The Decision To Seek or To Be Sought

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Abstract

A one-shot market with two sides is considered where everybody can be matched with at most one person. Individuals have to find trading partners on their own. Whether searching or waiting is the optimal strategy is the central question of this paper. In a market where searching and waiting are done exclusively by one market side, it is more efficient if the long market side searches. In a market where on both sides some individuals search and others stay put, there are also mixed equilibria which are even more efficient. The matching friction due to uncoordinated search by individuals implies that larger market are in general less efficient than a collection of smaller markets.
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1 Introduction

Existing markets differ in many ways. Among other criteria, markets are characterized by how trading partners meet and how prices are determined. In decentralized markets no central institution or mechanism takes care of it. Individuals have to find trading partners on their own and they have to determine prices themselves. This paper concentrates on the first aspect, the uncoordinated search for trading partners by several individuals. It focuses on the most basic question in this context: who should search and who should wait to be found by someone else. This question is addressed in a simple one-shot market with two sides where everybody can be matched with at most one person and where individuals either search or wait. Searching for trading partners consists of approaching others randomly. If a searcher is accepted as trading partner, a match occurs; if not she remains unmatched. If someone waited to be found and nobody came forward, then also this person remains unmatched. Uncoordinated search may therefore lead to inefficiencies of the matching process; it causes a matching (or search) friction. In what follows the impact of this matching friction will be analyzed. Market efficiency will be evaluated as a function of who searches and as a function of market size; market equilibria will be characterized in terms of in matching behaviour.

The paper analyzes two different types of markets: markets where searching and waiting is done exclusively by members of one market side, and markets where on both sides some individuals search and others wait. As an illustration, imagine a dancing event (like a high school prom), where boys sit on one side of the room and girls on the other. As the band sounds the note to play, the boys stand up and walk across the room, each approaching one girl (or vice versa). This is an example of the first type of market. There may be some boys who fancy the same girl. She will choose one of these boys to dance with, the others miss this dance. There may also be some unfortunate girls no boy wishes to dance with. These girls are left on their own too. The second type of market is comparable to a dancing event where some boys and some girls get up at the same time and look for a partner. When they cross the room, they may run into each other and dance together, or they approach a girl (or boy) at the other side of the room as before. In both market types individuals can either search or wait. In the first kind of market matchings occur only between searching and waiting individuals. In the second market type there may also be matches between two searching individuals.

There are different existing markets which function like this. For instance, many insurance markets belong to that category. Insurance agents either wait for customers in their office or they visit homes to solicit contracts; customers wait for an insurance agent to come by or they contact an insurance agent to get coverage. Neither of them can do both - contacting the other party and being contacted - at once. In many existing markets individuals from only one market side search whilst individuals from the other side wait. This division of labour is often also postulated in economic models. There seems to be a "natural equilibrium" where either

- the long side searches and the short side waits or
- buyers search and sellers wait.

Whether such a "natural equilibrium" amounts to "casual empiricism" only, or whether there is any justification for it, is a central point of this paper. The answer is sought in this model in the *process of decentralized exchange per se*, contrary to Burdett et al (1995) who asked a similar question but based their answer on characteristics of the "process of monetary exchange per se". In their model it is decisive that there are different agents, some hold a monetary unit, others hold a unit of a good, and that their exchange options differ. Here, it only matters who searches and who stays put, and how large the two market sides of searchers and stayers are. Taylor (1995) has focused on the latter, i.e. on the impact of the size of two different market sides on prices, but he basically ignores matching behaviour. Here, price formation is ignored and the main focus is on the impact of matching behaviour. Since market size and matching behaviour are the only relevant criteria, the economic roles played by the agents on the two market sides are basically irrelevant. They could be sellers and buyers each holding or demanding one unit of a homogeneous good. The roles could also be phrased in terms of right and left gloves that have to be matched, cf. Shapley (1969).
The main results of the paper are as follows. In terms of efficiency it is best, if on both sides some individuals search and others wait. This is an equilibrium only if the numbers of searchers and stayers is "right" on both market sides: most agents of the long market side should be searchers and most agents of the short market side should be stayers. If the likelihood that two searchers meet when they cross the field is low enough, then exclusive search by either of the two market sides are also equilibria. With exclusive search by one market side it is more efficient if the long market side searches and the short side stays put. The expected number of matchings is higher and its variance is lower. The same applies to the individual level, matching probabilities are higher if the long market side searches. These findings are valid also for large markets, unless one market side grows infinitely larger than the other. Large markets are actually worse than smaller markets in the sense that dividing a large market into several smaller markets increases efficiency, i.e. the expected total number of matches.

The rest of the paper is organized as follows. The next section presents results on a market with exclusive search by agents of one market side. It focuses on efficiency aspects. Section three extends the analysis to markets where there may be searchers and stayers on both market sides. It considers mainly strategic aspects. The paper is concluded by a discussion of related approaches and some suggestions for further research.

2 Searchers and Stayers on Separate Market Sides

Consider an economy with a finite number of individuals. These individuals are of either of two types. All individuals of one type are assumed to be identical. In the sequel the two types will be labelled as searchers and stayers or as belonging to the short and the long market side. Obviously, one of these two types could be buyers and the other could be sellers. However, the model will not be phrased in terms of buyers and sellers, because the difference between these two roles is of no relevance for the model. Players of both types meet at most one matching partner during only one period and they form one-to-one matchings. It is assumed that any two players of the two different types benefit from a match between them. All individuals of one type are identical.This implies that price determination can be ignored[1]. Despite the absence of prices, the focus of the model is on efficiency, where - in the standard sense - efficiency measures how well a market clears. Here, it is expressed in terms of matching probabilities or equivalently the expected number of matchings in the market.

In the market of this model there is no central institution that matches the two groups of individuals. They have to find each other on their own. Assume that each individual has the choice between two possible actions, *search* or *stay* (and wait to be found). For this section assume also that all individuals of one type pursue the same action; they are all either *searchers* or *stayers*. So, agents on one market side are searchers, those on the other market side are stayers. The number of stayers is denoted by m, the number of searchers by n. Either of the two market sides may be the larger one. Assume that there are b individuals on the larger (long) market side and s individuals on the smaller (short) market side, i.e. always $b \geq s$. This notation distinguishes between the actions or strategies of agents and the size of the two market sides; either $b = n$ and $s = m$, or the opposite.

Moreover, assume that all searchers go out simultaneously and without discrimination ap-

[1]The model can easily be phrased in standard market terms: Assume that there is a finite number of buyers and sellers. Each seller has one unit of an indivisible homogeneous good for sale, and each buyer wants to acquire exactly one unit of that good. Buyers have a reservation value of zero and sellers have a reservation value of two. Therefore, a surplus of two can be divided between the players of a matched pair. Assume that they split the surplus equally. The expected gain in the market (with the matching behaviour as described in the text) corresponds to the matching probability (times the surplus share of one).

proach one stayer randomly. When doing so they are not aware of other searchers' intentions and actions. It may therefore happen that more than one searcher approaches the same stayer. It may also happen that some stayers are not approached by anyone. Once this *initial assignment* is reached, it is the stayers' turn. Those who were approached by several searchers pick one of them randomly (again with equal probabilities) as their (only) matching partner. Stayers who were contacted by exactly one searcher are matched to that person. These matches form the *final assignment*[2]. Stayers who were not approached by any searchers in the initial assignment and searchers who were rejected (not picked) by the stayer they had contacted in the initial assignment remain unmatched. Typically, such an uncoordinated search by searchers leads to a final assignment with some matches and also some unmatched individuals on both market sides. The implications of the fact, that there may be beneficial matches which do not materialize due to coordination failure, is the central focus of this model. The main point of this section is to analyze the effect and magnitude of this matching friction. The following section will then concentrate on optimal behaviour in a market with this kind of coordination failure.

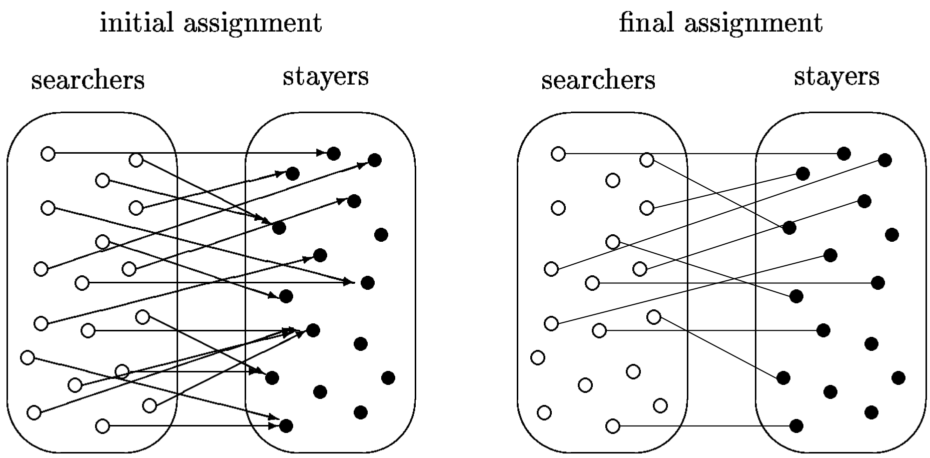

Figure 1: Initial and Final Matching Assignments

[2]All initial assignments with the same number k of stayers who were approached are equally likely, $k \leq m$. There are m^n possible initial assignments.
With k stayers approached by searchers, there are $\binom{m}{k}$ ways to select those stayers. Since each stayer picks one searcher, there are k matched pairs. In those k matched pairs there are also k searchers, which can be selected in $\binom{n}{k}$ different ways. The k stayers and k searchers can be assigned to each other in $k!$ different permutations. Therefore, the total number of possible final assignments is

$$\sum_{k=1}^{\eta} \binom{n}{k} \binom{m}{k} k! = \sum_{k=1}^{\eta} \frac{n!}{(n-k)!} \frac{1}{k!} \frac{m!}{(m-k)!},$$

where $\eta = \min\{m, n\}$. Each final assignment with k matched pairs is equally likely: $1/\left[\binom{n}{k}\binom{m}{k}k!\right]$.

Matching probabilities in such a market are

$$p_0(n,m) \;=\; 1 - \left(\frac{m-1}{m}\right)^n \tag{1}$$

for a stayer and

$$p_1(n,m) \;=\; \frac{m}{n}\left(1 - \left(\frac{m-1}{m}\right)^n\right) \tag{2}$$

for a searcher (the first argument always refers to the searching market side, the second to the stayers). An index 0 refers to a stayer and an index 1 indicates a searcher.

If there were a centralized institution which assured that all possible matches take place (without discrimination between individuals), then all individuals on the short market side would be matched with probability one and all individuals on the long market side would be matched with a probability of s/b. The matching behaviour assumed in this models implies that matching probabilities are determined in relation to the actions in the market (searcher or stayer) and not in relation to which market side is larger. Obviously, both matching probabilities depend on the size of m and n, but the functions are the same irrespective of whether $m \geq n$ or the opposite. The matching friction manifests itself in the subtracted term in equations (1) and (2). For a stayer it represents the probability that all searchers contact other stayers. If that happens this stayer remains unmatched. In the opposite case, the stayer is matched. With m stayers the expected number of matchings is

$$E_{n,m}(\Delta) \;=\; m \cdot p_0(n,m) \;=\; m \cdot \left(1 - \left(\frac{m-1}{m}\right)^n\right), \tag{3}$$

where Δ the number of matchings. The expected number of matchings can also be calculated as $E_{n,m}(\Delta) = n \cdot p_1(n,m)$, from which the searchers' matching probability of equation (2) can be derived[3].

In a market with a central matching institution that exhausts all matching opportunities the expected number of matchings would equal the size of the short market side. There would be no uncertainty about exactly how many matched pairs there are. With the matching friction of this model any number of matchings between 1 and s may happen. Obviously, they are not all equally likely to happen. The following probability distribution characterizes the matching process[4]

$$P_{n,m}(\Delta = k) \;=\; \frac{\binom{m}{k} \cdot Z_n(k)}{m^n}, \quad Z_n(k) = k^n - \sum_{j=1}^{k-1} \binom{k}{j} Z_n(j) = \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} \cdot i^n, \tag{4}$$

with $Z_n(1) = 1$, $Z_n(0) = 0$. The probabilities are determined as the ratio between the number of final assignments with exactly k matched pairs and all possible initial assignments (see

[3]This is always a well-defined probability by derivation, in particular, it also holds for $m > n$.

[4]Derivations can be found in appendices A and B.

footnote 2). $Z_n(k)$ counts the number of possible matchings[5] with exactly k of n searchers matched; $\binom{m}{k}$ counts the number of possibilities to pick k of m stayers, and m^n is the total number of possible initial assignments (see footnote 2).

The uncertainty about the exact number of matchings is also reflected in the variance of the number of matchings[6]

$$Var_{n,m}(\Delta) \;=\; m\left(\frac{m-1}{m}\right)^n \cdot \left[1 - m\left(\frac{m-1}{m}\right)^n + (m-1)\left(\frac{m-2}{m-1}\right)^n\right] \qquad (6)$$

Hence, the matching friction in such a market reduces matching probabilities and the expected number of matchings, and it introduces an uncertainty (and therefore a variance) about the number of matchings. The first question is here whether this effect of coordination failure is independent of market size or not. Obviously, matching probabilities and the expected number of matchings depend on the number of individuals on both market sides, but in a highly unlinear fashion. The following theorem clarifies that the impact of the matching friction is the worse the larger a market.

Theorem 1 (Decreasing Returns to Scale of Markets) *If a market is increased by a factor $\lambda \in \mathbb{R}$, the expected number of matchings in that market increases by less than λ:*

$$E_{\lambda n,\lambda m}(\Delta) \;<\; \lambda\, E_{n,m}(\Delta), \qquad (7)$$

$n, m \in \mathbb{N}$.

Theorem 1 implies the following results for matching probabilities:

Corollary 1 (Decreasing Returns to Scale of Matching Probabilities) *If a market is increased by a factor $\lambda \in \mathbb{R}$, matching probabilities of searchers and stayers increase by less*

[5] The number of possible matchings with k of n searchers matched is determined by the k^n possibilities to distribute n searchers in k matchings minus all those combinations which result in less than k matchings (the sum term) because different searchers were assigned to the same matching. For more, see appendix A.

[6] For the derivation see appendix D.

Observe that the matching process does *not* correspond to a Binomial distribution, since there is always at least one matching. The expected number of matchings is the same in both distributions, $E(\Delta) = m \cdot \theta$ (with θ the matching probability); the variance of the distribution (4) is lower than that of a Binomial distribution. The variance of a Binomial distribution is

$$Var_B(\Delta) = m \cdot \theta \cdot (1-\theta) = m \cdot \left(\frac{m-1}{m}\right)^n \cdot \left(1 - \left(\frac{m-1}{m}\right)^n\right)$$

and therefore

$$Var_{n,m}(\Delta) = Var_B(\Delta) - m(m-1) \cdot \left(\frac{m-1}{m}\right)^n \cdot \left[\left(\frac{m-1}{m}\right)^n - \left(\frac{m-2}{m-1}\right)^n\right]. \qquad (5)$$

As equation (13) shows, for $m \to \infty$ and $n \to 0$ the variance of the matching process approaches that of a Binomial distribution.

than λ*:*

$$p_0(\lambda n, \lambda m) \quad < \quad \lambda p_0(n,m) \tag{8}$$

$$p_1(\lambda n, \lambda m) \quad < \quad \lambda p_1(n,m) \tag{9}$$

Proof: For inequality (7) it suffices to show that

$$\lambda \cdot m \cdot \left(1 - \left(\frac{m-1}{m}\right)^n\right) \quad > \quad \lambda m \cdot \left(1 - \left(\frac{\lambda m - 1}{\lambda m}\right)^{\lambda n}\right)$$

or

$$\left(\frac{\lambda m - 1}{\lambda m}\right)^{\lambda n} \quad > \quad \left(\frac{m-1}{m}\right)^n$$

for $\lambda > 1$. The derivative of the LHS w.r.t. λ is

$$\left(\frac{\lambda m - 1}{\lambda m}\right)^{\lambda n} \cdot n \cdot \left[\frac{1}{\lambda m - 1} - \ln\left(1 + \frac{1}{\lambda m - 1}\right)\right]$$

where the term in the squared brackets is always > 0 due to the properties of the logarithmic function. Hence, the LHS of the inequality increases in λ; obviously also for $\lambda > 1$.
The corollary equations are obtained by dividing inequality (7) by λ and by m or n. □

Theorem 1 implies that a decentralized market with a certain number of buyers and sellers is more efficient, if the market is split into smaller identical submarkets. In these smaller markets the ratio of searchers to stayers is the same as in the large market. Usually, such a division into smaller markets would require central organization, which - by its very definition - does not exist in decentralized markets. However, the theorem explains why there are markets where individuals divide themselves into groups along seemingly irrelevant criteria like age, first letter of the last name, etc. The criterion is indeed irrelevant, but it serves as coordination device to achieve smaller but otherwise identical markets.
These decreasing returns to scale are more pronounced for small markets and for markets where the number of searchers and stayers are not equal as the examples in table 1 indicate.

