# Optimal Social Security with Moral Hazard


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**Abstract**

Social security is a form of insurance that protects individuals against interruption or loss of earning power. However, some analyses suggest that the structures of social security benefits lead people to seek early retirement. To consider this incentive problem and to design the optimal benefit structures, a model of dynamic insurance against the risk of permanent shocks is developed by modifying Atkeson and Lucas' (1995) model of repeated principal agent problem. The existence and the feasibility of the optimal contract are established. It is shown that this contract involves a decreasing social security tax and an increasing annuity with tenure. We find that it is optimal for people to retire after a finite period of work even if they are still able to work. It is also shown that the presence of hidden saving does not completely upset the efficiency of the optimal contract over autarky with saving. Simulation results are presented that suggest that the gains could be made by changing the current systems to the optimal program.



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## 1. Introduction

Individuals in an economy face the risk of losing their ability to work because of disability or old age, thereby losing earning power. Naturally, the provision of social security to provide for these people is a feature of all developed economies. The social security program is the largest entitlement program in the U.S. and social security benefit payments in 1993 were over 4% of the U.S. GDP (Diamond and Gruber (1997)). In Germany, social security income represents about 80% of household income of households headed by a person aged 65 and over (Borsch-Supan and Schnabel (1997)). Japan's social security expenditure in 1994 amounted to 11.9% of its GDP in that year (Oshio and Yashiro (1997)).

In spite of widespread use of social security systems, most countries face the problem of financial viability of these systems due to demographic trends, rapidly aging populations, and longer life spans. The financial pressure on social security is compounded by another trend. In virtually every country, employees are leaving the labor force at younger and younger ages. The Economist (1998) has reported that at present, most state systems offer built-in incentives to retire early, and many workers have taken the hint. Labor-force participation of men aged 60-64 has declined from over 80% in most rich countries in 1960 to 50% in America and below 35% in Germany, Italy, and France. Recently, studies by Gruber and Wise (1997) and Borsch-Supan and Schnabel (1997) found that social security in most countries places a heavy implicit tax on working beyond the age when an employee becomes eligible for early retirement and, therefore, discourages people from working after reaching this age.

Social security is an insurance against the loss of ability to work. However, often it is very difficult to verify whether someone is really not able to work and thus is entitled to be a beneficiary of social security. Therefore, generous benefits to the retired may give working people incentive to retire to enjoy these benefits. The early retirement trend can be interpreted as evidence of the presence of a moral hazard problem in the current systems. In fact, this problem is well recognized: many recent reforms have been designed to reduce this moral hazard problem by making early retirement less attractive. But excessive measure on early retirement may not provide enough protection to people who have lost their earning power. Thus, the key public policy problem is how to structure the benefit system to balance financial security with an incentive to work.

The purpose of this paper is to examine more formally the problem of incentive based social security design. We make two important contributions. First, we develop a model of dynamic insurance with the risk of permanent shock by modifying Atkeson and Lucas' (1995) model of repeated principal agent problem. We characterize the optimal contract under this environment and show the existence of an equilibrium in a steady state. We also show that in our model the efficiency of the optimal contract over autarky is not completely eradicated with the introduction of hidden saving, which is usually assumed away in most repeated principal agent models such as ours. The second contribution is that by applying the model to the design of social security systems we present a theoretical method to consider potential incentive problems in social security such as early retirement trend. We also execute a simulation to compare the current system to the optimal contract.

Our model considers a simple economy in which people have the risk of permanently losing their ability to work, but being able to work is not observable to the public so that an individual can pretend not to be able to work if it is in his interest. To risk averse individuals, it would be first best to eliminate the risk by pooling income completely; however, complete pooling is not feasible informationally since then everyone would claim that he is not able to work and receive the benefits. Therefore, any efficient arrangement to reduce the risk must recognize the need to be incentive compatible. An insurance contract specifies a time sequence of transfers between the principal (e.g., government, intermediaries) and the agent (e.g., employed workers), conditional on tenure. The optimal contract minimizes the expected discounted value of these net transfers–the budget–subject to providing the agent with a prespecified welfare level in an incentive compatible way.

We show that the optimal contract bears meaningful implications to social security programs. In the optimal contract, annuity must increase with tenure, but the social security tax must decrease with tenure. It is also found that in the optimal contract people are forced to retire after a finite period of work even if they are still able to work. Therefore, a mandatory retirement age is necessary. The logic behind this is clear. To induce people to work, the program must reward the people who continue to work by increasing their entitlement. The increase in entitlement will be realized by an increase in the level of consumption spread over time since this saves the costs of a given entitlement increase. Thus, as people continue to work, they will have higher current consumption levels (higher net current income or lower social security tax) and higher future consumption levels through higher future entitlement (higher annuity). But beyond a threshold, the incentive cost to induce people to work outweighs the productivity they possess. At this point they should be forced to retire.

The optimal contract characterized is in the one-on-one principal agent contract framework studied by Green (1987), Spear and Srivastava (1987), Thomas and Worral (1990), Atkeson and Lucas (1992), and many others. But any social security system is a collection of all such individual contracts. The next issue we address is the existence of steady state feasible social security system as an aggregation of all optimal contracts to individuals. For the system to be feasible, at all dates the aggregate consumption given to agents based on the transfers of all individual contracts must be equal to the aggregate production by agents. We show that for any discount rate close to 1, there is an equilibrium initial entitlement given to the newly born generation under which the system is feasible.

As most repeated principal agent models do, we assume that the agent's consumption level is observable. The intermediary is thus able to control the intertemporal decision of agents, which gives an extra degree of freedom to the intermediary to save incentive costs compared to static models. The resulting optimal contract has a feature that is common to all these models: along the optimal contract the agent is saving constrained (Rogerson (1985)). Therefore, people have an incentive to deviate from the optimal contract instruction if they can save some of the current consumption assigned to them by the intermediary. This can potentially upset the optimal contract. In certain environments, especially in a hidden income environment, this assumption is critical to achieve the efficiency of the optimal contract over autarky with saving, allowing the optimal contract to provide consumption smoothing across the states as well as across time (Cole and Kocherlakota (1998) and Allen (1985)). We show that in our model there will be some efficiency loss due to the hidden saving since the optimal contract cannot command saving constrained consumption any more. But it is also shown that the efficiency is not completely lost. The observability of income distinguishes our model from the models of no-insurance results by giving extra information to the intermediary.

We also execute a simulation of the model and compare the implications of the optimal contract with the current social security systems. The studies by Gruber and Wise (1997) and Borsch-Supan and Schnabel (1997) used the concept of "social security accrual" to measure the incentive for early retirement. Negative (positive) values of social security accrual tend to provide incentives (disincentives) for early retirement. They found that most current social security systems have negative values of social security accrual; therefore, they tend to encourage early retirement. Our simulation shows that social security accrual is likely to be positive at the optimal contract. This result suggests that gains can be made by changing to the optimal plan from the current systems.

We model the optimal contract problem as a dynamic programming problem. The formulation of the economy's efficiency problem is a modification of the Atkeson and Lucas model (1995). Atkeson and Lucas studied optimal unemployment insurance in an economy in which workers face idiosyncratic employment risk every period. In our model the agent faces permanent productivity shocks rather than period-by-period shocks. The permanent nature of shocks distinguishes this model from other repeated principal agent models, except that of Hopenhayn and Nicolini (1997), in that our model provides a framework to analyze dynamic insurance against permanent shocks. Hopenhayn and Nicolini (1997) dealt with permanent employment shocks in a unemployment insurance model. However, the permanent nature of the shock is for convenience rather than a genuine permanent shock. In their model, the source of private information is hidden action, and they use the first order approach in dealing with the incentive compatible constraint. In our model the source of private information is hidden information, so we use screening in dealing with the incentive compatible constraint.

Our model is also distinct from macroeconomic approaches that analyze the social security system as an exogenously given fiscal policy in a complete information economy rather than as an endogenous policy (Feldstein (1985) and Imrohoroglu *et al.* (1995)). The issues are mainly the policy implications of social security systems (especially pay-as-you-go systems) on the savings of the private sector and thus on the growth of the economy in various environments.

We apply the model of dynamic insurance with permanent shock to social security design. By focusing on the insurance aspect of the social security system, the problem can be posed and the solution characterized in such a way as to deal with the incentive problem of early retirement. Our approach is to find the optimal mechanism in a given environment rather than to find a best policy under a given particular mechanism. The social security system emerges as an optimal mechanism, not as a given fiscal policy, and variables such as the retirement age are not given exogenously but determined endogenously. The properties of the optimal social security in our model exactly confirm Diamond and Mirrlees (1978) which also considered potential moral hazard problem in social security.[1] They use a continuous time model without discounting in an open system. Our model is a discrete time model using dynamic programming with discounting. And our economy is a closed system so that we can discuss the feasibility of the program in a general equilibrium.

The paper proceeds as follows. In section 2 we describe the model and pose the problem, and in section 3 the problem is formulated recursively. In section 4 we state the main theoretical results on the properties of the optimal allocation. The budget balancing and hidden saving issues are analyzed in sections 5 and 6, respectively. In section 7, we simulate the model and compare the results with current systems. Section 8 is the conclusion.

## 2. Model

This section describes the theoretical model that captures the insurance aspect of social security and potentially generates the moral hazard problem discussed in the previous section. We model these aspects as follows:

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[1] Our work is done independently without knowing their results.

The economy consists of overlapping generations of identical agents and an intermediary. Each generation of agents consists of a continuum of agents and is of equal size (unity) at birth. Each agent is born with a job and the ability to work. He can work $l \in [0,1]$ unit of time at the job and $l$ hour of work produces $ly$ unit of perishable good. However, he faces the risk of permanent loss of ability to work at some point in his life. At every period with probability $1-\pi$ the agent permanently loses his ability to work and thus is forced to retire. With probability $\pi$ the agent preserves his ability to work intact. The fact that the agent is able to work is not observable. Thus, an agent who is able to work can continue to work or pretend not to be able to work if it is in his interest. At the end of each period, agent is assumed to die with probability $(1-\delta)$, independent of age. Thus, the size of the t-aged generation equals $\delta^t$ and the size of the work force is $\pi^{t+1}\delta^t$. It is assumed at the first calendar date that there exists the steady state number of agents $1/(1-\delta)$ and the steady state work force $\pi/(1-\delta\pi)$.

An agent who consumes resource $c$ and works $l$ hour within the current period obtains flow utility $U(c) - lv$, where $U : I\!R_+ \to D \subset I\!R$, $U(0) = 0$, and $v > 0$, the disutility of work, is a fixed parameter. Let $C(u)$, $C : D \to I\!R_+$ be the inverse of the flow utility function $U(c)$. We assume that $C$ is continuously differentiable, strictly increasing, and strictly convex with $\inf_{u \in D} C'(u) = 0$.

At the beginning of his life, each agent enters into a contract with the intermediary. The agent sends an unverifiable message to the intermediary regarding his ability to work. At each date the intermediary assigns the agent some current level of consumption $C(x_t)$ (some current flow utility from consumption $x_t$) and assigns some hours of work $l_t$ to the agent based on his history of message and the initial entitlement. We assume that the agent is precluded from lending and borrowing so that the intermediary can directly control the agent's consumption. Let $w_0 \in D$ be the initial entitlement and $h^t = (h_0, \ldots, h_t)$ be the reported history of ability to work, where $h_i \in \{0,1\}$. We use 0 to indicate the loss of ability to work. We assume that once the agent reports he is not able to work, the agent will never be allowed to go back to work. Thus if $h_i = 0$, then $h_j = 0$ for all $j$, $i \le j \le t$.

