Voting by Committees under Constraints*

by

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ABSTRACT: We consider social choice problems where a society must choose a subset from a set of objects. Specifically, we characterize the families of strategy-proof voting procedures when not all possible subsets of objects are feasible, and voter's preferences are separable or additively representable.

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1 Introduction

Many problems of social choice take the following form. There are n voters and a set $K = \{1, ..., k\}$ of objects. These objects may be bills considered by a legislature, candidates to some set of positions, or the collection of characteristics which distinguish a social alternative from another. The voters must choose a subset of the set of objects.

Sometimes, any combination of objects is feasible: for example, if we consider the election of candidates to join a club which is ready to admit as many of them as the voters choose, or if we are modelling the global results of a legislature, which may pass or reject any number of bills. It is for these cases that Barberà, Sonnenschein, and Zhou (1991) provided characterizations of all voting procedures which are strategy-proof and respect voter's sovereignty (all subsets of object may be chosen) when voters' preferences are additively representable, and also when these are separable. For both of these restricted domains, voting by committees turns out to be the family of all rules satisfying the above requirements. Rules in this class are defined by a family of winning coalitions, one for each object; agents vote for sets of objects; to be elected, an object must get the vote of all members of some coalition among those that are winning for that object.

Most often, though, some combinations of objects are not feasible, while others are: if there are more candidates than positions to be filled, only sets of size less than or equal to the available number of slots are feasible; if objects are the characteristics of an alternative, some collections of characteristics may be mutually incompatible, and others not. Our purpose in this paper is to characterize the families of strategy-proof voting procedures when not all possible subsets of objects are feasible, and voters' preferences are separable or additively representable. Our main conclusions are the following. First: all rules that satisfy strategy-proofness must still be voting by committees, but now voters must vote for feasible sets of objects. Second: the committees for different objects must be interrelated, in precise ways which depend on what families of sets of objects are feasible. Third: unlike in Barberà, Sonnenschein, and Zhou (1991), the class of strategy-proof rules when preferences are additively representable can be substantially larger that the set of rules satisfying the same requirement when voter' preferences are separable.

Specifically, let $\mathcal{R}_F$ be the range of alternatives voters can choose from. Based on the form of this range, we propose a decomposition of the set K of objects into cylindric sections and define certain subsets of each section to be active. Then we prove that, when preferences are additively representable, the committees for all objects in the same cylindric section must be the same, that these committees can take any form if the corresponding section only contains two sets, and that these committees must be dictatorial otherwise, with (possibly) different dictators for different sections. One cannot describe this characterization result as either positive or negative, because it has different consequences depending on the exact shape of the range of feasible choices. We shall provide examples which are quite negative, as well as positive ones. At any rate, the results in Barberà, Sonnenschein, and Zhou (1991) refer to a particular case which lies on the positive side, while the Gibbard-Satterthwaite theorem can also be obtained as a corollary of our result, and would certainly lie on the negative side. On the other hand, we also obtain a characterization result for separable preferences. Here, unfeasibilities quickly turn any non-dictatorial rule into a manipulable one, except for very limited ranges. The contrast between these two characterization results is a striking conclusion of our research, because until now the results regarding strategy-proof mechanisms for these two domains went hand to hand, even if they are, of course, logically independent.

Other than Barberà, Sonnenschein, and Zhou (1991), the closest reference to the present paper is our preceding article on Voting under Constraints (Barberà, Massó, and Neme (1997)). As far as the set of alternatives is concerned, the setting there is more general. In the present paper we can identify sets of objects with their characteristic function, and our objects of choice as (some of) the vertices of a k-dimensional hypercube. Thus our framework here is restricted (like in Barberà, Sonnenschein, and Zhou (1991)), to allow for only two values in each dimension. In Barberà, Massó, and Neme (1997) we study situations where the objects of choice are subsets of a cartesian product of any interger intervals, not only binary ones. On the other hand, our present paper analyses the question of voting under constraints for a richer and more natural set of admissible preferences. Here, we consider all additively representable preferences (respectively, all separable preferences) on the power set of K. In particular, we allow for preferences whose bliss point is not feasible, even if then voters must vote for feasible alternatives. This is a considerable improvement over the assumption made in our preceding work, which was marred by the restrictive assumption that all agents' most preferred set was a feasible one.

The paper is organized as follows. In Section 2 we present the preliminary notation and definitions. In Section 3 we define voting by committees and give its main partial characterizations. The characterization with additive preferences is given in Section 4, while Section 5 contains the characterization with separable preferences.

