual macroeconomics interpretation of the overlapping generations economy as an economy in which young agents produce, save their wage and consume it when old (see, for instance, Azariadis and Guesnerie (1986)).

[33] It may seem incongruous at first sight that the price $p_{is'}$ may depend on the entire array of sunspot signals while consumer i is not able to infer from its observation the state of the world realized. As a matter of fact, $p_{is'}$ will not depend at equilibrium on any other sunspot signal than s_i, as it has already remarked, in such a way that the absurdity is only apparent (see footnote 35).

Proof.

(1) If $\{(c^i_{is_i},(c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ and $\{(p_{is})^n_{i=1}\}_{s\in S}$ constitute an equilibrium of the cyclical economy (u,e,n) then, for all $i=1\ldots,n$ and all $s_i=1,\ldots,k$, $(c^i_{is_i},(c^i_{i+1s'})_{s'|s'_i=s_i})$ is the solution to the problem (40) above, i.e. there exist positive multipliers $\lambda^{is_i}_{s'}$, one for each s' such that $s'_i=s_i$, such that

$$\begin{aligned}\sum_{s'|s'_i=s_i}\pi^{s'}D_1u(c^i_{is_i},c^i_{i+1s'})-\sum_{s'|s'_i=s_i}\lambda^{is_i}_{s'}p_{is'}=0\\ \pi^{s'}D_2u(c^i_{is_i},c^i_{i+1s'})-\lambda^{is_i}_{s'}p_{i+1s'}=0\ ,\ \forall s'|s'_i=s_i\\ p_{is'}(c^i_{is_i}-e_1)+p_{i+1s'}(c^i_{i+1s'}-e_2)=0\ ,\ \forall s'|s'_i=s_i.\end{aligned}\tag{43}$$

Multiplying the first equation by $(c^i_{is_i}-e_1)$, each equation in the second line by the corresponding $(c^i_{i+1s'}-e_2)$ and adding all them up taking into account the budget constraints in the third line, the condition (41) follows.

(2) In order to produce a set of prices p_{is}, one for each commodity $i=1,\ldots,n$ in each state of the world $s\in\{1,\ldots,k\}^n$, supporting the allocation $\{(c^i_{is_i},(c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ as an equilibrium, let p_{1s}, for each state of the world $s\in\{1,\ldots,k\}^n$, be any positive price in such a way that, for all[34] $s,s'\in S$,

$$\frac{p_{1s}}{p_{1s'}}=\frac{c^1_{1s'_1}-e_1}{c^1_{1s_1}-e_1}.\tag{44}$$

Then define[35], for every $i=1,\ldots,n$ and $s\in S$,

$$p_{is}=-\frac{c^1_{1s_1}-e_1}{c^{i-1}_{is}-e_2}p_{1s}\tag{45}$$

and let moreover, for all $i=1,\ldots,n$, all $s_i=1,\ldots,k$ and all s',

$$\lambda^{is_i}_{s'}=-\pi^{s'}D_2u(c^i_{is_i},c^i_{i+1s'})\frac{c^i_{i+1s'}-e_2}{c^1_{1s'_1}-e_1}\frac{1}{p_{1s'}}.\tag{46}$$

Then the allocation of resources satisfies the feasibility condition by assumption, as well as the first order conditions: the budget constraints are satisfied since, for any given $s\in S$ and any s' such that $s'_i=s_i$,

$$\frac{p_{is'}}{p_{i+1s'}}=\frac{-\frac{c^1_{1s'_1}-e_1}{c^{i-1}_{is'}-e_2}p_{1s'}}{-\frac{c^1_{1s'_1}-e_1}{c^i_{i+1s'}-e_2}p_{1s'}}=\frac{c^i_{i+1s'}-e_2}{c^{i-1}_{is'}-e_2}=-\frac{c^i_{i+1s'}-e_2}{c^i_{is_i}-e_1}\tag{47}$$

[34] Actually just one p_{1s} needs to be fixed arbitrarily, all the other prices of commodity 1 at each state of the world being determined by the normalization (44).