A contract is a sequence of functions

$$\sigma = \{x_t(w_0, h^t), l_t(w_0, h^t)\}_{t=0}^{\infty},$$

where $x_t$ maps the agent's initial entitlements $w_0$ and tenure $h^t$ into levels of current utility in $D$, while $l_t$ maps these same variables into the interval $[0, 1]$.

Given a contract $\sigma$, an agent chooses a reporting strategy (or a retirement strategy) $h = \{h_t(\theta^t)\}_{t=0}^{\infty}$, where $\theta^t = (\theta_0, \ldots, \theta_t)$, $\theta_t \in \{0, 1\}$ for all $t \geq 0$, which denotes the agent's true ability to work. Thus, if the agent is not able to work at $t$, $\theta_t = 0$, then he is not able to work then on either, $\theta_\tau = 0$ for all $\tau \geq t$. The agent's initial discounted expected utility can be written as a function of $w_0, \sigma$ and $h$:

$$U(w_0, \sigma, h) = E \sum_{t=0}^{\infty} \beta^t \delta^t [x_t(w_0, h_t(\theta^t)) - l_t(w_0, h^t(\theta^t))v],$$

where $\beta$ is the discount rate of the agent.

The intermediary is not free to offer any contract. We impose two conditions on contracts that the intermediary can offer. Let $h^* = \{h_t^*(\theta^t)\}_{t=0}^{\infty}$ denote the truthful reporting strategy, where $h_t^*(\theta^t) = \theta_t$ for all $t \geq 0$ and $\theta^t$. The first requires that $\sigma$ delivers $w_0$ to those entitled to $w_0$:

$$(PK) \qquad w_0 = U(w_0, \sigma, h^*),$$

for all $w_0 \in D$. Since the messages are unverifiable, the intermediary must act on them in such a way that the agent is provided with incentives to report truthfully. Thus secondly, contracts must be incentive compatible:

$$(IC) \qquad U(w_0, \sigma, h^*) \geq U(w_0, \sigma, h),$$

for all $w_0 \in D$ and all reporting strategies $h$.

A contract $\sigma = \{x_t(w_0, h^t), l_t(w_0, h^t)\}_{t=0}^{\infty}$ that delivers $w_0$ to the agent costs the intermediary

$$E \sum_{t \geq 0} \beta^t \delta^t [C(x_t(w_0, \theta^t)) - l_t(w_0, \theta^t)y].$$

We assume for simplicity that the discount rate of the intermediary is $\beta$, the same as the agent's.[2] For a given promised level of utility, the objective of the intermediary is to minimize the costs (or maximize the profits) of the contract subject to (PK) and (IC). We define this contract as the optimal contract.[3]

**Definition :** *A contract $\sigma$ is an optimal contract to achieve $w_0$ if $\sigma$ minimizes*

$$E\sum_{t\geq 0}\beta^t\delta^t[C(x_t(w_0,\theta^t))-l_t(w_0,\theta^t)y]$$

*subject to (IC) and (PK).*

## 3. A Recursive Formulation of the Problem

Both this section and section 4 are devoted to characterizing the optimal contract by applying a dynamic programming approach. In this section, we establish a recursive formulation of the original intermediary's problem by defining and analyzing a Bellman equation. The Bellman equation will be simplified by showing that the incentive constraint holds with equality at the solution. The usual argument relies on convexity of the value function. The slack in the incentive constraint is not consistent with the convex value function since the value could have been improved by narrowing the gap. However, in our model, because of the permanent nature of shocks, the decompositions of the total costs (represented by the value function) into the present costs and the future costs are not symmetric in the events with and without the shock. Thus, the usual convexity argument is not applicable. We will find the differentiability and strict convexity of the value function first directly without using the incentive constraint holding with equality (Lemma 3.4 and Lemma 3.5) and then show the equality by these properties (Proposition 3.5).

The original sequential problem of the intermediary is given by

[2] We can take the discount rate of the intermediary arbitrary. The results are basically the same with more complication.

[3] The intermediary can be interpreted in many ways. It can be literally a profit maximizing entity. It can also be the coalition of all agents or the benevolent social planner whose goal is to maximize the agent's welfare subject to physical and informational feasibility. The cost minimizing goal can be justified since it is a dual problem of the utility maximization of the agent subject to (IC) and feasibility. This method was initiated by Green (1987).

$$(SP) \qquad \inf_{\sigma} \quad E\sum_{t\geq 0}[\beta^t\delta^t(C(x_t) - l_t y)]$$

$$s.t.$$

$$w_0 = E\sum_{t\geq 0}\beta^t\delta^t[x_t - l_t v],$$

$$U(w_0, \sigma, h^*) \geq U(w_0, \sigma, h).$$

If the agent reports that he is not able to work, then he will be retired forever by the assumption in the previous section. Therefore, upon retirement it is optimal to the intermediary to give a constant level of consumption to the agent afterward. Thus, if the entitlement to the retired agent from today on is $w^r$, then $w^r$ will be achieved by a constant level of consumption $c^r$ every period. Since $w^r = U(c^r)/(1-\beta\delta)$, and $U^{-1} = C$, this costs to the intermediary $C(w^r(1-\beta\delta))$, each period. It simplifies the original problem, and then, accordingly, the recursive formulation of the original problem will be given by

$$(FE) \qquad V(w_0) = \inf_{u^e, l, w^e, w^r} \pi[C(u^e) - ly + \beta\delta V(w^e)] + (1-\pi)\frac{C(w^r(1-\beta\delta))}{(1-\beta\delta)}$$

$$s.t.$$

$$w_0 = \pi[u^e - lv + \beta\delta w^e] + (1-\pi)w^r$$

$$u^e - lv + \beta\delta w^e \geq w^r.$$

The costs to deliver $w_0$, $V(w_0)$ consist of the costs in two events. When the agent is able to work, the intermediary's costs can be decomposed by the current period transfers and the future costs, represented by the first term of the Bellman equation. But when the agent is not able to work, the cost of the intermediary is just the discounted sum of the constant level of consumption to the agent, represented by the second term. Note that the decompositions are not symmetric with the shocks and without the shocks. Thus, it is not obvious how the convexity of $V$ would be helpful to eliminate slack in the incentive compatible constraint at the solution.

In defining our Bellman equation, our approach will be to set an upper bound $\bar{w}$ on entitlements and to consider cost functions on the bounded set $\bar{D} = [0, \bar{w}]$. We formulate and analyze the Bellman equation on $\mathcal{C}(\bar{D})$, where $\mathcal{C}(\bar{D})$ is the space of bounded, continuous functions on $\bar{D}$, and obtain the corresponding optimal policy functions. Then we will show that if the chosen bound $\bar{w}$ is large enough, it will not be binding practically for any initial entitlement $w_0 \in \bar{D}$ so that our optimal policy functions are also cost minimizing for the original, unbounded problem for any such $w_0$.

The Bellman equation is specified as follows. Define the operator $T$ on $\mathcal{C}(\bar{D})$ by

$$(BE) \qquad (TV)(w) = \inf_{u^e, l, w^e, w^r} \pi[C(u^e) - ly + \beta\delta V(w^e)] + (1-\pi)\frac{C(w^r(1-\beta\delta))}{(1-\beta\delta)}$$

$$s.t.$$

$$w = \pi[u^e - lv + \beta\delta w^e] + (1-\pi)w^r$$

$$u^e - lv + \beta\delta w^e \geq w^r.$$

**Lemma 3.1:** *The operator $T$ has a unique fixed point $V^*$ in $\mathcal{C}(\bar{D})$ and for all $V \in \mathcal{C}(\bar{D})$, $\lim_{n\to\infty} T^n V = V^*$. The function $V^*$ is increasing and convex. For all $w_0 \in D$, the infimum of the right-hand side of (BE) is attained.*

**Proof:**

Applying $T$ involves minimizing a continuous function over a compact set. Theorems 4.6-4.8 in Stokey and Lucas (1989) apply to get the existence, uniqueness, and convergence. Let $\mathcal{C}'[0, \bar{w}]$ be a space of increasing, convex, continuous function over $[0, \bar{w}]$. Since the cost function $C$ is strictly increasing and strictly convex, $T(\mathcal{C}'[0, \bar{w}]) \subset \mathcal{C}'[0, \bar{w}]$. Since $\mathcal{C}'[0, \bar{w}]$ is complete, $V^* \in \mathcal{C}'[0, \bar{w}]$. ∎

We can obtain further properties about the value function $V^*$ and about minimizing policies $u^e, l, w^e, w^r$ by studying the first order conditions of $V^*$. To this end, we need establish the differentiability of $V^*$.

For any strictly increasing, strictly convex, and differentiable $V \in \mathcal{C}(\bar{D})$, the first order conditions that characterize the minimum choice of $u^e, l, w^e, w^r$ include

$$
\begin{aligned}
\lambda\pi v + \mu v &\geq \pi y, \quad \text{if} \quad l = 0 \\
\lambda\pi v + \mu v &= \pi y, \quad \text{if} \quad l \in (0,1) \qquad (1)\\
\lambda\pi v + \mu v &\leq \pi y, \quad \text{if} \quad l = 1.
\end{aligned}
$$

$$\pi C'(u^e) - \lambda\pi - \mu \geq 0, \tag{2}$$

$$\pi V'(w^e) - \lambda\pi - \mu \geq 0, \tag{3}$$

$$(1-\pi)C'(w^r(1-\beta\delta)) - \lambda(1-\pi) + \mu \geq 0, \tag{4}$$

where (2), (3), and (4) hold with equalities if $u^e, w^e$, and $w^r$ are positive and where $\lambda$ and $\mu$ are Lagrangian multipliers to (PK) and (IC), respectively. The constraints of (IC) and (PK) are

$$w_0 = \pi[u^e - lv + \beta\delta w^e] + (1-\pi)w^r \tag{$PK$}$$

$$u^e - lv + \beta\delta w^e \geq w^r. \tag{$IC$}$$

The following lemma shows that the (BE) operator $T$ preserves the differentiability. And it will be useful to establish the differentiability of $V^*$.

**Lemma 3.2:** *Let $u^e, l, w^e$, and $w^r$ be the solution to (BE) with $V$. For any strictly increasing, strictly convex, and differentiable $V \in \mathcal{C}(\bar{D})$, the followings are satisfied:*

1. *$TV$ is strictly increasing and strictly convex.*
2. *$u^e > 0$, for any $w$.*
3. *$w^r > 0$, for any $w > 0$.*
4. *$TV$ is differentiable and $(TV)'(w) = \pi C'(u^e) + (1-\pi)C'(w^r(1-\beta\delta))$.*

**Proof:**

1. It follows from the assumptions that $C$ and $V$ are all strictly increasing and strictly convex.
2. Suppose $u^e = 0$, then $C'(u^e) = 0$. By (2), $\lambda\pi + \mu \leq 0$. Then (1) implies $l = 1$. By (IC), $0 \leq w^r \leq -v + \beta\delta w^e$, and thus $w^e > 0$. Since $V$ is strictly increasing and strictly convex, $V'(w^e) > 0$ for any $w^e > 0$. However, by (3), $0 < V^*(w^e) = \lambda\pi + \mu$. It is contradictory to $C'(u^e) = 0$ and thus to $\lambda\pi + \mu \leq 0$.

3. Consider the cases with $\mu = 0$ and $\mu > 0$. Suppose $\mu = 0$, then by (2) and $u^e > 0$, $\lambda$ must be greater than 0. Then by (4) $w^r > 0$. Suppose the other, *ie*, $\mu > 0$. Then (IC) is binding, $u^e - lv + \beta\delta w^e = w^r$. Since $w = \pi[u^e - lv + \beta\delta w^e] + (1-\pi)w^r$ by (PK), $w^r = w > 0$.