2 Preliminary notation and definitions

Agents are the elements of a finite set $N = \{1, 2, ..., n\}$. The set of objects is $K = \{1, ..., k\}$. We assume that n and k are at least 2. Generic elements of N will be denoted by i and j and generic elements of K will be denoted by x, y, and z. *Alternatives* are subsets of K which will be denoted by X, Y, and Z. Subsets of N will be represented by I and J. Calligraphic letters will represent families of subsets; for instance, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ will represent families of subsets of alternatives and $\mathcal{W}$, $\mathcal{I}$, and $\mathcal{J}$ families of subsets of agents (coalitions).

Preferences are binary relations on alternatives. Let $\mathbf{P}$ be the set of complete, transitive, and asymmetric preferences on 2^K. We will always consider the empty set as belonging to the power set of any set. Preferences in $\mathbf{P}$ are denoted by P_i, P_j, P_i', and P_j'. For $P_i \in \mathbf{P}$ and $\mathcal{X} \subseteq 2^K$, we denote the alternative in $\mathcal{X}$ most-preferred according to P_i as $\tau_{\mathcal{X}}(P_i)$, and we call it *the top of* P_i *on* $\mathcal{X}$. We will use $\tau(P_i)$ to denote the top of P_i on 2^K. Generic subsets of preferences will be denoted by $\hat{\mathbf{P}}$ and given a family $\mathcal{X} \subseteq 2^K$, $\hat{\mathbf{P}}|_{\mathcal{X}}$ will denote the set of preferences on $\mathcal{X}$ obtained by restricting each preference P_i in $\hat{\mathbf{P}}$ with the property that $\tau(P_i) \in \mathcal{X}$.

Preference profiles are n-tuples of preferences. They will be represented by $P = (P_1, ..., P_n)$ or by $P = (P_i, P_{-i})$ if we want to stress the role of agents i's preference.

A *social choice function on* $\hat{\mathbf{P}}$ is a function $F : \hat{\mathbf{P}}^n \to 2^K$.

Definition 1 *The social choice function* $F : \hat{\mathbf{P}}^n \to 2^K$ respects voter's sovereignty on $\mathcal{X}$ *if for every* $X \in \mathcal{X}$ *there exists* $P \in \hat{\mathbf{P}}^n$ *such that* $F(P) = X$.

The range of a social choice function $F : \hat{\mathbf{P}}^n \to 2^K$ is denoted by $\mathcal{R}_F$; that is,

$$\mathcal{R}_F = \left\{ X \in 2^K \mid \text{there exists } P \in \hat{\mathbf{P}}^n \text{ such that } F(P) = X \right\}.$$

Denote by R_F the set of chosen objects; namely,

$$R_F = \{x \in K \mid x \in X \text{ for some } X \in \mathcal{R}_F\}.$$

Definition 2 *A social choice function* $F : \hat{\mathbf{P}}^n \to 2^K$ *is* manipulable *if there exist* $P = (P_1, ..., P_n) \in \hat{\mathbf{P}}^n$, $i \in N$, *and* $P_i' \in \hat{\mathbf{P}}$ *such that* $F(P_i', P_{-i}) P_i F(P)$. *A social choice function on* $\hat{\mathbf{P}}$ *is* strategy-proof *if it is not manipulable.*

Definition 3 *A social choice function* $F : \hat{\mathbf{P}}^n \to 2^K$ *is* dictatorial *if there exists* $i \in N$ *such that* $F(P) = \tau_{\mathcal{R}_F}(P_i)$ *for all* $P \in \hat{\mathbf{P}}^n$.

The Gibbard-Satterthwaite theorem states that any social choice function on $\mathbf{P}$ will be either dictatorial or its range will have only two elements. However, there are many situations were agents' preferences have specific structure due to the nature of the set of objects. This will impose a particular structure on the way agents extend preferences on objects to preferences on subsets of objects. We will be interested in the following two natural domains of preferences.