[35] Notice that since, firstly, from the feasibility condition (42), c^{i-1}_{is} cannot depend at equilibrium on any other sunspot signal that s_i and, secondly, $(c^1_{1s_1}-e_1)p_{1s}$ does not depend on s according to the normalization (44), then p_{is} cannot depend on any other sunspot signal than s_i and, thus, equilibrium prices do not convey any information on the signal observed by anybody else, as we had already anticipated in footnote 33.

where the last equality follows from the feasibility condition; the partial derivatives of the lagrangian with respect to $c^i_{i+1s'}$, for all $s' \in S$ such that $s'_i = s_i$, are all zero by the very definition of the multipliers $\lambda^{is_i}_{s'}$, and the partial derivative with respect to $c^i_{is_i}$ are zeroed too since

$$\sum_{s'|s'_i=s_i} \pi^{s'} D_1u(c^i_{is_i}, c^i_{i+1s'}) - \sum_{s'|s'_i=s_i} \lambda^{is_i}_{s'} p_{is'} =$$

$$\sum_{s'|s'_i=s_i} \pi^{s'} \Big(D_1u(c^i_{is_i}, c^i_{i+1s'}) + D_2u(c^i_{is_i}, c^i_{i+1s'}) \frac{c^i_{i+1s'} - e_2}{c^1_{1s'_1} - e_1} \frac{1}{p_{1s'}} \cdot - \frac{c^1_{1s'_1} - e_1}{c^{i-1}_{is'} - e_2} p_{1s'}\Big) =$$

$$\frac{1}{c^i_{is_i} - e_1} \sum_{s'|s'_i=s_i} \pi^{s'} \Big(D_1u(c^i_{is_i}, c^i_{i+1s'})(c^i_{is_i} - e_1) + D_2u(c^i_{is_i}, c^i_{i+1s'})(c^i_{i+1s'} - e_2)\Big) = 0 \tag{48}$$

where the last two equalities follow from the feasibility condition (42) and the condition (41) respectively.

□

Notice that an immediate consequence of the feasibility condition is that, at equilibrium, $c^i_{i+1s'} = c^i_{i+1s''}$ whenever $s'_{i+1} = s''_{i+1}$. Thus, if we denote by $c^i_{i+1s_{i+1}}$ the common value of c^i_{i+1s} for all the states $s \in S$ with the same s_{i+1}, then an equilibrium allocation of the cyclical economy (u, e, n) consists of $\{(c^i_{s_i}, (c^i_{i+1s_{i+1}})^k_{s_{i+1}=1})^k_{s_i=1}\}^n_{i=1}$ such that, for all $i = 1, \dots, n$ and all $s_i = 1, \dots, k$,

$$\sum_{s_{i+1}=1}^{k} \frac{\sum_{\substack{s'|s'_i=s_i \\ s'_{i+1}=s_{i+1}}} \pi^{s'}}{\sum_{s''|s''_i=s_i} \pi^{s''}} \Big(D_1u(c^i_{is_i}, c^i_{i+1s_{i+1}})(c^i_{is_i} - e_1) + D_2u(c^i_{is_i}, c^i_{i+1s_{i+1}})(c^i_{i+1s_{i+1}} - e_2)\Big) = 0 \tag{49}$$

and for all $i = 1, \dots, n$ and all $s_{i+1} = 1, \dots, k$,

$$c^i_{i+1s_{i+1}} + c^{i+1}_{i+1s_{i+1}} = e_1 + e_2. \tag{50}$$

In the next example an equilibrium is produced for the simple case of an economy with two agents and signals taking two values each.

Example 2. Consider a 2-cyclical economy as depicted in Figure 2. An equilibrium allocation of this economy consists of same-label commodity consumptions for each agent contingent their signals, c^1_{11}, c^1_{12} and c^2_{21}, c^2_{22}, and the like for the other commodity but contingent to the other agent signal c^1_{21}, c^1_{22} and c^2_{11}, c^2_{12}, such that (49) and (50) above are satisfied.

The feasibility condition (50) asks for the four points (c^1_{11}, c^1_{22}), (c^1_{11}, c^1_{21}), (c^1_{12}, c^1_{21}) and (c^1_{12}, c^1_{22}) to form a box as shown in Figure 2, although in general this box needs not be neither square nor laying on the diagonal. Nonetheless, the allocation has to be necessarily such that there exist probabilities π^s for which (49) holds. This condition leads to a non-homogeneous system of five linear equations in the four π^s's that, in order to avoid overdeterminacy, requires the allocation to satisfy

$$D^1_{11} D^2_{12} D^2_{21} D^1_{22} = D^2_{22} D^1_{21} D^1_{12} D^2_{11} \tag{51}$$

where $D^i_{s_i s_{i+1}}$, for $i = 1, 2$, and $s_i, s_{i+1} = 1, 2$, stands for the inner product of consumer i's gradient with his excess demand (notice that in the case of a symmetrical allocation as the one depicted in Figure 2, this condition is satisfied).