4. By the Benveniste and Scheinkman theorem (1979), we need to show that for any $w_0 > 0$, there is a neighborhood $A$ of $w_0$ and a convex, differentiable function $W : A \to I\!R$ such that $W(w_0) = TV(w_0)$, $W(w) \geq TV(w)$ for any $w \in A$. Then $TV$ is differentiable and $TV'(w_0) = W'(w_0)$.

   Let $(u_0^e, l_0, w_0^e, w_0^r)$ be a solution to $TV(w_0)$. Define a function $W : A \to I\!R$ by

$$W(w) = \pi[C(u_0^e + w - w_0) - l_0 v + \beta\delta V(w_0^e)] + (1-\pi)\frac{C((w_0^r + w - w_0)(1-\beta\delta))}{(1-\beta\delta)}.$$

   $W$ is differentiable and convex since $C$ and $V$ are differentiable and convex. In addition, $W(w_0) = TV(w_0)$. If we take $\rho = (u^e, l, w^e, w^r)$ by

$$u^e = u_0^e + w - w_0, l = l_0, w^e = w_0^e, \text{ and } w^r = w_0^r + w - w_0,$$

   respectively, then $\rho$ satisfies the (IC) and (PK) constraints with entitlement $w$ as follows:

$$\begin{aligned}
&\pi[u^e - lv + \beta\delta w^e] + (1-\pi)w^r \\
&\quad = \pi[u_0^e + w - w_0 - l_0 v + \beta\delta w_0^e] + (1-\pi)(w_0^r + w - w_0) \\
&\quad = w - w_0 + \pi[u_0^e - l_0 + \beta\delta w_0^e] + (1-\pi)w_0^r \\
&\quad = w
\end{aligned}$$

   and

$$\begin{aligned}
u^e - lv + &\beta\delta w^e - w^r \\
&= (u_0^e + w - w_0 - l_0 v + \beta\delta w_0^e) - (w_0^r + w - w_0) \\
&= u_0^e - l_0 v + \beta\delta w_0^e - w^r \geq 0.
\end{aligned}$$

   Thus, by the definition of $TV$, $W(w) \geq TV(w)$. By the Benveniste and Scheinkman theorem $W'(w) = TV'(w)$ and,

$$\frac{d(TV)}{dw}(w) = \pi C'(u^e) + (1-\pi)C'(w^r(1-\beta\delta)).$$

Since $w^r = 0$ at $w = 0$, for $w = 0$, $C'(w^r(1-\beta\delta)) = 0$. In addition, since $C'(u^e) \leq V'(w^e)$, the relation holds with $TV'(0) = \pi C'(u^e(0))$. ∎

Since any $V \in \mathcal{C}(\bar{D})$ converges to $V^*$ uniformly under the operator $T$, the differentiation is preserved in the limit.

**Lemma 3.3:** *The fixed point $V^*$ of the operator $T$ is differentiable, and the policy functions $\rho = (u^e, l, w^e, w^r)$ are continuous.*

**Proof:**

Let's take a strictly increasing, strictly convex, and differentiable function $V^0 \in \mathcal{C}(\bar{D})$. Define $V^{n+1}(w) = (TV^n)(w)$ for all $w$ and $n \geq 0$. On each iteration $n$, we denote the optimal policy functions by $(u_n^e, l_n, w_n^e, w_n^r)$. By the theorem of maximum and the strict convexity of $C$ and $V$, $(u_n^e, l_n, w_n^e, w_n^r)$ is continuous. By theorem 3.8 of Stokey and Lucas (1989), the sequence $\{u_n^e, l_n, w_n^e, w_n^r\}$ defined above converges uniformly to $(u^e, l, w^e, w^r)$ and thus $\rho = (u^e, l, w^e, w^r)$ are continuous. Since by Lemma 3.2.1, $V^n$ is strictly increasing and strictly convex for any $n$, $\frac{d(TV^n)}{dw}(w) = \pi C'(u_n^e) + (1-\pi)C'(w_n^r(1-\beta\delta))$ by Lemma 3.2.4. Thus, it follows that the sequence of derivatives converges uniformly to $\pi C'(u^e) + (1-\pi)C'(w^r(1-\beta\delta))$. Since $\{V^n\}$ converges uniformly to $V^*$, $\pi C'(u^e) + (1-\pi)C'(w^r(1-\beta\delta))$ is the derivative of $V^*$ and, it is also continuous. ∎

Since $V^*$ is differentiable, we can use the first order conditions. They are the same as (1), (2), (3), (4), (PK), and (IC) with $V^*$. The next lemma shows the strict convexity and strict monotonicity are also preserved in the limit.

**Lemma 3.4:** *$V^*$ is strictly increasing and strictly convex.*

**Proof:**

1. Since $V^*$ is convex and nondecreasing, if it has an interval over which it is constant, then this interval must start at $w = 0$. So let $w' := \max\{w : V^*(w) = V^*(0)\}$ and let $(u^e, l, w^e, w^r)$ be policy functions of the entitlement $w'$. Now suppose $w' > 0$. By the assumption of differentiability and by the definition of $w'$, we know that $V^{*\prime}(w') = 0$. But by Lemma 3.2, for $w' > 0$, $u^e$ and $w^r$ must be positive and thus

$$V^{*\prime}(w') = \pi C'(u^e) + (1-\pi)C'(w^r(1-\beta\delta)) > 0.$$

This is a contradiction. Therefore, $w'$=0 and $V^*$ is strictly increasing.

2. Suppose that $V^*$ is not strictly convex. Then for some $w_1 < w_2$, $V^*$ is linear over $[w_1, w_2]$. Suppose that $w_2$ is the maximum of this property. Let $w_t = tw_1 + (1-t)w_2$ for some $t \in (0,1)$ and let $\rho_i = (w_i^e, w_i^r, u_i^e, l_i)$ be policy functions to the entitlement $w_i$, $i = 1, 2$. Since $V^*$ is linear over $[w_1, w_2]$,

$$V^*(w_t) = tV^*(w_1) + (1-t)V^*(w_2). \tag{6}$$

Also, define $\rho_t = (w_t^e, w_t^r, u_t^e, l_t)$ as the convex combinations of corresponding variables, $t\rho_1 + (1-t)\rho_2$. By the linearity of constraints, $\rho_t = (w_t^e, w_t^r, u_t^e, l_t)$ must satisfy (PK) and (IC). By the definition of $V^*$,

$$V^*(w_t) \le \pi[C(u_t^e) - l_t y + \beta\delta V^*(w_t^e)] + (1-\pi)\frac{C(w_t^r(1-\beta\delta))}{(1-\beta\delta).}$$

Since $\rho_t$ is a convex combination of $\rho_1$ and $\rho_2$ and since $C$ and $V^*$ are convex,

$$\begin{aligned}
&\pi[C(u_t^e) - l_t y + \beta\delta V^*(w_t^e)] + (1-\pi)\frac{C(w_t^r(1-\beta\delta))}{(1-\beta\delta)} \\
&\le t[\pi[C(u_1^e) - l_1 y + \beta\delta V^*(w_1^e)] + (1-\pi)\frac{C(w_1^r(1-\beta\delta))}{(1-\beta\delta)}] \\
&+(1-t)[\pi[C(u_2^e) - l_2 y + \beta\delta V^*(w_2^e)] + (1-\pi)\frac{C(w_2^r(1-\beta\delta))}{(1-\beta\delta)}] \\
&= tV^*(w_1) + (1-t)V^*(w_2),
\end{aligned}$$

where the last relation is by the definition of $\rho_i$'s. Since (6) is satisfied and $l_t = tl_1 + (1-t)l_2$, by strict convexity of $C$, necessarily $u_1^e = u_2^e$ and $w_1^r = w_2^r$ hold. Thus, it must be the case that

$$V^*(w_t^e) = tV^*(w_1^e) + (1-t)V^*(w_2^e).$$

Since $w_t^e = tw_1^e + (1-t)w_2^e$ and $V^*$ is convex, $V^{*\prime}(w_1^e) = V^{*\prime}(w_2^e)$. Since the Lagrangian multiplier $\mu$ is nonnegative, by (2) and (4), $C'(u_1^e) \ge C'(w_i^r(1-\beta\delta))$. Thus,

$$V^{*\prime}(w_i) = \pi C'(u_i^e) + (1-\pi)C'(w_i^r(1-\beta\delta)) \le C'(u_i^e) \le V^{*\prime}(w_i^e).$$

Therefore, $w_i^e \geq w_i$ by the convexity of $V^*$. Especially since $w_2$ is the maximum of such property, $w_2^e = w_2$. But (PK) implies

$$w_2 - w_1 = \pi\beta\delta(w_2^e - w_1^e) \geq \pi\beta\delta(w_2 - w_1).$$

This is a contradiction. Therefore, $V^*$ is strictly convex. ∎

**Proposition 3.5:** *(IC) must hold with equality where the minimum of the Bellman equation with $V^*$ is achieved.*

**Proof:**

Suppose that there is slack in (IC) at the optimal solution. Since $C$ is strictly convex, current period consumption in either event must be the same, *i.e.*, $u^e = w^r(1-\beta\delta)$.[4] Therefore, (IC) with slack implies $\beta\delta w^e > lv + \beta\delta w^r$. Thus, $w^e > w^r \geq 0$. However, since $u^e = w^r(1-\beta\delta)$ and $V^{*\prime}(w) = \pi C'(u^e) + (1-\pi)C'(w^r(1-\beta\delta))$, we must have $V^{*\prime}(w) = C'(u^e)$. And by (2), (3) and by $w^e > w^r \geq 0$, $C'(u^e) = V^{*\prime}(w^e)$. Since $V^*$ is strictly convex, $w = w^e$. With $u^e = w^r(1-\beta\delta)$ and (PK), $w = \pi[u^e - lv + \beta\delta w^e] + (1-\pi)w^r$, we get

$$\begin{aligned} w^e = w =& \pi w^r(1-\beta\delta) - \pi lv + \pi\beta\delta w^e + (1-\pi)w^r \\ =& w^r - \pi lv(1-\pi\beta\delta)^{-1}. \end{aligned}$$

It follows that $w^e \leq w^r$. This is a contradiction to an implication of non-binding (IC), $w^e > w^r$. ∎

In view of the proposition, two constraints can be replaced by the equalities:

$$w = \pi[u^e - lv + \beta\delta w^e] + (1-\pi)w^r, \quad \text{and} \quad u^e - lv + \beta\delta w^e = w^r.$$

Thus, $w = w^r$. Our problem is simplified to

$$V^*(w) = \min_{l,w^e} \quad \pi[C(w + lv - \beta\delta w^e) - ly + \beta\delta V^*(w^e)] + (1-\pi)\frac{C(w(1-\beta\delta))}{1-\beta\delta}.$$

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[4] If they are not the same, we can find another policy that costs less by decreasing the difference between the current consumption levels of the two event. Since (IC) holds with inequality, we can take such policy without violating (IC) or (PK). Since $C$ is strictly convex, this policy will cost less. This is a contradiction to the assumption that the minimum is achieved.

The first term on the right-hand side is the costs to agents who are able to work, and the second term is the costs to the retired agents.

The first order conditions that characterize the minimum choice of $l, w^e$ include

$$\begin{aligned} C'(w + lv - \beta\delta w^e)v - y &\geq 0, \quad \text{if} \quad l = 0 \\ C'(w + lv - \beta\delta w^e)v - y &= 0, \quad \text{if} \quad l \in (0,1) \\ C'(w + lv - \beta\delta w^e)v - y &\leq 0, \quad \text{if} \quad l = 1. \end{aligned} \tag{7}$$

The intertemporal first order condition is given by

$$C'(w + lv - \beta\delta w^e) \leq V^{*\prime}(w^e), \tag{8}$$

where (2) holds with equality when $w^e > 0$. In addition, by Lemma 3.3 on differentiability of $V^*$,

$$\frac{d(V^*)(w)}{dw} = \pi C'(w + lv - \beta\delta w^e) + (1-\pi)C'(w(1-\beta\delta)). \tag{9}$$

## 4. Characterization of the Policy Functions

In this section we present the main results of the paper. We establish that the solution to the Bellman equation is equivalent to the solution to the original intermediary's problem (optimal contract) (Proposition 4.4). By characterizing the policy functions of the Bellman equation, therefore, we can derive properties of the optimal contract (Proposition 4.6), which hold meaningful policy implications for the social security system.

