and

$$\pi_i^f(b^*(v_i), w_i) = \frac{1}{4}[v_i - b^*(v_i)] \tag{41}$$

following which equation (11) takes the form

$$\frac{1}{2}(v_i - b^*(v_i) - e) + \frac{1}{4}(v_i - b^*(v_i)) = -e \tag{42}$$

whose unique solution is

$$b^*(v_i) = v_i + \frac{2}{3}e \tag{43}$$

It remains to verify that $b^*(v_i)$ is strictly increasing over the interval $\bar{v} - e \leq v_i < \bar{v} - \frac{2}{3}e$. To do so, note that the right side of (43) is strictly concave (globally) and $\frac{db^*(v_i)}{dv_i}\big|_{v_i = \bar{v} - \frac{2}{3}e} = \frac{1}{2} > 0$.

$$\frac{3}{4}(v_i - b^*(v_i))(b^*(v_i) - v_i) + \frac{1}{2}\int_{b^*(v_i)}^{v_i+e}(v_i - z)dz + \frac{1}{2}\{(v_i + e) - v_i\}e = 0 \tag{36}$$

The unique solution to (36) satisfying $v_i \leq b^*(v_i) < v_i + e$ is

$$b^*(v_i) = v_i + e\sqrt{0.5} \tag{37}$$

<u>Case 2: </u>$\bar{v} - e \leq v_i < \bar{v} - \frac{2}{3}e$

Equation (14) now takes the form

$$\frac{3}{4}(v_i - b^*(v_i))(b^*(v_i) - v_i) + \frac{1}{2}\int_{b^*(v_i)}^{\bar{v}}(v_i - z)dz + \frac{1}{2}\{\bar{v} - v_i\}e = 0 \tag{38}$$

The unique solution to (38) satisfying $v_i \leq b^*(v_i) < v_i + e$ is

$$b^*(v_i) = v_i + \sqrt{e(\bar{v} - v_i) - \frac{1}{2}(\bar{v} - v_i)^2} \tag{39}$$

Notice that $b^*(\bar{v} - \frac{2}{3}e) = \bar{v}$.

<u>Case 3: </u>$\bar{v} - \frac{2}{3}e \leq v_i \leq \bar{v}$

In this case,

$$\pi_i^f(b^*(v_i), s_i) = \frac{1}{2}[(v_i - b^*(v_i)) - e] \tag{40}$$

Case 2: $\bar{v} - e \leq v_i < b^{*-1}(\bar{v})$

In this case

$$\frac{\partial \pi_i^f(b_i)}{\partial v_i} = \begin{cases} \frac{3}{8}[F(b_i - e \mid b_i) + F(b_i \mid b_i)] \\ +\frac{1}{4}[\int_{b_i - e}^{v_i} f(z \mid b_i)dz + \int_{b_i}^{\bar{v}} f(z \mid b_i)dz] \\ +\frac{1}{4}ef(v_i \mid b_i) \end{cases} \tag{34}$$

which is strictly positive due to the same arguments as in Case 1, together with the fact that $b^*(v_i) \leq \bar{v} \quad \forall v_i \in [\bar{v} - e, b^{*-1}(\bar{v})]$.

Case 3: $\bar{v} \geq v_i \geq b^{*-1}(\bar{v})$

In this case

$$\frac{\partial \pi_i^f(b_i)}{\partial v_i} = \begin{cases} \frac{3}{8}[F(b_i - e \mid b_i) + F(b_i \mid b_i)] \\ +\frac{1}{4}\int_{b_i - e}^{v_i} f(z \mid b_i)dz \\ +\frac{1}{4}ef(v_i \mid b_i) \end{cases} \tag{35}$$

which is strictly positive due to the arguments made for case 1.Ξ

Proof of Example 1

To solve equation (14) for $F \equiv U[\underline{v}, \bar{v}]$, we need to consider three separate cases since $\pi_i^f(b_i)$ is not smooth in v_i.

Case I: $\underline{v} \leq v_i < \bar{v} - e$

Using $F \equiv U[\underline{v}, \bar{v}]$, equation (14) can be written as

$$\frac{\partial^2 \pi_i(b_i)}{\partial b_i \partial v_i} = 2\frac{\partial \pi_i^f(b_i)}{\partial v_i}\{1 - F(b^{*-1}(b_i))\} f(b^{*-1}(b_i)) b^{*-1'}(b_i) \tag{31}$$

In view of the arguments used to prove Proposition 1, we need only show that $\frac{\partial \pi_i^f(b_i)}{\partial v_i} > 0$.

Incorporating the expressions in (4), (5), (6) and (7) into the right side of (9), we can write

$$\pi_i^f(b_i) = \begin{cases} \frac{3}{8}(v_i - b_i)[F(b_i - e \mid b_i) + F(b_i \mid b_i)] \\ +\frac{1}{4}[\int_{b_i-e}^{v_i}(v_i - z) f(z \mid b_i)dz + \int_{b_i}^{v_i+e}(v_i - z) f(z \mid b_i)dz] \\ -\frac{1}{4}e[\{1 - F(v_i \mid b_i)\} + \{1 - F(v_i + e \mid b_i)\} \end{cases} \tag{32}$$

As explained in the proof of Proposition 2, since the function $\pi_i^f(b_i)$ is not smooth in v_i, we need to consider 3 separate cases.

Case 1:$\underline{v} \leq v_i < \bar{v} - e$

Differentiating (32) with respect to v_i and cancelling terms, we get

$$\frac{\partial \pi_i^f(b_i)}{\partial v_i} = \begin{cases} \frac{3}{8}[F(b_i - e \mid b_i) + F(b_i \mid b_i)] \\ +\frac{1}{4}[\int_{b_i-e}^{v_i} f(z \mid b_i)dz + \int_{b_i}^{v_i+e} f(z \mid b_i)dz] \\ +\frac{1}{4}e f(v_i \mid b_i) \end{cases} \tag{33}$$

First note that $v_i + e > b^*(v_i) \geq v_i$ and $e > 0$ implies that $F(b_i \mid b_i) > 0$ and the integrals in the right side of (33) are non-negative. Therefore, we conclude that $\frac{\partial \pi_i^f(b_i)}{\partial v_i} > 0$.

where the last inequality follows from the fact that $b > v$.

For case (ii): $b > \bar{v}$, we have

$$\lambda(v,b) = \frac{3}{2}(v-b)[1-F(v)] + [1-F(v_i)]e \tag{30}$$

since $b < v + e$. Differentiating (30) with respect to b, we have

$$\partial\lambda(v,b)/\partial b = -\frac{3}{2}[1-F(v)] < 0$$

Thus, we have shown that $\partial\lambda(v,b)/\partial b$ exists and is not equal to zero a.e. on the domain of b. This concludes our proof.Ξ

Proof of Theorem 1

The proof follows a pseudo-concavity argument borrowed from Matthews[1995] (p.23). Suppose $\frac{\partial^2 \pi_i(b_i)}{\partial b_i \partial v_i} > 0 \quad \forall v_i, \forall b_i \in (0, b^*(\bar{v}))$. Then we can show that $\frac{\partial \pi_i(\hat{b})}{\partial b_i} \geq 0$ for all $\hat{b} \in [0, b^*(v_i)]$ and $\frac{\partial \pi_i(\hat{b})}{\partial b_i} \leq 0$ for all $\hat{b} \in [b^*(v_i), b^*(\bar{v})]$. The first of the proposed inequalities is proved as follows. Note that since $b^*(.)$ is strictly increasing, $b^{*-1}(\hat{b}) \leq v_i \quad \forall \hat{b} \in [0, b^*(v_i)]$. This implies that $\frac{\partial \pi(v_i, \hat{b})}{\partial b_i} \geq \frac{\partial \pi(b^{*-1}(\hat{b}), \hat{b})}{\partial b_i} = 0$ (where the last equality is the first order necessary condition satisfied by $b^*(.)$). A similar argument proves $\frac{\partial \pi_i(\hat{b})}{\partial b_i} \leq 0$ for all $\hat{b} \in [b^*(v_i), b^*(\bar{v})]$. Hence, we conclude that if $\frac{\partial^2 \pi_i(b_i)}{\partial b_i \partial v_i} > 0 \quad \forall v_i, \forall b_i \in (0, b^*(\bar{v}))$, then $b^*(.)$ indeed maximizes (9).

It therefore remains to show that $\frac{\partial^2 \pi_i(b_i)}{\partial b_i \partial v_i} > 0$. First note from (9) that

$$
\begin{aligned}
\frac{\partial \sigma(v,b)}{\partial v} &= F(b) - F(v) - vf(v) + bf(v) - \frac{2}{3}ef(v+e) + \frac{2}{3}[F(v+e) - F(b)] \\
&\quad + \frac{2}{3}[f(v+e) - f(v)]e \\
&= F(b) - F(v) + (b - v - \frac{2}{3}e)f(v) + \frac{2}{3}[F(v+e) - F(b)] \\
&> 0
\end{aligned}
$$

where the last inequality follows from the fact that $v + e > b > v$ and $b \geq v + \frac{2}{3}e$ (lemma 1).

For case (ii), $\frac{\partial \sigma(v,b)}{\partial v} = 1 - F(v) + (b - v - \frac{2}{3}e)f(v) > 0$ by lemma 1.

It remains to show that the solution $b(v)$ is differentiable in v a.e. Call the LHS of (14), $\lambda(v, b)$.