$\lambda n/\lambda m = 30/20$				$\lambda n/\lambda m = 30/30$				$\lambda n/\lambda m = 20/30$			
submarkets			$E_{\lambda n,\lambda m}(\Delta)/$	submarkets			$E_{\lambda n,\lambda m}(\Delta)/$	submarkets			$E_{\lambda n,\lambda m}(\Delta)/$
λ	m	n	$\lambda E_{n,m}(\Delta)$	λ	m	n	$\lambda E_{n,m}(\Delta)$	λ	m	n	$\lambda E_{n,m}(\Delta)$
10	2	3	0.8976	10	3	3	0.9071	10	3	2	0.8863
5	4	6	0.9554	5	6	6	0.9598	5	6	4	0.9510
2	10	15	0.9890	2	15	15	0.9901	2	15	10	0.9880
$\lambda n/\lambda m = 60/40$				$\lambda n/\lambda m = 60/60$				$\lambda n/\lambda m = 40/60$			
submarkets			$E_{\lambda n,\lambda m}(\Delta)/$	submarkets			$E_{\lambda n,\lambda m}(\Delta)/$	submarkets			$E_{\lambda n,\lambda m}(\Delta)/$
λ	m	n	$\lambda E_{n,m}(\Delta)$	λ	m	n	$\lambda E_{n,m}(\Delta)$	λ	m	n	$\lambda E_{n,m}(\Delta)$
10	4	6	0.9502	10	6	6	0.9551	10	6	4	0.9454
5	8	12	0.9781	5	12	12	0.9803	5	12	8	0.9760
2	20	30	0.9946	2	30	30	0.9951	2	30	20	0.9941

Table 1: Magnitude of Decreasing Returns to Scale

A market may also be split into non-identical submarkets, i.e. submarkets where the ratio between searchers and stayers is not identical to the large market. As long as the ratio between searchers and stayers in the smaller markets is sufficienctly similar to that of the large market, the expected sum of matches in the smaller markets is larger than the expected number of matchings in the large market. Figure 2 shows the relative expected total number of matchings for all possible partitions of a market with $m = 20$ and $n = 30$ into two smaller markets. Any partition along the main diagonal between the two contour lines labelled "+1" represents two smaller markets that are more efficient than the large market; i.e. for them

$$\frac{E_{i,j}(\Delta) + E_{n-i,m-j}(\Delta)}{E_{n,m}(\Delta)} > 1 \tag{10}$$

holds with $i \in [1, n-1]$ and $j \in [1, m-1]$. For instance, splitting the large market into two smaller markets with 25 (5) searchers and 17 (3) stayers respectively is more efficient than a market with 30 searchers and 20 stayers. The expected number of matches is 1% higher. Also for non-identical submarkets it holds, that the relative inefficiency of larger markets is more pronounced if overall market size is small and if the short side searches. The expected number of matches increases the more smaller markets there are. It is more efficient to divide a large market into many small markets than to split it only a few times. Obviously, the expected number of matchings is maximal if matching frictions are reduced as much as possible. This is the case if there are exactly as many markets as there are players on the short market side and if there is at least one long side individual in each of these markets. In each small market there will be exactly one match and therefore $E_{n,m}(\Delta) = s$.

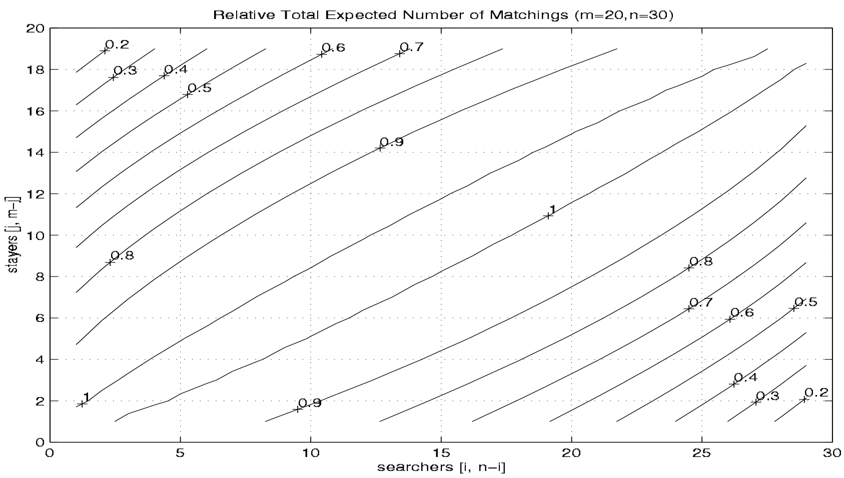

Figure 2: Partition of Large Market into Two Smaller Markets
(see equation (10))

If a large market is split into submarkets in order to increase efficiency, it also affects the variance and therefore risk. Numerical simulations show that there is a critical value for the relative size of the two market sides of approximately 3/2. If both sides are sufficiently similar in size, then there is a relatively lower variance in a larger market than there is in a collection of smaller markets. If there are many more searchers than stayers, or the other way round, then it is best in terms of variance to split a large market into several smaller markets. Stated formally:

If a market is increased by a factor $\lambda \in \mathbb{R}$, the variance of the number of matchings in the market may increase by more or less than λ depending on the relative number of searchers and stayers:

$$\begin{aligned} Var_{\lambda n,\lambda m}(\Delta) &> \lambda\, Var_{n,m}(\Delta) \quad \text{if } \tfrac{n}{m} > 1.5 \text{ and } \tfrac{m}{n} > 1.6 \\ Var_{\lambda n,\lambda m}(\Delta) &< \lambda\, Var_{n,m}(\Delta) \quad \text{if } \tfrac{n}{m} < 1.5 \text{ and } \tfrac{m}{n} < 1.6 \end{aligned} \tag{11}$$

for all $n, m \in \mathbb{N}$. Although the model abstracts completely from risk attitudes, this result permits to draw the following conclusions.

Corollary 2 (Risk Attitudes and Market Size) *If the two market sides of searchers and stayers are of similar size, then there is a trade-off between efficiency and risk. Risk loving and risk neutral individuals prefer smaller markets to larger markets. Also risk averse individuals prefer smaller markets, if the expected return is valued sufficiently relative to the variance. Otherwise, they prefer larger markets.*
If there are sufficiently more searchers than stayers (or the opposite), then efficiency and risk effects are aligned. Risk averse and risk neutral individuals prefer smaller markets to larger markets. Also risk loving individuals prefer smaller markets if they value expected return enough relative to risk. Otherwise they prefer larger markets.

If markets become very large but the relative size of the two market sides stays constant, then risk considerations become irrelevant as the following results implies.

Theorem 2 (Large Market Approximations) *The scale-independent expected number of matchings in large markets is*

$$\lim_{\lambda\to\infty} E_{\lambda n,\lambda m}(\Delta)/\lambda \;=\; m\cdot\left(1-e^{-n/m}\right) \tag{12}$$

The scale-independent variance of the number of matchings in large markets is

$$\lim_{\lambda\to\infty} Var_{\lambda n,\lambda m}(\Delta)/\lambda \;=\; m\cdot e^{-n/m}\cdot\left(1-\frac{m+n}{m}\cdot e^{-n/m}\right) \tag{13}$$

$n, m \in \mathbb{N}$.

Theorem 2 implies the following:

Corollary 3 *In large markets a stayer's matching probability is*

$$\lim_{\lambda\to\infty} p_0(\lambda n,\lambda m) = 1-e^{-n/m} \tag{14}$$

and a searcher's matching probability is

$$\lim_{\lambda\to\infty} p_1(\lambda n,\lambda m) = \frac{1-e^{-n/m}}{n/m}. \tag{15}$$

Proof: The derivation of equation (12), and therefore also of (14) and (15), is straightforward

$$\begin{aligned}\lim_{\lambda\to\infty} m\cdot\left(1-\left(\frac{\lambda m-1}{\lambda m}\right)^{\lambda n}\right) &= m\cdot\left(1-\exp\left\{\lim_{\lambda\to\infty}\left[\lambda n\cdot\ln\left(\frac{\lambda m-1}{\lambda m}\right)\right]\right\}\right)\\ &= m\cdot(1-\exp(-n/m)).\end{aligned}$$

For equation (13), note that equation (6) can be written as

$$Var_{n,m}(\Delta) = m\left(\frac{m-1}{m}\right)^n \cdot \left[1 - m\left(\frac{m-2}{m-1}\right)^n - m\left\{\left(\frac{m-1}{m}\right)^n - \left(\frac{m-2}{m-1}\right)^n\right\}\right]$$

Replacing m by λm and n by λn and taking the limit w.r.t. $\lambda \to \infty$ yields equation (13). It relies on equation (12) and the following approximation for the $m\{\}$ term on the right

$$\lim_{\lambda\to\infty}\left(\frac{\lambda m-2}{\lambda m-1}\right)^{\lambda n} \cdot \lambda m \cdot \underbrace{\left\{1-\left(1-\frac{1}{\lambda m-1}\right)^{\lambda n}\right\}}_{\lim_{\lambda\to\infty}\bullet = n/m^2} = e^{-n/m}\cdot\frac{n}{m} \qquad \square.$$

Hence, also in very large markets matching probabilities are bounded away from one for both types of individuals due to the matching friction. Only if one market side becomes infinitely larger than the other, then all agents on the short market side are matched with probability one and the matching probability on the long market side converges to zero. Only in this case matching probabilities are determined by the size of the market side an individual belongs to and not by her actions (search or stay). In a balanced market ($b = s$) a central matching institution would usually succeed in matching everybody. In a large decentralized and balanced market with uncoordinated search as analyzed here, individual matching probabilities are only $1-1/e = 0.63$. The matching friction accounts for a reduction in efficiency of more than a third.
Equation (12) reveals that large markets exhibit constant returns to scale[7]: $E_{\lambda n,\lambda m}(\Delta) = \lambda E_{n,m}(\Delta)$. Hence, the effect of uncoordinated search by all individuals from one market side exists also in infinitely large markets and the relative importance of that friction increases the larger the market. But the relative importance of the friction increases at a decreasing rate.

As already mentioned, table 1 shows that the returns of scale are decreasing more if the short market side searches (RHS of table), i.e. if $n = s$ and $m = b$. This seems to suggest, that the effect of uncoordinated search by one market side is worse if it is the short market side that does the searching (and the long market side stays put). This impression is confirmed by figure 3, which shows the expected number of matches and the variance in a market with total size of 60 individuals. The vertical line indicates an equal split of the market into searchers and stayers. The other straight lines connect expected values or variance with the same ratio of long to short side but with opposite roles. For expected values these lines slope downward, i.e. there are on average less matches if the short side searches. For the variance these lines slope upward, i.e. there is less variance if the long side searches. Hence, the expected number

[7]This was already noticed by Kultti (1998) in his claim 1.

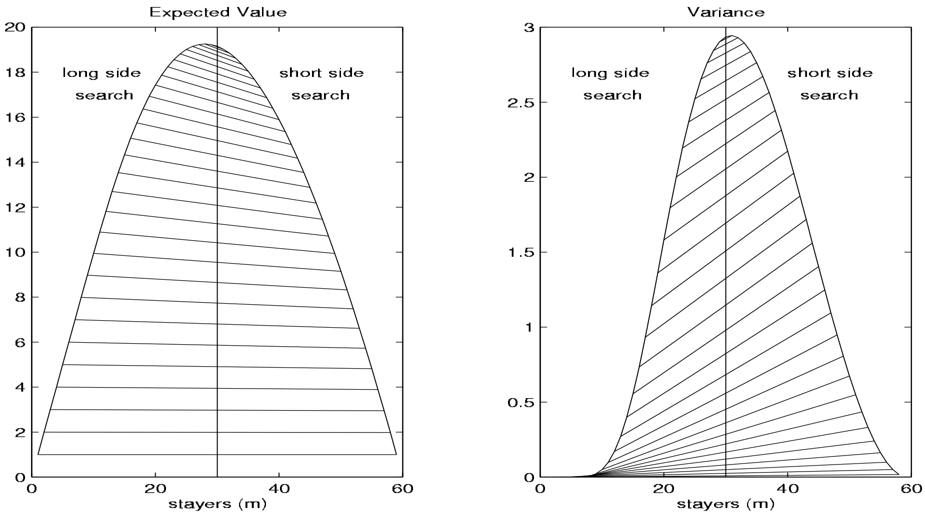

Figure 3: Expected Number of Matches and Variance
(Total Market Size: 60)

of matchings is higher and the variance of the number of matchings is lower if the long side searches[8]. This holds in general for all $b \geq s$:

Theorem 3 (Long Side Search More Efficient) *The expected number of matchings $E(\Delta)$ is higher*

$$E_{b,s}(\Delta) \quad > \quad E_{s,b}(\Delta) \tag{16}$$

and the variance of the number of matchings $Var(\Delta)$ is lower

$$Var_{b,s}(\Delta) \quad < \quad Var_{s,b}(\Delta) \tag{17}$$

if the long market side instead of the short market side searches.

The same result holds at the individual level:

Corollary 4 *Matching probabilities are higher with long side search than they are with short side search:*

$$\underset{\textit{(short side)}}{p_0(b,s) \; > \; p_1(s,b)} \qquad \textit{and} \qquad \underset{\textit{(long side)}}{p_1(b,s) \; > \; p_0(s,b)} \tag{18}$$

[8]To simplify terminology, long side search will stand for a market where the long side searches and the short side stays put; and vice versa for short side search.

Therefore, efficiency of a market with decentralized and uncoordinated search does not only depend on the size of the market, but also on who searches in the market. The larger a market the less efficient it is; and if the shorter market side searches the market is less efficient, too.

Both market sides are equally affected by the efficiency difference (in terms of expected matches) depending on which market side searches. By definition there are fewer individuals on the short market side. Consequently, for each of them the difference in matching probability is larger than for individuals on the long market side if the two sides swap roles (of searchers and stayers). This yields the following corollary.

Corollary 5 *Matching probabilities on the short side differ more than those on the long side between markets with long side search and short side search:*

$$p_0(b,s) - p_1(s,b) \quad > \quad p_1(b,s) - p_0(s,b) \tag{19}$$

Proof: Observe that inequalities (18) and (19) are the same as inequality (16). Hence it is sufficient to show that $E_{b,s}(\Delta) > E_{s,b}(\Delta)$ and that $Var_{b,s}(\Delta) < Var_{s,b}(\Delta)$. The proof relies on the appropriate expressions for large markets[9], equations (12) and (13). With $x = b/s \geq 1$, observe that inequality (16) is equivalent to

$$\frac{1-e^{-x}}{1-e^{-1/x}} > x \qquad \text{or} \qquad \underbrace{\frac{1}{x} \cdot \frac{e^{1/x}}{e^{1/x}-1}}_{\lim_{x\to\infty} \bullet = 1} > \underbrace{\frac{e^x}{e^x - 1}}_{\lim_{x\to\infty} \bullet = 1} \tag{20}$$

This is the same as

$$\frac{1}{x}\coth\left(\frac{1}{2x}\right) - \coth\left(\frac{x}{2}\right) > \frac{x-1}{x}$$

At $x = 1$ ($b = s$) the two sides are equal, and the slope[10] of the LHS is 1.5187 and the slope of the RHS is 1. At $x = 1$ the LHS increases faster than the RHS and for $x \to \infty$ the LHS and the RHS converge both to one (from below). The properties of the hyperbolic cotanges

[9]It can also be done with equations (3) and (6) directly, however it turns out very cumbersome and is therefore omitted. See appendix E for more detail.