The permanent nature of the shock gives the monotone properties of the policy functions with tenure.

**Lemma 4.1:** *Let $(w^e, w^r, l, u^e)$ be policy functions. Then $w^e, w^r$, and $u^e$ satisfy the followings:*

1. *For any $w$, $w^e > 0$.*
2. *$w^e \geq w = w^r$.*
3. *$u^e \geq u^r = w^r(1 - \beta\delta)$.*

4. $w^e$ *and* $u^e$ *are increasing functions of* $w$.

**Proof:**

1. Suppose $w^e = 0$. Then by (9), $V^{*\prime}(0) = \pi C'(w + lv - \beta\delta w^e)$. But by Lemma 3.2 and by (8),

$$0 < C'(w + lv - \beta\delta w^e) \leq V^{*\prime}(0) = \pi C'(w + lv - \beta\delta w^e).$$

   This is a contradiction.

2. By 1. and (8), $C'(w + lv - \beta\delta w^e) = V^{*\prime}(w^e)$. Suppose $w^e < w$. Then since $V^*$ is strictly increasing and strictly convex, $V^{*\prime}(w) > V^{*\prime}(w^e)$. Since $\frac{d(V^*)(w)}{dw} = \pi C'(w + lv - \beta\delta w^e) + (1 - \pi)C'(w(1 - \beta\delta))$,

$$C'(w + lv - \beta\delta w^e) < \pi C'(w + lv - \beta\delta w^e) + (1 - \pi)C'(w(1 - \beta\delta)).$$

   Therefore, $C'(w + lv - \beta\delta w^e) < C'(w(1 - \beta\delta))$. Since $C$ is strictly increasing and strictly convex, $w + lv - \beta\delta w^e < w(1 - \beta\delta)$. Or $0 < lv < \beta\delta(w^e - w)$. This is a contradiction.

3. By 2, $V^{*\prime}(w^e) \geq V^{*\prime}(w)$. Since $V^{*\prime}(w^e) = \pi C'(w + lv - \beta\delta w^e) + (1 - \pi)C'(w(1 - \beta\delta)$ and $C'(w + lv - \beta\delta w^e) = V^{*\prime}(w^e)$, it must be the case that $\pi C'(w + lv - \beta\delta w^e) + (1 - \pi)C'(w(1 - \beta\delta) \leq C'(w + lv - \beta\delta w^e)$. Therefore, $C'(w + lv - \beta\delta w^e) \geq C'(w(1 - \beta\delta))$ and $u^e \geq w^r(1 - \beta\delta)$.

4. This follows from $C'(w + lv - \beta\delta w^e) = V^{*\prime}(w^e)$. Suppose that there is an interval of entitlements over which $w^e$ is strictly decreasing. Since $V^*$ and $C$ are strictly convex, the argument of $C'$, $w + lv - \beta\delta w^e$, will decrease strictly over the interval. However, as $C'(w + lv - \beta\delta w^e)$ decreases, $l$ is nondecreasing by (7). Then, $w + lv - \beta\delta w^e$ should be nondecreasing. This is a contradiction. ∎

The next lemma shows that there is a threshold entitlement beyond which agents are assigned retirement even if they are able to work. As the entitlement increases, it is more costly to induce agents to work due to the concavity of the utility function and constant disutility. Beyond a certain point, incentive costs outweigh the productivity. Therefore, retiring the work force saves more costs than giving them incentive to work. This discussion leads to the following lemma:

**Lemma 4.2:** *Let $w^0$ be the solution to $C'(w^0(1-\beta\delta))v = y$. If $w^e(w) = w$, then $w \geq w^0$.*

**Proof:**

$w^e(w) = w$ implies $C'(w + lv - \beta\delta w) = V^{*\prime}(w)$. Since $V^{*\prime}(w) = \pi C'(w + lv - \beta\delta w^e) + (1-\pi)C'(w(1-\beta\delta))$ and $C$ is strictly convex, $w + lv - \beta\delta w = w(1-\beta\delta)$. Thus, $l = 0$. If $l = 0$, then $C'(w(1-\beta\delta))v \geq y$ by (7). Therefore, since $C'$ is increasing, $w \geq w^0$. ∎

Let $w_1$ be the maximum entitlement with which full time work ($l = 1$) is assigned. Then $w_1$ is the solution of $C'(w_1 + v - \beta\delta w^e(w_1))v = y$. Since $w^e, l$, and $C$ are continuous, the solution exists by the intermediate value theorem. Since $u^e = w + lv - \beta\delta w^e$ is an increasing function of $w$, for any $w \in [0, w_1]$, $l(w) = 1$.

**Lemma 4.3:** *For any $w \in [w_1, w^0]$, $w^e(w) = w^0$.*

**Proof:**

For any $w \in [w_1, w^0]$, since $u^e$ is an increasing function of $w$,

$$w_1 + v - \beta\delta w^e(w_1) \leq w + lv - \beta\delta w^e(w) \leq w^0 - \beta\delta w^0.$$

Since $C'(w_1 + v - \beta\delta w^e(w_1)) = C'(w^0(1-\beta\delta)) = y/v$, we have $C'(w + lv - \beta\delta w^e(w)) = y/v$. Since $C'(w + lv - \beta\delta w^e(w)) = V^{*\prime}(w^e(w))$, for all $w$,

$$V^{*\prime}(w^e(w)) = C'(w^0(1-\beta\delta)) = V^{*\prime}(w^0).$$

Since $V^*$ is strictly increasing, therefore, $w^e(w) = w^0$. ∎

We now relate the Bellman equation (FE) to the original problem (SE). As Lemma 4.2 shows, if $w \geq w^0$, then

$$V^*(w) = \frac{C(w(1-\beta\delta))}{(1-\beta\delta)}. \tag{10}$$

Since $V^*$ is the value function restricted to the compact domain $[0, \bar{w}]$, it is not exactly the solution to the original Bellman equation. However, by (10), as long as $\bar{w} > w^0$, there is virtually no difference between $V^*$ and the fixed point of the original Bellman equation. Now we define the fixed point of the (FE) $\bar{V}$ by

$$\bar{V}(w) = \begin{cases} V^*(w), & \text{if } w \leq w^0; \\ \frac{C(w(1-\beta\delta))}{(1-\beta\delta)}, & \text{if } w > w^0. \end{cases}$$

The following proposition establishes that the solution to the Bellman equation is equivalent to the solution to the original intermediary's problem. Thus, the optimal contract can be characterized through the analysis of policy functions of the value function. Define the contract rule $\sigma$ as a function that maps each initial entitlement to a contract. First note that policy functions $\rho = (u^e(w), l(w), w^e(w))$ can be used to generate a contract rule $\sigma$ in the following manner: Let $x_0(w_0, 1) = u^e(w_0),\ x_0(w_0, 0) = w_0(1 - \beta\delta),\ l_0(w_0) = l(w_0)$ for any $w_0 \in D$. Define $w_1(w_0, 1) = w^e(w_0)$ and $w_1(w_0, 0) = w_0$ for any $w_0 \in D$. Iterating on this procedure to complete the definition, set

$$x_t(w_0, h^{t-1}, 1) = u^e(w_t(w_0, h^{t-1})),\ x_t(w_0, h^{t-1}, 0) = w_t(w_0, h^{t-1})(1 - \beta\delta),$$

$$l_t(w_0, h^t) = h_t l(w_t(w_0, h^{t-1})),$$

$$w_{t+1}(w_0, h^{t-1}, 1) = w^e(w_t(w_0, h^{t-1}, 1)),\ w_{t+1}(w_0, h^{t-1}, 0) = w^e(w_{t-1}(w_0, h^{t-1})),$$

where $h_{t-1} = 1$ and for $h^{t-1}$ with $h_{t-1} = 0$

$$x_t(w_0, h^t) = w_t(w_0, h^{t-1})(1 - \beta\delta),\ l_t(w_0, h^t) = 0,$$

$$w_{t+1}(w_0, h^t) = w^e(w_{t-1}(w_0, h^{t-1})).$$

We establish the equivalence of the solutions by showing that the constraints of the problem (SE) and problem (FE) are equivalent.

**Proposition 4.4:** *Let $C'(\bar{w}(1 - \beta\delta))v > y$. In addition, $(u^e, l, w^e)$ be a policy function for $\bar{V}$. If $\sigma^*$ is a contract rule generated by $(u^e, l, w^e)$, then for any $w_0$ $\sigma^*(w_0)$ minimizes costs subject to (IC) and (PK) of the original problem (SE).*

**Proof:**

By Theorems 4.3-4.5 of Stokey and Lucas (1989), for any $w_0 \in D$, $\bar{V}(w_0)$ equals the infimum of (SE) over the set of contracts $\sigma(w_0)$ that satisfy

$$\begin{aligned} U_t(w_0, \sigma, h^*, \theta^{t-1}) =& \pi[x_t(w_0, (\theta^{t-1}, 1)) - l_t(w_0, (\theta^{t-1}, 1))v + \beta\delta U_{t+1}(w_0, \sigma, h^*, (\theta^{t-1}, 1)] \\ &+ (1 - \pi)[x_t(w_0, (\theta^{t-1}, 0)) + \beta\delta U_{t+1}(w_0, \sigma, h^*, (\theta^{t-1}, 0)] \end{aligned} \tag{11}$$

$$\begin{aligned} &x_t(w_0, (\theta^{t-1}, 1)) - l_t(w_0, (\theta^{t-1}, 1))v + \beta\delta U_{t+1}(w_0, \sigma, h^*, (\theta^{t-1}, 1) \\ &\geq x_t(w_0, (\theta^{t-1}, 0)) + \beta\delta U_{t+1}(w_0, \sigma, h^*, (\theta^{t-1}, 0) \end{aligned} \tag{12}$$

for all $t, \theta^{t-1}$ with $\theta_{t-1} = 1$, and

$$U_t(w_0, \sigma, h^*, \theta^t) = U_{t+s}(w_0, \sigma, h^*, \theta^{t+s}) \tag{13}$$

for all $t, s \geq 0$, and $\theta^t$ with $\theta_t = 0$. Moreover, the contract $\sigma^*(w_0)$ uniquely attains the minimum of the set satisfying (11)-(13). Obviously, any contract satisfying (IC) and (PK) of (SE) satisfies (11)-(13). Therefore, it remains to show that any contract satisfying (11)-(13) satisfies (IC) and (PK) of (SE).

Let $\sigma(w_0)$ be a contract satisfying (11)-(13). Clearly by (11), (PK) is satisfied. Since the (IC) constraint doesn't matter at any history $h^t$ with $h_t = 0$, we need to consider only histories whose components are all 1's. Suppose that $\sigma(w_0)$ is not incentive compatible, then there is another strategy $h$ that is not a truth-telling strategy such that

$$U(w_0, \sigma(w_0), h^*) < U(w_0, \sigma(w_0), h).$$

Since $h$ is not a truth-telling strategy, there is a period $t$ at which the agent will report the loss of productivity falsely, $h_t(\theta^t) = 0$ and $\theta_t = 1$. But this is contradictory to (12) at $t$. Therefore, any contract satisfying (11)-(13) satisfies (IC). Thus, $\sigma^*(w_0)$ uniquely attains the minimum of (SE) as well. ∎

The following proposition shows that the autarkic contract is dominated by an optimal contract with 0 net transfer. Thus, the welfare of agents can be improved upon autarky without any net subsidy.