From the Implicit Function Theorem, we know that $\frac{\partial b}{\partial v}$ exists and is equal to $-\frac{\partial \lambda(v,b)/\partial v}{\partial \lambda(v,b)/\partial b}$ whenever $\partial \lambda(v,b)/\partial v$ and $\partial \lambda(v,b)/\partial b$ exist and the latter is not equal to zero. It is straightforward to verify that $\partial \lambda(v,b)/\partial v$ exists over the entire domain of v except at the point $v = \bar{v} - e$. Similarly, $\partial \lambda(v,b)/\partial b$ exists over the entire domain of b except at the point $b = \bar{v}$. For case (i): $b \leq \bar{v}$, differentiating the LHS of (14) we have

$$
\begin{aligned}
\partial \lambda(v,b)/\partial b &= -\frac{3}{2}[F(b) - F(v)] + \frac{3}{2}(v - b)f(b) - (v - b)f(b) \\
&= -\frac{3}{2}[F(b) - F(v)] + \frac{1}{2}(v - b)f(b) \\
&< 0
\end{aligned}
$$

(c1) $\phi'(b) = F(b) + bf(b)$

(c2) $\frac{\partial\sigma(v,b)}{\partial b} = vf(b) + F(v) - \frac{2}{3}(v-b)f(b) = F(v) + (\frac{1}{3}v + \frac{2}{3}b)f(b)$

But since $b > v$, we conclude that (c) $\frac{\partial\sigma(v,b)}{\partial b} < \phi'(b) \quad \forall v$ such that $b \leq \bar{v}$.

(a) and (b) together with the fact that $\phi(b)$ and $\sigma(v,b)$ are both continuous in b imply that there must be at least one $b \in (v, v+e)$ that satisfies (28). However (c) implies that $\phi(b)$ can intersect $\sigma(v,b)$ only from below, which together with continuity of both functions in b implies that the solution to (28) is unique.

Case (ii) $b > \bar{v}$

In this case,

$$\sigma(v,b) = v[1-F(v)] + bF(v) + \frac{2}{3}[1-F(v)]e \tag{29}$$

As before, to prove the existence of a unique solution to (14), we verify the following relationships:

(a) $\sigma(v,v) = v + \frac{2}{3}[1-F(v)]e \geq vF(v) = \phi(v)$

(b) $\sigma(v,v+e) = v[1-F(v)] + (v+e)F(v) + \frac{2}{3}[1-F(v)]e = v + e[\frac{1}{3}F(v) + \frac{2}{3}] \leq v + e = \phi(v+e)$

(c2) $\frac{\partial\sigma(v,b)}{\partial b} = F(v) < 1 = \phi'(b)$.

Continuity of the solution $b(v)$ follows from the continuity of $\sigma(v,b)$ in v. To prove that $b(v)$ is a strictly increasing function of v, it would suffice to show that $\sigma(v,b)$ is strictly increasing in v. For case (i), we get

$$\frac{3}{2}(v_i - b^*(v_i))[F(b^*(v_i)) - F(v_i)] + \int_{b^*(v_i)}^{v_i+e} (v_i - z) f(z) dz + [F(v_i + e) - F(v_i)]e = 0 \tag{27}$$

Ξ

Proof of Proposition 3:

Dropping all superscripts and subcripts, we can rewrite (14) as

$$bF(b) = v[F(b) - F(v)] + bF(v) + \frac{2}{3}[\int_b^{v+e} (v - z) f(z) dz + \{F(v + e) - F(v)\}e] \tag{28}$$

Denote the LHS of (28) by $\phi(b)$ and the RHS by $\sigma(v, b)$. Note that $\phi(b)$ is continuous in b while $\sigma(v, b)$ is continuous in both b and v. Also, recall from lemma 1 that $v \le b \le v + e$.

Observe that $\sigma(v, b)$ is not differentiable in b at $b = \bar{v}$. Accordingly, we consider two separate cases: (i) $b \le \bar{v}$ and (ii) $b > \bar{v}$.

Case (i) $b \le \bar{v}$

First note the following relationships:

(a) $\sigma(v, v) = vF(v) + \frac{2}{3}[\int_v^{v+e}(v - z) f(z) dz + \{F(v + e) - F(v)\}e] > vF(v) = \phi(v)$, where the inequality follows by noting that $\int_v^{v+e}(v - z) f(z) dz > -e \int_v^{v+e} f(z) dz = -e[F(v + e) - F(v)]$

(b) $\sigma(v, v + e) = v[F(v + e) - F(v)] + (v + e)F(v) + \frac{2}{3}[F(v + e) - F(v)]e$

$= vF(v + e) + e[\frac{2}{3}F(v + e) + \frac{1}{3}F(v)] < (v + e)F(v + e) = \phi(v + e)$

Appendix

Proof of Proposition 2:

Incorporating beliefs (13), we can simplify expressions (4) - (7) as follows:

$$\pi_i^s(b^*(v_i), s_i; v_i) = -e \tag{22}$$

$$\begin{aligned}\pi_i^w(b^*(v_i), s_i; v_i) = (v_i - b^*(v_i))[\tfrac{F(b^*(v_i))-F(v_i)}{1-F(v_i)}] + \textstyle\int_{b^*(v_i)}^{v_i+e}(v_i - z)\tfrac{f(z)}{[1-F(v_i)]}dz \\ +[\tfrac{1-F(v_i+e)}{1-F(v_i)}](-e)\end{aligned} \tag{23}$$

$$\pi_i^s(b^*(v_i), w_i; v_i) = 0 \tag{24}$$

$$\pi_i^w(b^*(v_i), w_i; v_i) = \frac{1}{2}(v_i - b^*(v_i))[\frac{F(b^*(v_i)) - F(v_i)}{1 - F(v_i)}] \tag{25}$$

Adding up the simplified expressions above, (11) can be written as

$$\begin{aligned}\tfrac{1}{4}[-e + \tfrac{3}{2}(v_i - b^*(v_i))[\tfrac{F(b^*(v_i))-F(v_i)}{1-F(v_i)}] \\ + \textstyle\int_{b^*(v_i)}^{v_i+e}(v_i - z)\tfrac{f(z)}{[1-F(v_i)]}dz + [\tfrac{1-F(v_i+e)}{1-F(v_i)}](-e)] + \tfrac{1}{2}e = 0\end{aligned} \tag{26}$$

Multiplying throughout by $4[1 - F(v_i)]$ and cancelling terms yields mechanism, average willingness to pay is a sufficient statistic to characterize bidding behavior

[12]A second price sealed bid auction differs from a first price sealed bid auction in that the winner pays a price equal to the second highest bid

[13]In a Dutch descending bid auction, starting from some high initial price, the seller continuously lowers the price till one of the bidders 'cries out' to stop the price clock. The first such bidder to cry out then wins the auction at the price at which she cried out

[14]This is unlike the equilibrium allocations in an auction without externalities, and it violates condition (1) in Myerson's Corollary (Revenue Equivalence Theorem) (see Myerson[1981], p.65-66)

[15]We simulate expected revenues from the two auction formats using 5000 random draws for use values at each value of e. e is varied in steps of .002 from 0 to .75

[16]Aside from revenue considerations, social efficiency may form part of a benevolent seller's explicit objective as was the case with the spectrum auctions conducted by the FCC (see, e.g., Cramton[1995], p.268)

nature of the open auction reveals the private information of other bidders, thereby mitigating winner's curse to some extent and causing bidders to bid more aggressively. In the context of identity dependent externalities, even in a IPV framework, the open auction reveals useful information by way of bidder identities

[6]Of course, the ex-ante probability distributions of this two-tuple private information is assumed to be symmetric across bidders

[7]see section 3 for a description of the game used to study the open auction

[8]Consider for example, the button model game between 2 bidders, each of who draws their private use value from some distribution with support $[\underline{v}, \overline{v}]$. The following asymmetric strategy profile constitutes a Nash equilibrium: bidder 1 bids 0 and bidder 2 bids $\overline{v}$. However, the seller's revenue (which equals 0 in this equilibrium) will not equal his expected revenue in the symmetric equilibrium to the FPA (the reader may refer to Riley and Samuelson[1981] for the symmetric equilibrium strategy and hence expected revenue in the FPA)

[9]Our interest in finding an equilibrium in symmetric strategies, together with the common priors, obviates the need for a s_i subscript for the belief probability

[10]In other words we do not need to update bidder i's belief about the random variable $\tilde{s_{s_i}}$

[11]This insight regarding sealed-bid auctions is due to JMS(1) who show in their general multi-dimensional framework that in any symmetric equilibrium of an anonymous

Footnotes

[1]A summary of results that follow from weakening assumptions (i)-(iii) can be found in McAfee and McMillan[1987] or Wilson[1992]; for (iv) see Che and Gale[1998]

[2]Jehiel and Moldovanu[1996] answer a different set of questions in the context of auctions with identity dependent externalities. Among other things, they show that in a complete information setting, some bidders may gain by credibly commiting to non-participation

[3]The second dimension of each bidder's private information, viz., the externality that she causes to the other bidders plays a role only off-the-equilibrium-path, where one of the bidders contemplates non-participation. The optimal mechanism punishes such non-participation by awarding the object to the bidder who causes the non-participant to receive the lowest payoff

