Compromises between Cardinality and Ordinality

with an application to the convexity of preferences

Abstract:

By taking sets of utility functions as a primitive description of agents, we define an ordering over the *measurability classes* of assumptions on utility functions. Cardinal and ordinal assumptions constitute two types of measurability classes, but several standard assumptions lie strictly between these extremes. We apply the ordering to arguments for the convexity of preferences and show that diminishing marginal utility, which implies convexity, is an example of a compromise between cardinality and ordinality. Moreover, Arrow's (1951) explanation of convexity, proposed as an ordinal theory, in fact relies on utility functions that lie in the cardinal measurement class. In addition, we show that transitivity and order-density (but not completeness) fully characterize the ordinal preferences that can be induced from sets of utility functions. Finally, we derive a more general cardinality theorem for additively separable preferences.

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1. Introduction

According to ordinalist methodology, only those properties of utility functions that are preserved under monotonically increasing transformations, the *ordinal properties of utility*, are the proper primitives of utility analysis. The rationale behind this rule is that any property p on utility functions that is not preserved under increasing transformations cannot be verified through observations of choice behavior: if a utility u satisfies p, there will exist another utility u' that does not satisfy p but that represents the same preferences that u represents. Put differently, any nonordinal property of utility p is needlessly restrictive: there will exist another property q that is weaker than p (i.e., the set of utilities that satisfy q contains the set that satisfies p) that has the same implications for choice behavior as p. Historically, ordinalism's first target was diminishing marginal utility – or its generalization, concavity – which had been highly prominent in preordinal utility theory as an argument for the convexity of preferences. Neither DMU or concavity is preserved under increasing transformations and hence both are inadmissible as ordinal axioms. Rather, when a utility theory analogue for the convexity of preferences is necessary, the ordinalist procedure is to assume that utility functions are quasiconcave. Many early ordinalists, e.g., Arrow (1951), claimed in addition that diminishing marginal utility is tantamount to assuming that utility is cardinal or "measurable." Arrow's 1950's position was typical and persists today: *either* an assumption on utility is ordinal *or* it is cardinal.

In this paper, I propose a finer gradation of measurability classes that takes arbitrary sets of utility functions as a primitive description of agents. On this view, ordinal preference theory, which takes the functions generated by all increasing transformations of a given utility function as primitive, lies at one extreme. The cardinalist view, which takes the functions generated by all increasing affine transformations of a given utility function as primitive advocates a much smaller set of utility representations and is therefore a "stronger" theory. Outside of economics, ratio scales, which are associated with the functions generated by all increasing linear transformations of a given utility function, are common. But in addition to these well-known cases, there is an infinity of other models. Specifically, we will see that diminishing marginal utility and concavity lie precisely in the middle ground between cardinality and ordinality; as a primitive assumption, diminishing marginal utility posits a set of utility functions that is larger than a cardinal set of utilities but smaller than an ordinal set. Thus, contrary to common belief, diminishing marginal utility does not depend on cardinalist foundations.

Given that ordinal properties of utility map precisely into testable features of choice behavior, what advantage can there be in taking nonordinal properties of utility as primitive? The prime benefit is that nonordinal properties can provide rationales for assumptions on (ordinal) preference relations. Diminishing marginal utility, for instance, gives a psychological rationale for why preferences should be convex. Declaring by fiat that preferences relations are convex or that utility is quasi-concave, in contrast, offers no psychological justification. This paper thus gives utility functions a purpose, whereas in ordinal theory they serve only as a convenient shorthand for preference relations.

To illustrate how our ordering of measurability classes can be applied, we consider another famous rationale for the convexity of preferences, Arrow's (1951) argument (following Koopmans) that an agent's leeway to determine the precise timing of consumption implies that preferences must be convex. Arrow reasoned that this rationale for convexity, unlike diminishing marginal utility, was free of any taint of cardinality. I show, however, that the utility structure that lies behind the Arrow/Koopmans position is cardinal. Bringing these results together, we see that the classical explanation, diminishing marginal utility, rests on less demanding primitives.

Our ordering of measurement classes draws principally on two literatures. The first is Krantz, Luce, Suppes, and Tversky (1971) and kindred work in measurement theory. KLST, following Stevens (1946), identify measurement classes with sets of admissible transformations, which in an economic context are applied to utility functions. For example, ratio scales are defined by transformations that are unique up to increasing linear transformations, interval scales are defined by transformations that are unique up to increasing affine transformations, and ordinal scales are unique up to all increasing transformations. Measurement theory implicitly defines an implicit ordering of measurement classes since the sets of transformations considered are often nested – that is, ordered according to set inclusion – as in the above examples. This implicit ordering of measurement classes via sets of transformations is similar to the ordering we propose; in fact, when both are well-defined, the two orderings coincide. The drawback of the measurement theory approach is that it considers only a few prominent cases and, as I will explain presently, cannot define a sufficiently rich array of measurement classes.

The second literature consists of social choice models that vary the set of admissible transformations of utility functions according to the desired degree of interpersonal comparability (Sen (1970), Roberts (1980)). These models, which employ multiple-agent profiles of utility functions, place restrictions on what transformations can be applied to any individual utility function and on whether transformations vary across individuals. Applying a smaller set of transformations imposes a tighter interpersonal comparability requirement.