[10]For $\lambda \in \mathbb{R}$ the derivatives are

$$\frac{d}{dx}\,\text{LHS} = \frac{1}{x^2}\coth\left(\frac{\lambda}{x}\right)\cdot\left[\frac{\lambda}{x}\coth\left(\frac{\lambda}{x}\right) - 1\right] + \lambda\left[\coth^2(\lambda x) - 1 - \frac{1}{x^3}\right] \quad \text{and} \quad \frac{d}{dx}\,\text{RHS} = \frac{1}{x^2}$$

function guarantee that there is no $x \in (1, \infty)$ for which the two sides are equal. Hence the inequality holds for all $x \in (1, \infty)$.
Inequality (17) is equivalent to

$$\frac{e^{-x} \cdot [1 - (1+x)e^{-x}]}{e^{-1/x} \cdot [1 - (1+1/x)e^{-1/x}]} < x. \tag{21}$$

For $x = 1$ the two sides of the inequality are equal. Dividing inequality (21) by x, it is straightforward to show that the LHS is strictly decreasing in x. Thus, inequality (21) holds for all $x \in (1, \infty)$. $\square$

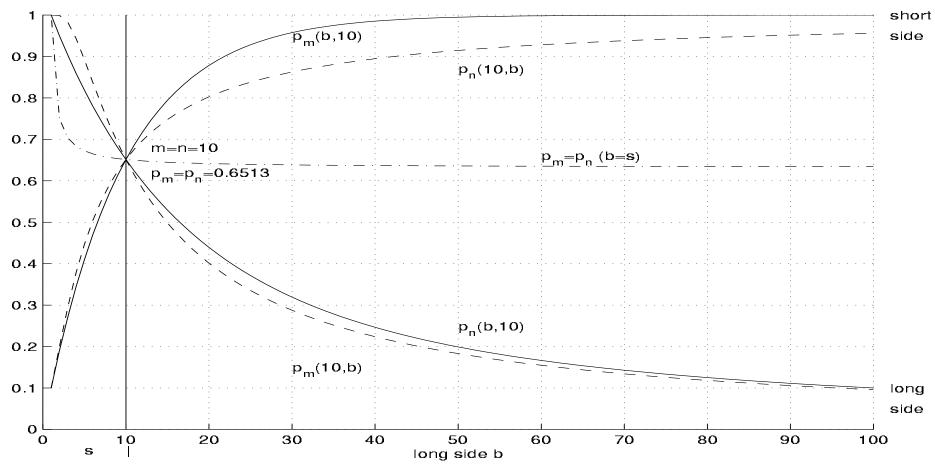

Figure 4: Matching Probabilities

Figure 4 illustrates the difference in individual matching probabilities depending on which side searches[11]. To the left of the vertical line at 10, $s < b = 10$ and to the right $b > s = 10$. The solid lines correspond to matching probabilities where the number of searchers (n) varies along the horizontal axis and the number of stayers is constant at $m = 10$. The dashed lines correspond to matching probabilities where the number of searchers stays constant at $n = 10$ and the number of stayers (m) varies along the horizontal axis. Therefore, matching

[11]To the left of the vertical line at 10 the functions are from top to bottom: $p_0(s, 10)$, $p_1(10, s)$, $p_0 = p_1$, $p_1(s, 10)$, $p_0(10, s)$.

probabilities in a market where the long side searches correspond to solid lines to the right of the vertical line at 10 and they correspond to dashed lines to the left of the vertical line at 10. Corollaries 4 and 5 can be observed directly in figure 4.

To illustrate this result further, consider as an example the smallest market that is of interest in this context, the market with $s = 2$ and $b = 3$:

	long side searches $n = b$ and $m = s$			short side searches $n = s$ and $m = b$	
s	$p_0(b, s) =$	21/24	$>$	20/24	$= p_1(s, b)$
b	$p_1(b, s) =$	21/36	$>$	20/36	$= p_0(s, b)$
	$P_{b,s}(\Delta = 1) =$	0.25	$<$	0.33	$= P_{s,b}(\Delta = 1)$
	$P_{b,s}(\Delta = 2) =$	0.75	$>$	0.66	$= P_{s,b}(\Delta = 2)$
	$E_{b,s}(\Delta) =$	21/12	$>$	20/12	$= E_{s,b}(\Delta)$
	$Var_{b,s}(\Delta) =$	27/144	$<$	32/144	$= Var_{s,b}(\Delta)$

Table 2: Example: Smallest Relevant Market, $s = 2$ and $b = 3$

Why is it efficient if the long side searches in this example? If the short side searches, then the two searchers approach the same stayer in 3 of 9 possible initial assignments. If the long side searches, then in all possible initial assignments, there are at least two searchers who approached the same stayer. More than one searcher approaching the same stayer is a coordination failure. This coordination failure always happens if the long side searches, whereas it happens only in some cases if the short side searches. Once a coordination failure has happened, the effect is different again. If the short side searches it is more likely that a large share of the searchers approaches the same (identical) stayer, whilst if the long side searches it is less likely that a large share of searchers approach the same identical stayer. This effect is extreme in this example, because if coordination failure happens when the short side searches, then necessarily both agents have approached the same stayer. If the three agents from the long side search, coordination failure does not always imply that they all approach the same stayer (this happens only in 2 of 8 cases). Generally, if the long side searches coordination failure is more likely to happen, but it has less severe consequences. Theorem 3 states that the latter effect is more important than the former.

As was shown before, the effect of the matching friction does not disappear in large markets unless one market side is infinitely larger than the other market side. Moreover, also in large markets it is more efficient if the long market side searches. To what extent the efficiency difference between a large market with long side search and a large market with short side search depends on the relative size of the two market sides is the point of the following corollary.

Corollary 6 (Relative Efficiency in Large Markets) *Define x by $b = x \cdot s$ and $X(sx,s) = E_{xs,s}(\Delta) - E_{s,xs}(\Delta)$ and therefore for s sufficiently large*

$$X(sx,s) \;=\; s \cdot \underbrace{\left(x \cdot e^{-1/x} - x + 1 - e^{-x}\right)}_{>0}. \tag{22}$$

The efficiency difference between long side search and short side search in terms of the expected number of matches is maximal if

$$\bar{x} \;=\; \frac{e^{-1/x}}{1 - e^{-x} - e^{-1/x}} \;=\; 3.2565 \qquad \textit{yielding} \qquad X(s\bar{x},s) = s \cdot 0.1004. \tag{23}$$

For individual matching probabilities the efficiency difference is maximal for the short side at

$$x \;=\; 3.2565 \qquad \textit{with} \qquad p_0(b,s) - p_1(s,b) \;=\; 0.1004 \tag{24}$$

and for the long side at

$$x \;=\; 2.1065 \qquad \textit{with} \qquad p_1(b,s) - p_0(s,b) \;=\; 0.0390 \tag{25}$$

Proof: See appendix F.

The difference between long and short market side search is zero[12] if both sides are of equal size ($x = 1$) or if one side is infinitely larger than the other ($x = \infty$). For all other $x \in [1, \infty)$ values, the difference is positive. It turns out to be maximal if the long side is 3.26 times as large as the short side. Corollary 6 states that - comparing long side with short side search - there is one additional matching for every 10 short side players (and every 33 long side players).

The corollary is stated only in terms of large markets. That the approximation is valid also for relatively small markets can be seen in the following table:

s	$\lim_s X(s\bar{x},s)$	$X(s\bar{x},s)$	$\max_x X(sx,s)$
1	0.1004	0.0000	0.0000
2	0.2009	0.1316	0.1379
10	1.0044	0.9510	0.9522
100	10.0436	9.9931	9.9932

[12] This is basically the proof of theorem 3.

The second column shows values for the large market approximation, i.e. for the RHS of equation (22). The third and fourth column show values for the exact small market equation (based on equation (3)). For markets with more than 10 individuals on the short side, the error of the approximation drops quickly to below 1%.

Theorem 3 has implications also for a market where individuals may choose which side they belong to, i.e. where individuals determine by their choice of strategy which market side they are part of (cf. figure 3).

Corollary 7 (Choice of Market Side) *With t individuals in a market, the expected number of matchings is maximal if*[13]

$$\left(\frac{m}{m-1}\right)^{t-m} = \frac{t-1}{m-1} + m \cdot \ln\left(\frac{m}{m-1}\right)$$

or equivalently in large markets ($m \to \infty$, $t \to \infty$ and m/t constant) if

$$\exp\left(\frac{1-m/t}{m/t}\right) = \frac{1+m/t}{m/t},$$

that is if $m/t = 0.4660$.

Optimally, searchers would constitute the long market side ($b = n$) with 53.3% and stayers the short market side ($s = m$) with 46.6% of all individuals. In large markets this implies a matching probability of $p_0(0.533, 0.466) = 0.6814$ for short side individuals and of $p_1(0.533, 0.466) = 0.5957$ for long side individuals; the maximal expected number of matchings is therefore 0.6814 times the number of stayers in the market.

The matching probabilities of searchers and stayers (see equations (2) and (1)) are both decreasing in the number of individuals with their own matching behaviour and increasing in the number of individuals with the other matching behaviour. Searchers do therefore prefer to become stayers if they are in the majority. Long side individuals have an incentive[14] to change matching behaviour as long as they are the long side agents, i.e. as long as $b = n > m = s$. Although it is better in welfare terms to have more searchers than stayers, from an individual

[13] It is straightforward to show that the implicitely defined m is smaller than $t/2$, i.e. that the LHS is smaller than the RHS at $t/2$.

[14] If matched pairs of individuals had to split a surplus, the expected value of entering the market would be the same for both types if searchers got 53.3% of the surplus:

$$(1-\delta)p_0(n,m) = \delta p_1(n,m) \quad \Longleftrightarrow \quad \delta = \frac{p_0}{p_0+p_1} = \frac{n}{n+m} = \frac{n}{t}$$

Observe that the expected value for both types is higher under this unequal spit than under an equal split with the same number of searchers and stayers: $0.3175 > 0.3160$. Any expected number of matchings between one and the maximal number can be achieved as equilibrium by setting the splitting rule appropriately and exogenously.

(strategic) point of view it does not pay to be a searcher if there are at least as many searchers as stayers. Searchers would need to be compensated for their lower matching probabilities to induce enough individuals to search[15]. Note that this compensation would be independent of any "real" search costs, like transportation costs.

This section has evaluated the effect of uncoordinated search by one market side. It leads to inefficiency in the matching process which is relatively worse in large markets and with short side search. Efficiency considerations, however, imply nothing for strategic behaviour. As corollary 7 showed, there is an incentive for individuals to behave differently. The next section evaluates individual incentives in a framework where individuals belong to one market side but can choose their matching behaviour (searcher or stayer) independently. Hence market sides are determined exogenously, but matching behaviour is determined endogenously. As in corollary 7 competition between individuals pursuing the same strategy will determine what strategy (not what market side) they choose.

3 Searchers and Stayers on Both Market Sides

Assume now that it is possible that on both market sides some individuals search and others wait. If there are searching agents on both market sides then they may encounter each other during that search process. Such a market can be imagined as a field[16], where at one end there are the players of one type and at the opposite end there are the players of the other type. There may be searchers and stayers among the players at both ends of the field. Stayers wait at their end of the field until a searcher approaches them - the so-called *stayers' market*. Searchers cross the field to the opposite side and approach a stayer over there. On their way to the opposite end of the field a searcher may meet another searcher - in the so-called *searchers' market*. With a probability relative to the size of the two populations of searchers, that person is a searcher from the opposite side of the field, i.e. of the opposite type. Assume that encounter probabilities in the searchers' market are indedependent and that the probability that two players from the same market side simultaneously meet the same player from the other market side is zero. Assume further that all searchers arrive at the opposite end of the field at the same time and that they are unaware of the decisions of their fellow searchers. In such a market matchings happen in two subsequent steps, first in the searchers' market and then in the stayers' markets. The matching procedure in the stayers' markets is exactly the same as in the preceding section. Each stayers' market (at both ends of the field) suffers from a matching friction due to coordination failure, as discussed earlier.

[15] Shapley/Shubik (1969) suggest a cooperative solution along those lines.
[16] The dancing floor of the example in the introduction.

In the searchers' market there is no coordination failure, still, there is a matching friction; players may meet the wrong person (same type), if they meet anyone at all.
Denote by s_0 the number of stayers and by s_1 the number of searchers on the short side of the market, and by b_0 and b_1 the number of stayers and searchers on the long market side. Assume that each searcher meets another person in the searchers' market with[17] probability $c \in [0,1]$. The variable c is a market characteristic, which, in terms of the illustration used before, can be imagined as the layout of the field that the searchers cross. Observe that $c = 0$ implies that there is no searchers' market. Yet, this case is of interest, since it is not obvious how many players on the short and on the long market side should choose to search if $c = 0$.
Define

$$q_{s1} = c \cdot \frac{b_1}{s_1 + b_1 - 1} \qquad \text{and} \qquad q_{b1} = c \cdot \frac{s_1}{s_1 + b_1 - 1} \tag{26}$$

as the probabilities that players from the short (index $s1$) and from the long market (index $b1$) side meet individuals of the other type when they cross the field. If $s_1 = 0$ then the probability q_{s1} is not defined, and similarly for q_{b1} if $b_1 = 0$. Players who meet a trading partner on the field form a match and leave the market; these searchers never reach the opposite end of the field. Individual matching probabilities do therefore depend on the number of searchers and stayers on both market sides. The number of searchers determines the expected number of matches in the searchers' market and therefore also the expected number of players in the two stayers' markets.
With

$$\theta(n,m) \;=\; 1 - \left(\frac{m-1}{m}\right)^n \qquad \text{and} \qquad Bin_k(n,q) = \binom{n}{k} q^k (1-q)^{n-k} \tag{27}$$

matching probabilities $p_. \;=\; p_.(s_0, s_1, b_1, b_0)$ are

$$\begin{aligned} p_{s0} &= \sum_{k=0}^{\eta} \left[Bin_k(\eta, q_\eta) \cdot \theta(b_1 - k, s - s_1)\right] \\ &= \frac{1}{s - s_1} \cdot \sum_{k=0}^{\eta} \left[Bin_k(\eta, q_\eta) \cdot E_{b_1 - k, s - s_1}(\Delta)\right] \end{aligned} \tag{28}$$

[17] The variable c could also be interpreted as the average number of people a searcher meets when she crosses the field. In the context of this model, any market where searchers are *certain* to meet a trading partner are irrelevant; there would be no stayers' market. As long as

$$c < \min\left\{\frac{s_1 + b_1 - 1}{s_1}, \frac{s_1 + b_1 - 1}{b_1}\right\}$$

where the RHS is minimal at 1 if $b_1 = 1$ or $s_1 = 1$, a searcher cannot count on meeting a trading partner in the searchers' market. Since all possible states of the market will be considered, including those where b_1 and s_1 may be 1, only values of $c < 1$ are relevant.