**Proposition 4.5:** *There is an optimal contract with 0 transfer that improves upon autarky.*

**Proof:**

Let $w_{aut} = \sum \pi^t \beta^{t-1} \delta^{t-1} (U(y) - v)$. Then $w_{aut}$ is an expected discounted utility with autarky, null insurance. We can find an allocation that improves upon autarky. Consider an allocation in which $\frac{\pi}{1-\pi}$ is taken from agent's $t$ period income and $\pi\tau$ is transferred to the state of productivity loss of the period. If we take the transfer small enough, (IC) is intact. Since $U$ is concave, the expected discounted utility $w'$ of the alternative allocation is greater than $w_{aut}$. Since $w'$ needs 0 net transfer, by the definition of $\bar{V}$, $\bar{V}(w') \leq 0$.

Since $\bar{V}$ is continuous, $\bar{V}(w') \leq 0$ and obviously $\bar{V}(\bar{w}) > 0$, by the intermediate value function theorem there is $w^* \geq w' > w_{aut}$ such that $\bar{V}(w^*) = 0$. ∎

Now we present our main results. The following proposition shows that policy functions, the future entitlement $w^e$, and the current utility $u^e$, increase strictly with tenure. The future entitlement is exactly the same as the entitlement of the retired next period. The current utility $u^e$ is achieved by the current consumption, and the level of current consumption is the same as the income after tax. In terms of social security, this implies that the annuity of the agent will strictly increase with tenure and that the social security tax on the agent must decrease with tenure. These are the implications from (IC) and the concavity of utility function. To induce people to work, the disutility from the work must be compensated by increasing consumption. Since the utility function is concave it is optimal to spread the consumption over the agent's lifetime. The current consumption increase will be by a decrease in the social security tax, and the increase in future entitlement will be given in the form of an annuity increase. Another interesting feature of the optimal contract is that after a finite period of work, it reaches the threshold $w^0$. Beyond the threshold entitlement $w_0$ it is more costly to induce the agent to work rather than to have him retired, as we saw in Lemma 4.2. Thus it is optimal that the agent retire eventually within a finite period of tenure even if he is able to work. This amounts to putting a mandatory retirement age clause in the contract.

Let $w^e_n(w) = w^e(w^e_{n-1}(w))$ denote the future entitlement after $n$ periods if the initial entitlement is $w$.

**Proposition 4.6:**

1. *$w^e$ increases strictly over $[0, w_1]$ and there exists $k$ such that $w^e(w) \geq w + k$ for all $w \in [0, w_1]$.*
2. *$u^e$ increases strictly, or $y - c^e$ decreases strictly.*
3. *There exists $N_w$ such that $w^e_{N_w}(w) \geq w^0$ for any $w$.*

**Proof:**

1. If $w^e$ is not strictly increasing, there is an interval $[w', w''] \in [0, w_1]$ such that $w^e(w') = w^e(w'')$. Since $C'(u^e) = \bar{V}(w^e)$, $u^e(w') = u^e(w'')$. Also $l(w') = l(w'')$. But this

violates (PK) since $w' < w''$.

For any $w \in [0, w_1]$, $l(w) = 1$. If $w = w^e$, then $u^e = w + v - \beta\delta w < w(1 - \beta\delta)$, which is a contradiction to Lemma 4.1.3. Thus, $w^e > w$. Since $w^e$ is continuous and $[0, w_1]$ is compact, there exists $k > 0$ such that $w^e(w) > w + k$ for all $w \in [0, w_1]$.

2. This is immediate by 1. and (7) of the first order condition. By (7), $C'(u^e) = \bar{V}(w^e)$. Since $w^e$ is strictly increasing, as $w$ increases, the left-hand side increases. Since $C'$ is strictly increasing, the conclusion follows.

3. By 1, $w^e_N \geq w + Nk$ if $w \in [0, w_1]$. Therefore, there exists the minimum $N_w - 1$ such that $w^e_{N_w - 1}(w) > w_1$. Then $w^e_{N_w} \geq w^0$. ∎

By the first, the entitlement increases strictly with tenure. Since $w^r = w$, the entitlement to the retired $w^r$ must increase with tenure as well. $C(u^e)$ increases with $u^e$. Since $y - C(u^e)$ can be interpreted as social security tax, the second implies the tax decreases with tenure. The third shows that after the finite periods of work, the threshold $w_0$ will be reached and the agent is retired.

## 5. Existence of an Equilibrium in Steady State

The previous section completes the characterization of the value function and policy functions that solve the one intermediary and one agent insurance problem. However, the way that the intermediary saves the costs is to pool the risk of a continuum of agents by transferring resources among them, based on such individual contracts. Therefore, the intermediary must design individual contracts in a physically feasible way at the aggregate level as well. The net total transfers have to be balanced each period. In other words, total current period consumption entitled to agents by individual contracts must be equal to the total current period production.

This section shows the existence of a feasible social security system in a steady state. By social security system, we mean the collection of individual optimal contracts. The steady state distribution of entitlements depends on both the initial entitlement and the discount rate. We show that for any discount rate close to 1,[5] there exists an equilibrium initial entitlement under which the system is feasible. This is an equilibrium in the sense that given an initial entitlement as a price, the intermediary is maximizing its profit by providing an optimal contract and given the contract the agents are maximizing their discounted utility by choosing the best reporting strategy. The resulting allocation is feasible. For the feasibility analysis, we will consider all possible values of discount rate $\beta$ to the intermediary and agents to carry out comparative statics.

Let $n^*(w,\beta)$ be the periods of work before the agent reaches the mandatory retirement age at the optimal contract if the agent starts with the initial entitlement $w$ and the discount rate is $\beta$. The higher the discount rate $\beta$ is, we find that the longer the agent has to work before he is forced to retire.

**Lemma 5.1:** *If $\beta_1 < \beta_2$, then $n^*(w,\beta_1) \leq n^*(w,\beta_2)$ for any $w$.*

**Proof:**

Let $(u_i^e, l_i, w_i^e)$ be the policy functions if the discount rate is $\beta_i$. Suppose that $u_2^e \geq u_1^e$. Then the first order conditions, (7) and (8), imply that $l_2 \leq l_1$ and $w_2^e \geq w_1^e$. But since $\beta_1 < \beta_2$,

$$w = u_2^e - l_2 + \beta_2 \delta w_2^e > u_1^e - l_1 + \beta_1 \delta w_1^e = w.$$

This is a contradiction. Therefore, if $\beta_1 < \beta_2$, then $u_2^e < u_1^e$. Thus, $w_2^e < w_1^e$ by first order condition (8). Also the strict monotonicity of $w^e$ function implies $w_n^e(w,\beta_1) > w_n^e(w,\beta_2)$ for any $n$.

On the other hand, by Lemma 4.2, since $C'(w^0(1-\beta\delta))v = y$, the critical value of the future entitlement $w^0$ at which the agent is assigned to retire increases in $\beta$. Since this holds for any $w$, $n^*(w,\beta_1) \leq n^*(w,\beta_2)$ for any $w$. ∎

The intermediary's problem is restated as

[5] "Close to 1" is not a limit concept here. It is rather the concept of neighborhood which does not have to be "close to 1".

$$V(w,\beta) = \min_{\sigma(w)} \quad E\sum_{t\geq 0}[\beta^t\delta^t(C(x_t) - l_t y)]$$

$$s.t.$$

$$w = E\sum_{t\geq 0}\beta^t\delta^t[x_t - l_t v],$$

$$U(\beta, w, \sigma, h^*) \geq U(\beta, w, \sigma, h),$$

where the function $U$ is denoted as a function of $\beta$ as well.

By the lemma, we know that $n^*(w,\beta)$ increases monotonically in $\beta$. Among the optimal contracts, the possible largest periods the agent has to work before he is forced to retire is therefore $n^*(0,1)$, that is, starting from 0 entitlement and at the lowest discount rate $\beta = 1$. This consideration with the property of "once-retired-forever-retired" effectively puts a restriction on constraints to the intermediary's problem. Since at the optimal contract the agent will have been retired after $n^*(0,1)$ and the level of consumption will be the same after this time, the set of meaningful history of tenure is finite with upper bound $n^*(0,1)$ periods. Therefore, without loss of generality we can restrict the constraint set to the set of contracts whose assignment $(x_t, l_t)$ after the period $n^*(0,1)$ is restricted to $(x_{n^*(0,1)}, l_{n^*(0,1)})$. Thus, the set of contracts $\Phi$ is in effect of finite dimension and compact:

$$\Phi = \{\sigma(\beta,w) : \beta \in [0,1], w \in [0,\bar{w}], \sigma(\beta,w) = \{x_t(w,h^t), l_t(w,h^t)\}_{t=0}^{n^*(0,1)}\}.$$

Let $\Psi$ be a incentive compatible (IC) and promise keeping (PK) contract correspondence $\Psi : [0,1] \times [0,\bar{w}] \to \Phi$ defined by

$$\Psi(\beta,w) = \{\sigma \in \Phi : w = U(\beta,w,\sigma,h^*), \text{ and } U(\beta,w,\sigma,h^*) \geq U(\beta,w,\sigma,h)\}.$$

By the compactness of $\Phi$ and continuity of the function $U$, we can show the continuity of the (IC) and (PK) contract correspondence.

**Lemma 5.2:** *For any $\beta$ and $w$, $\Psi(\beta,w)$ is a nonempty and compact subset of $\Phi$.*

**Proof:**

Since $U$ is continuous over $\Phi$, the set of contracts satisfying $w = U(\beta,w,\sigma,h^*)$ is a closed set. The set of contracts satisfying (IC) is

$$K(\beta,w) = \cap_h\{\sigma \in \Phi : U(\beta,w,\sigma,h^*) \geq U(\beta,w,\sigma,h)\}.$$

The set $K(\beta, w)$ is closed since each set is closed. The autarky arrangement with initial entitlement $w$ is an element of $K(\beta, w)$. Thus, $\Psi(\beta, w)$ is a nonempty closed subset of $\Phi$. ∎

**Lemma 5.3:** *The (IC) and (PK) contract correspondence is continuous.*

**Proof:**

1. The upper hemi continuity of $\Psi$ follows from the continuity of the function $U$. If $(\beta_n, w_n) \to (\beta, w), \sigma_n \in \Psi(\beta_n, w_n)$, and $\sigma_n \to \sigma$, then by the continuity of $U$, $\sigma$ satisfies (PK). If $\sigma$ does not satisfy (IC), then by the continuity of $U$, for sufficient large $n$, $\sigma_n$ will not satisfy (IC) either, which is a contradiction to $\sigma_n \in \Psi(\beta_n, w_n)$.
2. The lower hemi continuity of $\Psi$ also follows from the continuity of the function $U$ as well. Let $\sigma \in \Psi(\beta, w)$ and $(\beta_n, w_n) \to (\beta, w)$. We can take a sequence of contracts $\sigma_n$ in the relative interior of the set of (IC) constraint contracts under $(\beta, w)$ such that each $\sigma_n$ satisfies (PK) with $(\beta_n, w_n)$. Since the function $U$ is continuous, $\sigma_n$ satisfies (IC) constraint of $(\beta_n, w_n)$ for sufficiently large $n$. ∎

The cost function of the intermediary can be rewritten as

$$V(\beta, w) = \min_{\sigma \in \Psi(\beta, w)} \quad E \sum_{t \geq 0} [\beta^t \delta^t (C(x_t) - l_t y)].$$

The intermediary's problem has all the elements to apply the theorem of maximum. Let $\sigma(\beta, w) = \{x_t(w, h^t, \beta), l_t(w, h^t, \beta)\}_{t=0}^{n^*(0,1)} := (x(\beta, w), l(\beta, w))$ be a solution to the intermediary's problem.