[4]Ascending bid auctions where bidder identities are kept secret (e.g., by admitting bids on-line or over the telephone) obviously do not fit this description of an open auction. However, this does not diminish the normative implications of our work since we show that in the presence of identity dependent externalities, the seller typically raises higher revenues and achieves a more efficient allocation by holding a non-anonymous mechanism

[5]This is somewhat reminiscent of the Milgrom and Weber[1982] result that in the presence of common value uncertainty, unlike a sealed bid auction, the progressive

References

[1998] Che, Y. and Gale, I., "Standard Auctions with Financially Constrained Bidders", *Review of Economic Studies*, 65 (1), 1-21

[1995] Cramton, P., "Money Out of Thin Air", *Journal of Economics and Management Strategy*, 4(2), 267-344

[1996a] Jehiel, P., Moldovanu, B., and Stachetti, E., "Multi-dimensional Mechanism Design for Auctions with Externalities", *CMSEMS Discussion Paper*, Northwestern University

[1996b] Jehiel, P., Moldovanu, B., and Stachetti, E., "How (Not) To Sell Nuclear Weapons", *American Economic Review*, 86(4), 814-829

[1996] Jehiel, P., and Moldovanu, B., "Strategic nonparticipation", *RAND Journal of Economics*, 27(1), 84-98

[1996] Maskin, E. and Riley, J., "Asymmetric Auctions", *working paper*, UCLA

[1995] Matthews, S.A., "A Technical Primer on Auction Theory I: Independent Private Values", *Northwestern University Discussion Paper # 1096*

[1987] McAfee, P., and McMillan, J., "Auctions and Bidding", *Journal of Economic Literature*, 25, 699-738

[1982] Milgrom, P., and Weber, R., "A Theory of Auctions and Competitive Bidding", *Econometrica* 50, 1089-1122

[1981] Myerson, R., "Optimal Auction Design", *Mathematics of Operations Research*, 6, 58-73

[1981] Riley, J., and Samuelson, W., "Optimal Auctions", *American Economic Review*, 71, 381-392

[1961] Vickrey, W., "Counterspeculation, Auctions and Competitive Sealed Tenders", *Journal of Finance*, 16, 8-37

[1992] Wilson, R., "Strategic Analysis of Auctions", Chapter 8 in R.J. Aumann and S. Hart ed. *Handbook of Game Theory,* Elsevier Science Publishers, 227-279

establish a general revenue ranking result owing to the lack of closed-form solutions for bid functions. Nonetheless, the simple model employed in this paper serves to verify the intuition that an open auction reveals more equilibrium information about the identity of the auction winner than a sealed bid auction, which in turn leads bidders to place higher values on the object. Moreover, the open auction is also shown to be more efficient in its allocative properties. These results indicate that in the presence of winner specific externalities, the seller may gain by making bidder identities observable during the course of an open auction. Of course, the merits of this recommendation must be weighed against the well known result that the openness of an auction makes it more susceptible to bidder collusion. The greater efficiency of an open auction makes it possible that it is ex-ante Pareto superior to the sealed bid auction. It may also be interesting to pursue the implications of the main intuition of this work for the revenue ranking between simultaneous and sequential auctions of multiple goods in the presence of externalities. We leave these questions for future work.

Notice that the open auction is not socially optimal since it is possible for the interim realizations of valuations and rivalries to be such that it would be most efficient for the object to be allocated to the bidder with the lowest use value. In contrast to the FPA however, the open auction is efficient with respect to the highest and second highest valued bidders. Theorem 4 thus contradicts the basic IPV model as well as its generalizations obtained by relaxing assumptions (i) and (ii), all of which imply that the FPA and the open auction are both socially optimal in terms of the allocations they achieve. Our result agrees partially with that from an asymmetric environment where the open auction is socially optimal while the FPA is not; while our result stands in sharp contrast to the model with budget constrained bidders which implies greater efficiency of allocation in an FPA.

7 Conclusion

The contribution of this paper has been to formalize the intuition that in the presence of identity dependent externalities, an open ascending bid auction might raise higher expected revenues compared to a sealed bid auction. The multi-dimensional nature of private information precludes a completely general analysis of the topic. In particular, increasing the number of bidders to more than three causes the bid functions to become multi-dimensional which in turn leads to well known problems for computing equilibrium strategies. Moreover for a part of the parameter space we are unable to

(c) the bidder with valuation $v_{(1)}$ is strong with respect to the bidder with valuation $v_{(2)}$

(d) the bidder with valuation $v_{(2)}$ is weak with respect to the bidder with valuation $v_{(1)}$.

In event A, for the open auction, the gross payoffs of the bidders with valuations $v_{(2)}$ and $v_{(1)}$ are, respectively, $v_{(2)}$ and 0, so that the sum of their payoffs is $v_{(2)}$. In the same event A, for the FPA, since the bidder with valuation $v_{(1)}$ wins, the sum of their gross payoffs is $-e + v_{(1)}$. But by (b), the former sum exceeds the latter sum. This shows that conditional on event A, the total expected surplus is greater in the open auction. As A is the only event in which the total surpluses in the two auctions are different, this proves that the total expected surplus in the open auction is greater than it is in the FPA. ∎

Observe that the bidder with the lowest valuation $v_{(3)}$ loses the auction with probability 1 in both the open auction and the FPA, and hence her expected surplus equals $-\frac{1}{2}e$ in both formats. However, in the event where (b)-(d) (see proof of Theorem 4) are true, the total surplus is clearly higher when the object is allocated to the bidder with valuation $v_{(2)}$ instead of the bidder with valuation $v_{(1)}$. Thus the open auction achieves a different allocation than that resulting from the FPA precisely when it is more efficient (as measured by the total surplus) to do so.

6 Allocative Efficiency

In the presence of externalities, an interesting difference between the FPA and the open auction lies in their allocative properties. While the FPA always allocates the object to the bidder with the highest use value, the open auction sometimes allocates the object to the bidder with the second highest use value (see the statement of Proposition 4). It may therefore be important to inquire as to which of these two auction forms generates a higher surplus.[16] Given that the price is simply a transfer payment and utilities are quasi-linear, we can define an efficient allocation to be one that maximizes the sum of the gross payoffs received by bidders. Denote by t_l^i the gross payoff to bidder l when the object is allocated to bidder i. In what follows we compare the expected value of $\sum_l t_l^i$ for each of the two auction formats where i denotes the identity of the auction winner for the format in question.

Theorem 4 *For every $e > 0$, the open auction yields a higher surplus than the FPA.*

Proof. First note that the surpluses resulting from the FPA and the open auctions are the same in any event in which the bidder with the highest valuation $v_{(1)}$ wins both auctions. Now consider the event in which this is not the case. Call this latter event A. From the equilibrium characterizations of the two auctions, we know that in event A the following must be true:

(a) the winner of the open auction is the bidder with valuation $v_{(2)}$

(b) $v_{(1)} < v_{(2)} + e$

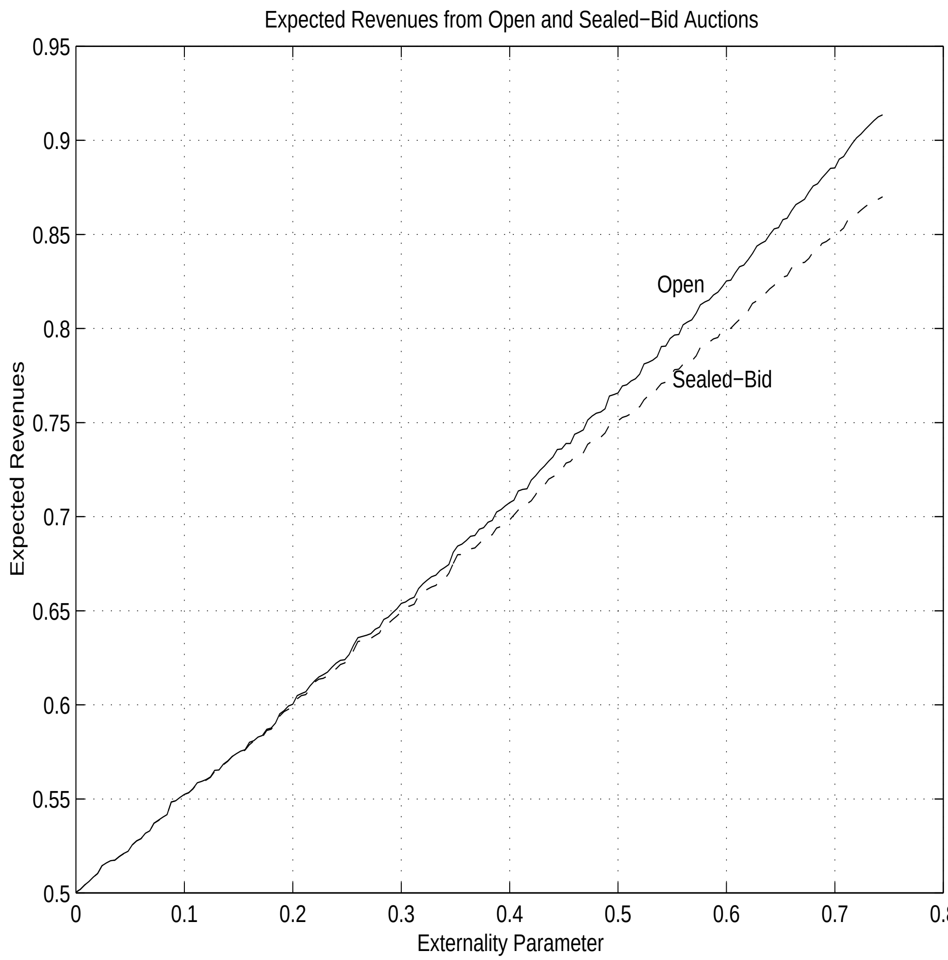
Expected Revenues from Open and Sealed−Bid Auctions
Open
Sealed−Bid
Expected Revenues
Externality Parameter
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