Moreover, as in the Example 1 in section 4.1 on k-SSE, the left-hand side of (49) can be interpreted as 0 being in the convex hull of the inner products of gradients and excess demands, the weights making 0 a convex linear combination of the latter being the probabilities of s conditional to the private information. Thus, in order to have the right signs for those inner products, agent 1's offer curve must separate the top and bottom corners of the allocation box, while agent 2's offer curve must separate the left and right corners (see Figure 2).

Yet for the allocation in Figure 2 to be at equilibrium, it remains to be checked that there exists indeed a probability distribution for the two private signals such that (49) is satisfied. This is particularly easy to see in this example because, as we have seen in the Example 1, this allocation determines a 2-SSE of the overlapping generations economy with agent 1 as representative agent. In other words, there exists a Markov matrix (m^{ij}) whose rows are the weights which put 0 as convex linear combination of the inner products of gradients and excess demands for agent 1 (and hence for agent 2 as well, because of the symmetry). The trick is now to see that there exists a joint distribution (π^s) for the two signals, inducing conditional distributions of each signal given the other which coincide with the Markov matrix of the 2-SSE, i.e. that there is a solution in π^{11}, π^{12}, π^{21} and π^{22} to

$$\begin{array}{c} \pi^{11} + \pi^{12} + \pi^{21} + \pi^{22} = 1 \\ \dfrac{\pi^{12}}{\pi^{11} + \pi^{12}} = m^{12} \qquad \dfrac{\pi^{21}}{\pi^{11} + \pi^{22}} = m^{21} \\ \dfrac{\pi^{21}}{\pi^{11} + \pi^{21}} = m^{12} \qquad \dfrac{\pi^{12}}{\pi^{12} + \pi^{22}} = m^{21}. \end{array} \tag{52}$$

In effect, this system has as its unique solution

$$\pi^{11} = \frac{(1 - m^{12})m^{21}}{m^{21} + m^{12}}, \quad \pi^{12} = \frac{m^{12} m^{21}}{m^{21} + m^{12}} = \pi^{21}, \quad \pi^{22} = \frac{m^{12}(1 - m^{21})}{m^{21} + m^{12}}. \tag{53}$$

This solution is actually the one provided in Forgès and Peck (1995) when the connection that they establish between the sunspot equilibria of an overlapping generations economy and the correlated equilibria of a market game mimicking it, is particularized to the case of 2-SSE, our agents 1 and 2 being their odd and even generations.

The procedure followed in the previous example, to link the equilibria of the cyclical economy with asymmetric information to 2-SSE of the overlapping generations economy, seems to be straightforward enough to expect it to go through for general k-SSE as well, as it was indeed conjectured in Maskin and Tirole (1987). Nevertheless, Dávila (1999) showed that this is typically not the case. The reason is that the class of k-SSE is not the right choice to try extend the connection, but rather that of markovian sunspot cycles, as Proposition 6 in the next section shows.

4.3 The connection with extrinsic uncertainty.

Recalling the conditions (35,36) characterizing the allocation of a markovian sunspot cycle of period n and order k, and dependent on the current sunspot only, of the overlapping generations economy (u, e), their similarity to the conditions (49,50) characterizing the equilibrium allocation of the corresponding cyclical economy (u, e, n) with private sunspot signals, hints at the following proposition, which establishes the connection between the markovian sunspot cycles of (u, e) and the equilibria of its associated cyclical economies.

Proposition 6. *Any markovian sunspot cycle of period n and dependent only on the current sunspot, of an overlapping generations economy (u, e) can be translated into an equilibrium of any of its cyclical economies (u, e, n') with private sunspot signals, with n' multiple of n, and conversely.*[36]

Proof. Eliminating $c^i_{2\sigma_2}$ from (35,36) and $c^i_{i+1 s_{i+1}}$ from (49,50), and letting[37] $c^i_{1\sigma_1} = c^i_{is_i}$ whenever $\sigma_1 = s_i$, the two sets of equations turn out to be the same one, but for the number of equations (n in (35), n' in (49)), if it happened to be the case that, for any $\sigma_1, \sigma_2 = 1, \ldots, k$ and all $i = 1, \ldots, n'$,

$$\frac{\displaystyle\sum_{\substack{s' | s'_i = s_i \\ s'_{i+1} = s_{i+1}}} \pi^{s'}}{\displaystyle\sum_{s'' | s''_i = s_i} \pi^{s''}} = m^{\sigma_1 \sigma_2} \tag{54}$$

whenever $(\sigma_1, \sigma_2) = (s_i, s_{i+1})$. Leaving aside for a moment whether (54) can actually hold true and assuming it can, since n' is a multiple of n, a solution to (35) becomes a solution to (49) by replication and, the other way round, a solution to (49) is at least a solution to (35) for the case $n = n'$, while it may well be also a replication of a solution to (35) for some n divisor of n'.[38]