**Lemma 5.4:** *$V$ is continuous over $[0, 1] \times [0, \bar{w}]$. And $\sigma(\beta, w) = (x(\beta, w), l(\beta, w))$ is continuous.*

**Proof:**

$\Psi$ is a continuous and compact valued correspondence. $E \sum_{t \geq 0} [\beta^t \delta^t (C(x_t) - l_t y)]$ is continuous over $\Phi$. By the maximum theorem, we know that $V$ is continuous over $[0, 1] \times [0, \bar{w}]$. And the solution $\sigma(\beta, w) = (x(\beta, w), l(\beta, w))$ is continuous since it is single valued. ∎

By the discussion of the previous section, we know that if everyone has the same initial entitlement, individuals with the same tenure history will follow the same monotone path of entitlements. Thus, the cross sectional distribution of the entitlement and work force in the steady state are the exact copy of the lifetime distribution of the entitlement and tenure of a newly born generation. If we know the initial entitlement of the newly born generation, we can calculate the optimal contract $\{(x_t(\beta, w, h^t), l_t(\beta, w, h^t)\}_{t\geq 0}$ to the newly born. This is in fact the cross sectional distribution of the optimal contract in a steady state. For instance, at the current period, the consumption and work hour of generation $t$ whose tenure history is $h^t$ can be represented exactly by $((x_t(\beta, w, h^t), l_t(\beta, w, h^t))$. With this information, the intermediary can calculate the total current period balance to run the entire system.

Let $B(\beta, w)$ be the current period balance if the initial entitlement of every newly born is $w$ and the discount rate to the intermediary and agents is $\beta$. Then,

$$B(\beta, w) = E\sum_{t\geq 0}[\delta^t(C(x_t(\beta, w)) - l_t(\beta, w)y)].$$

Note that the difference between $B(\beta, w)$ and $V(\beta, w)$ is the presence of the discount rate. We know that for any $\beta \in [0, 1]$, $V(\beta, w_{aut}) < 0$, since there always exists an optimal contract that improves upon autarky with 0 transfer. In particular, at $\beta = 1$, $V(1, w_{aut}) < 0$. But at $\beta = 1$, the current balance is exactly the same as the lifetime cost of a newly born generation, $V(1, w_{aut}) = B(1, w_{aut})$ and therefore $B(1, w_{aut}) < 0$. We can conclude at least that if the discount rate is 1, then there is an initial entitlement $w^*$ such that the current period budget is balanced and, moreover, everyone is better off than in autarky. The next theorem shows that the same argument applies to the discount rates close to 1.

**Proposition 5.5:** *There exists $\beta^* \in [0, 1)$ such that for any $\beta \in [\beta^*, 1]$, there is an initial entitlement $w^*$ under which $B(\beta, w^*) = 0$ and $w^* > w_{aut}$.*

**Proof:**

Since $\sigma(\beta, w)$ and $C$ are continuous and $[0, 1] \times [0, \bar{w}]$ is compact, $B$ is continuous on $(\beta, w)$. If there is no $\beta \in [0, 1)$ with $B(\beta, w_{aut}) > 0$, then trivially we can set $\beta^* = 0$. If there is such $\beta$, then since $B(1, w_{aut}) < 0$, by the intermediate value function theorem, there is $\hat{\beta} \in [0, 1)$ such that $B(\hat{\beta}, w_{aut}) = 0$. Take the maximum $\hat{\beta}$ with this property and denote it by $\beta^*$. Then $\beta^*$ satisfies the property. For any $\beta \in [\beta^*, 1]$, with the same logic as Proposition 4.5, there exists $w^*$ such that $B(\beta, w^*) = 0$. ∎

We showed that for any discount rate close to 1, there is an initial entitlement as a value of the contract under which the steady state has an equilibrium. Therefore, the properties of optimal contract derived from one-on-one contracts are viable under the closed system as well.

## 6. Hidden Saving

We characterized the optimal solution to our repeated principal agent problem based on the assumption that the agents are not allowed to access the capital market as most of the repeated principal agent models assume. The importance of this assumption on the efficiency of the optimal contract has been considered and it has been known that in some environments, this assumption is critical to achieve the efficiency of the optimal contract over the optimal allocation in autarky with saving (Allen (1985), Cole and Kocherlakota (1998) and Fudenberg *et. al.* (1990)). The allocation in autarky with saving smoothes consumption over the periods but cannot smooth consumption across the states. In other words, the insurance for the consumption smoothing over the states is absent in the allocation of autarky with saving.

In this section we will show that in our model the efficiency of the optimal contract over autarky with saving is not completely lost in the presence of a hidden saving environment. To this end, we will first characterize the agent's decision problem when the only means for the agent to smooth consumption is to save some consumption good. Suppose that the agent can purchase the risk free bonds anonymously, which will deliver 1 unit of good in the next period at the price of $\beta\delta$ today.[6] The agent's problem is to maximize the discounted utility subject to the budget constraint,

[6] This particular price simplifies the analysis. The same implications can be obtained with more complication if we use an arbitrary price.

$$\max_{(c,s,l)} \quad E[\sum_{t\geq 0} \beta^t \delta^t (U(c_t) - l_t v)]$$

$$s.t.$$

$$c_t + \beta\delta s_{t+1} = l_t y + s_t.$$

In particular, if the agent whose current saving level is $\hat{s}$ is not able to work, then since the agent must live up to his saving, he will consume the constant amount of consumption $\hat{s}(1-\beta\delta)$ every period. Therefore, the recursive formulation of the saving decision problem is given by

$$W(s) = \max_{(c,l,s^e)} \quad \pi[U(c) - lv + \beta\delta W(s^e)] + (1-\pi)\frac{U(s(1-\beta\delta))}{1-\beta\delta}$$

$$s.t.$$

$$c + \beta\delta s^e = ly + s.$$

With the similar argument we made in section 3, it can be shown that $W$ has the same properties as $V$. Especially there is a threshold $\bar{s}$ such that the agent retires after he reaches this level. In addition, it takes a finite period of work to reach this level.

**Proposition 6.1:** *Let $(c, l, s^e)$ be the policy functions of the value function $W$.*

1. *$W$ is strictly increasing, strictly convex, and differentiable:*

$$W'(s) = \pi U'(c) + (1-\pi)U'(s(1-\beta\delta)).$$

2. *There is $\bar{s}$ such that for any $s \geq \bar{s}$, $l(s) = 0$.*
3. *For any $s < \bar{s}$, $s^e$ is strictly increasing and $s^e(s) > s$.*
4. *Let $s^e_n(s) = s^e(s^e_{n-1}(s))$ be the saving decision after $n$ periods if the current saving level is $s$. For any $s$, there is $N^*_s$ such that for any $n \geq N^*_s$, $l(s^e_n(s)) = 0$.*

To show that the intermediary can improve the welfare of the agent upon the pure bond trading ex ante, we will consider one particular transfer policy that involves only the first period transfer. At the first period, the agent pays transfer $\tau > 0$ when he works. He receives $\frac{\pi}{1-\pi}\tau$ when he reports that he is not able to work. The transfers of the other periods are all 0's. If the policy induces truth telling, then the expected cost of this transfer system will be 0. Thus, if we find some $\tau$ with which this policy induces truth telling and improves the expected discount utility, we know that the optimal contract with hidden saving will strictly make the agent better off.

Suppose that the parameters of the bond trading economy guarantee that full time work, $l = 1$, is strictly preferred in the first period. Now consider a contract that assigns full time work and tax $\tau$ if the agent is able to work and that pays $\frac{\pi}{1-\pi}\tau$ to the agent if he is not able to work. Suppose that the agent reports truthfully, then the agent's saving problem is given by

$$\begin{aligned} &\max_s \pi[U(c) - v + \beta\delta W(s)] + (1-\pi)U(\frac{\pi}{1-\pi}\tau(1-\beta\delta)) \\ &\quad s.t. \\ &\quad c + \beta\delta s = y - \tau. \end{aligned}$$

If we plug the constraint into the objective function, we get,

$$\max_s \ \pi[U(y - \tau - \beta\delta s) - v + \beta\delta W(s)] + (1-\pi)U(\frac{\pi}{1-\pi}\tau(1-\beta\delta)).$$

Let $s^*(\tau)$ be the maximizer and $W^*$ be an indirect utility function defined by

$$W^*(\tau) = \pi[U(y - \tau - \beta\delta s^*(\tau)) - v + \beta\delta W(s^*(\tau))] + (1-\pi)U(\frac{\pi}{1-\pi}\tau(1-\beta\delta)).$$

Now compare $W^*(\tau)$ with $W^*(0)$, which is the same as $W(0)$, the ex ante discounted utility in autarky with saving.

$$\begin{aligned} W^*(\tau) - W^*(0) = \beta\delta(W(s^*(\tau) - W(s^*(0))+ \\ (1-\pi)U(\frac{\pi}{1-\pi}\tau(1-\beta\delta)) - \pi(U(y-\tau-\beta\delta s^*(\tau)) - U(y - \beta\delta s^*(0))). \end{aligned}$$

We know that $s^*$ and $W$ are continuous by the theorem of maximum. As $\tau$ is close to 0, $s^*(\tau)$ gets close to $s^*(0)$. The derivatives of $W$ and $U$ are finite at $s^*(0)$ and $y - \beta\delta s^*(0)$. However, the derivative of $U$ at $\frac{\pi}{1-\pi}\tau(1-\beta\delta)$ goes to infinity as $\tau$ goes to 0. Therefore, for sufficiently small $\tau$, the first and the third term on the right-hand side will be negligible compared to the second term, implying $W^*(\tau) > W^*(0)$.[7] Now we only need to show that the (IC) constraint of the policy holds. By the assumption at $\tau = 0$, full time work is strictly preferred. Since $W$ and $s^*$ are continuous, for sufficiently small $\tau$, full time work is still preferred. Therefore, the agent who is able to work will report truthfully.

**Proposition 6.2:** *An optimal contract with hidden saving strictly improves upon the allocation with bond trading.*

The main reason that our result is different from no insurance results such as Allen (1985) is due to the difference in environments, especially the observability of income. In our model the ability to work is private information but since one has to work to produce income and working is publicly observable, the income realization is publicly observable as well. Thus, if one reports the loss of ability to work, he also loses the income from the work. This is extra information given to the intermediary compared to the hidden income model where the agent's report itself does not reveal the actual income.

The problem of the optimal contract with hidden saving imposes an additional constraint to the constraints of the problem without hidden saving. The consumption plan chosen by the agent must satisfy the Euler condition. Unless the Euler condition is satisfied at the optimal contract, the agent may have incentives to save or dissave from the consumption level dictated by the intermediary, which can potentially upset the optimal plan. However, as we will see, the optimal contract without hidden saving does not satisfy the Euler condition and, moreover, it shows saving constrained, which is a common feature of these kinds of repeated principal agent models. The agent will have incentive to deviate from the consumption level allocated by the optimal contract.

**Proposition 6.3:** *The optimal contract without hidden saving does not satisfy the Euler condition and, moreover, it is saving constrained.*

**Proof:**

By (7) and (8), we know that $C'(u^e) = V'(w^e)$ and $V'(w) = \pi C'(u^e) + (1-\pi)C'(u^r)$.

[7] This also can be seen by the envelope property, $W^{*\prime}(\tau) = -\pi U'(y - \tau - s^*) + \frac{\pi}{1-\pi}(1-\pi)(1-\beta\delta)U'(\frac{\pi}{1-\pi}\tau(1-\beta\delta))$. Since the second term goes to infinity whereas the first term converges to the finite number as $\tau$ goes to 0, $W^{*\prime}$ goes to infinity in the limit.