Figure 1:

$$b^*(v_i) = \begin{cases} v_i + e\sqrt{0.5}, & \underline{v} \leq v_i < \overline{v} - e \\ v_i + \sqrt{e(\overline{v} - v_i) - \frac{1}{2}(\overline{v} - v_i)^2}, & \overline{v} - e \leq v_i < \overline{v} - \frac{2}{3}e \\ v_i + \frac{2}{3}e, & \overline{v} - \frac{2}{3}e \leq v_i \leq \overline{v} \end{cases} \tag{21}$$

Proof. See Appendix ■

Given the complexity of the functional form above, we do not attempt to analytically compute prices from the open auction even for this simple case of uniformly distributed valuations. Instead, we simulate expected prices from the two auctions (using random draws of valuations from a uniform distribution) for different values of the parameter e. We use $F \equiv U[0,1]$ as the distribution of use values for this exercise and plot the expected revenues from the open and sealed bid auctions[15] in Figure 1.

Figure 1 shows that the open auction and the FPA yield the same expected revenues approximately up to the point $e = 0.25$. For $e > 0.25$, the open auction revenue dominates the FPA. This finding should however be interpreted with caution because (i) we do not have any economic insights as to why the FPA does not yield higher expected revenues compared to the open auction for small values of e and (ii) because of the special assumption of uniformly distributed valuations. However, given that the revenue equivalence theorem holds regardless of the number of bidders and the distribution of bidder valuations, this numerical calculation complements Theorem 3 in highlighting the critical nature of assumption (v).

than offsets the positive difference $E(p^f) - E(p)$. Therefore, the revenue superiority of the open auction over the FPA (for sufficiently large e) is a consequence of the fact that the open auction reveals more payoff-relevant information to the bidders before final bids (that determine the price) are due. The ability of the bidders who are still active at the start of phase f to incorporate this additional payoff-relevant information in their subsequent strategies causes bidders to value the object more at the start of an open auction (compared to the FPA), leading to higher equilibrium dropout points in phase $r1$ and thus higher prices on average.

We next turn to evaluating the revenue ranking between the two auctions for small values of e. Analytical comparison of expected revenues from the open auction with that from a FPA for $e \leq 3\left[E(v_{(2)}) - E(v_{(3)})\right]$ requires us to have closed form solutions for the equilibrium strategy in $r1$. However, inspection of (14) reveals that for most distributions of bidder valuations, $b^{r1}(.)$ may not possess a closed form solution. The uniform is the only class of distributions for which we have been able to solve (14) closed form.

Example 1 *For $F \equiv U[\underline{v}, \overline{v}]$, the unique solution to (14) consists of the following piece-wise, continuous and strictly increasing function:*

It is then easily verified that $E(p^o) > E(p^f)$ for $e > 3\left[E(v_{(2)}) - E(v_{(3)})\right]$. ■

Theorem 3 shows that for sufficiently large externalities, expected revenues from the open auction strictly exceed that from the FPA. Of course $e > 3\left[E(v_{(2)}) - E(v_{(3)})\right]$ is an overly strong sufficient condition for the revenue superiority of the open auction. The necessary condition for this revenue superiority will be much weaker and in particular will depend on the distribution of bidder valuations.

Notice that in our model the equilibrium allocation in an open auction might differ from that in a FPA. In particular, in an open auction, the bidder with the second highest use value of the object wins with positive probability while in an FPA the bidder with the highest use value wins with probability one.[14] Note also from its probability distribution in the statement of Proposition 4, that the expected unconstrained price from the open auction is strictly less than the expected price from the FPA i.e., $E(p) < E\{v_{(2)} + \frac{1}{2}e\} = E(p^f)$ (since $\Pr(v_{(1)} < v_{(2)} + e) > 0$). Therefore, as the logic of the proof of Theorem 3 suggests, the revenue superiority of the open auction (for sufficiently large e) is a consequence of the fact that the constraint $p^o \geq b^{r1}(v_{(3)})$ binds frequently enough. Recall our previous remarks following Theorem 1 regarding the option value to bidders of staying active till the start of the frozen phase. This option value is increasing in the magnitude of the externality parameter. For sufficiently large e, the bid or the total valuation of the object (i.e., average willingness to pay + option value) for the bidder with the lowest use value binds often enough and when it does, on average it more *the potential winner before the auction ends. This implies revenue equivalence between the Dutch auction and the sealed bid auctions.*

Observe that for $e = 0$, the price from an open auction equals $v_{(2)}$ (since $b^{r1}(v_{(3)}) = v_{(3)}$ for $e = 0$) and that the open auction and FPA yield the same expected revenue in this case, as we would expect. The revenue ranking of the two formats for $e > 0$ will likely depend on the exact value of e. Our strategy in comparing revenues will be to first find regions of the parameter space for which we can theoretically prove the corresponding revenue ranking results. For the rest of the parameter space, we will appeal to numerical methods to determine revenue rankings.

Theorem 3 *If* $e > 3\left[E(v_{(2)}) - E(v_{(3)})\right]$*, then the open auction yields strictly greater expected revenues than the FPA.*

Proof. First note that $\max\{v_{(2)}, b^{r1}(v_{(3)})\} \geq b^{r1}(v_{(3)})$. Similarly, $\max\{\min\{v_{(2)} + e, v_{(1)}\}, b^{r1}(v_{(3)})\} \geq b^{r1}(v_{(3)})$.

From the statement of Proposition 4 and Lemma 1, it then follows that

$$E(p^o) \geq \frac{3}{4}\{E(v_{(3)}) + \frac{2}{3}e\} + \frac{1}{4}\{E\left(v_{(2)}\right) + e\}$$

while we already know that

$$E(p^f) = E\left(v_{(2)}\right) + \frac{1}{2}e$$

respect to each other, and (ii) the second highest valued bidder is strong with respect to the highest valued bidder, while the reverse is not true. The unconstrained price equals $v_{(2)} + e$ in the event that both the two highest valued bidders are strong with respect to each other, an event that occurs with probability $\frac{1}{4}$. Finally in the event that the highest valued bidder is strong with respect to the second highest valued bidder, while the reverse is not true, the unconstrained price equals the minimum of $v_{(2)} + e$ and $v_{(1)}$. The actual price, for any given realization of mutual rivalries, is the maximum, of the unconstrained price for that realization, and the frozen price $b^{r1}(v_{(3)})$. ■

The expected price from a FPA is relatively simpler to characterize. From the similarity of the equilibrium to that in a standard IPV model, it is immediate that the expectation of the price p^f equals $E(v_{(2)}) + \frac{1}{2}e$.

Remark 1 *JMS describe the equilibrium to a second price sealed bid auction (SPA)*[12] *in their framework. In their equilibrium, each bidder bids her average willingness to pay for the object and the price therefore equals the second highest order statistic from the distribution of average willingness to pay. In our framework, this amounts to a SPA price of* $v_{(2)} + \frac{1}{2}e$*, implying revenue equivalence between the two sealed bid auction formats. This conclusion is expected since both the FPA and the SPA differ from the open auction in a similar fashion viz., in their sealed bid feature. Also, note that the Dutch descending bid auction*[13] *continues to be strategically equivalent to the FPA in the presence of externalities since neither format reveals any information about the identity of* question. Recall the equilibrium strategies for the open auction and note that since strategies in $r1$ are symmetric, strictly increasing functions of use value alone, the first bidder that quits the auction is the one with the lowest use value of the object. The price is then determined by a standard open auction between the two highest valued bidders subject to the constraint that it cannot be any less than the price at which the lowest valued bidder quit the auction. Let us call the price from an English auction between the two highest valued bidders the "unconstrained" price. In the two bidder open auction that follows $r1$, the unconstrained price is therefore determined by the two highest use values and the nature of their mutual rivalry. Let us define $v_{(l)}$ $(l = 1, 2, 3)$ as the $\underline{lth.}$ highest order statistic of three independent random draws from $F(.)$.

Proposition 4 *The price from an open auction p^o equals* $\max\left\{b^{r1}(v_{(3)}), p\right\}$ *where p follows the distribution*

Price(p)	Probability
$v_{(2)}$	$\frac{1}{2}$
$v_{(2)} + e$	$\frac{1}{4}$
$\min\left\{v_{(2)} + e, v_{(1)}\right\}$	$\frac{1}{4}$

Proof. Since strategies in $r1$ are a function of use values alone, each of the two highest valued bidders is strong with respect to the other with probability $\frac{1}{2}$. The unconstrained price equals $v_{(2)}$ in the following two events, each of which therefore occurs with probability $\frac{1}{4}$: (i) none of the two highest valued bidders are strong with

$$\frac{\partial^2 \Pi_i(b_i)}{\partial b_i \partial v_i} = F(b^{*-1}(b_i))f(b^{*-1}(b_i))b^{*-1'}(b_i) \tag{20}$$

which is strictly positive for all $b_i \in (0, b^*(\bar{v})]$ since $F(.)$ is strictly increasing and $b^*(.)$ is both strictly increasing and continuous. ■

In the symmetric strategy equilibrium, the sealed bid nature of the FPA essentially causes bidder i to behave as if her "type" is $v_i + \frac{1}{2}e$. Each bidder's bid takes the familiar form of the being the expected value of the second highest order statistic from the distribution of "types", conditional on the bidder's own "type" being the highest. In contrast to the open auction, where conditional on being active at the start of the frozen phase, a bidder gets a second chance to revise her bid in response to the observed history of play, the sealed bid auction is instantaneous and offers no such opportunity. As a result, for each bidder, her average willingness to pay for the object turns out to be a sufficient statistic to characterize her equilibrium bidding strategy.[11] Recall that in an open auction this is not the case and bidder strategies (for the entire game) are multi-dimensional, depending on both v_i and s_i.