Now let us see that, whichever is the $k \times k$ Markov matrix (m^{ij}) driving the sunspot signal in the overlapping generations economy (u, e), there is a probability distribution (π^s) for $s \in \{1, \ldots, k\}^{n'}$ in the cyclical economy (u, e, n'), such that (54) holds. In effect, the equations (54) together with the necessary condition

$$\sum_{s \in \{1, \ldots, k\}^{n'}} \pi^s = 1 \tag{55}$$

form a system of $n'(k^2 - k) + 1$ linear equations (for each agent $i = 1, \ldots, n'$, there is one condition for each off-diagonal entry of the Markov matrix and, moreover, all the probabilities have to add up to one), in $k^{n'}$ variables (the probabilities π^s, for all $s \in \{1, \ldots, k\}^{n'}$). The number of degrees of freedom of the system is then

[36] As a matter of fact, in the case $n' = 2$ the statement actually holds true if $k = 2$ (see Example 2 in section 4.2), but not for $k \geq 3$ typically (see Dávila (1999)).

[37] Again, as in the certainty case, a more careful notation has been sacrificed for the sake of the readability of the argument and thus remarks similar to those in footnote 13 apply. Namely, in the case $i = n$, $c^n_{1s_1} = c^n_{n\sigma_n}$, with $\sigma_n = s_1$, does not refer to any constraint on the choice of the n-th consumer of the cyclical economy, but to the identification of his consumption of commodity n when observing σ_n $(= s_1)$ to the n-th consumption when young if s_1 $(= \sigma_n)$ is observed in the sunspot cycle of the overlapping generations economy. As in the certainty case too, if $n < n'$, then for any right-hand side i exceeding n, $c^i_{i\sigma_i}$ is to be identified to $c^{i \bmod k}_{1s_1}$ whenever $\sigma_i = s_1$.

[38] See footnote 14 in the proof of Proposition 3.

$k^{n'} - n'(k^2 - k) - 1$, which is positive for any $n' > 2$ and $k \geq 2$.[39] This implies that the linear subspace of solutions to (54) has positive dimension and, therefore, in order to show that the system of equations has a solution, it suffices to show that this linear subspace meets the interior of the positive orthant, because in that case it necessarily meets the affine space (55) as well.

In effect, each subspace associated to an equation in (54) meets the strictly positive orthant, otherwise it would be possible to separate these two convex sets by a hyperplane. However, this hyperplane can only be the subspace itself. But then, necessarily, a normal vector of such hyperplane would have to be in the strictly positive orthant too. Nevertheless, it can be easily seen that the normal vector of each subspace in (54) has positive (of the form $1 - m^{\sigma_1\sigma_2}$) as well as negative (of the form $-m^{\sigma_1\sigma_2}$) coordinates. So the separation is not possible and thus the non trivial intersection subspace determined by the system (54) does meet the positive orthant indeed. Hence it meets the affine space (55) and a solution to the system exists[40] □

5. Some examples of properties following from the connection

We conclude showing how the previous results may be used to derive properties of one of the frameworks studied by means of importing known properties of the other. The statements below follow readily from the identifications established in Propositions 3 and 6. Property 1 is a simple and straightforward application of Proposition 3. Properties 2 and 3 have more interesting on their own and I'll comment on them below. Property 4 illustrates the problems that the introduction of asymmetric information on sunspots may generate for the indeterminacy of finite economies.

Property 1. *Any single-commodity overlapping generations economy with representative agent has generically a finite number of cycles of any given period.*

In effect, since a cycle of period n of (u, e) is an equilibrium of (u, e, n) and, by Debreu (1970), generically there are finitely many of the later, the conclusion follows.

Property 2. *Any generic single-commodity overlapping generations economy allowing for heterogeneity across generations is indistinguishable during any arbitrarily long period of observation of an economy with no cycle at all.*[41]

In order to see this property, let $\{(u^t, e^t)\}_{t \in Z}$ be an overlapping generations economy allowing for different preferences and endowments across generations, and let t_0, t_1 and n be integers such that $t_0 < t_1$ and $t_1 - t_0 < n$.

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[39] It can easily be checked that 1 is, for every integer n', both a root and a critical point of the polynomial in k defining the number of degrees of freedom $f_{n'}(k) = k^{n'} - n'(k^2 - k) - 1$. Moreover its curvature is non-negative at $k = 1$ and strictly increasing at every positive k, for every $n' \geq 3$. Therefore $f_{n'}(k)$ is strictly convex at every $k \geq 1$, for every $n' \geq 3$ and, thus, $f_{n'}(k)$ has no root bigger than 1, which guarantees $f_{n'}(k) > 0$ for all $k \geq 2$ and $n' > 2$. As for the case $n' = 2$, see the footnote 36.