The weakness of both literatures is that they identify a standard of measurability or interpersonal comparability with a set of utility transformations. At first glance, this appears to be an advantage: any utility function can be a member of any of the standard measurement classes. But taking arbitrary sets of utility functions as primitive admits a greater variety of measurement classes and is more flexible. For instance, the set of continuous functions defines a measurement class that cannot be characterized by a set of transformations whose domain is the set of all utility functions. Moreover, as this example indicates, using utility functions as a primitive allows us to identify the implicit measurement class of assumptions on utility functions and hence to compare the measurability demands of different assumptions. To be precise, we define the measurement class of an assumption on utility functions as the set of utility functions that satisfy that assumption and that are ordinally equivalent on the domains for which the agent has well-defined preferences. For example, for agents with complete preferences, concavity is associated with the set of concave utility functions that ordinally agree. Obviously this measurement class cannot be applied to agents whose preferences do not have any concave utility representations. As we will see, the apparent drawback of this approach – that measurement classes cannot be used to compare agents who utilities satisfy different assumptions – can easily be avoided without sacrificing the advantage that measurement classes are calibrated quite finely. Seemingly nonsensical claims – e.g., "'diminishing marginal utility' is a weaker assumption than 'additive separability'" – are made rigorous.

The ordinalist approach to cardinal utility has always been a puzzle. Considerable work (e.g., KLST (1971), Debreu (1960)) has gone into specifying axioms on binary preference relations that ensure that preferences can be represented by a utility unique up to an increasing affine transformation. But the significance of such representation results remains limited. If the primitives of theory are indeed the binary relations, the cardinal utility whose existence is established has no significance beyond the notational. The purpose of once again taking utility functions as primitive, on the other hand, is immediate: nonordinal properties of utility are frequently associated plausible psychological theories and can thereby offer justifications for the preference relations they induce.

Some other work, notably Basu (1979), has also explored room for compromise between ordinal and cardinal utility theory. In the same spirit that we do, Basu contends that DMU resides in this middle ground and remarks on the advantages of taking nonordinal assumptions as primitive. But Basu sticks to method of characterizing measurement classes via utility transformations. Furthermore, as Basu (1982) indicates, the middle ground that Basu (1979) linked to DMU ends up being equivalent to full-scale cardinality in classical commodity spaces. Basu concludes that utility theory prior to the ordinal revolution used assumptions that were tantamount to cardinality (even when, as in the case of Oscar Lange, they were attempting to rid themselves of cardinalist foundations). A characterization of measurement classes via utility functions, in contrast, permits compromises between cardinality and ordinality that are robust to the specification of the commodity space.

Finally, we mention a companion paper, Mandler (1999), that applies our measurement classes to social choice. Here too, we derive compromises between ordinality and cardinality. The paper is laid out as follows. Section 2 defines psychologies, our ordering of psychologies, and cardinal and ordinal properties of utility. We also characterize the ordinal preferences that can be induced by psychologies. Outside of a technical requirement, any transitive preference relation can be induced by some psychology. We thus have a general representation result for incomplete preferences. Section 3 establishes that concavity is weaker than any cardinal property of utility and stronger than any ordinal property. Section 4 shows that the Arrow/Koopmans utility structure is cardinal. In section 5, we define extended psychologies, which cover the intransitive preferences that cannot be represented by simple psychologies. Extended psychologies also provide a more natural setting in which to take diminishing marginal utility as a primitive.

2. Psychologies

Let X be a nonempty set of consumption options and, for any nonempty $A \subset X$, let $\mathscr{F}_A$ be the set of functions from A to R. An agent is characterized by a nonempty set $\boldsymbol{U} \subset \mathscr{F}_X$, which lists the utility functions that accurately depict the agent's psychological reactions to the options in X. We say that $\boldsymbol{U}$ is a *psychology* and that X is the *domain* of $\boldsymbol{U}$.

Preference relations on X emerge straightforwardly from $\boldsymbol{U}$. Call $A \subset X$ *decisive* for $\boldsymbol{U}$ if for all $u, v \in \boldsymbol{U}$ and $x, y \in A$, $u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$. Define the binary relation $\succeq_{\boldsymbol{U}} \subset X \times X$, *the induced preference relation* of psychology $\boldsymbol{U}$, by $x \succeq_{\boldsymbol{U}} y$ if and only if there exists a decisive $A \supset \{x, y\}$ and a $u \in U$ such that $u(x) \geq u(y)$. Psychologies can be endowed with most of the standard properties of ordinal preferences. For instance, we define $\succeq_U$ or U to be *complete* if X is decisive. On the other hand, psychologies as we have defined them plainly cannot be intransitive. As the theorem below reports, however, outside of transitivity and a standard technical requirement, any preference relation can induced by a psychology. In section 5, we introduce extended psychologies, which can cover the intransitive cases omitted here.

A binary relation $\succeq$ on X is *countably order-dense* if there exists a countable $Y \subset X$ such that for all $x, z \in X$ with $x \succeq z$ and not $z \succeq x$, there exists a $y \in Y$ such that $x \succeq y \succeq z$.

Theorem 2.1 The binary relation $\succeq$ on X is transitive and countably order-dense if and only if there exists a psychology U with domain X whose induced preference relation is $\succeq$.