5 Revenue Comparison

The main objective of this paper is to examine whether the open and sealed bid auctions continue to be revenue equivalent when bidders suffer identity dependent externalities. We now turn to computing prices from the two auction formats so as to answer this

$$-F(b^{*-1}(b_i))^2 + (v_i - b_i + \frac{1}{2}e)F(b^{*-1}(b_i))f(b^{*-1}(b_i))b^{*-1'}(b_i) = 0 \tag{18}$$

Setting $b_i = b^*(v_i)$ and rearranging, we get

$$b^{*'}(v_i) = (v_i + \frac{1}{2}e - b^*(v_i))\frac{f(v_i)}{F(v_i)} \tag{19}$$

Also, in a symmetric equilibrium in strictly increasing strategies, the lowest type to submit a bid must be bidding the reserve price and she must also be indifferent to participating in the auction (i.e., making zero expected profits). Notice that in a symmetric equilibrium, type $\underline{v}$'s willingness to pay (difference between her use value and her expected payoff from not submitting a bid) equals the minimum allowed bid, i.e., $\underline{v}+\frac{1}{2}e$. Hence with a pre-announced reserve price of $\underline{v}+\frac{1}{2}e$, all types would actually submit bids and $b^*(\underline{v}) = \underline{v} + \frac{1}{2}e$.

Theorem 2 $b^*(v_i) = \int_{\underline{v}}^{v_i} \frac{zf(z)dz}{F(v_i)} + \frac{1}{2}e$ *(with $b^*(\underline{v}) = \underline{v} + \frac{1}{2}e$) constitutes a symmetric equilibrium strategy for the FPA.*

Proof. First note that $b^*(.)$ as defined in the statement of the Theorem, satisfies the first order condition (19) and is strictly increasing, continuous (note $b^*(\underline{v}) = \underline{v} + \frac{1}{2}e$) and differentiable. Now in view of the pseudo-concavity argument presented in the proof of Theorem 1, it would suffice for existence if we can show that $\frac{\partial^2\Pi_i(b_i)}{\partial b_i \partial v_i} > 0$. But case of the open auction, we will confine attention to BNE of the FPA in symmetric, strictly increasing and differentiable strategies. Also, as in the case of the open auction, observe that in a symmetric equilibrium, we can drop the argument s_i from the bid function since conditional on losing, each bidder assigns equal probability to each of her rivals winning the auction. Given the ex-ante symmetry of the model, bidders have no basis to discriminate amongst their rivals in terms of their equilibrium strategies.

Let $b^*(.)$ denote the symmetric strategy employed by bidders s_i and w_i. Bidder i's objective function can then be formulated as

$$\Pi_i(b_i; v_i) = (v_i - b_i)\{F(b^{*-1}(b_i))^2\} + [1 - F(b^{*-1}(b_i))^2](-\frac{1}{2}e) \tag{16}$$

The first term in (16) is the expected payoff of bidder i corresponding to the event that she wins the auction and the second term, corresponding to the event that she loses. Notice that if bidders s_i and w_i employ symmetric strategies, then bidder i assigns equal probability to each of her rivals winning the auction and hence her expected payoff conditional on losing is simply $-\frac{1}{2}e$. Bidder i then solves

$$\underset{b_i}{Max}\,\Pi_i(b_i; v_i) \tag{17}$$

subject to $b_i(v_i) \geq \underline{v} + \frac{1}{2}e$.

The first order necessary condition for an interior solution to bidder i's problem is value in addition to her average willingness to pay of $v_i + \frac{1}{2}e$.

Moreover, since $b^*(v_i)$ is strictly greater than $v_i + \frac{1}{2}e$, the equilibrium strategies for the open auction derived above (and hence the seller's expected revenue) would remain unchanged if the seller were to start the price-clock from some price less than or equal to$\underline{v} + \frac{1}{2}e$ instead of a price of 0 as implicitly assumed earlier. In the next section on sealed-bid auctions, we assume that the seller uses a reserve price of $\underline{v} + \frac{1}{2}e$.

4 First Price Sealed Bid Auction

In a first price sealed bid auction (FPA), each participating bidder secretly submits a dollar amount by way of a sealed bid, to the seller. The seller then awards the object to the highest bidder at a price that equals her bid, provided this highest bid exceeds a pre-announced reserve price (say R). First price auctions are frequently used for public procurement and sale of natural resources e.g., the Department of Interior's sale of drilling rights for off-shore oil and natural gas reserves, the US Forest Service's sales of timber harvesting rights on federally owned forestland etc. A strategy for each bidder in a FPA consists of a map from her private information to a positive dollar amount (at least equal to the reserve price) or the decision not to submit a bid, i.e., $b_i : [\underline{v}, \overline{v}] \times S \backslash i \rightarrow [R, \infty) \cup \{\text{no bid}\}$. A Bayesian Nash Equilibrium (BNE) for the FPA consists of a strategy for each bidder such that given the strategy of her opponents, her own strategy maximizes her expected payoff. We assume that $R = \underline{v} + \frac{1}{2}e$. As in the externalities. Recall that the nature of private information in this model is such that in $r1$, strategies are a function only of the first element of private information (i.e., use values). Hence for purposes of computing the expected profits at each frozen price, beliefs need to be updated only along the first dimension of private information. Strategies in f are functions of both elements of private information, but the existence of a dominant strategy for each active bidder in $r2$ obviates the need to further update beliefs following the observed outcome of f. In this way the model avoids the complications associated with multi-dimensional belief updation. Also note that this simplification works only in a 3 bidder model. If the number of bidders exceeds 3, then in all phases except the first and the last set of rising and frozen phases, both elements of beliefs would need to be updated, since strategies in the preceding phases would be two-dimensional.

An interesting feature of $b^*(v_i)$ is that it is strictly greater than $v_i + \frac{1}{2}e$. While the significance of this will become clear during our comparison of expected revenues between the open and sealed bid auctions, it is worth pointing out at this stage that the open auction allows active bidders the option to revise their bids in response to the information that is revealed at the end of the first rising phase. In particular, each active bidder gets to see the identity of her other active rival at the start of the frozen phase and bid accordingly. Therefore there is an option value to being active at the start of the frozen phase. Bidder i's bid in the first rising phase reflects this option increasing, continuous and differentiable a.e. in v_i. We next show that (15) indeed has such a fixed point and furthermore it is unique. The following lemma will be useful for our proof.

Lemma 1 *If b^* is a solution to (15) for some v_i, then $b^* \geq v_i + \frac{2}{3}e$*

Proof. Observe that $\int_{b^*}^{v_i+e}(v_i-z)f(z)dz \geq -e\int_{b^*}^{v_i+e} f(z)dz = -e[F(v_i+e)-F(b^*)]$. Substituting this inequality into (15), we have

$$b^* \geq v_i + \tfrac{2}{3}\tfrac{e[F(v_i+e)-F(v_i)]-e[F(v_i+e)-F(b^*)]}{[F(b^*)-F(v_i)]} = v_i + \tfrac{2}{3}e \quad \blacksquare$$

Proposition 3 *The solution to (15) that satisfies $v_i \leq b^*(v_i) \leq v_i + e$ is unique and is strictly increasing, continuous and differentiable a.e in v_i.*

Proof. See Appendix. ■

In order to complete the equilibrium characterization, we are now left to show that $b^*(v_i)$ indeed maximizes the objective function in (9). This is done in the following theorem.

Theorem 1 *$b^*(v_i)$ defined in (15) is the symmetric equilibrium bid function for phase $r1$.*

Proof. See Appendix. ■

We have thus proved that there exists a unique equilibrium in symmetric strategies for the "button model" of an open auction where bidders suffer identity dependent some additional probability. By doing this, she gains in expected payoff by an amount $\pi_i^f(b; v_i)$ and loses by an amount of $-\frac{1}{2}e$ (her expected payoff on being the first quitter). At the optimal bid, the gain must exactly balance the loss.