$$
\begin{aligned}
p_{s1} &= \sum_{k=0}^{\eta}\left[Bin_k(\eta,q_\eta)\cdot\left\{\frac{k}{s_1}\cdot 1+\frac{s_1-k}{s_1}\cdot\frac{b-b_1}{s_1-k}\cdot\theta(s_1-k,b-b_1)\right\}\right] \\
&= q_{s1}+\frac{1}{s_1}\cdot\sum_{k=0}^{\eta}\left[Bin_k(\eta,q_\eta)\cdot E_{s_1-k,b-b_1}(\Delta)\right] \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
p_{b1} &= \sum_{k=0}^{\eta}\left[Bin_k(\eta,q_\eta)\cdot\left\{\frac{k}{b_1}\cdot 1+\frac{b_1-k}{b_1}\cdot\frac{s-s_1}{b_1-k}\cdot\theta(b_1-k,s-s_1)\right\}\right] \\
&= q_{b1}+\frac{1}{b_1}\cdot\sum_{k=0}^{\eta}\left[Bin_k(\eta,q_\eta)\cdot E_{b_1-k,s-s_1}(\Delta)\right] \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
p_{b0} &= \sum_{k=0}^{\eta}\left[Bin_k(\eta,q_\eta)\cdot\theta(s_1-k,b-b_1)\right] \\
&= \frac{1}{b-b_1}\cdot\sum_{k=0}^{\eta}\left[Bin_k(\eta,q_\eta)\cdot E_{s_1-k,b-b_1}(\Delta)\right] \qquad (31)
\end{aligned}
$$

with $\eta = \min\{s_1, b_1\}$ and for given s, b, and therefore s_0 and b_0. The indices $s0$, $s1$, $b1$ and $b0$ denote stayers and searchers on the short and on the long market side in the obvious way. These matching probabilities are influenced by both steps, the searchers' and the stayers' market. A searcher's matching probability is determined by the probability to meet someone in the searchers' market and, if this does not happen, by the probability to find a matching partner in the appropriate stayers' market. A stayer's matching probability is basically the same as p_0 in the preceeding section adjusted by what happened in the searchers' market. Therefore, for both types, searchers and stayers, the matching probabilities have to be adjusted by the number of matchings in the searchers' market. The number k counts the number of searchers who met a trading partner in the searchers' market. Consequently, the binomial coefficient determines the probability that there are exactly k matchings in the searchers' market. If a searcher happens to be one of those k matched players (likelihood of k/s_1 and k/b_1), his matching probability is one, otherwise (likelihood of $(s_1-k)/s_1$ and $(b_1-k)/b_1$), his matching probability is determined by p_1 (of the preceding section) in the stayers' market at the opposite end of the field.
The expected number of matchings is

$$
\begin{aligned}
E(\Delta) &= s_0\cdot p_{s0} = s_1\cdot p_{s1} = b_1\cdot p_{b1} = b_0\cdot p_{b0} \\
&= b_1 q_{b1}+\sum_{k=0}^{\eta}\left[Bin_k(\eta,q_\eta)\cdot\{E_{s_1-k,b-b_1}(\Delta)+E_{b_1-k,s-s_1}(\Delta)\}\right], \qquad (32)
\end{aligned}
$$

which is the expected number of matchings in the searchers' market ($b_1 q_{b1} = s_1 q_{s1}$) plus the expected number of matchings in the two stayers' market weighed by the appropriate

probabilities.
Define as a *state* any strategy profile, i.e. any vector where the elements correspond to a strategy per player. Hence, any market situation where a specific number of players on the long and on the short market side search (and stay) constitutes a state of the market. A *Nash equilibrium* in such a market is a state where no player wants to change her matching strategy (searcher or stayer) given the matching strategies of the other players. Non-equilibrium states, i.e. states where at least one player would be better off in terms of matching probability by changing strategy are said to be strictly dominated (or equivalently, the strategy of that player is strictly dominated in that state). All possible states constitute a grid, like the one in figure 5, where an edge corresponds to one individual changing her strategy. The dominance relation determines in which direction the vectors along the edges point that connect the vortices (states) on the grid. A vector always points from a dominated to a dominating state. A Nash equilibrium is a state which dominates all four neighbouring states, i.e. it is a vertex to which the four corresponding vectors point. As a convention, the long market side determines horizontal and the short market side determines vertical comparison of states.

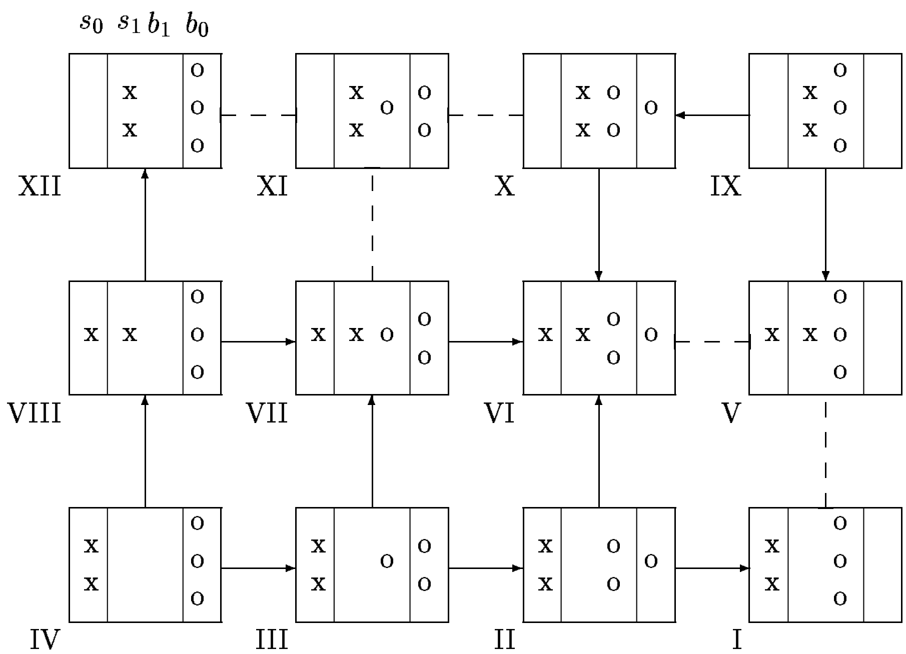

Figure 5: Searchers' and Stayers' Markets

The fact that in a dominated state there is at least one player who can improve her situation by changing strategy, does not imply that it is trivial to identify a player unless there is exactly one player of that type on that market side. If by switching from stayer to searcher a player's matching probability increases, the matching probability of the other stayers may increase too and the player would have been better off, had another stayer of his type decided to switch strategy. Since all players on one market side are assumed to be identical, there is no way to distinguish between those players. It is therefore assumed that one randomly drawn player switches strategy[18].

Figure 5 illustrates the smallest relevant market with $s = 2$ and $b = 3$ as an example. Each box corresponds to one state of the market. Circles are agents on the long market side, the x stand for players on the short market side. The two outer rectangles of each box are the stayers' markets. On the right the number of the stayers on the long market side and on the left the number of stayers of the short market side are shown. The central rectangle of each box is the searchers' market, where the number of searchers on both market sides is shown. State I is the market where all agents on the long side search and all agents on the short side are stayers. State XII describes the opposite. In state IV nobody searches and in state IX everybody searches. In this specific market matching probabilities and the expected number of matchings are as follows[19]:

state	s_0,s_1,b_1,b_0	p_{s0}	p_{s1}	p_{b1}	p_{b0}	$E(\Delta)$
I	$2,0,3,0$	$7/8$	-	$7/12$	-	$7/4$
II	$2,0,2,1$	$3/4$	-	$3/4$	0	$3/2$
III	$2,0,1,2$	$1/2$	-	1	0	1
IV	$2,0,0,3$	0	-	-	0	0
V	$1,1,3,0$	1	c	$(1+c)/3$	-	$1+c$
VI	$1,1,2,1$	1	1	$(1+c)/2$	$1-c$	2
VII	$1,1,1,2$	$1-c$	1	1	$(1-c)/2$	$2-c$
VIII	$1,1,0,3$	0	1	-	$1/3$	1
IX	$0,2,3,0$	-	$3c/4$	$c/2$	-	$3c/2$
X	$0,2,2,1$	-	$(9+12c-4c^2)/18$	$2c/3$	$(9-4c^2)/9$	$(9+12c-4c^2)/9$
XI	$0,2,1,2$	-	$(3+c)/4$	c	$(3-c)/4$	$(3+c)/2$
XII	$0,2,0,3$	-	$5/6$	-	$5/9$	$5/3$

Table 3: Matching Probabilities, $s = 2$ and $b = 3$

For instance, comparing state III and state IV, the one player on the long market side who searches in state III ($p_{b1} = 1$) is obviously strictly better off as if he were a stayer ($p_{b0} = 0$) in state IV. So, the arrow in figure 5 points to the right. Between states X and XI, this type

[18]See Shapley/Shubik (1969) for a possible cooperative solution to this problem
[19]For calculations see appendix G.

of probability comparison depends on c: $p_{b1} = 2c/3$ versus $p_{b0} = (3-c)/4$. If dominance depends on $c \in [0,1)$, edges are dashed lines.

Comparing matching probabilities in this manner, all Nash equilibria can be found. If $c < 7/8$ then exclusive long side search is a Nash equilibrium (V $\longrightarrow$ I). If $c < 5/9$ then also a market where only the short side searches is a Nash equilibrium (XII $\longleftarrow$ XI). Both inequalities imply that state XI dominates state X (the opposite is the case if $c > 9/11$). However, they do not determine the relationship between V and VI, and VII and XI. If $c < 1/2$ then VI dominates V and there are three Nash equilibria: I, VI, and XII. Otherwise, the only two equilibria are the states where only one market side searches. To complete the characterization, if $c > 1/5$, then XI dominates VII. This is irrelevant for the number of Nash equilibria. It is relevant, however, for the basins of attraction of the different equilibria, i.e. to determine the set of states from which a specific equilibrium is reached.

The analysis of this example is concluded with the following general observation. If the market is in a dominated state where there are too many searchers for a given c value, then players on both sides may leave the market to alleviate "congestion" in the searchers' market. For instance in state XI in figure 5, if c is sufficiently low ($c < 1/5$), matching probabilities improve if either a player on the long side (state X) or a player on the short side (state VII) leaves the market. The same holds for states with too few searchers, like state II, which is dominated by states I and VI, both states with one more searcher in the market.

In general, horizontal arrows are determined by comparing

$$p_{b1}(s_0, s_1, b_1+1, b_0-1) \qquad \text{with} \qquad p_{b0}(s_0, s_1, b_1, b_0). \tag{33}$$

If the LHS is larger than the RHS, then searching is a dominant strategy for the one agent on the long side who considers switching from being a stayer to becoming a searcher. If being a searcher is a dominant strategy, then the arrow points to the right.

Equivalently, vertical arrows are determined by comparing

$$p_{s1}(s_0-1, s_1+1, b_1, b_0) \qquad \text{with} \qquad p_{s0}(s_0, s_1, b_1, b_0). \tag{34}$$

If the LHS is larger than the RHS, then searching is a dominant strategy for an agent on the short market side and the arrow points upward.

Observe that all arrows in the bottom row always point to the right and in the left column always point upwards. If one market side consists of stayers only, then the other market side is better off by searching (otherwise they remain unmatched with certainty). Whether the opposite is true too, i.e. whether it is better to stay if all individuals on the other market side search and all individuals on the own market side stay, is determined by the following result:

Theorem 4 (Nash Equilibria) *The two states with exclusive search by all players on one market side are Nash equilibria if*

$$c \leq 1 - \left(\frac{s-1}{s}\right)^b \qquad \textit{and} \qquad c \leq 1 - \left(\frac{b-1}{b}\right)^s. \tag{35}$$

If only the first condition holds, then exclusive short side search is no Nash equilibrium. It is never a Nash equilibrium that all players in a market are searchers.

Theorem 4 states that the probability of meeting someone in the searchers' market determines how many players search and also on which market side. If c is high, then it is worth for a stayer to enter the searchers' market since the likelihood to be matched out there is sufficiently high. If c is too low then this incentive vanishes and it is better to wait for searchers from the other market side who did not find a matching partner in the searchers' market. However, even if c is very high, i.e. even if one is fairly certain to meet someone in the searchers' market, it never pays to be a searcher if everybody else is also one. There is too much "congestion" in the searchers' market.

Proof: In order to show that exclusive search by one market side is a Nash equilibrium, it suffices to show that

$$p_{b1}(s,0,b,0) \ > \ p_{b0}(s,0,b-1,1) \qquad \text{and} \qquad p_{s0}(s,0,b,0) \ > \ p_{s1}(s-1,1,b,0)$$

if the long side searches, and that

$$p_{s1}(0,s,0,b) \ > \ p_{s0}(1,s-1,0.b) \qquad \text{and} \qquad p_{b0}(0,s,0,b) \ > \ p_{b1}(0,s,1,b-1)$$

if the short side searches.
The LHS conditions do always hold since it is always better to start searching if the other side consists of stayers only ($p_{b0}(0,\cdot) = p_{s0}(\cdot,0) = 0$). The RHS conditions are those stated in the theorem with

$$p_{s1}(s-1,1,b,0) = q_{s1} = c \cdot \frac{b}{1+b-1} = c \qquad \text{and} \qquad p_{b1}(0,s,1,b-1) = q_{b1} = c \cdot \frac{s}{1+s-1} = c$$

$$p_{s0}(s,0,b,0) = \theta(b,s) = 1 - \left(\frac{s-1}{s}\right)^b \qquad \text{and} \qquad p_{b0}(0,s,0,b) = \theta(s,b) = 1 - \left(\frac{b-1}{b}\right)^s.$$

All players searching cannot be a Nash equilibrium if either of the two conditions holds

$$p_{s0}(s-1,b) = 1 \ > \ c \cdot \frac{b}{s+b-1} = p_{s1}(s,b) \tag{36}$$

$$p_{b0}(s,b-1) = 1 - \left(c \cdot \frac{b-1}{s+b-2}\right)^s \ > \ c \cdot \frac{s}{s+b-1} = p_{b1}(s,b) \tag{37}$$

The first one is always true: by assumption $c < 1$. The second one holds also[20]. □

Observe that inequalities (36) and (37) in the proof imply that the states with one stayer on one of the two market sides dominate the state in which everybody searches (the upper right most point in figure 5). The theorem states that - provided conditions (35) hold - it is better to stay if all individuals on the other market side search and all individuals on the own market side stay. If only some of one's peers stay and all agents on the other market side search (the upper most row and the right most column in figure 5), also then it is better to stay than to search if conditions (35) hold. This can be derived in a straightforward way from the matching probabilities (with $k = 0$). The intuition is that, if competition between individuals on one market side is not too high if *all* of them stay, i.e. if none of them wants to become a searcher (conditions (35)), then competition between a lower number of stayers is even lower and therefore there is no incentive to become a searcher.

There may be no searchers' market because there is exclusive search by one market side ($k = 0$). It may also be the case that $c = 0$ and therefore that there are no encounters in the searchers' market. Even if individuals on both market sides search, they meet nobody in the searchers' market. The following theorem deals with the latter case:

Theorem 5 (Nash Equilibria without a Searchers' Market) *If $c = 0$, then all Nash equilibria are along the diagonal*

$$s_1 = \frac{s}{b}(b - b_1)$$

In all Nash equilibria with $s_1 \geq 1$, the expected number of matchings is decreasing in s_1. It is maximal at $s_1 = 1$ (and $b_1 = b - b/s$) with

$$s - (s-1) \cdot \left(\frac{s-2}{s-1}\right)^{b-b/s}$$

Obviously, the number of players, s_1 and b_1, are discrete numbers. The associated $b_1 = (s - s_1) \cdot b/s$ may fail to be integers; if so, the number of searchers on the long market side is rounded to the next integer number.

It was determined in the preceding section, that in a market with exclusive search by one market side it is more efficient if the long market side searches. If there is no searchers' market at all, this seems to suggest that it is best if everybody on the long side searches and the short side stays put. However, it has also been determined that larger markets are less

[20] Despite its irrelevance at this point, see appendix H for the proof. Both properties are used in the following paragraph.

efficient than a collection of smaller markets. This could imply that "naturally splitting" the market by having some individuals on each market side search and other stay, i.e. splitting the market in two smaller market, increases efficiency. Theorem 5 confirms this intuition to some degree (a particular kind of partition).