Let $u^e_f$ and $u^r_f$ denote the current utility given to the agent who is able to work and to the retired agent, respectively when the entitlement to the agent is $w^e$. Then

$$C'(u^e) = V'(w^e) = \pi C'(u^e_f) + (1-\pi)C'(u^r_f).$$

Since $U^{-1} = C$,

$$\frac{1}{U'(c^e)} = \frac{\pi}{U'(c^e_f)} + \frac{(1-\pi)}{U'(c^r_f)},$$

where $U(c^e_f) = u^e_f$ and $U(c^r_f) = u^r_f$. Since the function $1/x$ is convex,

$$\frac{\pi}{U'(c^e_f)} + \frac{(1-\pi)}{U'(c^r_f)} > \frac{1}{\pi U'(c^e_f) + (1-\pi)U'(c^r_f)}.$$

Consequently, $U'(c^e) < \pi U'(c^e_f) + (1-\pi)U'(c^r_f)$, which violates the Euler condition and shows savings constrained since $U$ is concave. ∎

The proposition shows that the Euler condition adds a nontrivial constraint to the planner's problem and thus the optimal contract with hidden saving achieves an ex ante utility level lower than the one without hidden saving. With these two propositions, we showed that the welfare level of the optimal contract with hidden saving is strictly greater than that of autarky with saving but strictly less than that of the optimal contract without hidden saving.

Our approach to these two propositions in this section is indirect in the sense that we did not get them from the characterization of the optimal contract with hidden saving. The characterization of the optimal contract with hidden saving in our model is still an open problem.

## 7. Simulation

We simulate the model with the following parameter values: $\bar{w} = 50, y = 1.98, v = 1, \beta = 0.97, \delta = 0.99$, and $\pi = 0.98$. With a mortality rate of 0.01 the average life span after the agent participates in the work force is 99. On average, agents lose their productivity 50 years after they join the work force. We assume that the cost function has the form of

$C(u) = u^\sigma/\sigma$, and we set the value of $\sigma$ to 2. Since it is an inverse function of the flow utility function, it amounts to giving an intermediate degree of risk aversion. The expected discounted utility of autarky is 16.45, and the entitlement of the optimal contract with 0 transfer gives 24.2 expected discounted utility. Forty-eight years after entering the labor force, the agent is forced to retire. By a simulation we generated the value function (Fig 1) and the policy functions such as the future entitlement function (Fig 2), consumption function to the current work force (Fig 3), work hour (Fig 4), flow utility to the workers (Fig 5), consumption to the retired (Fig 6), and normal retirement age (Fig 8). We also get implications for the optimal allocation on social security accrual (Fig 9), the variance of log consumption (Fig 11), and the replacement rate (Fig 12).

The policy functions show certain properties likely at optimal allocations. Future entitlement grows at an almost constant rate (Fig 2). Therefore, the level of consumption to the retired will grow at an increasing rate (Fig 6) with tenure. The consumption function to the current work force (Fig 3) shows a convex shape. These two observations give the increasing inequality in consumption with age (Fig 11). This resembles the empirical study by Deaton and Paxon (1994) because they found a rather convex shape of variance of log consumption. The work hour (Fig 4) shows that typically right before retirement, part time work is assigned.

At the optimal allocation, the replacement rate increases at an increasing rate (Fig 12). If the annuity increases faster than the average net income, the replacement rate increases. This is likely to happen in our model since the annuity grows at faster rate than the consumption of the current work force. Germany formerly had a constant replacement rate from age 60 to the normal retirement age of 65 before it reformed its system in 1992. Most countries have the replacement rate increase at a constant rate. Therefore, the current systems seem to have a slow replacement rate increase with tenure. More serious calibration will be helpful to determine whether replacement rate increase in current systems is slower than it should be.

Gruber and Wise (1997) and Borsch-Supan and Schnabel (1997) used a concept "benefit accrual" to measure the incentive for early retirement. $SSW_a(a+t)$ denotes the net present discounted value of social security benefits upon retirement at age $a + t$ evaluated at age $a$. The key consideration for the retirement decision is how this wealth will evolve with continued work. The difference between $SSW_a(a)$ and $SSW_a(a+1)$, $SSW_a(a+1) - SSW_a(a)$, is called "SSW accrual." It is the marginal benefit to social security wealth if the agent decides to work one more year. By working another year, the agent has to give up this year's social security benefit and pay social security tax, but his future benefit may be raised by an increase in the annuity. Negative accrual means that the net present value of annuity decreases as the agent postpones retirement one more year and thus tends to discourage continuation in the labor force and positive accrual tends to encourage continued work force participation. They found that pension accrual is typically negative at older ages in most countries showing an early retirement trend. Since the consumption of the retired is convex and the social security tax decreases with tenure, given near linear future entitlement (Fig 2), it is likely that SSW accrual should be positive at the optimal allocation as shown in Fig 8. Again qualitatively, the current systems seem to discourage continuation in the labor force by increasing annuity at a slower pace than it should be increased.

## 8. Conclusion

In this paper, we considered the problem of optimal design of social security programs when the ability to work is not observable to the public. By introducing an incentive problem to the model explicitly, we could theoretically present policy implications on the structure of social security benefits that can prevent abusive early retirement. We established that at the optimal contract the annuity increases with tenure and the social security tax decreases with tenure. We also showed that there must be a mandatory retirement age after a finite period of work even if the agent is still able to work. The individual contract we considered is a contract between one agent and one principal in an open system. We closed the system to consider the existence of a feasible system and we found that there is a range of discount rates under which the optimal system balances the budget every period.

As in other models of repeated moral hazard, we have assumed that the agent cannot access intertemporal transactions so that the the intermediary can monitor the consumption of the agent. But the agent is always saving constrained at the optimal contract, which violates Euler's condition. Thus, when the agent is allowed to engage in intertemporal trade privately, the optimal contract will be disrupted and thus the welfare level of the optimal contract cannot be achieved. We showed, however, that efficiency is not completely lost in our model. A further investigation of the optimal allocation in hidden saving would be an interesting topic, especially when privatizing social security system is an important issue.

We considered only steady states of the economy. It would also be a possible extension of the model to consider a non-steady state to deal with issues of demographic change and its impact on the solvency of the social security system.

The simulations we have executed are just the beginning of a potentially fruitful quantitative analysis. Our simulation has already suggested that there can be potential welfare gains from switching the current systems to the optimal contract. Further substantial investigation of these gains will provide an interesting policy implications. It would also be useful to measure the welfare loss from hidden saving. If the welfare loss is relatively large, it would be desirable to test the wealth of the agent when he decides to retire rather than to test the earnings as many countries do.