Definition 4 *A preference* P_i *on* 2^K *is* additive *if there exists a function* $u_i : 2^K \to \Re$ *such that* $u_i(\emptyset) = 0$ *and for all* $X, Y \subseteq K$

$$X P_i Y \text{ if and only if } \sum_{x \in X} u_i(x) > \sum_{y \in Y} u_i(y).$$

The set of additive preferences will be denoted by $\mathbf{A}$. An agent i has separable preferences if the division between *good* objects ($xP_i\emptyset$) and *bad* objects ($\emptyset P_i x$) guides the ordering of subsets in the sense that adding a good object leads to a *better* set, while adding a bad object leads to a *worse* set. Formally,

Definition 5 *A preference P_i on 2^K is* separable *if for all $X \subseteq K$ and all $y \notin X$*

$$X \cup \{y\} P_i X \text{ if and only if } yP_i\emptyset.$$

Let $\mathbf{S}$ be the set of all separable preferences on 2^K. Additivity implies separability but the converse is false with more than two objects. We can give a geometric interpretation to the set of separable preferences by identifying each object with a coordenate and the set of objects as the vertices of a k-dimensional cube where each subset of objects X corresponds to the k-dimensional vector of zeros and ones where x belongs to X if and only if the vector has a one in the coordinate that corresponds to object x. Sometimes we will make use of this geometric interpretation. For instance, given $X, Y \subseteq K$ the *minimal box on X* and Y is the smallest subcube containing the vectors corresponding to X and Y; namely,

$$MB(X,Y) = \left\{Z \in 2^K \mid (X \cap Y) \subseteq Z \subseteq (X \cup Y)\right\}.$$

Following with this interpretation, it is easy to see that a preference P_i is separable if it is multidimensional single-peaked; that is, YP_iZ for all $Y \in MB(\tau(P_i), Z) \setminus \{Y\}$.

3 Voting by Committees and its Characterizations

To define voting by committees as in Barberà, Sonnenschein, and Zhou (1991) we need the concept of a committee.

Definition 6 *A* committee *is a pair $C = (N, \mathcal{W})$, where $N = \{1, ..., n\}$ is the set of agents, $\mathcal{W}$ is a nonempty family of nonempty coalitions of N, which satisfies coalition monotonicity in the sense that if $I \in \mathcal{W}$ and $I \subseteq J$, then $J \in \mathcal{W}$. Coalitions in $\mathcal{W}$ are called* winning. *A coalition $I \in \mathcal{W}$ is a* minimal winning coalition *if for all $J \subsetneq I$ we have that $J \notin \mathcal{W}$.*

Given a committee $C = (N, \mathcal{W})$, we will denote by $\mathcal{W}^m$ the set of minimal winning coalitions. A committee $C = (N, \mathcal{W})$ is *dictatorial* if there exists $i \in N$ such that $\mathcal{W}^m = \{\{i\}\}$. Now, we can define a special subclass of social choice functions.

Definition 7 *A social choice function* $F : \hat{\mathbf{P}}^n \to 2^K$ *is* voting by committees*, if for each* $x \in K$*, there exists a committee* $C_x = (N, \mathcal{W}_x)$ *such that for all* $P = (P_1, ..., P_n) \in \hat{\mathbf{P}}^n$*,*

$$x \in F(P) \text{ if and only if } \{i \in N \mid x \in \tau_{\mathcal{R}_F}(P_i)\} \in \mathcal{W}_x.$$

We state, as Proposition 1 below, Barberà, Sonnenschein, and Zhou (1991)'s characterization of voting by committees as the class of strategy-proof social choice functions on $\mathbf{S}$ satisfying voter's sovereignty on 2^K.

Proposition 1 *A social choice function* $F : \mathbf{S}^n \to 2^K$ *is strategy-proof and satisfies voter's sovereignty on* 2^K *if and only if it is voting by committees.*[1]

To cover social choice problems with constraints we have to drop the voter's sovereignty condition of Proposition 1. Below, we state, as Proposition 2, a result in Barberà, Massó, and Neme (1997) which says that committees are also a consequence of strategy-proofness.

Proposition 2 *Assume* $F : \mathbf{S}^n \to 2^K$ *is strategy-proof. Then, F is voting by committees.*[2]

4 A characterization with additive preferences

4.1 The statement and one example

To state Theorem 1 below we need the following notation and definitions. Given two families of subsets of objects $\mathcal{X}$ and $\mathcal{Y}$ we denote by $\mathcal{X} + \mathcal{Y}$ the sum of the two; namely, $\mathcal{X} + \mathcal{Y} = \{X \cup Y \in 2^K \mid X \in \mathcal{X} \text{ and } Y \in \mathcal{Y}\}$.