If one individual alone on the short side searches, i.e. he goes to to the other end of the field, then this person is *certain* to find a matching partner if there is anyone at the other end of the field. Obviously, it also pays for one individual on the long market side to stop searching and to stay and wait, since this one searcher from the other side approaches her. Hence, the matching probability of the new searcher on the short market side is one and therefore higher than it was in the role of a stayer. The same holds for the new stayer on the long market side. Theorem 5 states that the matching probability is higher (but no longer one) for all long market side stayers than their matching probability would be as searchers in a market with exclusive long side search. It is optimal for more than one individual on the long market side to become a stayer, since the ratio of individuals on the long market side to individuals on the short market side did deteriorate in the "old" market by the departure of one member of the short market side. Theorem 5 states also that by having one searcher on the short side and the appropriate number of stayers on the long market side, the gain in the expected number of matchings, namely one, in this additional and small stayers' market is larger than the loss in the expected number of matchings, namely less than one, in the "old" and large stayers' market.

Observe that both stayers' markets have the same ratio of long side to short side players along the diagonal, but in opposite roles: $s_1/s = 1 - b_1/b$. Hence, in all Nash equilibrium markets along the diagonal there is one stayers' market with more searchers than stayers, and there is another stayers' market with more stayers than searchers. The latter stayers' market would be more efficient if the two groups switched their roles (theorem 3). However, if they were playing opposite roles in this framework, then by definition they would be part of the other stayers' market, i.e. there would be only one market where the long side searches. This is less efficient than being on their own in the "wrong" roles (theorem 5), if the stayers' market with more stayers than searchers ($b_0 > s_1$) is sufficiently small (always for $s_1 = 1$, but generally also for some $s_1 > 1$).

Proof: A state is a Nash equilibrium if no player wants to change strategy given all other players' strategies. Although the number of players is discrete, the matching probabilities are continuous functions of the number of players. Therefore the Nash equilibrium conditions are simply the following:

$$p_{b0} = 1 - \left(\frac{b_0 - 1}{b_0}\right)^{s_1} = \frac{s_0}{b_1} \cdot \left(1 - \left(\frac{s_0 - 1}{s_0}\right)^{b_1}\right) = p_{b1}$$

and

$$p_{s0} = 1 - \left(\frac{s_0 - 1}{s_0}\right)^{b_1} = \frac{b_0}{s_1} \cdot \left(1 - \left(\frac{b_0 - 1}{b_0}\right)^{s_1}\right) = p_{s1}$$

Mutliplying the latter equality by s_1/b_0 and equating the LHS of the first and the RHS of the second equality yields the condition $s_1 b_1 = s_0 b_0$ or equivalently $s_1 = \frac{s}{b}(b - b_1)$.
The expected number of matchings is maximal if the sum of the expected number of matchings in the two stayers' markets

$$E(\Delta) = (b - b_1) \cdot \left(1 - \left(\frac{b - b_1 - 1}{b - b_1}\right)^{s_1}\right) + (s - s_1) \cdot \left(1 - \left(\frac{s - s_1 - 1}{s - s_1}\right)^{b_1}\right)$$

is maximal. It can be shown that the expected number of matchings is larger at $s_1 = 1$ than at $s_1 = 0$ and also that it is larger at $s_1 = s - 1$ than at $s_1 = s$. Moreover, the expected number of matchings is a falling function in $s_1 \in [1, s-1]$. For details see appendix I. □

If $c > 0$ then there is also a searchers' market and equilibria have to be determined by comparing matching probabilities as indicated in (33) and (34). It has proved impossible to derive closed form solutions for these comparisons. The following results are based on explicit calculations[21] of matching probabilities for all markets up to $s = 100$ and $b = 400$. Unless stated otherwise[22], all results are determined by assuming $c = 1 - ((b-1)/b)^s$.

Observation 1 *With*

$$c = 1 - \left(\frac{b-1}{b}\right)^s \tag{38}$$

there are more than two Nash equilibria for $s > 2$ and $b \geq 3$.

There are always the two stable Nash equilibria which correspond to a market where only one market side searches; they are called *sorted* Nash equilibria. The other Nash equilibria correspond to markets where on both sides some players search and some stay put; they are called *mixed* Nash equilibria. As was noted before when analyzing the example of a market with $s = 2$ and $b = 3$, if c is not low enough, there may be no mixed equilibrium. However,

[21] Appendix K contains a general description of equilibrium and stability properties, including a description of the associated phase diagram.
[22] The same calculations were performed for an identical c for all markets. Qualitative findings are the same.

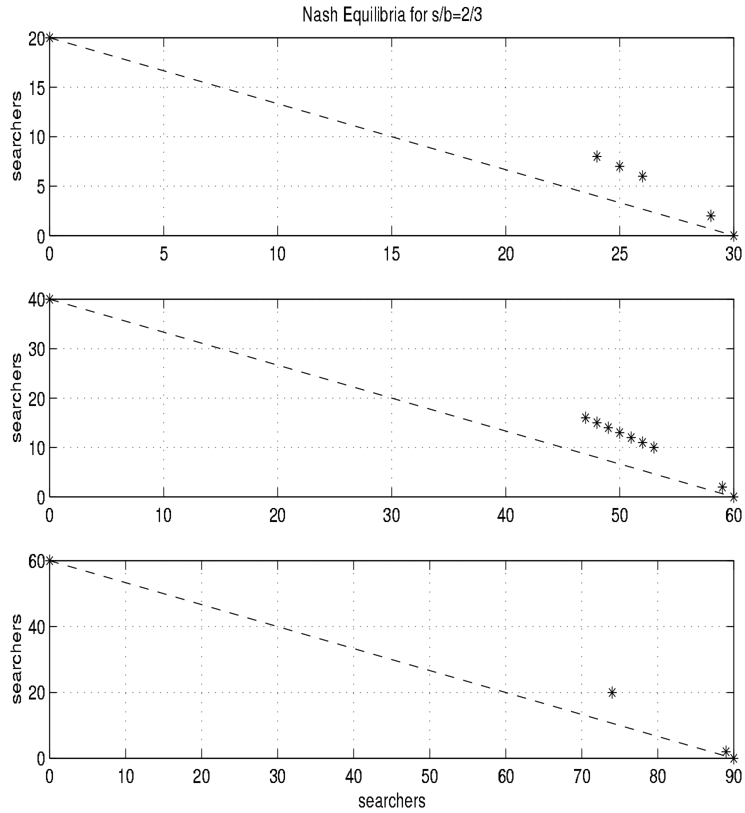

Figure 6: Nash Equilibria for $s/b = 2/3$

this phenomenon exists *only* in this specific market. All other markets[23] have at least one mixed equilibrium.

Observation 2 *If the long market side is at most twice as large as the short side, then there are several mixed equilibria. If the long market side is larger than twice as large as the short side, then there are few mixed equilibria. In very unbalanced markets there is only one mixed equilibrium:* $s_1 = 1$ *and* $b_0 = b/s - 1$*. The number of mixed equilibria decreases the larger the total market size.*

[23] Also the market with $s = b = 3$ has a mixed equilibrium with $(s_0, s_1, b_1, b_0) = (1, 2, 2, 1)$. The expected number of matchings is 2.4981 (with $c = 0.7937$), where on average there are 0.9383 matchings in the searchers' market. Hence, in most cases the two stayers' markets consist of one searcher and one stayer each. If one market side alone were searching then the expected number of matchings would be only 2.1111.

Table 4 shows the number of mixed equilibria for some markets. In general[24], the lower c the more pronounced the difference between markets with $b/s < 2$ and those with $b/s \geq 2$. This suggests that there is a cut-off value for the ratio of the size of the two market sides. If the two market sides are of sufficiently different size, then markets are basically large, i.e. all Nash equilibria are sorted or almost sorted (there is the one mixed equilibrium described in observation 2). If the two market sides are of sufficiently similar size, then markets display typical features of small markets; in addition to the two sorted equilibria there are (several) mixed equilibria.
Obviously, the number of mixed equilibria depends on the composition of the market and in particular on the composition of the two markets of a mixed equilibrium. The odds have to be sufficiently similar in the two markets for a mixed equilibrium to exist. Observation 3 will address this issue.

	ratio b/s				
s	4	3	2	3/2	1
2	1	1	1	0	0
4	1	1	2	2	1
6	1	1	2	2	3
10	1	1	2	4	4
20	1	1	3	4	5
30	1	1	3	5	4
40	1	1	3	8	3
50	1	1	4	5	2
60	1	1	4	2	1

Table 4: Number of Mixed Nash Equilibria

As figure 6 indicates[25], mixed equilibria are always such that a majority of the individuals on the long market side search and that some, but less than 50%, of the short side individuals search. There is one large stayers' market with more than half of the short side individuals as stayers and with more than half of the long side individuals as searchers. A majority of the long side individuals are searchers, however, usually not all of them reach the stayers' market at the other end. In the smaller stayers' market, there are a few long side individuals as stayers and some short side individuals as searchers. In the first market there may be more searchers than stayers, in the latter market this is not clear. The following observation[26]

[24]This finding is similar to that of Kultti (199?). In a related framework with markets with infinitely many individuals on both market sides, he finds that there is only one unstable mixed equilibrium (if there are two sorted equilibria). It exists only if one market side is at most 1.72 times as large as the other market side.

[25]Appendix J shows further graphs for different market size.

[26]In appendix K a table with average numbers of searchers can be found.

addresses these two points.

Observation 3 *In mixed Nash equilibria, the number of searchers is relatively larger on the long side than on the short side:*

$$\frac{b_1}{b} \geq \frac{s_1 + b_1}{s + b} \geq \frac{s_1}{s}$$

The stayers' market on the short side has relatively more seachers than stayers than the stayers' market on the long side:

$$\frac{b_1}{s_0} > \frac{s_1}{b_0}$$

The stayers' market on the long side has more stayers than searchers ($s_1/b_0 < 1$) if c is sufficiently low.

The stayers' market on the long side exists only, i.e. the appropriate mixed Nash equilibrium exists, if it helps to reduce the effect of the matching friction. Even if one market side is much larger than the other, it pays to have a stayers' market on the long side, where one matching occurs with certainty. There is a substantial amount of miscoordination in the stayers' market on the short side due to its large size. Thus, it pays for more than one of the long side searchers to become a stayer in the stayers' market on the long side, if there is also a searcher from the short market side. If the two market sides are of very different size, then it is basically impossible to have two markets with more searchers than stayers. Of the few individuals on the short market side, not too many can be searchers because otherwise there are not enough stayers in the stayers' market on the short side. It does not pay to have a large stayers' market on the long side, because that market is fairly inefficient with more stayers than searchers. If the two market sides are of similar size, then it is possible to generate two markets with more searchers than stayers. In this case there is a trade-off between having one fairly efficient stayers' market with many searchers relative to stayers and having another less efficient, but not too inefficient market, with hardly more searchers than stayers.

If c is very low, then there is basically no searchers' market and the existence of mixed Nash equilibria depends only on the two stayers' market. This implies that a second stayers' market on the long side exists in an unbalanced market ($b \gg s$) only if it increases the matching probability at a sufficiently low "cost" to the other stayers' market on the short side, i.e. if $s_1 = 1$. If c is high, then there are sufficient matching opportunities also in the searchers' market. Individual incentives are no longer only determined by a relative comparison of the two stayers' market, but also by the searchers' market. Searching is more attractive and more players search, even if the two market sides are of very different size.

Contrary to markets where all individuals on one market side search and the others stay, the two smaller markets do not have the same ratio of players from the two market sides. In markets with exclusive search by one market side, the ratio is basically the same in both markets because individuals continue to "play the same role" (searcher or stayer) also in the smaller markets. With search on both market sides individuals "play different roles" (searcher or stayer) in the two smaller markets. The efficiency advantage of a market with long side search implies that there is one larger market with many searchers from the long side and some stayers from the short side, and a smaller market with - if possible - more searchers from the short side than stayers from the long side.

As an example compare a market with a long side of $b = 30$ and a short side of $s = 20$ between the two scenarios, exclusive search by one market side and search on both market sides. For exclusive search by the long market side, an example of two smaller markets is of one market with 25 searchers and 17 stayers, and another market with 5 searchers and 3 stayers (cf. figure 2). The expected total number of matches is 1% higher than in the large market. If the same market is split by letting some individuals on both sides search, then one optimal partition[27] would be such that in one market there are 25 searchers (b_1) and 13 stayers (s_0), and in the other market there are 7 searchers (s_1) and 5 stayers (b_0). In this case the expected total number of matches is 5% higher than in the one large market with exclusive long side search. However, on average 17% of all matches occur in the searchers' market. Thus, mixed equilibria are more efficient but to a large extent this efficiency gain is based on the presence of the searchers' market and not on the composition of the two stayers' markets. Theorem 5 showed that even if there is no searchers' market, there is at least one mixed Nash equilibrium that is more efficient than the sorted Nash equilibria. Hence, the efficiency difference between mixed and sorted Nash equilibria relies on two aspects, on the additional matching opportunities in the searchers' market and the implied lower numbers in the stayers' markets (some searchers left the market because they met someone in the searchers' market), and also on the partition of the large market. So, the relatively higher efficiency of smaller markets matters because the market is partitioned and because there are fewer individuals in the two stayers' markets. This yields the last result:

Observation 4 (Pareto Efficiency of Mixed Nash Equilibria) *Mixed Nash equilibria are more efficient than sorted Nash equilibria. Both, the expected number of matchings* $E(\Delta)$

[27]The other mixed Nash equilibria are $(s_0, s_1, b_1, b_0) \in \{(18, 2, 29, 1), (14, 6, 26, 4), (12, 8, 24, 6)\}$ with $c = 0.4924$.

and individual matching probabilities for all types of players $(p_{s0}, p_{s1}, p_{b1}, p_{b0})$*, are unambiguously higher in a mixed than in the sorted Nash equilibria.*

Mixed equilibria Pareto dominate the sorted ones, since they combine the matching opportunities of the two subsequent markets. Table 5 shows the magnitude of this effect. The larger one market side relative to another, the more similar to a sorted Nash equilibrium is the mixed Nash equilibrium. Hence efficiency gains are smallest in markets where the two sides are of very different size.

	ratio b/s				
s	4	3	2	3/2	1
4	1.01/1.09	1.02/1.12	1.04/1.14	1.08/1.14	1.13/1.13
10	1.00/1.10	1.01/1.12	1.02/1.12	1.06/1.13	1.14/1.14
30	1.00/1.11	1.00/1.12	1.01/1.11	1.04/1.11	1.13/1.13
50	1.00/1.11	1.00/1.12	1.01/1.11	1.04/1.11	1.13/1.13

Table 5: Average Expected Number of Matchings in Mixed Nash Equilibria (relative to $E(\Delta)$ at $(0, s, 0, b)$ and at $(s, 0, b, 0)$)

This section has shown that exclusive search by one market side is indeed an equilibrium if the probability to encounter someone in the searchers' market is sufficiently low. In any case[28], there are always mixed equilibria with searchers on both market sides. Matching probabilities are higher in mixed equilibria. There are two reasons for this, one is the existence of the searchers' market, i.e. the additional opportunity to meet a matching partner; the second reason is that by having searchers (and stayers) on both market sides, the market is split and efficiency is increased. The market is split in a different way if there are searchers and stayers on both market sides and if searching is done exclusively by one market side. The exact ratios of searchers to stayers and of the relative size of the smaller markets depends on the size and composition of the market. In an optimal mixed equilibrium, i.e. in one with the highest possible expected number of matches, the matching behaviour is determined by the trade-off between having one market where the imbalance between long and short side is as large as possible and another market, that contributes to the efficiency gain without removing to many individuals from the first market and thereby rendering it less efficient.

[28]The only exemption is a market with $s = 2$ and $b = 3$ if c is sufficiently high.