Fact 1 *In an equilibrium in symmetric strictly increasing strategies*

$$F(z \mid b^*(v_i)) = \begin{cases} 0 \quad \forall z \in (-\infty, v_i] \\ \frac{F(z)-F(v_i)}{1-F(v_i)} \quad \forall z \in (v_i, \bar{v}] \\ 1 \quad \forall z \in (\bar{v}, \infty) \end{cases} \tag{13}$$

Proposition 2 *$b^*(v_i)$ must satisfy the following equation*

$$\frac{3}{2}(v_i - b^*)[F(b^*) - F(v_i)] + \int_{b^*}^{v_i+e} (v_i - z)f(z)dz + [F(v_i + e) - F(v_i)]e = 0 \tag{14}$$

Proof. See Appendix. ∎

Rearranging (14), we get

$$b^* = v_i + \frac{2}{3}\frac{\int_{b^*}^{v_i+e}(v_i - z)f(z)dz + [F(v_i + e) - F(v_i)]e}{[F(b^*) - F(v_i)]} \tag{15}$$

The equilibrium strategy for the first rising phase of the auction is thus a fixed point of the right hand side (RHS) of (15), if one exists and satisfies the hypotheses of our equilibrium, i.e., it satisfies the bounds $v_i \leq b^*(v_i) \leq v_i + e$, is monotonically under the hypothesis $v_i \leq p < v_i + e$ are all that we need in order to solve the first order condition that results from (10).

Proposition 1 *If $b^*(.)$ is differentiable a.e. on its domain, then it satisfies the following necessary condition a.e.*

$$\pi_i^f(b^*(v_i); v_i) + \frac{1}{2}e = 0 \tag{11}$$

Proof. If $b^*(.)$ is continuous, strictly increasing and differentiable a.e. on the support of v_i, then $b^{*-1}(.)$ must also be continuous, strictly increasing and differentiable a.e. on the range of $b^*(.)$. Setting the derivative of the right side of (9) with respect to b_i equal to zero then gives

$$2\{\pi_i^f(b_i; v_i) + \frac{1}{2}e\}\{1 - F(b^{*-1}(b_i))\}f(b^{*-1}(b_i))b^{*-1'}(b_i) = 0 \tag{12}$$

Next, note that since $b^{*-1}(.)$ is strictly increasing everywhere on its domain, $b^{*-1'}(b_i) > 0$. Also, note that since $F(.)$ is strictly increasing and $b^*(.)$ is continuous, the bidders' equilibrium bid distributions cannot have any gaps (i.e., $f(b^{*-1}(b_i)) > 0$ for all b_i in the range of $b^*(.)$). Then setting $b_i(.) \equiv b^*(.))$ gives (11). ■

Equation (11) has the following interpretation. If bidder i increases her bid from b by a little bit, then she gets to stay active till the start of the frozen phase with

$$\pi_i^f(p; v_i) = \frac{1}{2}[\pi_i^f(p, s_i; v_i) + \pi_i^f(p, w_i; v_i)] \tag{8}$$

Now consider bidder i's problem at the start of the first rising phase of the auction. If none of bidder i's rivals quit the auction in $r1$ at any price lower than b_i^{r1}, then bidder i can expect each one of her rivals to go on to win the auction with probability $\frac{1}{2}$ (since strategies are symmetric), which leaves bidder i with a payoff of $-\frac{1}{2}e$. Then (omitting the $r1$ superscript on the bid function) we can formulate bidder i's objective function at the start of the auction as

$$\pi_i(b_i; v_i) = \int_0^{b_i} \pi_i^f(p; v_i) d[1 - \{1 - F(b^{*-1}(p))\}^2] + [1 - F(b^{*-1}(b_i))]^2(-\frac{1}{2}e) \tag{9}$$

where $b^*(.)$ denotes the symmetric, strictly increasing and continuous equilibrium bid function being solved for.

Bidder i's bid in the first rising phase is a solution to the optimization problem

$$\underset{v_i \leq b_i \leq v_i + e}{Max.} \pi_i(b_i; v_i) \tag{10}$$

Note that even though the integral in the objective function (9) has a lower limit of 0, our only use of this objective function will be to solve $b^*(v_i)$ which we have already argued must satisfy $v_i \leq b^*(v_i) \leq v_i + e$. Hence the expressions (4) - (7) that we derived

Similarly, if bidders i and w_i are active at the start of the frozen phase, then bidder i's expected payoff at price p can be written as $\pi_i^f(p, w_i; v_i) = \frac{1}{2}[\pi_i^s(p, w_i; v_i) + \pi_i^w(p, w_i; v_i)]$ where

$$\begin{aligned} \pi_i^s(p, w_i; v_i) &= [1 - F(p - e \mid p)].[0] + [\frac{1}{2}(v_i - p) + \frac{1}{2}.[0]]F(p - e \mid p) \qquad (6) \\ &= 0 \quad (\text{since } F(p - e \mid p) = 0) \end{aligned}$$

corresponds to the case where bidder i is strong with respect to w_i, and

$$\pi_i^w(p, w_i; v_i) = [1 - F(p \mid p)].[0] + [\frac{1}{2}(v_i - p) + \frac{1}{2}.[0]]F(p \mid p) \qquad (7)$$

to the case where bidder i is weak with respect to w_i.

Again, to derive (6), observe from (1) that bidder i will quit in f . Since in this case i is strong with respect to w_i, bidder w_i will also quit the auction in $f(p, j)$ only if $p > \tilde{v}_{w_i} + e$. Hence bidder i loses the auction outright, to bidder w_i with probability $[1 - F(p - e \mid p)]$ and ties for the object with bidder w_i, with probability $F(p - e \mid p)$. Similar arguments, with $\tilde{v}_{w_i} + e$ replaced by $\tilde{v}_{w_i}$, lead to the expression in (7).

Thus at the start of the first rising phase, conditional on one her rivals quitting the auction at some price $v_i \leq p \leq v_i + e$, bidder i considers her expected payoff at the start of the frozen phase to be win the auction at frozen price p with probability $F(p-e \mid p)$, she will win the auction at price $\tilde{v}_{s_i} + e$ $(\tilde{v}_{s_i} < v_i)$ with corresponding probability density $f(\tilde{v}_{s_i} \mid p)$, and lose the auction to bidder s_i with probability $[1 - F(v_i \mid p)]$. In the last case bidder i's payoff is $-e$. Notice that since no bidder would ever bid greater than the sum of their use value and e, $F(p-e \mid p) = 0$. Incorporating this (3) simplifies to

$$\pi_i^s(p, s_i; v_i) = \int_{p-e}^{v_i} (v_i - z - e) f(z \mid p) dz + [1 - F(v_i \mid p)](-e) \tag{4}$$

Similarly, in the event bidder i is weak with respect to s_i, bidder i's expected payoff at frozen price p is

$$\pi_i^w(p, s_i; v_i) = (v_i - p) F(p \mid p) + \int_p^{v_i+e} (v_i - z) f(z \mid p) dz + [1 - F(v_i + e \mid p)](-e) \tag{5}$$

The derivation of (5) follows the same arguments that were offered to explain (4) with the exception that bidder s_i is now going to quit the auction in f if $\tilde{v}_{s_i} \leq p$, otherwise she will stay in the auction till the price reaches $\tilde{v}_{s_i}$.

Because $b_{s_i}^{r1}$ does not depend on the identity of bidder s_i's strong rival, bidder i continues to assign probability $\frac{1}{2}$ to herself being strong with respect to bidder s_i.[10] Hence, if bidders i and s_i are active at the start of the frozen phase, bidder i's expected payoff is $\pi_i^f(p, s_i; v_i) = \frac{1}{2}[\pi_i^s(p, s_i; v_i) + \pi_i^w(p, s_i; v_i)]$.

of the frozen phase $f(b_i^{r1}, j)$ equals her expected payoff from being the first bidder to quit at price b_i^{r1}. Therefore, in view of the dominance argument presented above, for purposes of computing b_i^{r1}, we can restrict attention to the interval $v_i \leq p < v_i + e$ while computing the expected payoff of a bidder at the start of $f(p, j)$ as a function of the history $H = (p, j)$. However, since $j \in \{s_i, w_i\}$ and strategies in $f(p, j)$ are dependent on j as well, we need to consider each of these possibilities separately.

Suppose bidders i and s_i are active at the start of the frozen phase (i.e., $j = w_i$). Note that since $\tilde{v}_{s_i}$ is a random variable, bidder s_i's bid in $r1$ is a random variable too. We denote this random variable by $\tilde{b}_{s_i}^{r1}$. Let us define $F(z \mid p)$ (with corresponding density function $f(z \mid p)$) to be the equilibrium probability[9] that $\tilde{v}_{s_i} \leq z$ conditional on $\tilde{b}_{s_i}^{r1}$ being greater than p. Since $F(.)$ is continuous and the hypothesized strategies in the first rising phase are strictly increasing functions of use value, $f(z \mid p)$ cannot have any mass points. In the event that i is strong with respect to s_i, bidder i's expected payoff at frozen price p can be written as

$$\pi_i^s(p, s_i; v_i) = (v_i - p)F(p - e \mid p) + \int_{p-e}^{v_i} (v_i - z - e) f(z \mid p) dz + [1 - F(v_i \mid p)](-e) \tag{3}$$

From (2) we know that bidder i will stay in the auction till the price reaches $v_i + e$. From (1) and (2) we also know that bidder s_i will quit the auction in f if $\tilde{v}_{s_i} \leq p - e$, otherwise she will stay in the auction till the price reaches $\tilde{v}_{s_i} + e$. Hence bidder i will

In view of the earlier remarks on the existence of a dominant strategy for each active bidder at the start of the frozen phase, it is straightforward to compute bidder i's equilibrium expected payoff (as a function of H) at the start of the frozen phase, taking as given her beliefs about her active opponent's private information. From (1) and (2), we know that the expression for the payoff we are about to calculate must depend on which of the following 3 intervals the value of p lies in (i) $0 \leq p < v_i$ (ii) $v_i \leq p < v_i + e$, and (iii) $v_i + e \leq p$. In addition, it must also depend on the identity of j. However, note that for bidder i, any bid in phase $r1$ that is less than v_i is dominated by the bid v_i, and similarly, any bid greater than $v_i + e$ is dominated by the bid $v_i + e$. To see the first part of the claim, observe from (1) that if bidder i were active at the start of $f(p, j)$, she would not quit in the frozen phase if $p < v_i$, regardless of the identity of j. Recall that starting with the frozen phase, the game is simply an English auction between the two remaining bidders. This means that bidder i is strictly better off winning the auction at any price (strictly) less than v_i rather than losing the auction, regardless of the identity of her rival she might be losing to. Since exit is irrevocable, bidding v_i in $r1$ must then strictly dominate bidding anything less than v_i. An analogous argument with respect to $p > v_i + e$ establishes the second part of the claim.