Given a social choice function $F : \hat{\mathbf{P}}^n \to 2^K$ and $B' \subseteq B \subseteq K$ define the *range complement of* B' *relative to* B as

$$\mathcal{C}_F^B(B') = \{C \subseteq R_F \backslash B \mid B' \cup C \in \mathcal{R}_F\}.$$

Given a subset B of R_F define the *active components of* B *in the range* as $\mathcal{AC}(B) = \left\{X \subseteq B \mid X \cup Y \in \mathcal{R}_F \text{ for some } Y \in \mathcal{C}_F^B(B)\right\}$. Notice that $\mathcal{AC}(B)$ can also be written as $\{X \subseteq B \mid X = Y \cap B \text{ for some } Y \in \mathcal{R}_F\}$.

Definition 8 *A subset of objects* $B \subseteq K$ *is a* section *of* R_F *if for all active components* $B', B'' \in \mathcal{AC}(B)$ *we have* $\mathcal{C}_F^B(B') = \mathcal{C}_F^B(B'')$.

[1] Barberà, Sonnenschein, and Zhou (1991) also showed that Proposition 1 still holds when the domain of separable preferences is replaced by the smaller domain of additive preferences. As we will see, the introduction of constraints will yield significant differences between both domains.

[2] It is easy to check that the proof of Proposition 2 in Barberà, Massó, and Neme (1997) also aplies to the smaller domain of additive preferences.

Remark 1 R_F is a section of R_F.

Lemma 9 *If B is a section of R_F, $B = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$, and B_1 is a section of R_F then B_2 is also a section of R_F.*

Definition 10 *A partition $\{B_1, ..., B_q\}$ of R_F is a* cylindric decomposition *of the range if for all $p = 1, ..., q$, B_p is a section of R_F. A cylindric decomposition is called* minimal *if there is no finer cylindric decomposition of the range.*

Lemma 11 *A range has a unique minimal cylindric decomposition.*

Theorem 1 *A social choice function $F : \mathbf{A}^n \to 2^K$ is strategy-proof if and only if it is voting by committees with the following properties: (1) $\mathcal{W}_x = \mathcal{W}_y$ for all x and y in the same section with two active components of the minimal cylindric decomposition of the range of F and (2) $\mathcal{W}_x$ is dictatorial for all x's belonging to sections in the minimal cylindric decomposition which contain strictly more than two active components.*

Example 1 Let $K = \{a, b, c, x, y, z, w, r, s, t\}$ be the set of objects and assume that the set of feasible alternatives is

$$\begin{aligned}&\{\{bc\}, \{bct\}, \{bcrs\}, \{bctrs\}, \{bcz\}, \{bczt\}, \{bczrs\}, \{bcztrs\}, \{bczw\},\\ &\{bczwt\}, \{bczwrs\}, \{bczwtrs\}, \{bcx\}, \{bcxt\}, \{bcxrs\}, \{bcxtrs\}, \{bcxz\},\\ &\{bcxzt\}, \{bcxzrs\}, \{bcxztrs\}, \{bcxzw\}, \{bcxzwt\}, \{bcxzwrs\}, \{bcxzwtrs\},\\ &\{bcy\}, \{bcyt\}, \{bcyrs\}, \{bcytrs\}, \{bcyz\}, \{bcyzt\}, \{bcyzrs\}, \{bcyztrs\},\\ &\{bcyzw\}, \{bcyzwt\}, \{bcyzwrs\}, \{bcyzwtrs\}\}.\end{aligned}$$

Namely, (1) object a cannot be chosen, (2) objects b and c have to be chosen always, (3) objects x and y are never chosen simultaneously, (4) object w is only chosen together with object z, and (5) objects r and s can only be chosen together. Therefore, we are interested in strategy-proof social choice functions $\dot{F} : \mathbf{A}^n \to 2^K$ whose range is equal to

$$\mathcal{R}_F = \{\{b, c\}\} + \{\emptyset, \{x\}, \{y\}\} + \{\emptyset, \{z\}, \{z, w\}\} + \{\emptyset, \{t\}\} + \{\emptyset, \{r, s\}\}.$$

Notice that the partition $\{\{a\} \{b, c\} \{x, y\} \{z, w\} \{t\} \{r, s\}\}$ of K is the minimal cylindric decomposition of $\mathcal{R}_F$. To see that we have to check that all of its elements are minimal sections.

First, $\{a\}$ is a minimal section trivially because $\{a\}$ has no active components; i.e., $\mathcal{AC}(\{a\}) = \emptyset$.

Second, $\{b, c\}$ is also a minimal section trivially because only has itself as active component; i.e., $\mathcal{AC}(\{b, c\}) = \{b, c\}$.