4 Related Literature

The literature presented in this section is related to this paper in that it addresses one or some of the aspects treated in the model discussed before. BCKW (Burdett et al 1995) analyze who should search in a monetary exchange model but without any matching friction. Kultti (199?) investigates market division with a matching friction but without any explicit search and only for large markets. Taylor (1995) focuses on the impact of the size and evolution of long and short market side on reservation prices and ignores the matching technology. Peters (1991, 1992) applies the same matching technology in large markets in order to compare different pricing institutions. Baye/Cosimano (1990) consider the choice of market side by individuals in a model without a matching friction and without search. The following paragraphs discuss these results in some more detail.
The question whether there is "some way to determine endogeneously which agents [should] invest resources in the process of active search for trading partner and which agents [should] prefer to wait passivly for trading partners to come to them" has been asked before. BCKW (Burdett et al (1995)) obtain results on who should be searching in equilibrium in a model of monetary exchange where buyers and sellers want to consume one unit of a good. Buyers hold a unit of money, sellers hold a unit of a good they do not want but can exchange for a good of their liking. Individuals meet repeatedly and pairwise. The results in this paper are mainly driven by features of monetary exchange: buyers are more "flexible" in their model since they possess the more convertible medium of exchange (money). The general gist of BCKW's model is that it is best - both in individual and in welfare terms - if buyers alone search, basically irrespective of the number of buyers and sellers. They find other equilibria as well: nobody searches, only sellers search, or players on both market sides search. There is no coordination failure in the matching process in BCKW's model and therefore the size of market sides has only a minor impact on equilibria. In general, the support for the relevant equilibria is the larger the more stayers[29] there are.

Kultti (199?) analyzes a decentralized market model where trading partners have to find each other in the same way as in the model presented here. His matching process stops after the initial assignment. There are no "searchers" in the sense of this paper since they cannot meet each other. A market is split into two markets because individuals choose either of two market locations. Price determination plays a role in his model. If a stayer is matched with exactly one searcher then they bargain. If a stayer is matched with several searchers, then an auction ensues. Kultti analyzes only large markets, where matching probabilities are

[29] See appendix L for a short illustration.

as in equations (14) and (15). In an evolutionary setting, there are three equilibria if the two market sides are not too different in size: exclusive search by either market side or a mixed equilibrium where some members of both market sides search. Only the two equilibria of exclusive search are stable, the mixed equilibrium is unstable. The more efficient mixed equilibria are missed because of the absence of active searching and because populations are infinitely large. If the two market sides are sufficiently different in size, then the only evolutionary stable state is that only the long market side searches.

Taylor (1995) considers a model with countably many buyers and sellers who want to trade one unit of a homogeneous good every round. In each round at most one buyer and one seller enter the market according to a random process. Trading offers are announced publicly. Matches are one-to-one without any matching friction. Successful pairs leave the market. Reservation prices in such a market are a function of both the imbalance between long and short side in a given round, and of the long term perspectives in the market. The longer the long side and the less likely it is that the long side may become the short side in the future, the more of the surplus is a long side player willing to sacrifice (and conversely for a short side player). Taylor considers volatility too, however it depends on an exogeneously given process and is by assumption unrelated to the matching process. Long side individuals prefer high volatility markets because it increases their likelihood to end up being on the short side.

Peters (1991) and Peters (1992) consider a model with infinitely many buyers and sellers. Buyers and sellers want to buy/sell one unit of a homogeneous good. There are more buyers than sellers, hence there is excess demand. Only buyers, the long side, can search. The matching technology is like in this model. Buyers and sellers leave the market once they bought/sold their unit. The market is repeated as long as there are some sellers left in the market. In this context different pricing institutions are compared, ex-ante public commitment to prices, ex-post take-it-or-leave-it offers, and ex-post alternating offer bargaining. If players have a choice in any round whether they want to participate in the market at all, then ex-ante pricing is more efficient, because it takes into account correctly the participation decision of individuals (otherwise there is a friction). If individuals cannot opt out, then ex-ante pricing is more flexible[30] to take into account the changing odds in the market due to exiting players. In this case, the actual price level depends on the relative size of the two market sides. If both sides are of almost the same size, then buyers get a larger share of

[30]With ex-ante pricing sellers can influence how the surplus is split. In this case matching probilities influence reservation prices and the surplus split, whereas with ex-post pricing only reservation prices depend on matching probabilities (and therefore the size of the surplus, but not its division).

the surplus, if there are many more buyers than sellers, then sellers get a larger share of the surplus.

Baye/Cosimano (1990) consider a model in the flavour of assortative matching. Individual valuations of a homogeneous good are distributed in the population according to a given distribution. Players want to find a trading partner with a compatible valuation. Unless they leave the market without ever seeking a match, they are a searcher by definition. There are no stayers in the sense of this model. Depending on the valuation distribution and the search costs for each type of player, individuals become either buyers, sellers, or they do not participate at all in the market. As long as search costs are not too high, there is an equilibrium where everybody participates in the market. Otherwise only the most competitive individuals enter the market.

5 Conclusion and Outlook

This paper has looked at a decentralized market where individuals have to find trading partners on their own. Individuals can either search or stay and wait to be found. Uncoordinated search by individuals causes inefficiency in the matching process. The exact impact of this inefficiency depends on how individuals organize themselves. The results of this paper allow two kinds of conclusions: different decentralized markets can be evaluated and compared in terms of their matching behaviour and guidelines for centralized matching rules can be derived.

It was shown that the inefficiency due to uncoordinated search is more pronounced in large markets; it is better for all market participants if markets are small. *If possible, markets should be split into several similar or identical smaller markets.* If individuals can choose which market side they want to belong to, then there as many searchers as there are stayers in a market. However, the market would be more efficient, i.e. the expected number of matches would be higher, if there were more searchers than stayers. *In order to have a maximally efficient market, individuals should be compensated if they are searchers to render the two roles equally attractive.* If individuals cannot choose which market side they want to belong to and if all individuals on one market side pursue the same strategy, then it is best for both types of individuals if the long market side searches. *Unless other criteria determine the roles of searchers and stayers, long side individuals should be the searchers.* If individuals belong to one market side but can choose their strategy independently, then *it is best for all market participants if some individuals on both market sides are searchers.* There is always at least one such "division of labour" that is more efficient than exclusive search by one market side only. However, markets where some individuals on both market sides search are viable only if the numbers of searchers and stayers is just "right". This makes it difficult to implement any rule in real markets with central organization unless the number of participants is known with a high degree of accuracy. Even if individuals on both market sides may choose their strategies, exclusive search by one market side is usually an equilibrium too. This can be easily implemented in any market. To yield the best possible outcome, the only necessary information is which market side is expected to be larger.

It would be interesting to apply the matching techology analyzed in this paper to different market and contract environments. The main focus of this analysis should be the extent to which the matching friction determines terms of trade and contract rules. It could also be interesting to extend the analysis to markets with more than two sides.

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A Number of Possible Matchings with k of n Searchers Matched: $Z_n(k)$

The number of possible matchings with k of n buyers matched

$$Z_n(k) = k^n - \sum_{j=1}^{k-1} \binom{k}{j} Z_n(j), \qquad Z_n(1) = 1,\ Z_n(0) = 0$$

is determined by k^n, the number of possibilities to distribute n buyers in k slots, excluding all combinations where there are less than k matchings (the sum term).

$$\begin{aligned}
Z_n(k) &= k^n - \sum_{j=1}^{k-1} \binom{k}{j} Z_n(j) \\
&= k^n - \sum_{j=1}^{k-2} \binom{k}{j} Z_n(j) - \binom{k}{k-1} Z_n(k-1) \\
&= k^n - k(k-1)^n - \sum_{j=1}^{k-2} \left[\binom{k}{j} - k\binom{k-1}{j} \right] Z_n(j) \\
&= k^n - k(k-1)^n + k \cdot \sum_{j=1}^{k-2} \binom{k-1}{j} \frac{k-j-1}{k-j} \cdot Z_n(j) \\
&= k^n - k(k-1)^n + \frac{k(k-1)}{2}(k-2)^n - \frac{k(k-1)}{2} \cdot \sum_{j=1}^{k-3} \binom{k-2}{j} \frac{k-j-2}{k-j} \cdot Z_n(j) \\
&= \dots \\
&= \sum_{i=0}^{k-1} \binom{k}{i} (-1)^i \cdot (k-i)^n \;=\; \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} \cdot i^n
\end{aligned}$$

B Probability $P_{n,m}(\Delta = k)$

For $P_{n,m}(\Delta = k)$ to be well-defined probabilities it has to hold that

$$\sum_{k=1}^{s} \binom{m}{k} \cdot Z_n(k) \;=\; m^n.$$

with $s = \min\{m, n\}$.
If $s = m$, by using

$$Z_n(k) \;=\; k^n - \sum_{j=1}^{k-1} \binom{k}{j} Z_n(j)$$

this is straightforward:

$$\sum_{k=1}^{m}\binom{m}{k}\cdot Z_n(k) \;=\; \sum_{k=1}^{m-1}\binom{m}{k}\cdot Z_n(k)+m^n-\sum_{j=1}^{m-1}\binom{m}{j}\cdot Z_n(j) \;=\; m^n.$$

If $s=n$ it has to be shown that

$$\sum_{k=1}^{n}\binom{m}{k}\cdot Z_n(k) \;=\; m^n.$$

The following result is used below:

$$\begin{aligned}
Z_n(k) &= \sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}\cdot i^n = \sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}\cdot i^{n-1}\cdot i \\
&= \sum_{i=1}^{k}\binom{k}{i}(-1)^{k-i}\cdot i^{n-1}+\sum_{i=2}^{k}\binom{k}{i}(-1)^{k-i}\cdot i^{n-1}+\ldots+\binom{k}{k}\cdot i^{n-1} \\
&= \sum_{j=1}^{k}\sum_{i=j}^{k}\binom{k}{i}(-1)^{k-i}\cdot i^{n-1} \;=\; k\cdot Z_{n-1}(k)-\sum_{j=1}^{k-1}\sum_{i=1}^{j}\binom{k}{i}(-1)^{k-i}\cdot i^{n-1} \\
&= k\cdot Z_{n-1}(k)-\sum_{i=1}^{k-1}(k-i)\cdot\binom{k}{i}(-1)^{k-i}\cdot i^{n-1} \\
&= k\cdot Z_{n-1}(k)+k\cdot\sum_{i=1}^{k-1}\binom{k-1}{i}(-1)^{k-1-i}\cdot i^{n-1} \;= \\
Z_n(k) &= k\cdot[Z_{n-1}(k)+Z_{n-1}(k-1)] \qquad (39)
\end{aligned}$$

Observe that $Z_n(n)=n!$. This and equation (39) yield

$$\begin{aligned}
&\sum_{k=1}^{n}\binom{m}{k}\cdot Z_n(k) \\
&= \binom{m}{1}Z_n(1)+\binom{m}{n}Z_n(n)+\sum_{k=2}^{n-1}\binom{m}{k}\cdot Z_n(k) \\
&= \binom{m}{1}Z_n(1)+\binom{m}{n}Z_n(n)+\sum_{k=2}^{n-1}\binom{m}{k}\cdot k\cdot[Z_{n-1}(k)+Z_{n-1}(k-1)] \\
&= \binom{m}{1}Z_n(1)+\binom{m}{n}Z_n(n)+2\binom{m}{2}\cdot Z_{n-1}(1)+(n-1)\binom{m}{n-1}\cdot Z_{n-1}(n-1) \\
&\quad +\sum_{k=2}^{n-2}\left[k\binom{m}{k}+(k+1)\binom{m}{k+1}\right]\cdot Z_{n-1}(k) \\
&= \binom{m}{1}Z_n(1)+\binom{m}{n}Z_n(n)+2\binom{m}{2}\cdot Z_{n-1}(1)+(n-1)\binom{m}{n-1}\cdot Z_{n-1}(n-1)
\end{aligned}$$

$$
\begin{aligned}
& +m\cdot\sum_{k=2}^{n-2}\binom{m}{k}\cdot Z_{n-1}(k) \\
= \;& \binom{m}{1}Z_n(1)+\binom{m}{n}Z_n(n)+2\binom{m}{2}\cdot Z_{n-1}(1)+(n-1)\binom{m}{n-1}\cdot Z_{n-1}(n-1) \\
& +2\binom{m}{2}\cdot m\cdot Z_{n-2}(1)+(n-2)\binom{m}{n-2}\cdot m\cdot Z_{n-2}(n-2) \\
& +m^2\cdot\sum_{k=2}^{n-3}\binom{m}{k}\cdot Z_{n-2}(k) \\
= \;& \ldots \\
= \;& \binom{m}{1}Z_n(1)+\binom{m}{n}Z_n(n)+2\binom{m}{2}\cdot\sum_{k=2}^{n-1}Z_k(1)\cdot m^{n-1-k}+\sum_{k=2}^{n-1}k\binom{m}{k}\cdot Z_k(k)\cdot m^{n-1-k} \\
= \;& m+\binom{m}{n}n!+(m-1)\cdot\sum_{k=1}^{n-2}m^k+\sum_{k=2}^{n-1}k\binom{m}{k}\cdot k!\cdot m^{n-1-k} \\
= \;& \binom{m}{n}n!+m^{n-1}+m^{n-1}\left[\sum_{k=1}^{n-1}k\binom{m}{k}\cdot k!\cdot m^{-k}-1\right] \\
= \;& m^{n-1}\left[\sum_{k=1}^{n-1}k\binom{m}{k}\cdot k!\cdot m^{-k}+\binom{m}{n}\cdot n!\cdot m^{-(n-1)}\right] \\
= \;& m^{n-1}\left[\sum_{k=1}^{n}k\binom{m}{k}\cdot k!\cdot m^{-k}+(m-n)\binom{m}{n}\cdot n!\cdot m^{-n}\right] \\
= \;& m^{n-1}\left[\sum_{k=1}^{n}k\binom{m}{k}\cdot k!\cdot m^{-k}+\binom{m}{n+1}\cdot(n+1)!\cdot m^{-n}\right] \\
= \;& m^{n-1}\left[\sum_{k=1}^{n+1}k\binom{m}{k}\cdot k!\cdot m^{-k}+\binom{m}{n+2}\cdot(n+2)!\cdot m^{-(n+1)}\right] \\
= \;& \ldots \\
= \;& m^{n-1}\left[\sum_{k=1}^{m-1}k\binom{m}{k}\cdot k!\cdot m^{-k}+\binom{m}{m}\cdot(m)!\cdot m^{-(m-1)}\right] \\
= \;& m^{n-1}\sum_{k=1}^{m}k\binom{m}{k}\cdot k!\cdot m^{-k} \;=\; m^{n-1}\sum_{k=1}^{m}k\cdot\frac{m}{m}\cdot\frac{m-1}{m}\cdot\ldots\cdot\frac{m-k+1}{m} \\
= \;& m^{n-1}\left[\frac{m}{m}\left(1+\frac{m-1}{m}\left(2+\frac{m-2}{m}\left(3+\ldots\frac{3}{m}\left(m-2+\frac{2}{m}\left(m-1+\frac{1}{m}\cdot m\right)\right)\ldots\right)\right)\right)\right] \\
= \;& m^{n-1}\cdot m \;=\; m^n. \quad\square
\end{aligned}
$$

C Expected Number of Matchings: $E_{n,m}(\Delta)$

It has to be shown that

$$E_{n,m}(\Delta) \;=\; \frac{1}{m^n}\sum_{k=1}^{s} k\cdot\binom{m}{k}\cdot Z_n(k) \;=\; m\cdot\left(1-\left(\frac{m-1}{m}\right)^n\right)$$

or equivalently that

$$\sum_{k=1}^{s} k\cdot\binom{m}{k}\cdot Z_n(k) \;=\; m\cdot\left(m^n-(m-1)^n\right).$$