Intuitively, bidder i's drop-out point b_i^{r1} in the first rising phase must be such, that on the margin, her expected continuation payoff from being active at the start conditional on being active at the start of the frozen phase, each bidder expects to be confronted by either one of her rivals with equal probability. Also, bidders are ex-ante identical, leaving them with no basis to discriminate amongst their rivals in terms of their equilibrium strategies in $r1$.

The above simplification is important from a technical standpoint and deserves some discussion. Note that in any asymmetric equilibrium, this simplification will not be available and b_i^{r1} must be allowed to depend on s_i in addition to v_i. In addition to the usual lack of closed-form solutions to asymmetric bidding strategies (see e.g., Maskin and Riley[1996]), multi-dimensional bids complicate the Bayesian belief updation exercise and typically make it difficult to solve for consistent strategy-belief pairs. The latter problem also arises if the second element of a bidder's private information consists of the magnitude of the externality she suffers (instead of s_i as we have assumed) in which case b_i^{r1} is again two-dimensional. Furthermore, multi-dimensional belief updation is also avoided by restricting the analysis to a three bidder model. With three bidders, the only multi-dimensionality in equilibrium strategies arises at the start of the frozen phase, at which point each of the remaining bidders is known to have a dominant strategy. However with more than three bidders, equilibrium strategies will be multi-dimensional starting with the first frozen phase (with three or more bidders still active), making it essential to update multi-dimensional beliefs in order to compute equilibrium actions in subsequent phases of the auction.

A Perfect Bayesian Equilibrium (PBE) of the button model consists of a strategy for each bidder such that in each phase, given her beliefs about her active opponents' private information, the action prescribed by the strategy in that phase maximizes her expected payoff from the auction, and the beliefs so used satisfy Bayes rule with respect to the opponent's strategies, and priors, for the observed history of the game up until that phase. In view of the dominant strategy for each bidder at the start of the frozen phase, we need only solve for equilibrium strategies for the first rising phase. In the spirit of sub-game perfection, we will first compute the reduced form profit functions at the start of the frozen phase, taking as given the updated beliefs, and then use these profits to derive the equilibrium bid functions for the first rising phase that are consistent with those beliefs.

Given the ex-ante symmetry of the model, we will focus on finding PBE of this game that are characterized by symmetric strategies which are strictly increasing and continuous in the private use value of the object. The restriction of attention to symmetric strategy equilibria follows from our objective of re-evaluating the revenue equivalence result, which holds for the IPV framework only when the symmetric equilibrium to the open auction is compared to the equilibrium of the FPA.[8] Focussing on symmetric strategies allows us to drop the argument s_i from b_i^{r1}, the strategy in the first rising phase. This is because in a symmetric equilibrium, each bidder, conditional on losing the auction, expects to receive payoffs of 0 and $-e$ with equal probability. Moreover, rising phase is a map $b_i^{r1} : [\underline{v}, \overline{v}] \times S \backslash i \rightarrow \Re_+$, her strategy in the frozen phase is a map $b_i^f : [\underline{v}, \overline{v}] \times S \backslash i \times H \rightarrow \{quit, stay\}$ where history H consists of the frozen price p and the identity of the first quitter j, and strategy in the second rising phase is a map $b_i^{r2} : [\underline{v}, \overline{v}] \times S \backslash i \times H \rightarrow [p, \infty)$.

At this point, observe that the continuation game that starts with the frozen phase is simply an English auction between the two remaining bidders with an opening price equal to the frozen price. In this subgame, each of the two remaining bidders has a dominant strategy viz., to quit in the frozen phase if the frozen price exceeds their current willingness to pay for the object (difference between their use value of the object and the payoff they receive if their active opponent wins), or else to quit the second rising phase when the price reaches this willingness to pay. In our notation,

$$b_i^f(v_i, s_i, H) = \begin{cases} stay \text{ if } H = (p < v_i + e, w_i) \\ quit \text{ if } H = (p > v_i + e, w_i) \\ stay \text{ if } H = (p < v_i, s_i) \\ quit \text{ if } H = (p > v_i, s_i) \end{cases} \tag{1}$$

and,

$$b_i^{r2}(v_i, s_i, H) = \begin{cases} \max(p, v_i + e) \text{ if } H = (p, w_i) \\ \max(p, v_i) \text{ if } H = (p, s_i) \end{cases} \tag{2}$$

they so wish. If one of the two remaining bidders accepts this offer and quits then the remaining bidder is declared the winner of the auction at the current price. If both bidders quit the auction then the object is awarded at random to one of them at the current price. If no bidder quits, then the price clock resumes again and the auction proceeds to the price at which the first of the two remaining active bidders quits. The still-active bidder then wins the auction at this price. There are potentially two phases of this auction during which the price rises. We refer to the phase when all three bidders are active as the first rising phase of the auction ($r1$). Following the first exit is the only frozen phase ($f(p, j)$) of this auction when the seller makes simultaneous offers to the two remaining bidders to quit the auction. If none of the two remaining bidders accept this offer, then the auction witnesses a second rising phase ($r2(p, j)$) which ends when one of the two active bidders quits.

A strategy for each bidder in the "button model" described above consists of the following three elements: (i) a price at which to quit the first rising phase, (ii) a decision on whether to quit the auction during the frozen phase, contingent on the history of the game thus far (i.e., frozen price p and the identity of the rival (j) whose exit marked the end of the first rising phase), and (iii) a price (higher than p) at which to quit the second rising phase of the auction, contingent on the frozen price p, the first quitter's identity j, and the fact that the active rival at the beginning of the frozen phase chose not to quit during the frozen phase. To be more precise, bidder i's strategy in the first

3 Open Ascending-Bid Auction

In an open ascending bid auction, starting from some initial low bid, bidders continually raise the standing high bid by at least an amount equal to a pre-specified bid increment. The auction ends when no bidder is willing to raise the standing high bid. The standing high bidder then wins the object at a price equal to her final bid. Open ascending bid auctions are frequently used for the sale of art, antiques, wine, cars and real estates, and sometimes by federal agencies like the US Forest Service for sale of natural resources. Milgrom and Weber[1982] have analyzed the so-called "button model" as an approximation to the open ascending bid auction. In the "button model", starting from some initial low price, the seller increases the price continuously and each bidder indicates her active bidding status by keeping a button pressed. Bidders can irrevocably exit from the auction at any price simply by removing their fingers from the button at that price. All active bidders observe the identity and the quitting price of each exiting bidder. The auction ends at the price at which the second last bidder quits the auction. The last remaining bidder wins the object at this price. We follow Milgrom and Weber in analyzing the "button model" in our context of identity dependent externalities amongst bidders. We make explicit the following assumption in our three bidder "button model". The seller stops the price clock at the price (p) at which the first irrevocable exit (by bidder j (say)) occurs. He then simultaneously offers an opportunity to each of the two remaining bidders to quit the auction at that price if refer to $\tilde{s}_i$ and $\tilde{w}_i$ as *strong* and *weak* respectively, with respect to bidder i. We further assume that ex-ante, $\tilde{s}_i$ can be either element of the set $S \backslash i$ with equal probability (in this case $\frac{1}{2}$).

Each bidder's private information consists of the two-tuple (v_i, s_i), which is the ordered pair consisting of the actual realizations of the random variables $\tilde{v}_i$ and $\tilde{s}_i$. All rivals of i share the common priors on $\tilde{v}_i$ and $\tilde{s}_i$ as described above.

Notice that contrary to the examples used to motivate this work, the model described above does not include an explicit post-auction game. Instead, the reduced form implications of asymmetries in post-auction interaction are allowed to manifest themselves within the model by way of the assumed identity dependent externalities. This allows us to analyze a static game (aside from the dynamics of the open ascending-bid auction) and in the process abstract away from signaling issues that typically arise in dynamic games of asymmetric information, since such issues are not the focus of this work. In this respect the model is similar to JMS(1) and JMS(2).

independence assumptions of our model make it possible to solve for the symmetric equilibrium of the (dynamic) open auction. We are able to show existence and uniqueness of symmetric equilibrium strategies for the open auction, and also characterize revenue ranking between the open and sealed bid auctions.

The remainder of the paper is organized as follows. Section 2 presents the model, section 3 presents the rules of the open auction and solves for the equilibrium. Section 4 presents the equilibrium to the FPA. Section 5 compares expected revenues and section 6 examines the allocative properties of the two auctions. Section 7 concludes. An appendix contains some of the longer proofs.