Proof: For any $u \in U$, let $\succeq_u$ denote the preference relation induced by $\{u\}$. Given that $\succeq_U \subset \succeq_u$ for any $u \in U$, the proof that $\succeq_U$ is countably order-dense is standard and we omit it.

If $\succeq$ is complete in addition to transitive and order-dense, the proof that there exists a U with $\succeq_U = \succeq$ is the standard existence theorem for utility functions. So assume that $\succeq$ is not complete. Let $X/\sim$ denote the indifference classes of $\succeq$ and define $\succ$ on $X/\sim$ by $I \succ J$ if and only if $x \succeq y$ for some $x \in I$ and $y \in J$. For any $x, y \in X$ such that neither $x \succeq y$ nor $y \succeq x$ holds, let $I(x)$ and $I(y)$ denote the indifference classes that x and y, respectively, belong to. Define two strict partial orders $\succ_x$ and $\succ_y$ on $X/\sim$ by $\succ \cup\, (I(x), I(y))$ and $\succ \cup\, (I(y), I(x))$ respectively. Let $\succ_x^t$ and $\succ_y^t$ denote the transitive closures of $\succ_x$ and $\succ_y$ respectively.

By assumption, there is a countable set of indifference classes, say Y, that is order-dense with respect to $\succ$. Let $Y' = Y \cup \{I(x), I(y)\}$. To see that Y' is order-dense with respect to $\succ_x^t$ and $\succ_y^t$, suppose not. Then, to take the case of $\succ_x^t$, there exist $I, J \in X/\sim \setminus Y'$ such that $I \succ_x^t J$ and such that for all $K \in Y'$, not $I \succ_x^t K \succ_x^t J$. That is, there are two indifference classes not in Y' that are unranked according to $\succ_x$ but that are ranked according to $\succ_x^t$. By the definition of a transitive closure, there must exist a finite set of indifference classes, say $I_1, \ldots, I_n$ such that $I \succ_x I_1 \succ_x \ldots \succ_x I_n \succ_x J$. But since $\succ_x$ is transitive, at least one of the elements I_1 to I_n has to be $I(x)$ or $I(y)$, which contradicts the assumption that Y' is not order-dense. By Theorem 3.2 of Fishburn (1979) (which generalizes Richter (1966)), there exists a utility function $u_{x,y}$ on $X/\sim$ such that $L \succ_x^t M$ implies $u(L) > u(M)$. Similarly, there exists a $u_{y,x}$ on $X/\sim$ such that $L \succ_y^t M$ implies $u(L) > u(M)$. Define the utility functions $v_{x,y}$ and $v_{y,x}$ by letting each element of any indifference class inherit the utility number of its indifference class given by $u_{x,y}$ and $u_{y,x}$ respectively. Let $\boldsymbol{U}$ be defined by $v \in \boldsymbol{U}$ if and only if $v \in \{v_{x,y}, v_{y,x}\}$ for some $x, y \in X$ such that not $x \succeq y$ and not $y \succeq x$. It is immediate that $\succeq_{\boldsymbol{U}} = \succeq$. ∎

Theorem 2.1's characterization of representable preferences differs in only one respect from standard utility representation results: we have dropped completeness as an assumption. Transitivity and countable order-density are retained without change.

Ok (1999) has also recently discussed the question of when an incomplete preference relation $\succeq$ can be represented by a vector-valued function $\mathbf{u}$. Ok's definition of representation is the same as ours, except that he requires that the range of $\mathbf{u}$ is finite-dimensional (specifically, $\mathbf{u}: X \to R^n$ represents $\succeq$ if $x \succeq y \Leftrightarrow \mathbf{u}(x) \geq \mathbf{u}(y)$, for all $x, y \in X$). Ok presents several conditions that are sufficient for representability, but a characterization of finite-dimensional representability remains elusive. As Theorem 2.1 indicates, infinite-dimensional representability is tackled more easily; clear-cut necessary and sufficient conditions are available.

We now introduce the key ordering of psychologies. Some of the definitions to follow are difficult on first reading. Since most of the complexities disappear when psychologies are complete, we use the complete case to illustrate some of the definitions. For any $A \subset X$, let $\boldsymbol{U}|A$ (the restriction of $\boldsymbol{U}$ to A) denote the set $\{w \in \mathscr{F}_A: w = u|A$ for some $u \in$

$U\}$. We say that psychologies U and V *have the same decisive sets* if, for all $A \subset X$, A is decisive for $U \Leftrightarrow A$ is decisive for V.

Definition 2.1 Psychology U is *weaker than* psychology V if U and V have the same decisive sets and, for each decisive A, $U|A \supset V|A$.

If U and V are both complete, Definition 2.1 reduces to U is weaker than V if $U \supset V$.

Remark. An alternative ordering of psychologies – namely that U is weaker than V if, for each $A \subset X$ that is decisive for U, $U|A \supset V|A$ – is sharper in the sense that more pairs of psychologies are ranked. The additional discrimination is unnecessary for our applications, however, and so, for simplicity, we use Definition 2.1 as stated.

It is immediate that the "weaker than" relation on psychologies is transitive and, when $|X| > 1$, incomplete. As usual, we define U to be *strictly weaker* than V if U is weaker than V and V is not weaker than U.

Our primary use of this ordering is as a device for comparing the strength of properties of utility functions.