Third, $\{x,y\}$ is a section because $\mathcal{AC}(\{x,y\}) = \{\emptyset, \{x\}, \{y\}\}$ (notice that the subset $\{x,y\}$ is not an active component of itself) and $\mathcal{C}_F^{\{x,y\}}(\emptyset)$, $\mathcal{C}_F^{\{x,y\}}(\{x\})$, and $\mathcal{C}_F^{\{x,y\}}(\{y\})$ are all equal to

$$\{\{b,c\}\} + \{\emptyset, \{z\}, \{z,w\}\} + \{\emptyset, \{t\}\} + \{\emptyset, \{r,s\}\}.$$

Moreover, this section is minimal since neither $\{x\}$ nor $\{y\}$ are sections because, for instance, $\mathcal{AC}(\{x\}) = \{\emptyset, \{x\}\}$ but

$$\mathcal{C}_F^{\{x\}}(\emptyset) = \{\{b,c\}\} + \{\emptyset, \{y\}\} + \{\emptyset, \{z\}, \{z,w\}\} + \{\emptyset, \{t\}\} + \{\emptyset, \{r,s\}\}$$

and

$$\mathcal{C}_F^{\{x\}}(\{x\}) = \{\{b,c\}\} + \{\emptyset, \{z\}, \{z,w\}\} + \{\emptyset, \{t\}\} + \{\emptyset, \{r,s\}\},$$

and hence, $\mathcal{C}_F^{\{x\}}(\emptyset) \neq \mathcal{C}_F^{\{x\}}(\{x\})$.

Fourth, $\{z,w\}$ is a section because $\mathcal{AC}(\{z,w\}) = \{\emptyset, \{z\}, \{z,w\}\}$ (notice that the subset $\{w\}$ is not an active component of $\{z,w\}$) and $\mathcal{C}_F^{\{z,w\}}(\emptyset)$, $\mathcal{C}_F^{\{z,w\}}(\{z\})$, and $\mathcal{C}_F^{\{z,w\}}(\{z,w\})$ are all equal to

$$\{\{b,c\}\} + \{\emptyset, \{x\}\{y\}\} + \{\emptyset, \{t\}\} + \{\emptyset, \{r,s\}\}.$$

Moreover, this section is minimal since neither $\{z\}$ nor $\{w\}$ are sections because, for instance, $\mathcal{AC}(\{w\}) = \{\emptyset, \{w\}\}$ but

$$\mathcal{C}_F^{\{w\}}(\emptyset) = \{\{b,c\}\} + \{\emptyset, \{x\}, \{y\}\} + \{\emptyset, \{z\}\} + \{\emptyset, \{t\}\} + \{\emptyset, \{r,s\}\}$$

and

$$\mathcal{C}_F^{\{w\}}(\{w\}) = \{\{b,c\}\} + \{\emptyset, \{x\}, \{y\}\} + \{z\} + \{\emptyset, \{t\}\} + \{\emptyset, \{r,s\}\},$$

and hence, $\mathcal{C}_F^{\{w\}}(\emptyset) \neq \mathcal{C}_F^{\{w\}}(\{w\})$.

Fifth, $\{t\}$ is a minimal section because $\mathcal{AC}(\{t\}) = \{\emptyset, \{t\}\}$ and

$$\mathcal{C}_F^{\{t\}}(\emptyset) = \mathcal{C}_F^{\{t\}}(\{t\}) = \{\{b,c\}\} + \{\emptyset, \{x\}, \{y\}\} + \{\emptyset, \{z\}, \{z,w\}\} + \{\emptyset, \{r,s\}\}.$$

Sixth, $\{r,s\}$ is a section because $\mathcal{AC}(\{r,s\}) = \{\emptyset, \{r,s\}\}$ and $\mathcal{C}_F^{\{r,s\}}(\emptyset)$ and $\mathcal{C}_F^{\{r,s\}}(\{r,s\})$ are both equal to

$$\{\{b,c\}\} + \{\emptyset, \{x\}, \{y\}\} + \{\emptyset, \{z\}, \{z,w\}\} + \{\emptyset, \{t\}\}.$$

Moreover, it is a minimal section because neither $\{r\}$ nor $\{s\}$ are sections since, for instance, $\mathcal{AC}(\{r\}) = \{\emptyset, \{r\}\}$ but

$$\mathcal{C}_F^{\{r\}}(\emptyset) = \{\{b,c\}\} + \{\emptyset, \{x\}, \{y\}\} + \{\emptyset, \{z\}, \{z,w\}\} + \{\emptyset, \{t\}\}$$

and

$$\mathcal{C}_F^{\{r\}}(\{r\}) = \{\{b,c\}\} + \{\emptyset, \{x\}, \{y\}\} + \{\emptyset, \{z\}, \{z,w\}\} + \{\emptyset, \{t\}\} + \{s\},$$

and hence, $\mathcal{C}_F^{\{r\}}(\emptyset) \neq \mathcal{C}_F^{\{r\}}(\{r\})$.