2 Payoffs and Information

Consider the auction of a single indivisible object amongst three bidders. Let the set of bidders be denoted by S. Bidder i's use value of the object is denoted by the random variable $\tilde{v}_i$ which is distributed according to $F(.)$ with corresponding density function $f(.)$. We assume that $F(.)$ is continuous, strictly increasing over its support $[\underline{v}, \overline{v}]$ ($\overline{v} > \underline{v} > 0$) and is common knowledge amongst all bidders. For each bidder i, there exists a bidder $\tilde{s}_i \in \{\ j \in S \mid j \neq i\}$, whose win gives i a payoff of $-e$ ($e > 0$). We denote the remaining rival of bidder i by the random variable $\tilde{w}_i \equiv S \backslash \{i, \tilde{s}_i\}$. We assume that if $\tilde{w}_i$ wins the auction, bidder i gets a payoff of 0. Bidder i's willingness to pay is therefore not unique but dependent on the identity of her rival in question. We the number of bidders. The minimum number of bidders required to model identity dependent externalities is 3. In the framework of JMS(1), the dynamic nature of the open auction would then necessitate the updating of multi-dimensional beliefs (of dimension at least 3), which turns out to be an intractable problem in general. The model is made tractable by restricting the environment in two ways. First, we reduce the dimensionality of the problem to a minimum by analyzing an open auction with only 3 bidders. Second, we assume that each bidder has one 'strong' and one 'weak' rival. In the event that a bidder's 'strong' rival wins, she receives a payoff that is strictly lower than the her payoff in the event that her 'weak' rival wins. The magnitude of the difference between these 2 possible payoffs (conditional on losing), is assumed to be commonly known. However, each bidder's private information is still two-dimensional, consisting of her use value of the object, and the identity of her 'strong' rival. This formulation has the advantage that even though private information is multi-dimensional, in any symmetric equilibrium, when bidding starts in the open auction, bidders are asymmetric only with respect to their use value of the object, and not with respect to the second element of their private information.[6] Therefore strategies in the first round of the open auction[7] are a function of only the use values and not the identities of the bidders' 'strong' rivals. This simplifies the task of solving for the equilibrium strategies in the first round of the open auction. Our formulation is thus quite different from those of JMS(1) and JMS(2). In particular, the slightly stronger symmetry and not useful because each bidder has a unique (exogenously realized) willingness to pay. However, with identity dependent externalities, the identity of the standing high bidder at each price during an open auction constitutes payoff-relevant information, and hence in equilibrium bidders condition their strategies on this additional information. As an open auction progresses, it offers bidders the option of adjusting their bids in response to changes in their beliefs as to the identity of the potential auction winner. We show that as a result of having this option, bidders value the object more in an open auction than in a sealed-bid auction.[5]

We find that the higher endogenous valuations that bidders place on the object in an open auction (compared to a sealed bid auction) leads to higher revenues from it. We prove this result for a generic value distribution, for the case in which the externality suffered by the bidders is sufficiently large. For smaller values of the externality parameter, we solve a numerical example (using uniformly distributed valuations) to show that the open auction could yield higher expected revenues compared to the FPA. We also find that in contrast to the standard IPV model, the open auction achieves a more efficient allocation compared to the sealed bid auction.

A few introductory remarks about our model is in order. We find that in the presence of identity dependent externalities, solving for the equilibrium of the open auction is quite complicated because of the multi-dimensional nature of private information. In the framework of JMS(1), the dimensionality of private information is the same as garding identity dependent externalities, the seller may do better in a non-anonymous mechanism where bidders can submit multi-dimensional bids and thereby convey more of their private information. Unlike a sealed bid auction, an open auction has the feature that at each point in the auction, bidders can observe the identity of the standing high bidder before they decide whether to raise the standing high bid or not. In other words, the open auction has the desirable feature that bids are effectively multi-dimensional and expected payments from bidders depend both on their rivals' bids as well as their identities. This raises the possibility that the open auction may raise higher revenues than the sealed bid auctions. In this paper, we formally examine the above conjecture.

To further clarify the main intuition, in the presence of identity dependent externalities, each bidder's willingness to pay for the object being auctioned is endogenous and depends on the identity of the highest bidder amongst her rivals. In the presence of asymmetric information a fundamental difference between sealed bid and open auctions is in the extent of information available to bidders before they must make their final bid. In an open auction, bidders can typically observe the identity of the standing high bidder at every point during the course of the auction.[4] In contrast, in a sealed bid auction, bidders must submit their bids not knowing who amongst their rivals would submit the highest bid. In the case of auctions without identity dependent externalities, this additional information that is revealed during the course of an open auction is consisting of their use value of the auctioned object and the externalities they inflict on the other bidders. Since the second element of a bidder's private information does not directly determine her willingness to pay, the seller's optimal mechanism makes no use of this in determining either the bidder's expected payment or her probability of winning.[3] JMS(1) on the other hand employs a more general model where each bidder's private information is an N-tuple denoting the ordered vector of gross payoffs that she receives for each possible allocation of the object. The optimal auction is quite different from standard auction formats that are observed in practice, and among other non-standard features, it requires losing bidders to make payments to the seller. While interesting in its characterization, the implementation of such an optimal auction might prove difficult and slow because of institutional imperfections and rigidities. It is therefore important to understand how identity dependent externalities may impact bidding behavior in standard auctions.

JMS(1) also analyze a class of mechanisms that they call "anonymous". An anonymous mechanism is defined to be one where (i) bidder reports are one dimensional, (ii) the expected payment and probability of win for any one bidder remains unchanged if two other bidders' reports are swapped, and (iii) swapping the reports of two bidders swaps their expected payments and win probabilities. JMS(1) shows that within this anonymous class of mechanisms, a second price sealed bid auction is optimal. However, intuition suggests that when bidders possess multi-dimensional private information re-

Assumption (v) may be rephrased to state that conditional on losing the auction, bidders are indifferent to the identity of the auction winner. This assumption may be unreasonable in many settings, especially when the auction is followed by subsequent interaction amongst the bidders. Examples of such ex-post interactions are abundant. Firms bidding for scarce inputs subsequently compete in product markets, bidders at auctions of art objects and antiques often meet again in resale markets, winners of defense contracts often subcontract part of the work to some of the losing bidders, while bidders at timber and oil auctions bid against each other repeatedly over time. Asymmetries in such post-auction interaction may create winner specific externalities for losing bidders. Thus, if firms compete in differentiated products in the output market, a losing bidder may be hurt more if the scarce inputs are won by a firm manufacturing close substitutes, or if a friendly firm wins a defense contract, a losing bidder may still be subcontracted a share of the work. Such asymmetries in ex-post interaction will then cause losing bidders to have preferences over the identity of the auction winner.

Such externalities have received little attention in the literature, especially in the context of standard auction mechanisms. Recent work by Jehiel, Moldovanu and Stachetti[1996a], [1996b] (JMS(1) and JMS(2) respectively, hereafter) has sought to relax this assumption in examining the nature of optimal auctions.[2] JMS(2) solves for the optimal auction in a model where bidders have two dimensional private information

1 Introduction

Auctions are frequently used as mechanisms of exchange in the presence of asymmetric information between buyers and sellers. Accordingly a large volume of research in auction theory has concentrated on examining bidding behavior in standard auction formats. One of the most remarkable results in this line of research is the theorem of revenue equivalence. It states that under the assumptions of (i) independent private use values (IPV), (ii) risk neutrality in bidder payoffs, (iii) ex-ante symmetry in bidder valuations, (iv) no budget constraints and (v) no identity dependent externalities, the *symmetric* equilibrium of the open ascending bid auction and the first price sealed-bid auction (FPA) yield the same expected revenues to the seller (see Vickrey[1961], Riley and Samuelson[1981] or Myerson[1981]). The revenue equivalence result follows in part from the fact that in their symmetric equilibria, both auction formats achieve the same allocation. The generality of this theorem derives from the fact that it does not rely on any assumptions either on the number of bidders (N) or on the distribution of bidder valuations. Subsequent research has re-evaluated this revenue equivalence result by relaxing individually, each one of assumptions (i)-(iv). In each case, the revenue equivalence result has been shown to break down making possible strict revenue rankings between the two auction formats.[1] Such results have had important policy implications for the design of optimal selling mechanisms under different economic environments.

Abstract

We analyze equilibrium bidding behavior in a three bidder open ascending-bid auction with identity dependent externalities. We prove the existence of a unique symmetric equilibrium and then show that for sufficiently large externalities, the open auction yields strictly higher expected revenues compared to a sealed bid auction. An open auction reveals to bidders more payoff-relevant information than a sealed bid auction and as a consequence, bidders are shown to have a higher willingness to pay in the early rounds of an open auction. The open auction is also shown to be more efficient than the sealed bid auction.

Keywords: auctions, externalities, revenue-ranking, allocative efficiency

JEL Classifications: D44, D62, D82

Running Head: Auctions with Externalities

Standard Auctions with Identity Dependent Externalities

Gopal Das Varma*
Fuqua School of Business
Duke University
Box 90120
Durham, NC 27708
Tel. (919) 660-1947
Fax (919) 684-2818
Email: dasvarma@mail.duke.edu

March, 1999

*I am indebted to Steve Matthews for providing very helpful suggestions on several previous drafts of this paper. Rob Porter provided useful comments on an early version. All remaining shortcomings are my own.