Definition 2.2 A psychology U with domain X *satisfies property* p if (1) for all $u \in U$ and all A that are decisive for U, there exists a $B \supset A$ that is decisive for U such that $u|B$ satisfies property p, and (2) there does not exist a larger psychology with the same decisive sets as U that meets condition (1) (i.e., there is no $V \supsetneq U$ with the same decisive sets as U where, for all $v \in V$ and all decisive A, there exists a decisive $B \supset A$ such that $v|B$ satisfies p).

In words, U satisfies p if it is largest among psychologies that share the same family of decisive sets and that, for each u in U and each decisive A, own a decisive B containing A such that u satisfies p on B. When psychologies are complete, Definition 2.2 reduces to: a complete U satisfies p if, for each $u \in U$, u satisfies property p, and there does not exist a larger (complete) psychology V such that each $v \in V$ satisfies p. In the general case, the need for the "containing" sets B in Definition 2.2 is unavoidable: since some properties (e.g., quasi-concavity or concavity) can only be satisfied on certain domains (convex sets), we cannot speak of those properties as satisfied on arbitrary decisive sets. Although immaterial in many economic applications, the dependence of "satisfying a property" in Definition 2.2 on the ambient domain X is also unavoidable. For example, if $|X|$ is finite, any psychology satisfies continuity, but if not, e.g., $X = R^n$, continuity imposes binding restrictions on which utility functions are allowable.

Given what it means to satisfy a property, our earlier ordering of psychologies induces an ordering of properties of utility functions.

Definition 2.3 Property p is *weaker than* property q on domain X if, for all U with domain X that satisfy p and all V with domain X that satisfy q such that (1) U and V have the same decisive sets and (2) $U|A \cap V|A \neq \varnothing$ for all decisive A, U is weaker than V. Property p is *strictly weaker than* property q if p is weaker than q and q is not weaker than p.

If we restrict ourselves to complete psychologies, property p is weaker than property q on domain X if, for all U with domain X that satisfy p and all V with domain X that satisfy q, $U \cap V \neq \varnothing$ implies $U \supset V$.

We can rephrase the strict part of Definition 2.3 as follows: p is strictly weaker than q if p is weaker than q and there exists a U that satisfies p and a V that satisfies q together meeting conditions (1) and (2) of the definition such that, for some decisive A, $U|A \supsetneq V|A$. It is easy to confirm that the relations "at least as strong as" and "stronger than" are each transitive and, when $|X| > 1$, incomplete.

Definition 2.4 (Ordinality) The functions u and v *agree* on $A \subset X$ if, for all $x, y \in A$, $u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$. A psychology $\boldsymbol{U}$ is *ordinal* if $u \in \boldsymbol{U}$ implies that, for all v with domain X such that u and v agree on all decisive A, $v \in \boldsymbol{U}$.

Equivalently, psychologies are ordinal if $u \in \boldsymbol{U} \Rightarrow (v \in \boldsymbol{U} \Leftrightarrow v \in \mathscr{F}_X$ and, for each decisive A, there exists an increasing transformation G such that $v|A = G \circ u|A)$.

Definition 2.5 (Cardinality) A function $G: E \subset R \to R$ is an *increasing affine transformation* if there exist $a > 0$ and b such that, for all $x \in E$, $G(x) = ax + b$. A psychology $\boldsymbol{U}$ is *cardinal* if $u \in \boldsymbol{U} \Rightarrow (v \in \boldsymbol{U} \Leftrightarrow v \in \mathscr{F}_X$ and, for each decisive A, there exists an increasing affine transformation G such that $v|A = G \circ u|A)$.

We can now define assumptions on utility functions as ordinal or cardinal.

Definition 2.6 A property p is *ordinal* (resp. *cardinal*) on X if any $\boldsymbol{U}$ with domain X that satisfies p is ordinal (resp. cardinal).

Most of the standard assumptions used nowadays in utility theory are ordinal properties. As an example, consider quasi-concavity. A function $u: Z \to R$ is *quasi-concave* if Z is convex and, for all $x, y \in Z$ and $\lambda \in [0, 1]$, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$. To confirm that quasi-concavity is an ordinal property on an arbitrary domain X, let $\boldsymbol{U}$ satisfy quasi-concavity, let u be an arbitrary element of $\boldsymbol{U}$, and suppose that, for all decisive A, $u|A$ and $v|A$ agree. For any decisive A, there exists a decisive $B \supset A$ such that $u|B$ satisfies quasi-concavity (due to (1) of Definition 2.2). Since B is decisive, $u|B$ and $v|B$ agree. Since $u|B$

and $v|B$ agree, there is an increasing transformation F: Range $u|B \to R$ such that $F \circ u|B = v|B$; since F is increasing, for all $x, y \in B$ and all $\lambda \in [0, 1]$, $v(\lambda x + (1-\lambda)y) \geq \min\{v(x), v(y)\}$. Since $u|B$ satisfies quasi-concavity, B is convex; hence $v|B$ satisfies quasi-concavity. So, by (2) of Definition 2.2, $v \in \boldsymbol{U}$.

As an example of a cardinal property, we consider additive separability, which will later be important in our examination of the convexity of preferences.

Definition 2.7 Let $Y_1, \ldots, Y_n$ be nonempty sets and let $Y = Y_1 \times \ldots \times Y_n$. A function $u: A \to R$, $A \subset Y$, satisfies *additive separability* if (1) there exist component functions $u_i: A_i \to R$, $i = 1, \ldots, n$, such that, for each $x \in A$, $u(x) = \sum_{i=1}^{n} u_i(x_i)$, (2) for each component i, cl Range u_i is an interval, and (3) two of these intervals have nonempty interior.