For

$$\begin{aligned}
s=m \quad : \quad & \sum_{k=1}^{m} k\cdot\binom{m}{k}\cdot Z_n(k) \;=\; \sum_{k=1}^{m-1} k\cdot\binom{m}{k}\cdot Z_n(k)+m\cdot Z_n(m) \;=\\
& \sum_{k=1}^{m-1} k\cdot\binom{m}{k}\cdot Z_n(k)+m\left(m^n-\sum_{j=1}^{m-1}\binom{m}{j}Z_n(j)\right) \;=\\
& m\cdot m^n-\sum_{k=1}^{m-1}(m-k)\cdot\binom{m}{k}\cdot Z_n(k) \;=\; m\cdot m^n-m\cdot\sum_{k=1}^{m-1}\binom{m-1}{k}\cdot Z_n(k) \;=\\
& m\cdot m^n-m\cdot(m-1)^n \;=\; m\cdot(m^n-(m-1)^n). \quad \square
\end{aligned}$$

$$s=n \quad : \quad \sum_{k=1}^{n} k\cdot\binom{m}{k}\cdot Z_n(k)$$

has to be equal to $m\cdot(m^n-(m-1)^n)$. Using the results of appendix B

$$\begin{aligned}
m^n-(m-1)^n \;&=\; \sum_{k=1}^{n}\binom{m}{k}Z_n(k)-\sum_{k=1}^{n}\binom{m-1}{k}Z_n(k)\\
&=\; \sum_{k=1}^{n}\binom{m}{k}\cdot\frac{k}{m}\cdot Z_n(k). \quad \square
\end{aligned}$$

D Variance of the Number of Matchings: $Var_{n.m}(\Delta)$

To determine the variance,

$$E_{n,m}(\Delta^2) \;=\; \frac{1}{m^n}\sum_{k=1}^{s} k^2\cdot\binom{m}{k}\cdot Z_n(k)$$

has to be calculated. It is for

$$s=m \quad : \quad \sum_{k=1}^{m} k^2\cdot\binom{m}{k}\cdot Z_n(k) \;=\; \sum_{k=1}^{m-1} k^2\cdot\binom{m}{k}\cdot Z_n(k)+m^2\cdot Z_n(m) \;=$$

$$\sum_{k=1}^{m-1} k^2\cdot\binom{m}{k}\cdot Z_n(k)+m^2\left(m^n-\sum_{j=1}^{m-1}\binom{m}{j}Z_n(j)\right) \;=$$

$$m^{n+2}-\sum_{k=1}^{m-1}(m^2-k^2)\cdot\binom{m}{k}\cdot Z_n(k) \;=$$

$$m^{n+2}-\sum_{k=1}^{m-1}(m+k)\cdot\binom{m-1}{k}\cdot m\cdot Z_n(k) \;=$$

$$m^{n+2}-m^2\sum_{k=1}^{m-1}\binom{m-1}{k}Z_n(k)-m\sum_{k=1}^{m-1}k\binom{m-1}{k}Z_n(k) \;=$$

$$m^{n+2}-m^2(m-1)^n-m(m-1)\cdot((m-1)^n-(m-2)^n) \;=$$

$$m^{n+2}-m(2m-1)(m-1)^n+m(m-1)(m-2)^n.$$

Therefore

$$\begin{aligned}
Var_{n,m}(\Delta) &= E_{n,m}(\Delta^2)-E_{n,m}(\Delta)^2\\
&= m^2-m(2m-1)\left(\frac{m-1}{m}\right)^n+m(m-1)\left(\frac{m-2}{m}\right)^n-m^2\left(1-\left(\frac{m-1}{m}\right)^n\right)^2\\
&= m\left(\frac{m-1}{m}\right)^n\cdot\left[1-m\left(\frac{m-1}{m}\right)^n+(m-1)\left(\frac{m-2}{m-1}\right)^n\right]. \quad \square
\end{aligned}$$

For

$$s=n \quad : \quad \sum_{k=1}^{n} k^2\cdot\binom{m}{k}\cdot Z_n(k)$$

is also equal to $m\cdot[m\cdot m^n-(2m-1)\cdot(m-1)^n+(m-1)\cdot(m-2)^n]$ as the following (using the results of appendix B) shows

$$\begin{aligned}
& m\cdot m^n-(2m-1)\cdot(m-1)^n+(m-1)\cdot(m-2)^n\\
&= \sum_{k=1}^{n}\left\{m\binom{m}{k}-(2m-1)\binom{m-1}{k}+(m-1)\binom{m-2}{k}\right\}Z_n(k)\\
&= \sum_{k=1}^{n}\left\{\binom{m}{k}\cdot\left[m-(2m-1)\frac{m-k}{m}+(m-1)\frac{(m-k)(m-k-1)}{m(m-1)}\right]\right\}Z_n(k)\\
&= \sum_{k=1}^{n}\binom{m}{k}\cdot\frac{k^2}{m}\cdot Z_n(k). \quad \square
\end{aligned}$$

E Long Side Search More Efficient

Inequality (16) can be rewritten as

$$X(b,s) \;=\; s\cdot\left(1-\left(\frac{s-1}{s}\right)^b\right)-b\cdot\left(1-\left(\frac{b-1}{b}\right)^s\right) \;>\; 0$$

Inequality (17) can be rewritten as

$$\begin{aligned} Z(b,s) \quad &= \quad b\left(\frac{b-1}{b}\right)^s\cdot\left[1-b\left(\frac{b-1}{b}\right)^s+(b-1)\left(\frac{b-2}{b-1}\right)^s\right] \\ &\quad - s\left(\frac{s-1}{s}\right)^b\cdot\left[1-s\left(\frac{s-1}{s}\right)^b+(s-1)\left(\frac{s-2}{s-1}\right)^b\right] \;>\; 0 \end{aligned}$$

The two functions can be seen in figure 7.

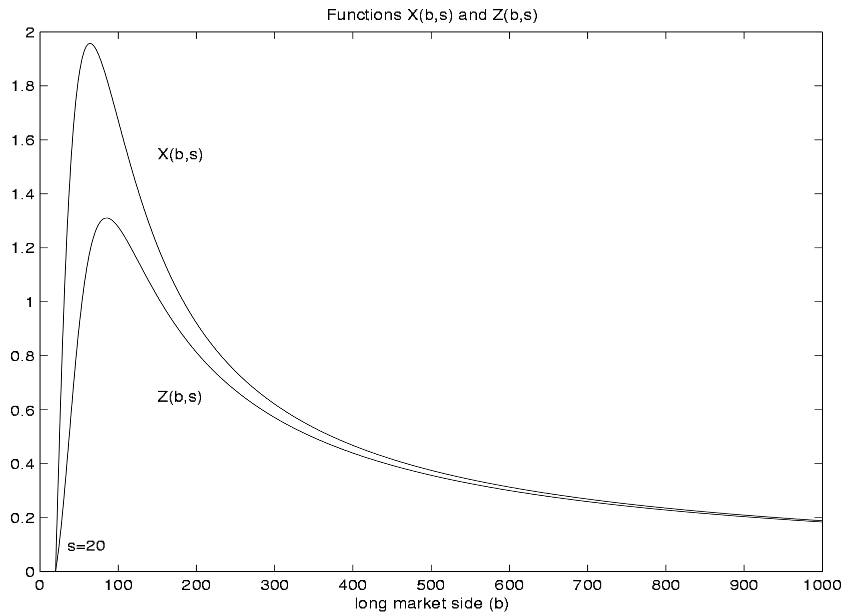

Figure 7: Functions $X(b,s)$ and $Z(b,s)$

F Relative Efficiency in Large Markets

Observe that

$$\lim_{s\to\infty}\left[xs\cdot\ln\left(\frac{s-1}{s}\right)\right] = xs\left(-\frac{1}{s}\right) = -x$$

$$\lim_{s\to\infty}\left[s\cdot\ln\left(\frac{xs-1}{xs}\right)\right] = s\cdot\left(-\frac{1}{xs}\right) = -\frac{1}{x}$$

Therefore for s sufficiently large

$$\left[s\cdot\left(1-\left(\frac{s-1}{s}\right)^{xs}\right)-xs\cdot\left(1-\left(\frac{xs-1}{xs}\right)^{s}\right)\right] \approx$$
$$\left[s\cdot\left((1-e^{-x})-x\cdot(1-e^{-1/x})\right)\right]$$

For

$$x=1: \qquad \left[s\cdot\left[\left(1-e^{-1}\right)-1\cdot\left(1-e^{-1}\right)\right]\right] = s\cdot 0 = 0$$

$$x\to\infty: \qquad [s\cdot(1-0-1)] = s\cdot 0 = 0$$

since via L'Hospital

$$\lim_{x\to\infty}\left[x\cdot\left(1-e^{-1/x}\right)\right] = \lim_{x\to\infty} 1\cdot\left(1+\frac{1}{x^2}e^{-1/x}\right) = 1$$

Both limits ($x=1$ and $x\to\infty$) converge to zero since the bracket converges faster to zero than $1/s$ and therefore the factor s to infinity.
For the maximum calculate

$$\frac{\partial}{\partial x}\left(1-e^{-x}-x\cdot(1-e^{-1/x})\right) = e^{-1/x}\left(1+\frac{1}{x}\right)-(1-e^{-x}) = 0$$

This yields

$$x = \frac{e^{-1/x}}{1-e^{-x}-e^{-1/x}},$$

which holds for $x\approx 3.2565$. The short side matching probabilities rely on the same result since $\lim_{s\to\infty}[p_0(b,s)-p_1(s,b)] = X(b,s)/s$. For $\lim_{s\to\infty}[p_1(b,s)-p_0(s,b)] = X(b,s)/b$ calculate

$$\frac{\partial}{\partial x}\left[\frac{1}{x}\cdot\left(x\cdot e^{-1/x}-x+1-e^{-x}\right)\right] = xe^{-x}-1+e^{-x}-xe^{-1/x}$$

which yields

$$x = \frac{1-e^{-1/x}-e^{-x}}{e^{-x}} \approx 2.1065.$$

G Matching Probabilities in Searchers' and Stayers' Market: $s = 2,\ b = 3$

The calculations are as follows:

$$\begin{aligned}
\text{state V} &: p_{b1} = c\left\{\frac{1}{3}\cdot 1+\frac{2}{3}\cdot\frac{1}{2}\right\}+(1-c)\cdot\frac{1}{3} = \frac{1+c}{3} \in \left(\frac{1}{3},\frac{2}{3}\right)\\
\text{state VI} &: p_{b1} = c\left\{\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 1\right\}+(1-c)\cdot\frac{1}{2} = \frac{1+c}{2} \in \left(\frac{1}{2},1\right)\\
\text{state X} &: p_{s1} = \binom{2}{0}\cdot\left(\frac{2c}{3}\right)^2\cdot 1+\binom{2}{1}\cdot\frac{2c}{3}\cdot\frac{3-2c}{3}\cdot\left\{\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 1\right\}+\binom{2}{2}\cdot\left(\frac{3-2c}{3}\right)^2\cdot\frac{1}{2}\\
&\quad = \frac{4c^2}{9}+2\cdot\frac{6c-4c^2}{9}+\frac{1}{2}\cdot\frac{9-12c+4c^2}{9} = \frac{9+12c-4c^2}{18} \in \left(\frac{1}{2},\frac{17}{18}\right)\\
\text{state XI} &: p_{s1} = c\left\{\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 1\right\}+(1-c)\cdot\frac{3}{4} = \frac{3+c}{4} \in \left(\frac{3}{4},1\right)\\
& \quad p_{b0} = c\cdot\frac{1}{2}+(1-c)\cdot\frac{3}{4} = \frac{3-c}{4} \in \left(\frac{1}{2},\frac{3}{4}\right)
\end{aligned}$$

The arrows in figure 5 that can point both ways are determined by:

$$\begin{aligned}
\text{V} \leftarrow \text{VI} &: p_{b1}(1,1,3,0) = \frac{1+c}{3} > 1-c = p_{b0}(1,1,2,1), \quad \text{for } c > 0.5\\
\text{X} \leftarrow \text{XI} &: p_{b1}(0,2,2,1) = \frac{2c}{3} > \frac{3-c}{4} = p_{b0}(0,2,1,2), \quad \text{for } c > \frac{9}{11} = 0.8182\\
\text{XI} \leftarrow \text{XII} &: p_{b1}(0,2,1,2) = c > \frac{5}{9} = p_{b0}(0,2,0,3), \quad \text{for } c > 0.5556\\
\text{VII} \rightarrow \text{XI} &: p_{s1}(0,2,1,2) = \frac{3+c}{4} > 1-c = p_{s0}(1,1,1,2), \quad \text{for } c > 0.2\\
\text{I} \rightarrow \text{V} &: p_{s1}(1,1,3,0) = c > \frac{7}{8} = p_{s0}(2,0,3,0), \quad \text{for } c > 0.8750
\end{aligned}$$

H All Players Searching is No Nash Equilibrium

To show that

$$1-\left(c\cdot\frac{b-1}{s+b-2}\right)^s > c\cdot\frac{s}{s+b-1}$$

and therefore

$$1 > c\cdot\frac{s}{s+b-1}+c^s\cdot\left(\frac{b-1}{s+b-2}\right)^s,$$

it suffices to show that the inequality holds for $c = 1$, since the RHS is increasing in c. In this case the inequality reduces to

$$1 > \frac{s}{s+b-1}+\left(\frac{b-1}{s+b-2}\right)^s, \tag{40}$$

where the left term (on the RHS) is increasing in s and the right term (on the RHS) falls sharply with s, therefore the RHS falls for small s and increases for s sufficiently large (once the right term is close to zero). The slope of the RHS is

$$\frac{b-1}{(s+b-1)^2} - \left(\frac{b-1}{s+b-2}\right)^s \cdot \left[\ln\left(\frac{s+b-2}{b-1}\right) + \frac{s}{s+b-2}\right]$$

Set equal to zero this is equivalent to

$$\frac{(s+b-2)^s}{(s+b-1)^2} = (b-1)^{s-1} \cdot \left[\ln\left(\frac{s+b-2}{b-1}\right) + \frac{s}{s+b-2}\right]$$

where the LHS increases faster in s than the RHS. As can be easily verified, there are small s for which the LHS is smaller than the RHS ($s = 1$ yields $(b-1)^2 < b^2$), and obviously, for large s the opposite holds. Therefore, in inequality (40) the RHS falls for small s values and increases for larger s values. Observe that inequality (40) holds for

$$\begin{aligned} s = 2 \quad : \quad & \frac{2}{b+1} + \left(\frac{b-1}{b}\right)^2 = \frac{b^3 - b^2 + b + 1}{b^3 + b^2} \\ & \frac{1 - 1/b + 1/b^2 + 1/b^3}{1 + 1/b} \to 1^- < 1 \qquad \text{for } b \to \infty, \\ s = b \quad : \quad & \frac{b}{2b-1} + \left(\frac{b-1}{2b-2}\right)^b = \frac{1}{2 - 1/b} + \frac{1}{2^b} < 1 \qquad \text{for } b > 2 \end{aligned}$$

which completes the proof. □

I Nash Equilibria without a Searchers' Market

The expected number of matchings is larger at $s_1 = 1$ than at $s_1 = 0$ if

$$\begin{aligned} s - (s-1) \cdot \left(\frac{s-2}{s-1}\right)^{b-b/s} &> s - s \cdot \left(\frac{s-1}{s}\right)^b \\ \left(\frac{s-1}{s}\right)^b &> \left(\frac{s-2}{s-1}\right)^{b\frac{s-1}{s}} \cdot \frac{s-1}{s} \end{aligned}$$

which holds trivially since

$$\left(\frac{s-1}{s}\right)^b > \left(\frac{s-2}{s-1}\right)^b > \left(\frac{s-2}{s-1}\right)^{b\frac{s-1}{s}} \cdot \frac{s-1}{s}$$

The expected number of matchings is larger at $s_1 = s - 1$ than at $s_1 = s$ if

$$\begin{aligned} (b - b/s) \cdot \left(1 - \left(\frac{b - b/s - 1}{b - b/s}\right)^{s-1}\right) + 1 &> b - b \cdot \left(\frac{b-1}{b}\right)^s \\ b \cdot \left(\frac{b-1}{b}\right)^s + 1 - b/s &> (b - b/s) \cdot \left(\frac{b - b/s - 1}{b - b/s}\right)^{s-1} \\ \left(1 - \frac{1}{b}\right) \cdot \left(1 - \frac{1}{b}\right)^{s-1} + \frac{1}{b} &> \left(1 - \frac{1}{s}\right) \cdot \left(1 - \frac{1}{b} \cdot \frac{s}{s-1}\right)^{s-1} + \frac{1}{s} \end{aligned}$$

It holds for $b = s$ since

$$\left(1 - \frac{1}{s}\right)^{s-1} > \left(1 - \frac{1}{s-1}\right)^{s-1}$$

The RHS increases faster in b than the LHS and for $b \to \infty$ both sides approach 1. To show that the slope of $E(\Delta)$ is falling in s_1 for $s_1 \in [1, s-1]$ rewrite $E(\Delta)$ as

$$E(\Delta) = \frac{b}{s}s_1 \cdot \left(1 - \left(\frac{\frac{b}{s}s_1 - 1}{\frac{b}{s}s_1}\right)^{s_1}\right) + (s - s_1) \cdot \left(1 - \left(\frac{s - s_1 - 1}{s - s_1}\right)^{\frac{b}{s}(s-s_1)}\right)$$

This can be rewritten as

$$E(\Delta) = \frac{b}{s}s_1 \cdot \left(1 - e^{-s/b}\right) + (s - s_1) \cdot \left(1 - e^{-b/s}\right)$$

with the large market approximations[31]. Taking the derivative w.r.t. s_1 and setting $x = b/s$ yields

$$\frac{b}{s} \cdot \left(1 - e^{-s/b}\right) - \left(1 - e^{-b/s}\right) < 0 \qquad \text{or} \qquad x < \frac{1 - e^{-x}}{1 - e^{-1/x}}$$

The latter is inequality (16) in theorem 3. □

[31] The same idea of the proof goes through with the original equation, but the proof is much longer.