Now, given a set of agents N, any voting by committees $\dot{F} : \mathbf{A}^n \to 2^K$ with the properties that: (1) $\mathcal{W}_a$,$\mathcal{W}_b$, and $\mathcal{W}_c$ are non-empty families of non-empty subsets of N; (2) $\mathcal{W}_x^m = \mathcal{W}_y^m = \{\{i_1\}\}$ and $\mathcal{W}_z^m = \mathcal{W}_w^m = \{\{i_2\}\}$ for some $i_1, i_2 \in N$; and (3) $\mathcal{W}_t$, $\mathcal{W}_r$, and $\mathcal{W}_s$ are any committees such that $\mathcal{W}_r = \mathcal{W}_s$, it will be strategy-proof because by Theorem 1.

Insert Figure 1, here

4.2 The proof

4.2.1 Necessity

The key step in this proof is Proposition 3 below which says that if the minimal cylindric decomposition of the range contains only one section with three or more active components, then all committees of the objects in the section are not only equal but also dictatorial.

Proposition 3 *Assume that the following properties of $\mathcal{R}_F$ hold:* (1) *its minimal cylindric decomposition has a unique section,* (2) $\#\mathcal{R}_F \geq 3$, *and* (3) $R_F = K$. *Then, there exists $i \in N$ such that for all $k \in K$, $\mathcal{W}_k = \{\{i\}\}$.*

Proof of Proposition 3. By condition (1) there exists $Z \in 2^K$ such that $Z \notin \mathcal{R}_F$. Without loss of generality assume that $Z = \emptyset$ and $\{x\} \in \mathcal{R}_F$. Let $y \in K$ be arbitrary. We will show that there exists $i \in N$ such that $\mathcal{W}_x = \mathcal{W}_y = \{\{i\}\}$. We will distinguish between two cases.

Case 1: There exists $D \in \mathcal{R}_F$ such that $y \in D$ and $MB(D, \emptyset) \cap \mathcal{R}_F = D$.

Subcase 1.1: Assume $MB(D \cup \{x\}, \emptyset) \neq \{\{x\}, D\}$. Since $MB\{D, \emptyset\} \cap \mathcal{R}_F = D$ there exists $B \cup \{x\} \in MB(D \cup \{x\}, \emptyset) \cap \mathcal{R}_F$.

Subcase 1.1.1: Assume $B \subsetneq D$. Without loss of generality assume that $MB(B \cup \{x\}, \{x\}) \cap \mathcal{R}_F = \{B \cup \{x\}, \{x\}\}$. Then we can generate, by an additive preference with top on $\emptyset$, the orderings $D \succ^1 \{x\} \succ^1 B \cup \{x\}$, $\{x\} \succ^2 D \succ^2 B \cup \{x\}$, and $\{x\} \succ^3 B \cup \{x\} \succ^3 D$, by an additive preference with top on B, the orderings $D \succ^4 B \cup \{x\} \succ^4 \{x\}$, $B \cup \{x\} \succ^5 \{x\} \succ^5 D$, and $B \cup \{x\} \succ^6 D \succ^6 \{x\}$. Therefore, we have a free-triple on the elements of the range D, $\{x\}$ and $B \cup \{x\}$, implying that there exists $i \in N$ such that $\mathcal{W}_x = \mathcal{W}_y = \{\{i\}\}$.

Subcase 1.1.2: Assume $B = D$. Because $MB(D \cup \{x\}, \emptyset) \neq \{\{x\}, D\}$ then $D \cup \{x\} \in \mathcal{R}_F$. Then $MB(D \cup \{x\}, \{x\}) = \{\{x\}, D \cup \{x\}\}$, $MB(D \cup \{x\}, D) = \{D, D \cup \{x\}\}$. Notice that $MB(D, \emptyset) = D$. Therefore we have a free triple on