Theorem 2.2 Additive separability is a cardinal property on Y.

Two features distinguish Theorem 2.2 from the existing literature on additive separability (see Debreu (1960) and KLST(1971)). First, since psychologies contains sets of functions – and those functions need not be ordinally identically – Theorem 2.2 extends classical cardinality results; specifically, incomplete preferences are covered. Second, the standard approach to additively separable functions proves cardinality as a by-product of existence theorems that specify conditions on ordinal preferences that imply the existence of an additively separable utility representation. Given the difficulty of the existence question, however, this technique ends up imposing overly strong restrictions. By separating cardinality from existence, Theorem 2.2 makes do with much weaker conditions relative to the literature (which usually supposes that utility functions are continuous).

Proof of Theorem 2.2: Let $\boldsymbol{U}$ with domain Y satisfy additive separability and let u be an arbitrary element of U. For any decisive A, let $B_A \supset A$ denote a decisive set such that $u|B_A$ satisfies additive separability. Given $v \in \mathscr{F}_Y$, suppose for each decisive A that there exists an increasing affine transformation G: Range $u|A \rightarrow R$ such that $G \circ u|A = v|A$. In particular, for the decisive set B_A, there will then exist an increasing affine transformation G such that $G \circ u|B_A = v|B_A$. Clearly, $v|B_A$ satisfies additive separability. So $v \in U$.

In the other direction, we must show, for any $v \in U$ and any decisive A, that there exists an increasing affine transformation G: Range $u|A \rightarrow R$ such that $v|A = G \circ u|A$. The remainder of the proof considers a fixed A and the associated $B \supset A$ such that $u|B$ satisfies additive separability. It is sufficient to show that if G is increasing and $G \circ u|B$ satisfies additive separability (i.e., $G \circ u|B \in U|B$), then G is affine. (For simplicity, we henceforth drop the notation "$|B$" that indicates the restriction of v, u, etc., to B.)

Note that since each cl Range u_i is an interval there exists some x' such that, for all i with u_i nonconstant, $u_i(x_i') \in$ Int cl Range u_i. By adding constants to the u_i, there exists an increasing affine transformation that, when applied to u, yields a g: $B \rightarrow R$ that is additively separable and that satisfies $g_i(x_i') = 0$ for all i. Clearly, there is also an increasing affine transformation that, when applied to g, yields u.

Consider an increasing transformation G that, when applied to u, yields an additively separable h: $B \rightarrow R$. For each i, define k_i: $B_i \rightarrow R$ by $k_i(x_i) = h(x_i) - h(x_i')$ and define k: $B \rightarrow R$ by $k(x) = \sum_{i=1}^{n} k_i(x_i)$. Since u is an increasing affine transformation of g, h is an increasing transformation of u, and k is an increasing affine transformation of h, (1) there is an increasing transformation F: Range $g \rightarrow R$ such that $F \circ g = k$, and (2) if F is linear, G is affine.

We first show that F is continuous. If not, let $\bar{x} \in B$ be a point such that F is not continuous at $g(\bar{x})$. Since there are at least two components such that cl Range u_i has nonempty interior, there is a component i such that $g_i(\bar{x}_i) \in$ Int cl Range g.

For any l, by setting $x_j = x_j'$ for $j \neq l$, we have $F(g_l(x_l)) = k_l(x_l)$ for all x_l. Hence,

(*) $\quad F(\sum_{l=1}^{n} g_l(x_l)) = \sum_{l=1}^{n} k_l(x_l) = \sum_{l=1}^{n} F(g_l(x_l))$ for all $x \in B$.

In particular, for any given set of $\hat{x}_l$, $l \neq i$,

(**) $\quad F(g_i(x_i)) + \sum_{l \neq i} F(g_l(\hat{x}_l)) = F(g_i(x_i) + \sum_{l \neq i} g_l(\hat{x}_l))$ for all $x_i \in B_i$.

Hence, using $\hat{x}_l = \bar{x}_l$, $\ell \neq i$, and given that cl Range g_i is a nondegenerate interval, the discontinuity of F at $g(\bar{x})$ implies that F is also not continuous at $g_i(\bar{x}_i)$.

Since F is increasing, $F(z+)$, the right hand limit of F at z, exists at any $z \neq \max_{x \in B} g(x)$ and $F(z-)$, the left hand limit of F at z, exists at any $z \neq \min_{x \in B} g(x)$. Define $J(z)$ to equal $F(z+) - F(z-)$ if z is a nonboundary point, $F(z+) - F(z)$ if $z = \min_{x \in B} g(x)$, and $F(z) - F(z-)$ if $z = \max_{x \in B} g(x)$. The discontinuity of F at $g_i(\bar{x}_i)$ implies $J(g_i(\bar{x}_i)) > 0$. Since there exists some component $j \neq i$ such that cl Range g_j is a nondegenerate interval containing 0 and given *, we can, by varying only $g_j(x_j)$ and setting $g_i(x_i) = g_i(\bar{x}_i)$ and $g_l(x_l) = 0$ for $l \neq i, j$, assemble a set $Q \subset$ Range g that is dense on a bounded nondegenerate interval and such that $g_i(\bar{x}_i) \in Q$. Given **, $J(g_i(x_i)) = J(g_i(x_i) + \sum_{l \neq i} g_l(\hat{x}_l))$ for any $g_i(x_i)$ and any set of $\hat{x}_l$, $l \neq i$. Hence, for any $z \in Q$, by setting $g_i(x_i) = g_i(\bar{x}_i)$ and $g_j(\hat{x}_j) = z - g_i(\bar{x}_i)$ (and the remaining $g_l(x_l) = 0$), we have $J(g_i(\bar{x}_i)) = J(z) > 0$. Given that Q has an infinite number of elements and F is increasing, $J(z) > 0$ for all $z \in Q$ contradicts the fact that Q is bounded. Hence F is continuous.