J Nash Equilibria in Stayers' and Searchers' Markets

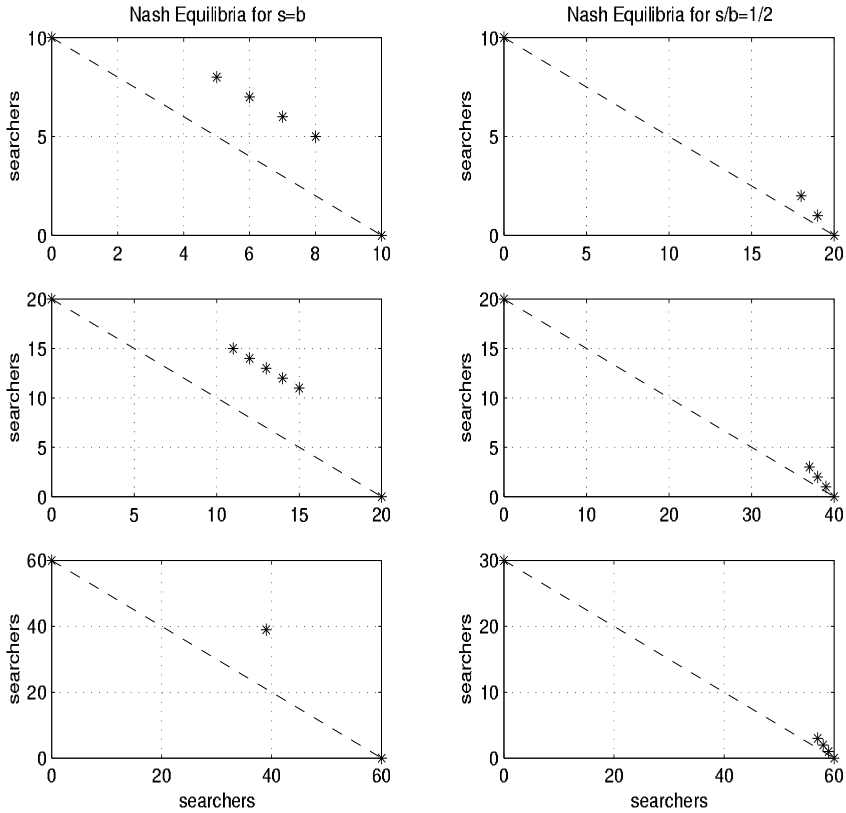

Figure 8: Nash Equilibria for $s = b$ and $s/b = 1/2$

K Nash Equilibria in Stayers' and Searchers' Markets

The following analysis is based on a graphical representation of a market like in figure 5. This means in particular that on the horizontal axis the number of searchers on the long market side increases and that on the vertical axis the number of searchers on the short market side increases. If $c \leq 1 - ((b-1)/b)^s$ then arrows along the edges are pointing rightward at the bottom, leftward at the top, upward on the left, and downward on the right. Moreover, all calculations have shown that the following properties hold

$$\frac{\Delta[p_{s1}(s_0 - 1, s_1 + 1, b_1, b_0) - p_{s0}(s_0, s_1, b_1, b_0)]}{\Delta s_1} < 0 \tag{41}$$

$$\frac{\Delta[p_{b1}(s_0, s_1, b_1+1, b_0-1) - p_{b0}(s_0, s_1, b_1, b_0)]}{\Delta b_1} < 0. \quad (42)$$

Hence, the more searchers there are on one's own side the less beneficial is it to be a searcher. In terms of the market diagram this implies that if arrows change along one row or column, then they change only once, always pointing to the center of the diagram (e.g. for a row this means, increasing s_1 the arrows may change from pointing to the right to pointing leftwards). Concerning the composition of the other market side, the following properties hold

$$\frac{\Delta[p_{s1}(s_0-1, s_1+1, b_1, b_0) - p_{s0}(s_0, s_1, b_1, b_0)]}{\Delta b_1} < 0 \quad (43)$$

$$\frac{\Delta[p_{b1}(s_0, s_1, b_1+1, b_0-1) - p_{b0}(s_0, s_1, b_1, b_0)]}{\Delta s_1} < 0 \quad (44)$$

The more searchers there are on the other market side the less beneficial is it to be a searcher. Therefore, the increased likelihood of meeting someone in the searchers' market is more than compensated by the decreased likelihood of meeting someone in the stayers' market at the other end.
Putting all this together implies that the locus of the vertical arrows falls and is steeper than the diagonal of the parameter space of the market. The locus of the horizontal arrows falls too; it is flatter than the diagonal of the parameter space of the market. The phase diagramm looks generically like (the vectors in the corners are (s_0, s_1, b_1, b_0)).

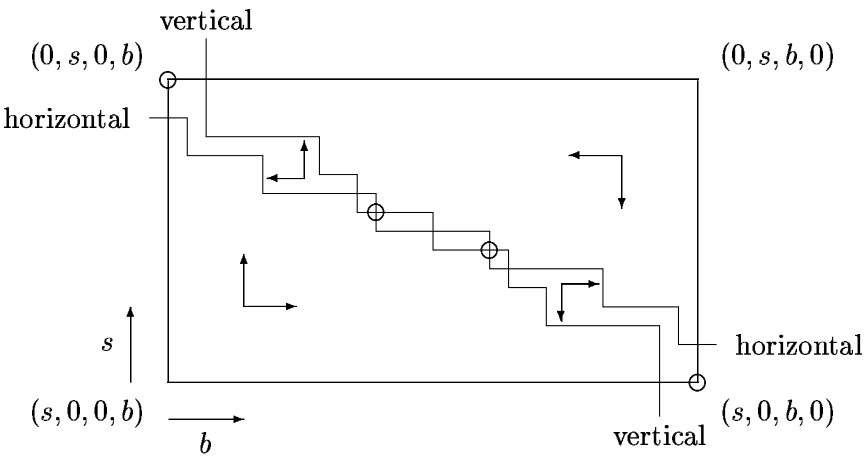

Figure 9: Phase Diagram of Stayers' and Searchers' Market

The circles indicate Nash equilibria. Depending on how exactly the loci cross, there may be an odd or an even number of mixed equilibria. The loci do always cross at least once. However, they may cross "between states", such that there is no mixed equilibrium at that point.
Mixed equilibria do not exist if the locus for the horizontal arrows is completely vertical or if the locus the vertical arrows is completely horizontal. This means that in one row (column) all arrows point into one direction and in the next row (column), they point in the other direction. This may happen in small markets as the example with $s = 2$ and $b = 3$ showed.

Observation 5 *Mixed Nash equilibria are always unstable.*

Mixed equilibra are stable if they uniquely dominate all four neighbouring states. They are unstable if at least one of their four neighbouring states is dominated by that equilibrium and another state. The following figure shows a case where the equilibrium is unstable because three of its neighbouring states ("u") are dominated by two different states. So, if the market is in one of those neighbouring states, optimal behaviour does not necessarily lead to the unstable equilibrium (circled). The line labelled "horizontal" depicts the locus for the horizontal arrows; similarly for the line labelled "vertical".

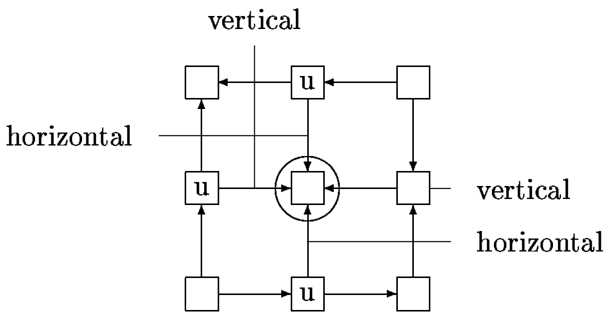

Figure 10: Unstable Nash Equilibrium in Stayers' and Searchers' Market

Observe that mixed Nash equilibria occur where the horizontal and the vertical locus intersect, and if the horizontal locus is vertical and if the vertical locus is horizontal (see figure 10 and 11). If two mixed Nash equilibria are next to each other on a falling diagonal, then the two states which are in the off-diagonal positions next to them are always unstable: both the lower left one and the upper right one are dominated by both mixed equilibria. Since both loci are weakly decreasing, the only possibility to get a stable mixed equilibrium is that horizontal arrows change direction also at the neighbouring state above and below and that vertical arrows change direction also at the neighbouring states to the right and to the left (cf. figure 11). If either of these four requirements does not hold, then there is at least one neighbouring state which is dominated by two different states. Therefore, for a mixed Nash equilibrium to be stable the locus for the horizontal arrows has to be almost vertical and the locus for the vertical arrows has to be almost horizontal at that equilibrium (otherwise the loci would not be straight lines for three rows or columns respectively). These two conditions can hold contemporaneously only if the Nash equilibria are located on a line which has a slope of approximately -1, as will be shown below. At that slope, however, the two loci have more or less the same slope and change direction frequently. If the loci cross at all, then these mixed Nash equilibria are unstable because some or all of the four requirements are violated.

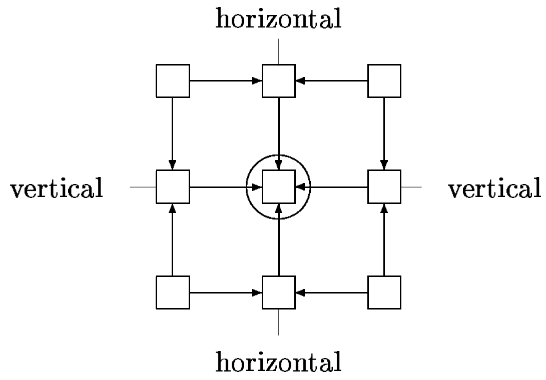

Figure 11: Characterization of a (Hypothetical) Stable Mixed Nash Equilibrium

If the equilibria of the market are connected by a line along which the difference in matching probabilities is zero (to be precise, where the change in sign occurs), then the numbers of searchers on both market sides have to change such that this property is preserved when moving from one equilibrium to the next. Starting at the equilibrium $(s, 0, b, 0)$, an increase in s_1 decreases the matching probability of a searcher on the long market side. This is compensated by a decrease in b_1, which in turn increases the matching probability of a searcher on the short market side. The relative changes in s_1 and b_1 determine where exactly the next equilibrium lies.

	Market $(s_1 + b_1)/(s + b)$					Long Side b_1/b				
	ratio b/s					ratio b/s				
s	4	3	2	3/2	1	4	3	2	3	1
2	0.60	0.62	0.67	-	-	0.62	0.67	0.75	-	-
4	0.70	0.69	0.67	0.65	0.75	0.81	0.83	0.81	0.67	0.75
6	0.73	0.71	0.67	0.67	0.67	0.88	0.89	0.88	0.83	0.67
10	0.76	0.72	0.67	0.64	0.65	0.92	0.93	0.92	0.83	0.65
20	0.78	0.74	0.67	0.64	0.65	0.96	0.97	0.95	0.87	0.65
30	0.79	0.74	0.67	0.62	0.65	0.98	0.98	0.97	0.88	0.65
40	0.79	0.74	0.67	0.63	0.65	0.98	0.98	0.98	0.85	0.65
50	0.79	0.74	0.67	0.63	0.65	0.98	0.99	0.98	0.86	0.65
60	0.79	0.75	0.67	0.62	0.65	0.99	0.99	0.98	0.91	0.65

Table 6: Relative Average Number of Searchers in Mixed Nash Equilibria

L The Role of Relative Size of the Two Market Sides in BCKW

In BCKW's model buyers are endowed with a unit of money while sellers are endowed with one unit of a good. Both types want to consume one unit of a specific good (for sellers it is a good different from the one they are endowed with), which someone else holds. Once a player has obtained his preferred good he consumes it and procudes one unit of a different good and then enters the market again. Buyers and sellers can both either search of stay put. Searchers always meet someone and stayers have a matching probability which is proportional to the fraction of searchers[32]. Both types of agents, buyers and sellers, incur a search cost during every round of the market. In this setting a variety of equilibria can occur, among them the degenerate equilibrium where nobody searches, and also the two equilibria where either one of the two market side searches. A market in which only sellers search is an equilibrium, if search costs for sellers are sufficiently low. The support for this equilibrium is the larger, the more buyers there are, i.e. the smaller the searching market side: The condition is

$$c \leq \bar{c} = \frac{m(m-1)(k-2)}{k(k-1)(rk+1-m)}$$

where $k \geq 3$ is the number of different goods/types, m the fraction of buyers in the population, r the discount rate. $\bar{c}$ is maximal at $m = \sqrt{1+rk} \cdot (\sqrt{1+rk} - \sqrt{rk})$ which is 1 for $rk = 0$ and approaches 1/2 from above for $rk \to \infty$, i.e. everywhere $m > 1/2$. See proposition 1 in BCKW for details. This does not imply, however, that the searching market side has to be the short side.

A market in which only buyers search is an equilibrium if the search costs are sufficiently high for sellers and sufficiently low for buyers. The support for this equilibrium does not depend on the size of the two market sides. One remarkable result of BCKW is that even if buyers face higher search costs it may be an equilibrium when they alone search. The support for this equilibrium is the larger, the more sellers there are, i.e. the smaller the searching market side. The support is

$$\frac{1-m}{k} - \frac{1-m}{k(k-1)} = \frac{k-2}{k(k-1)}(1-m).$$

Again, this does not imply that the searching market side has to be the short side. This finding is due to the asymmetric role of buyers and sellers. Buyers can only obtain their preferred good via a monetary exchange, whereas sellers can get their good via monetary exchange and via barter. However, if sellers exchange the good they hold against money, then they have not yet managed to get their preferred good. To enter a barter deal they have to find someone who holds the good they want and vice versa, an event which is less likely than that a buyer finds an appropriate trading partner. Money is therefore more convertible than a good and buyers benefit more from searching than sellers.

There is also an equilibrium where both sides search. This occurs if search costs for either side are not too high. If such an equilibrium exists then it is one of three multiple equilibria, where another one is always such that all buyers search (cf Burdett et al (1993)).

BCKW show that the support for the equilibrium where only buyers search is larger[33] than that where only sellers search. Moreover, if the search costs for buyers are not too large, total welfare in the population is larger in the equilibrium where only the buyers search than where either both sides or only the sellers search. A market where both sides search is strictly worse in welfare terms than one where only one side searches. The existence of barter in addition to monetary exchange (i.e. some sellers search) may be beneficial, if search costs for buyers are too high.

[32] This is a normalization, any process where the stayers' matching probility is proportional to the fraction of searcher and the searchers' matching probability is independent of that number yields the same result.

[33] For k, the number of types/goods large. Barter is too inefficient if there are too many different goods.