Fig 1

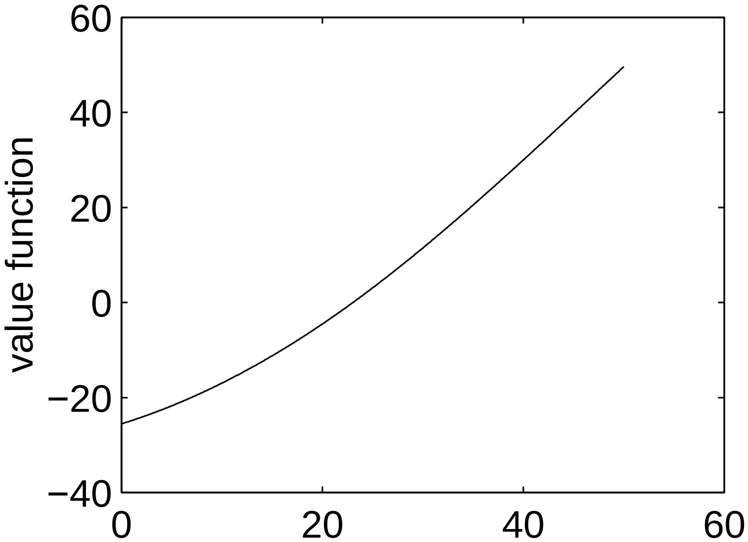



Fig 2

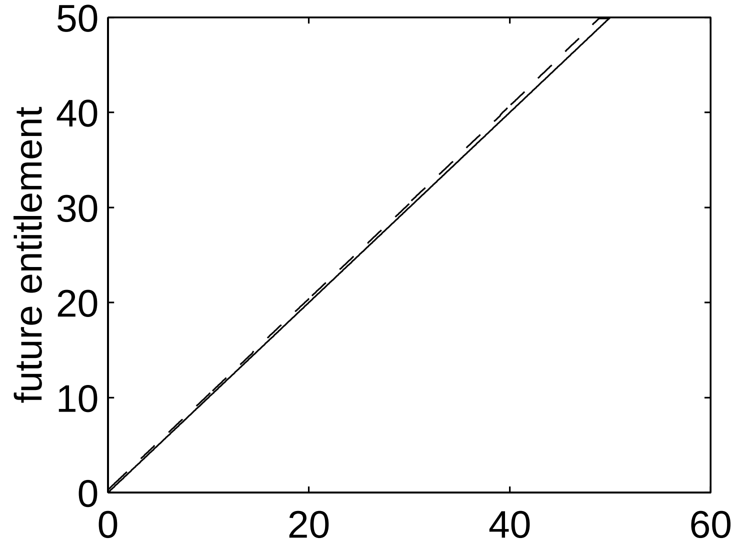



Fig 3

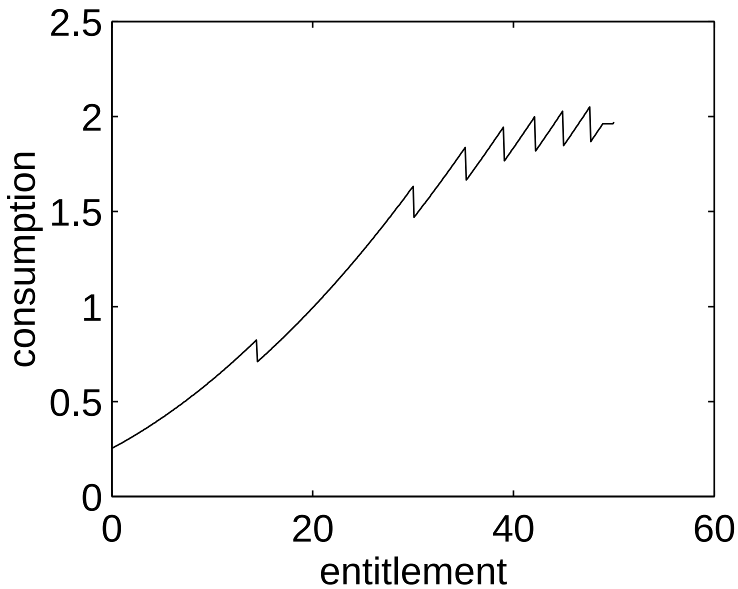



Fig 4

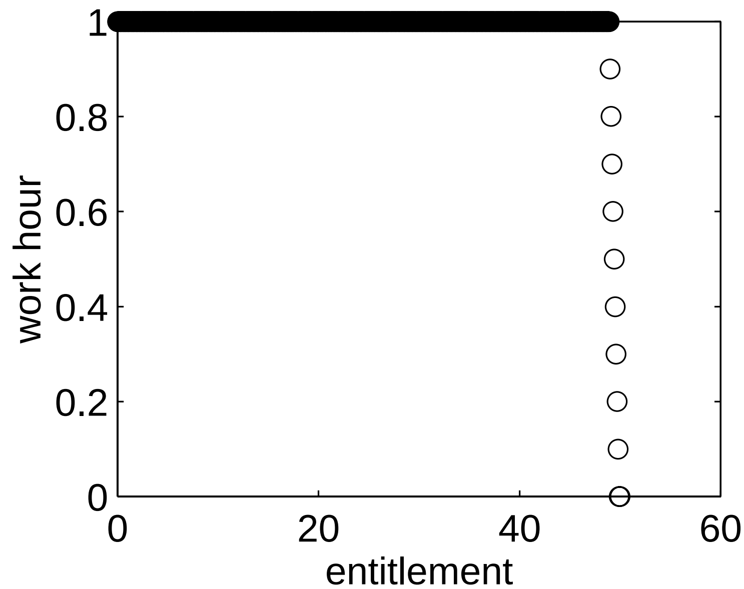


Fig 5

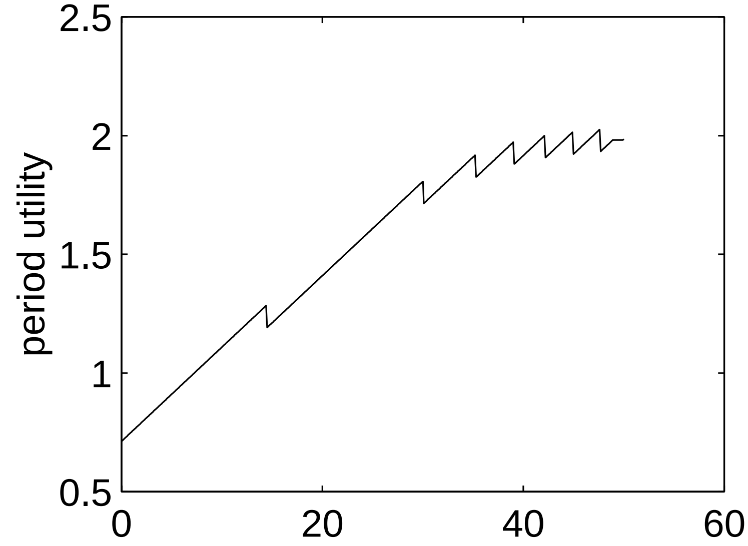



Fig 6

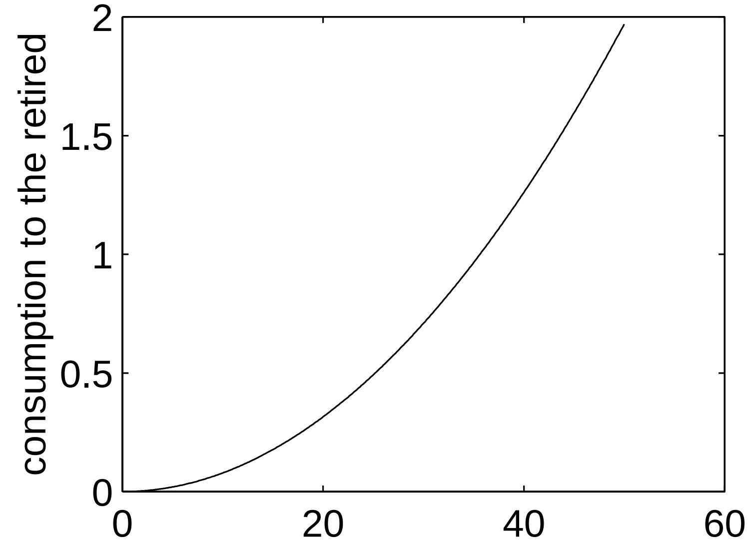



Fig 7

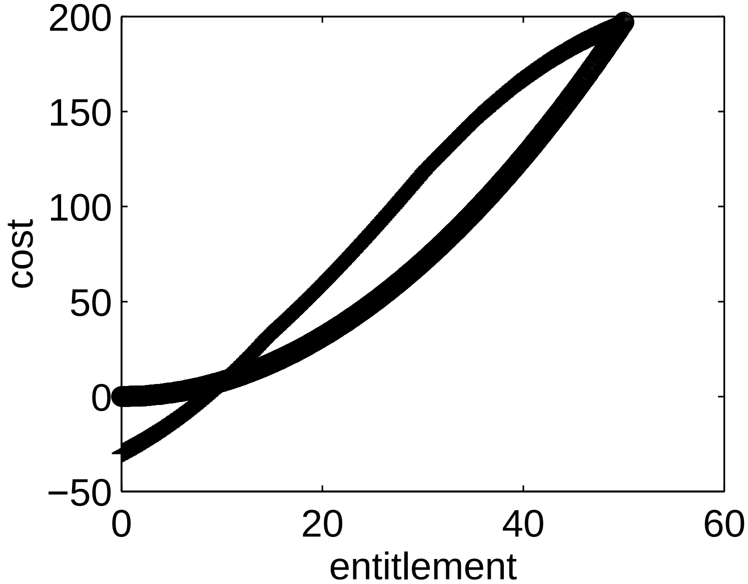



Fig 8

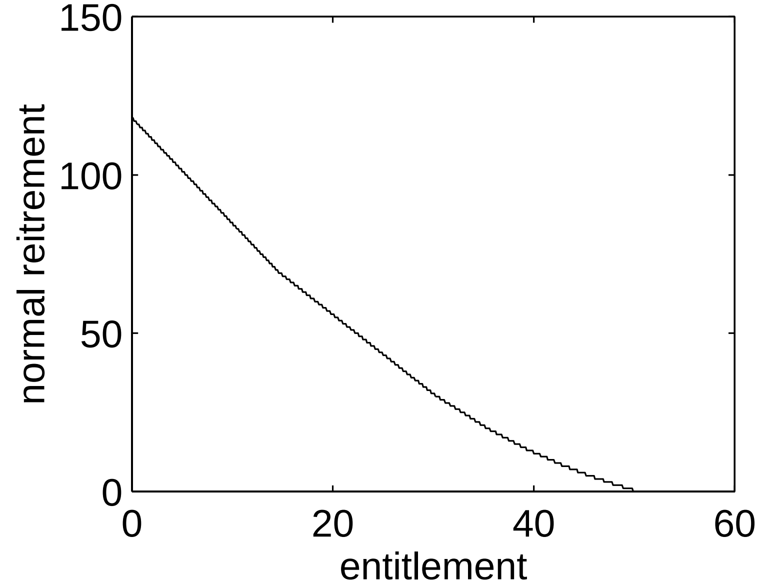


Fig 9

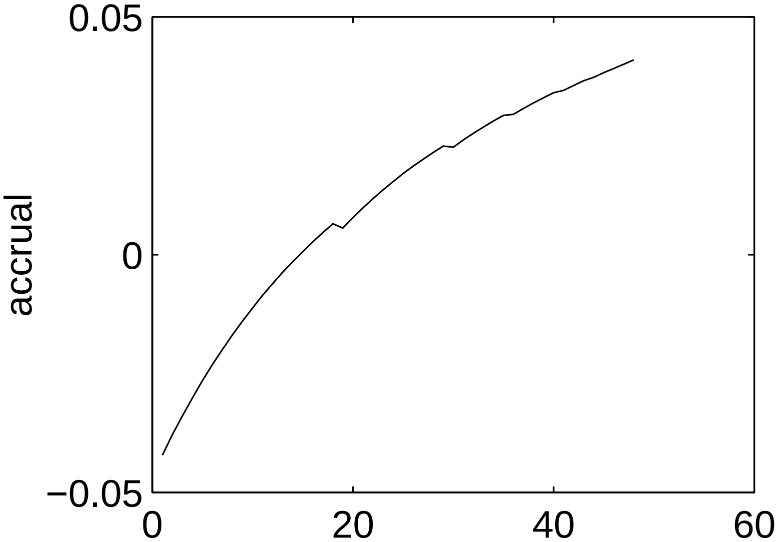



Fig 10

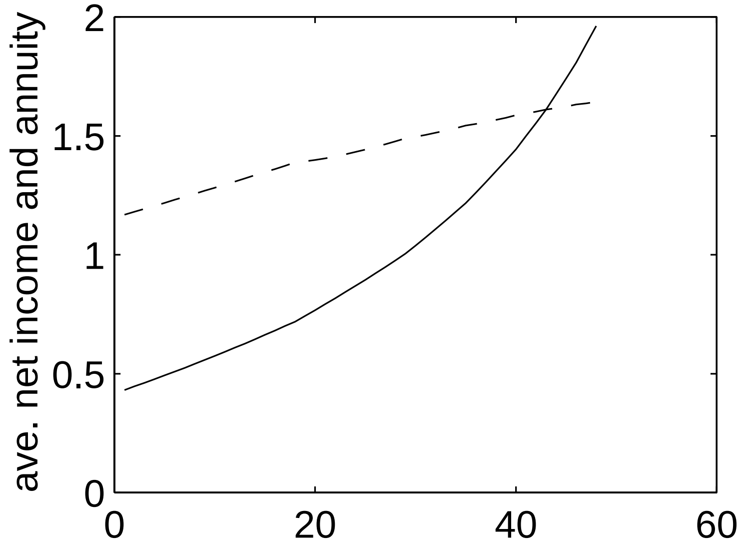



Fig 11

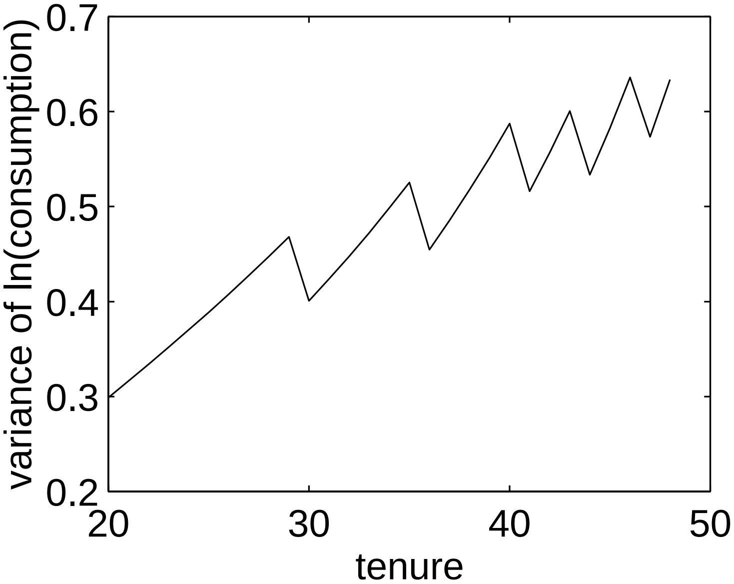
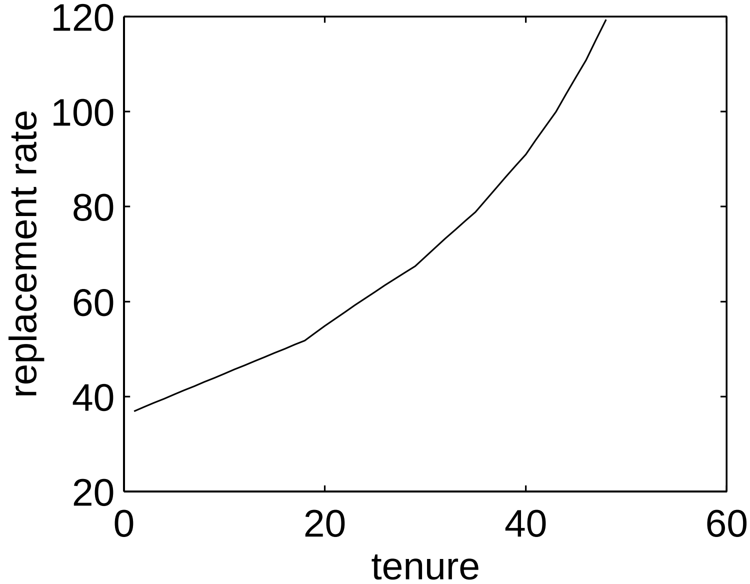



Fig 12



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