Given the continuity of F, F has a unique continuous extension on cl Range g, say F_e. Given ** we have,

(***) $$F_e(\sum_{l=1}^{n} a_l) = \sum_{l=1}^{n} F_e(a_l)$$

for all $a \in$ cl Range $g_1 \times \ldots \times$ cl Range g_n.

We turn to the linearity of F. Fix some $d > 0$ that satisfies $d \in$ cl Range g_i for all i such that g_i is not a constant function. (d exists since, for all i, $g_i(x_i') = 0$, cl Range g_i is an interval, and, when cl Range g_i is not a singleton, $0 \in$ Int cl Range g_i.) Consider any $e' > 0$ that is an element of Range g_i for some i. For all $\epsilon > 0$, there exists a $e \in$ cl Range g_i and rational r such that $dr = e$ and $|e - e'| < \epsilon$. Let s and t be positive integers such that $r = s/t$.

We have $d/t \in$ cl Range g_i for any i such that g_i is nonconstant. Let l and j be coordinates of nonconstant g_i. Choosing $a \in R^n$ such that $a_l = a_j = d/t$ and the remaining $a_i = 0$, *** implies $F_e((d/t) + (d/t)) = 2\, F_e(d/t)$. Iterating this argument t times, we have $F_e(d) = tF_e(d/t)$. (Since, for any positive integer $m \le t$, $(md)/t \le d$, $(md)/t \in$ cl Range g_l and $(md)/t \in$ cl Range g_j, which permits each stage of the iteration.) Changing the coordinate l if necessary so that $e \in$ cl Range g_l, apply the same iteration argument to conclude $F_e((sd)/t) = sF_e(d/t)$. (We now have, for any positive integer $m \le s$, that $(md)/t \le e$ and hence $(md)/t \in$ cl Range g_l.) So $F_e(e) = sF_e(d/t) = F_e(d)(s/t) = F_e(d)r = (F_e(d)/d)e$. The continuity of F_e implies that F, when restricted to $e \in \bigcup_{i=1}^{n}$ Range g_i such that $e > 0$, is linear. Now consider any $e \in$ Range g such that $e > 0$. For any such e, there exists a $a \in R_+^n$ such that each $a_i \in$ Range g_i, $e = \sum_{i=1}^{n} a_i$, and $F(e) = \sum_{i=1}^{n} F(a_i)$. Hence, $F(e) = \sum_{i=1}^{n} a_i(F(d)/d) = e(F(d)/d)$. So F is linear on positive points of its domain.

Since we can repeat the argument of the previous paragraph for $d < 0$ and $e' < 0$, F can be locally nonlinear only at 0 (and G therefore is locally nonaffine only at $u(x')$). By repeating our construction with some $\hat{x}$ such that $u(\hat{x}) \neq u(x')$ we can define new functions $\hat{k}$, $\hat{g}$, and $\hat{F}$ (where $\hat{k}$ is additively separable, $\hat{g}$ is an increasing affine transformation of u, and $\hat{F}$ is increasing) that satisfy $\hat{F} \circ \hat{g} = \hat{k}$. Just as with F, $\hat{F}$ can be locally nonlinear only at 0 and G can be locally nonaffine only at $u(\hat{x})$. G is therefore locally affine at $u(x')$ and so F is locally linear at 0. ■

3. Convexity of preferences I: concavity as a primitive

This section and the next present rationales for why a preference relation $\succeq$ should be convex, i.e., why given a convex domain X, the set $\{x \in X: x \succeq y\}$ should be convex for all y.

Definition 3.1 A function $u: Z \to R$ satisfies *concavity* if Z is convex and, for all $x, y \in Z$ and all $\lambda \in [0, 1]$, $u(\lambda x + (1-\lambda)y) \ge \lambda u(x) + (1-\lambda)u(y)$.

Theorem 3.1 If psychology $\boldsymbol{U}$ with domain X satisfies concavity, $\succeq_U$ is convex.

Proof: Let $\boldsymbol{U}$ be a psychology with domain X that satisfies concavity and let u be an arbitrary element of $\boldsymbol{U}$. For any $y \in X$, let $y_p = \{x \in X\text{: for all } v \in \boldsymbol{U}, v(x) \geq v(y)\}$. Since $\boldsymbol{U}$ satisfies concavity, there exists a decisive $B \supset y_p$ such that $u|B$ satisfies concavity. Since B is convex, $u|\text{co } y_p$ satisfies concavity (where co is the convex hull). Hence, for all $x \in \text{co } y_p$ and all $v \in \boldsymbol{U}$, $v(x) \geq v(y)$. So co $y_p = y_p$, i.e., $\{x \in X\text{: } x \succeq_U y\}$ is convex. ■

Definition 3.2 A property p is *nonconstant* on X if for any $\boldsymbol{U}$ with domain X that satisfies p, there exists a decisive A and $u \in \boldsymbol{U}$ such that $u|A$ is nonconstant.