elements of the range $D, \{x\}$ and $D \cup \{x\}$, implying that there exists $i \in N$ such that $\mathcal{W}_x = \mathcal{W}_y = \{\{i\}\}$.
<u>Subcase 1.2</u>: Assume $MB(D \cup \{x\}, \emptyset) = \{\{x\}, D\}$.
<u>Subcase 1.2.1</u>: There exists $C \in \mathcal{R}_F$, such that $C \cap (D \cup \{x\}) \notin \{\{x\}, D\}$. Let $\overline{C} = C \cap D \cup \{x\}$ and w.l.o.g. assume $MB\left\{\overline{C}, C\right\} \cap \mathcal{R}_F = C$. Since $MB\left\{\overline{C}, \{x\}\right\} \cap \mathcal{R}_F = \{x\}$and $MB\left\{\overline{C}, D\right\} \cap \mathcal{R}_F = D$ we have a free triple on elements of the range $D, \{x\}$ and C, implying that there exists $i \in N$ such that $\mathcal{W}_x = \mathcal{W}_y = \{\{i\}\}$, because $y \in D$.
<u>Subcase 1.2.2</u>: For all $C \in \mathcal{R}_F$, $C \cap D \cup \{x\} \in \{\{x\}, D\}$.

Claim 1 Assume that for all $C \in \mathcal{R}_F$ either $\{x\} \subseteq C$ or $D \subseteq C$. Then, there exists $A, B \in \mathcal{R}_F$ and $Z \in \{\{x\}, D\}$ such that:
(1.1) $MB(A, B) \cap \mathcal{R}_F = \{A, B\}$.
(1.2) $Z \subseteq A \cap B$.
(1.3) $MB\left(\overline{A}, \overline{B}\right) \cap \mathcal{R}_F = \overline{A}$,
where $\overline{A} = (A \cup (\{x\} \cup D)) \backslash Z$ and $\overline{B} = (B \cup (\{x\} \cup D)) \backslash Z$.
Proof of Claim 1: Since $\mathcal{R}_F$ has the property that its minimal cylindric decomposition has a unique section there exits $G \in \mathcal{R}_F$ and $Z \in \{\{x\}, D\}$ such that $Z \subseteq G$ and $\overline{G} = (G \cup (\{x\} \cup D)) \backslash Z \notin \mathcal{R}_F$. Define

$$\overline{MB}(H, Z) = \left\{\overline{E} \in 2^K \mid \overline{E} = (E \cup (\{x\} \cup D)) \backslash Z \text{ for } E \in MB(H, Z) \cap \mathcal{R}_F\right\}.$$

Denote $\sim Z = x$ if $Z = D$ or $\sim Z = D$ if $Z = x$. Because $G \in MB(G, Z) \cap \mathcal{R}_F$, then $\overline{G} \in \overline{MB}(G, Z)$. Since $\overline{G} \notin MB\left(\overline{G}, \sim Z\right) \cap \mathcal{R}_F$ then $\overline{MB}(G, Z) \nsubseteq MB\left(\overline{G}, \sim Z\right) \cap \mathcal{R}_F$. Let B be the element in the range with minimal L_1-distance to Z with the property that $\overline{MB}(B, Z) \nsubseteq MB\left(\overline{B}, \sim Z\right) \cap \mathcal{R}_F$. This implies that

$$\overline{MB}(B, Z) \backslash \overline{B} = MB\left(\overline{B}, \{\sim Z\}\right) \cap \mathcal{R}_F. \tag{1}$$

Let $A \in MB(B, Z) \backslash B$ be such that $MB(A, B) = \{A, B\}$. Condition (1) implies that $\overline{A} \in \mathcal{R}_F$ and $MB\left(\overline{A}, \overline{B}\right) \cap \mathcal{R}_F = \overline{A}$. ■

Let $A, B \in \mathcal{R}_F$ and $Z \in \{\{x\}, D\}$ be such that conditions (1.1), (1.2), and (1.3) of Claim 1 hold. Then we can generate, by an additive preference with top on $A \cup \{\sim Z\}$, the orderings $A \succ^1 B \succ^1 \overline{A}$, $A \succ^2 \overline{A} \succ^2 B$, and $\overline{A} \succ {}^3 A \succ^3 B$, by an additive preference with top on $B \cup \{\sim z\}$, the orderings $B \succ^4 A \succ^4 \overline{A}$ and $B \succ^5 \overline{A} \succ^5 A$, and by an additive preference with top on $\overline{B}$, the ordering $\overline{A} \succ^6 B \succ^6 A$. Therefore, we have a free-triple on the elements of the range A, B, and $\overline{A}$, implying that here exists $i \in N$ such that $\mathcal{W}_x = \mathcal{W}_y = \{\{i\}\}$.