[40] As a matter of fact, there is a continuum of them whenever the number of degrees of freedom of the system is bigger than 1, which is always the case but for $k = 2$ and $n' = 3$.

[41] Not even steady state.

For all $i = 1, \ldots, n$ let $\tilde{u}^i = u^{t_0+i}$ and $\tilde{e}^i = e^{t_0+i}$, and consider the n consumers, n commodities cyclical economy[42] $\{(\tilde{u}^i, \tilde{e}^i)\}_{i=1}^n$. Generically, this economy has an odd number of equilibria and none of them exhibits any kind of symmetry (i.e. none is invariant to any power strictly divisor of n, of the $n \times n$ permutation matrix ρ whose typical entry ρ_{ij} is 1 if $j = i + 1$, and 0 otherwise). Since Propositions 1 through 3 still hold true if the utility function differs across agents (that is to say, substituting u^t and u^i to u in the overlapping generations and cyclical frameworks respectively), then the overlapping generations economy $\{(\tilde{u}^t, \tilde{e}^t)\}_{t \in Z}$ such that $(\tilde{u}^t, \tilde{e}^t) = (\tilde{u}^i, \tilde{e}^i)$ for any $t \in \mathbb{Z}$ and $i = 1, \ldots, n$ satisfying $t = i \mod n$, which cannot be distinguished from $\{(u^t, e^t)\}_{t \in Z}$ during the period $[t_0, t_1]$, has no cycle of any period smaller than n, and hence none of period smaller than the length of the period of observation $t_1 - t_0$.

Property 3. *There exist robust single-commodity overlapping generations economies allowing for heterogeneity across generations which exhibit a behavior arbitrarily close to a cycle during any given period of observation, without having cycles at all during that period.*

In effect, for any given integers $t_0 < t_1$, let n be a divisor of $t_1 - t_0$. Let also $\{(u^i, e^i)\}_{i=1}^n$ be an n-cyclical economy allowing for heterogeneity across agents, and $\{(u^t, e^t)\}_{t \in Z}$ be the corresponding overlapping generations economy (i.e. such that $(u^t, e^t) = (u^i, e^i)$ for all $t \in \mathbb{Z}$ and $i = 1, \ldots, n$ such that $t = i \mod n$).

Consider any integer n' multiple of n and bigger than $t_1 - t_0$, and also the cyclical economy $\{(u^i, e^i)\}_{i=1}^{n'}$ such that $u^i = u^{i \mod n}$ and $e^i = e^{i \mod n}$ for all $i = 1, \ldots, n'$. Since Propositions 1 through 3 still hold true with heterogeneous agents, then, on the one hand, any equilibrium of $\{(u^i, e^i)\}_{i=1}^{n'}$ is a cycle of period n' of the corresponding overlapping generations economy and conversely, while on the other hand, any equilibrium of $\{(u^i, e^i)\}_{i=1}^{n}$ is a cycle of period n of the corresponding overlapping generations economy and conversely. But the two overlapping generations economies are the same one: $\{(u^t, e^t)\}_{t \in Z}$. Hence, there is a one-to-one mapping between the equilibria of $\{(u^i, e^i)\}_{i=1}^{n}$ and those of $\{(u^i, e^i)\}_{i=1}^{n'}$, in such a way that necessarily every equilibrium of the latter is invariant to the n'/n power of the permutation ρ. That is to say, the cycles of period n' of $\{(u^i, e^i)\}_{i=1}^{n'}$ are actually cycles of period n.

Now, arbitrarily close to the economy $\{(u^i, e^i)\}_{i=1}^{n'}$ there is another economy $\{(\tilde{u}^i, \tilde{e}^i)\}_{i=1}^{n'}$ whose equilibria are not invariant any more to any permutation, and correspond thus to true cycles of period n'. Nevertheless, by continuity the equilibria of the economy $\{(\tilde{u}^i, \tilde{e}^i)\}_{i=1}^{n'}$ are still close to those of $\{(u^i, e^i)\}_{i=1}^{n'}$. Therefore, the cycles of period n' (longer than the period of observation $[t_0, t_1]$) of the corresponding overlapping generations economy $\{(\tilde{u}^t, \tilde{e}^t)\}_{t \in Z}$, look approximately like cycles of period n, although this economy has no cycles at all. The robustness follows from the fact that the same property holds for any overlapping generations economy close enough to this one.