Theorem 3.2 Any ordinal property is weaker than concavity and concavity is weaker than any cardinal property. If these ordinal and cardinal properties are in addition nonconstant, then "weaker" may be replaced by "strictly weaker."

Theorem 3.2 confirms a simple intuition about concavity. Along a line, concavity as a psychology assumes that an agent sees each successive unit of consumption as delivering a diminishing utility increment. But concavity does not require that that increment is a specific fraction of the previous utility increment. Agents experience diminishing marginal utility but no additional extra-ordinal precision. In contrast, cardinality requires that agents experience any pair of utility increments to equal a precise ratio. Cardinality thus imposes considerably more – indeed, implausibly more – psychological structure.

Proof of Theorem 3.2: Consider first concavity and cardinality. Let $\boldsymbol{U}_C$ be cardinal, let $\boldsymbol{U}_{CV}$ satisfy concavity, let $\boldsymbol{U}_C$ and $\boldsymbol{U}_{CV}$ have the same decisive sets, and assume for all decisive A that there exist $u \in \boldsymbol{U}_C$ and $v \in \boldsymbol{U}_{CV}$ such that $u|A = v|A$. For any decisive A, let $B_A \supset A$ denote a convex and decisive set such that $v|B_A$ satisfies concavity. Since B_A is itself decisive, by assumption there exist $\hat{u} \in \boldsymbol{U}_C$ and $\hat{v} \in \boldsymbol{U}_{CV}$ such that $\hat{u}|B_A = \hat{v}|B_A$. Since $\hat{v}|B_A \in \boldsymbol{U}_C|B_A$, for any $u' \in \boldsymbol{U}_C$ there exists an increasing affine transformation G such that $G \circ \hat{v}|B_A = u'|B_A$. Since an increasing affine transformation of a concave function is concave, $u'|B_A$ satisfies concavity. Moreover, since $u'|B_A$ satisfies concavity for the B_A that corresponds to any decisive A, $u' \in \boldsymbol{U}_{CV}$. Hence, concavity is weaker than any cardinal property. Now assume in addition that $\boldsymbol{U}_C$ contains a u'' that is nonconstant on some decisive A'. Let g: $R \to R$ be strictly concave and let v be an element of $\boldsymbol{U}_{CV}$. For any decisive A there exists a decisive $B \supset A$ such that $v|B$ and hence $g \circ v|B$ satisfy concavity. So $g \circ v \in \boldsymbol{U}_{CV}$. Since $B_{A'}$ is convex and g is continuous, Range $u''|B_{A'}$ is a nontrivial interval. So $g|(\text{Range } u''|B_{A'})$ is not affine and therefore $g \circ u'' \notin \boldsymbol{U}_C$. Hence, concavity is strictly weaker than any nonconstant cardinal property.

As for concavity and ordinality, any ordinal property is weaker than any property. To show that any nonconstant ordinal property is strictly weaker than concavity, let $\boldsymbol{U}_O$ be ordinal, let $\boldsymbol{U}_{CV}$ satisfy concavity and have the same decisive sets, suppose $u \in \boldsymbol{U}_O$ is nonconstant on some decisive A', and suppose for all decisive A that there exist $o \in \boldsymbol{U}_O$ and $v \in \boldsymbol{U}_{CV}$ such that $o|A = v|A$. For some convex and decisive $B \supset A'$, $v|B$ satisfies concavity and Range $v|B$ is a nontrivial interval. Let $x, z \in B$ satisfy $v(x) < v(z)$ and define $D = \{t \in B\colon t = \lambda x + (1 - \lambda)z$ for some $\lambda \in [0, 1]\}$. Let $C \subset D$ be a connected set such that v is monotonic on C and let $y = (x + y)/2$. So $v(x) < v(y) < v(z)$. Hence there exists an increasing transformation g: $R \to R$ such that $g(v(y)) < (1/2)g(v(x)) + (1/2)g(v(z))$. We then have $g \circ v \notin \boldsymbol{U}_{CV}$, but since g is increasing, $g \circ v \in \boldsymbol{U}_O$. ■

Remark. By strengthening our ordering of properties somewhat, we can tighten the "strictness" part of Theorem 3.2. Let property p be *definitively weaker* than q on domain X if, for all $\boldsymbol{U}$ with domain X that satisfy p and all $\boldsymbol{V}$ with domain X that satisfy q such that (1) $\boldsymbol{U}$

and V have the same decisive sets, and (2) there exists $u \in U$ and $v \in V$ with $u|A = v|A$ for all decisive A, U is strictly weaker than V. (Whereas property p is strictly weaker than q if it is merely the case that p is weaker than q and there is some U that satisfies p and some V that satisfies q such that (1) and (2) are satisfied and U is strictly weaker than V.) The above proof then establishes that concavity is definitively weaker than nonconstant cardinality and nonconstant ordinality is definitively weaker than concavity.

In closing this section, we note that concavity can be ranked relative to some other classical assumptions of utility theory.