<u>Case 2</u>: Assume that for every $D \in \mathcal{R}_F$ such that $y \in D$, there exists $B \in MB(D, \emptyset) \cap \mathcal{R}_F$.

Let D be such that $MB(D, \{y\}) \cap \mathcal{R}_F = D$ and let $B \neq D$ be such that $MB(B, \emptyset) \cap \mathcal{R}_F = B$. If $y \in B$ then we are back to case 1. Therefore, assume

that $y \notin B$. For each $z \in B$ we can apply case 1 and obtain that there exists $i \in N$ such that $\mathcal{W}_x = \mathcal{W}_z = \{\{i\}\}$.
Subcase 2.1: Assume that $\{x, y\} \in \mathcal{R}_F$. By hypothesis (1) saying that the minimal cylindric decomposition of $\mathcal{R}_F$ has a unique section we have that $MB(\{y\}, B) \cap \mathcal{R}_F = B$. Moreover, since $MB(\{y\}, D) \cap \mathcal{R}_F = D$ and $MB(\{y\}, \{x, y\}) \cap \mathcal{R}_F = \{x, y\}$ we can generate all orderings on $D, B, \{x, y\}$ and therefore there exists $i \in N$ such that $\mathcal{W}_x = \mathcal{W}_y = \{\{i\}\}$.
Subcase 2.2: Assume that $\{x, y\} \notin \mathcal{R}_F$. First suppose that $MB(\{y\}, B) \cap \mathcal{R}_F = B$. Since $MB(\{y\}, D) \cap \mathcal{R}_F = D$ and $MB(\{y\}, \{x\}) \cap \mathcal{R}_F = \{x\}$ we can generate all orderings on D, B, and $\{x\}$ and therefore there exists $i \in N$ such that $\mathcal{W}_x = \mathcal{W}_y = \{\{i\}\}$. Now, if $MB(\{y\}, B) \cap \mathcal{R}_F = \{B, D\}$ we can also generate all orderings on D, B, and $\{x\}$ with two preferences: one with top on y (orderings $D \succ^1 B \succ^1 x$, $D \succ^2 x \succ^2 B$, and $x \succ^3 D \succ^3 B$) and the other with top on $\emptyset$ (orderings $x \succ^4 B \succ^4 D$, $B \succ^5 D \succ^5 x$, and $B \succ^6 x \succ^6 D$). ■

Once Proposition 3 is established, to finish this part of the proof of Theorem 1 we only have to show that by additivity of the their preferences, voters care on the outcomes of each section independently on what happens in the other sections. Hence, by Proposition 3, to must have dictators in each section with strictly more than two active components, but these dictators may be different voters across sections.

4.2.2 Sufficiency

The easiest part. To be written.

5 A characterization with separable preferences

5.1 The statement

To state Theorem 2 below we need the following definition

Definition 12 *Given a social choice function $F : \hat{\boldsymbol{P}}^n \to 2^K$ we say that its range is a* subcube *if there exists $\{Q_0, Q_1\}$ a partition of K and $A \subseteq Q_0$ such that $\mathcal{R}_F = A + 2^{Q_1}$.*

Theorem 2 *Let $F : \mathbf{S}^n \to 2^K$ be a non-dictatorial social choice function with $\#\mathcal{R}_F \geq 3$. Then, F is strategy-proof if and only if F is voting by committees and its range is a subcube.*

5.2 The proof

Let $F : \mathbf{S}^n \to 2^K$ be a non-dictatorial social choice function with $\#\mathcal{R}_F \geq 3$. The proof that all voting by committees whose range is a subcube are strategy-proof is trivial. Therefore, assume that F is strategy-proof. By Proposition 2, F is voting by committees. Therefore, we only have to prove that $\mathcal{R}_F$ is a subcube. Since all additive preferences are separable, Theorem 1 applies to the subdomain of additive preferences. Therefore, the committees associated to F satisfy properties (1) and (2) of Theorem 1. However, because F is non-dictatorial the minimal cylindric decomposition of the range can not consists of just one section with strictly more than two active components. Moreover, the ordering of the active components of a section, under separable preferences, depends on the choice of objects in the other sections, therefore we can not have either two or more sections with strictly more than two active components. Now, using separable but non-additive profiles it is easy to verify that all sections have either only one active components (the objects that are always selected) or they just two active components and they are of the form $\{\emptyset, \{x\}\}$. Hence, the partition of a subset of R_F has sections consisting on singleton elements, implying that the range is a subcube. ∎

6 References

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