The positive import of the last two properties is not completely uninteresting: if the overlapping generations model is still to be taken as an stylized account of actual economies but, nevertheless, the existence of a representative agent is an admittedly unrealistic assumption, there is little hope for observing pure cycles no matter how long the period of observation is, while approximate cycles (i.e. recurrent

[42] As usual, consumer i derives utility from, and is endowed with commodities i and $i + 1$ only.

but irregular behavior, very much as in actual business cycles) can nonetheless be observed.

Property 4. *In a finite economy, the use by the agents of private information completely unrelated to the fundamentals may lead to the existence of a continuum of equilibria.*

To illustrate this point, consider the finite economy in Figure 2. As we have seen in the Example 2 in section 4.2, any 2-SSE of the overlapping generations economy whose representative agent is agent 1 is also an equilibrium of this economy. Since there is uncountably many 2-SSE of the former, there is also a continuum of equilibria of the latter.

Similar statements could be made on the number of equilibria of any cyclical economy (u, e, n) when there are private sunspots, if one considers the set of sunspot cycles of (u, e), which has the power of the continuum as well. Of course, the cyclical economies have a very special structure. Nonetheless, the point should be general enough. For instance, in the economy in Figure 2, if the agents are not required to share the same beliefs on the joint distribution of the sunspots, then the symmetry between the two agents can be readily disposed of: continua of equilibria depending on private sunspots still appear as long as there is an "allocation box" (not necessarily square now) whose corners are separated by the agents' offer curves qualitatively in the same way they are in Figure 2. Obviously, this equilibria cannot be rational expectations equilibria anymore, in the sense that there is at least one agent who is mistaking the joint distribution of the signals. But since the signals are private sunspots, who can be sure about what is in everybody else's mind?

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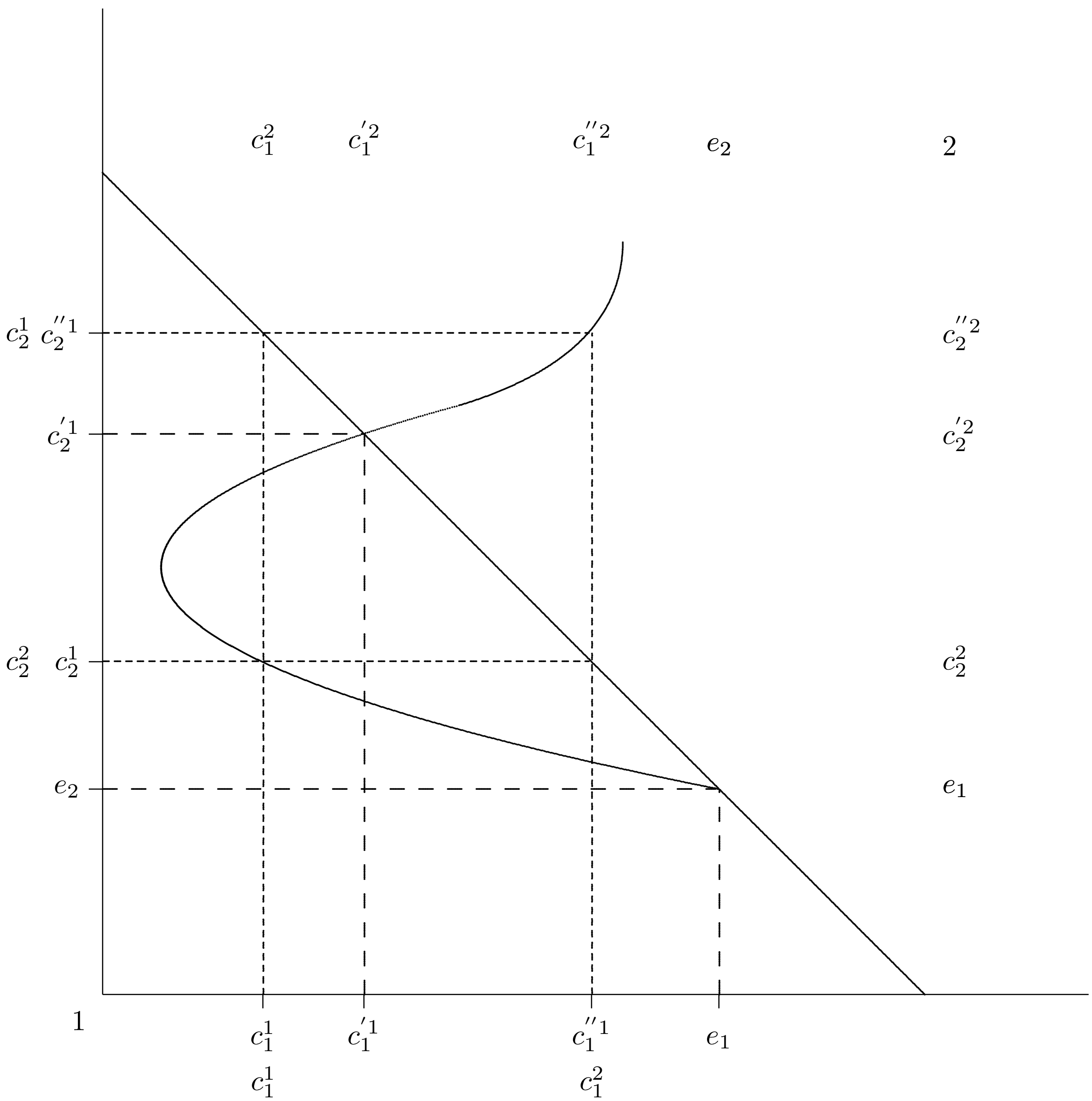
(The complete figure includes the symmetric image of the offer curve and the axes with respect to the line with slope -1)

Figure 1



(The complete figure includes the symmetric image of the offer curve and the axes with respect to the line with slope -1)

Figure 2

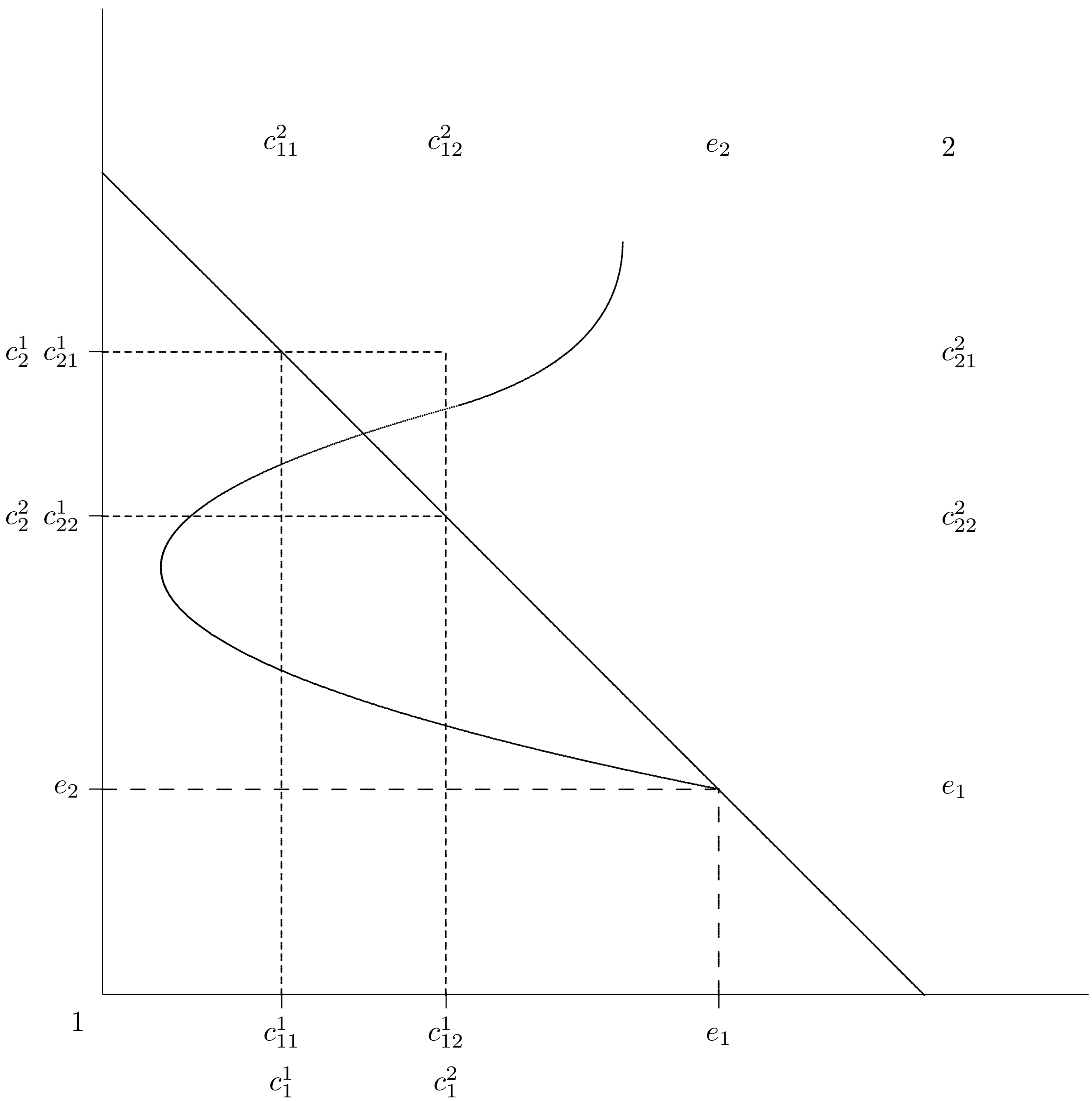


Figure 3

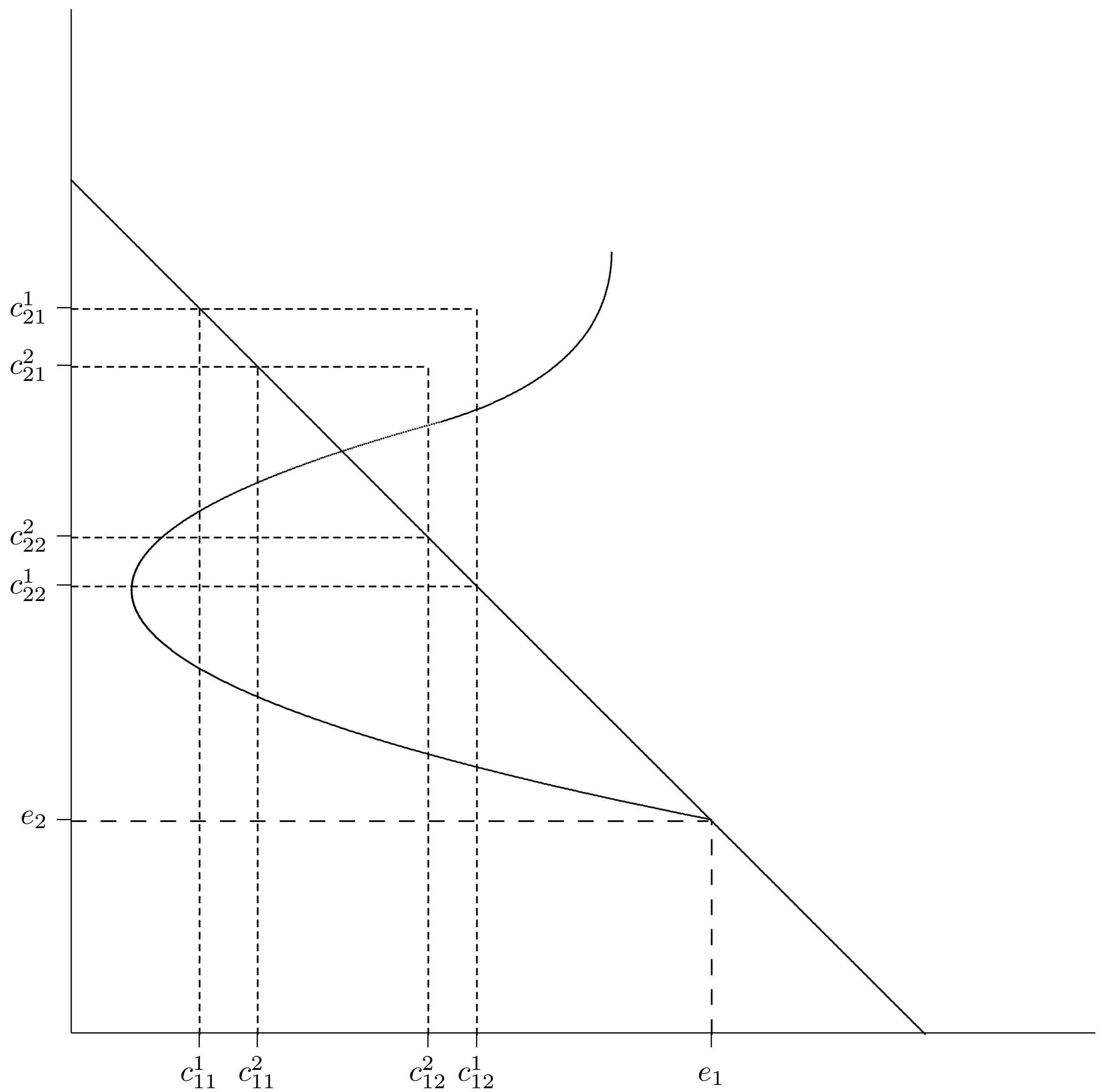


Figure 4