Theorem 3.3 Continuity is weaker than concavity and ordinality is weaker than continuity. If we replace "continuity" with "continuity and nonconstancy," we may replace "weaker" with "strictly weaker."

Theorem 3.3 follows from the fact that any concave function is continuous, but not vice versa and the fact that any continuous increasing transformation preserves continuity, but noncontinuous increasing transformations do not preserve continuity. We omit the details, which vary only slightly from the proof of Theorem 3.2. Given that concavity is weaker than any cardinal property (and the transitivity of the ordering of properties), Theorem 3.3 implies continuity is also a middle ground between cardinality and ordinality.

The measurement classes of concavity and continuity are each associated with a set of utility transformations, namely increasing concave and increasing continuous functions from R to R. This association is not shared by all measurement classes. Moreover, even for the cases at hand, the measurement classes should not be confused with their associated transformations: the transformation must usually be applied to a function from within the measurement class. For instance, an increasing concave transformation of an arbitrary function obviously need not be concave.

4. Convexity of preferences II: the Arrow/Koopmans theory

Arrow (1951), following unpublished remarks by Koopmans, argued that if agents hold a consumption bundle for a period of time, say $[0, T]$, and can decide on the timing of how that bundle is consumed, preferences must be convex. Arrow argued that this explanation of convexity, unlike the supposedly cardinalist explanations that rely on diminishing marginal utility, is free from any taint of cardinality. We follow Grodal's (1974) formalization of the Arrow/Koopmans theory.

Assume that a binary relation $\succeq$ on R_+^n can be represented by a utility function U: $R_+^n \to R$ that takes the form

$$U(z) = \sup_x \int_0^T u(x(t), t)\, d\mu(t) \quad \text{s.t.} \quad \int_0^T x_i(t)\, d\mu(t) \leq z_i,\ i = 1, \ldots, n,$$

where x: $[0, T] \to R^n$, u: $R_+^{n+1} \to R$, μ is Lebesgue measure, $t \to u(x(t), t)$ is (Lebesgue) integrable, $z \in R_+^n$, and where the supremum is taken over all integrable x such that $x_i(t) \geq 0$ for all t, $i = 1, \ldots, n$.

Theorem 4.1 U is concave and therefore $\succeq$ is convex.

Proof: Grodal (1974). ▮

We turn to the measurability class of the utility function, $\int_0^T u(x(t), t)\, d\mu(t)$, that underlies the above maximization problem. We generalize somewhat.

Definition 4.1 Let X be a set of (Lebesgue) measurable functions from $[0, T]$ to R^n such that if $z(t) = x(t)$ for some $x \in X$ for a.e. $t \in [0, T]$, then $z \in X$.

The constraint set $\{x: x_i(t) \geq 0$ for all i and t, and $\int_0^T x_i(t)\, d\mu(t) \leq z_i$ for all $i\}$ that underlies the definition of $U(z)$ is a sample case satisfying Definition 4.1..

Definition 4.2 A function $F: X \to R$ satisfies *utility integrability* if there exists a $u: R_+^{n+1} \to R$ with $t \to u(x(t), t)$ integrable such that $F(x) = \int_0^T f(x(t), t)\, d\mu(t)$ and $|\text{Range } F| > 1$.

Theorem 4.2 Utility integrability is a cardinal property on X.

Proof: Let $\boldsymbol{U}$ with domain X satisfy additive separability and let F be an arbitrary element of $\boldsymbol{U}$. We must show (1) if $V: X \to R$ is such that for all A that are decisive for $\boldsymbol{U}$ there exists an increasing affine transformation G that satisfies $G \circ F|A = V|A$, then V satisfies integrability, and (2) for any $V \in \boldsymbol{U}$ and any decisive A there exists an increasing affine transformation G such that $V|A = G \circ F|A$. The proof of (1) is identical to the beginning of the proof of Theorem 2.2. As for (2), we consider henceforth a fixed A and the associated $B \supset A$ such that $u|B$ satisfies utility integrability. It is sufficient to show that if G is an increasing transformation and $G \circ F|B$ satisfies integrability (i.e., $G \circ F|B \in \boldsymbol{U}$), then G is affine.

Observe that since there exist $x, x' \in X$ such that $F(x) > F(x')$, there also exists, for any $\epsilon > 0$, a measurable $C_1 \subset [0, T]$ such that $0 < \int_{C_1} (f(x(t), t) - f(x'(t), t))\, d\mu(t) < \epsilon$. By setting ϵ sufficiently small, we can partition $[0, T]$ into sets C_1 and C_2 such that $\int_{C_2} (f(x(t), t) - f(x'(t), t))\, d\mu(t) > 0$. For $i = 1, 2$, let X_i be the restriction of X to C_i (i.e., X_i is the set of functions from C_i to R^n defined by $x_i \in X_i$ if and only if there exists $x \in X$ such that $x_i(t) = x(t)$ for all $t \in C_i$), and let $F_i: X_i \to R$ be defined by $F_i(x_i) = \int_{C_i} f(x_i(t), t)\, d\mu(t)$. We have $X = X_1 \times X_2$, and $F(x) = F_1(x_1) + F_2(x_2)$ for all $x \in X$. Since, for each i, $F_i(x_i) > F_i(x_i')$,

$|\text{Range } F_i| > 1$.

We use the following classical result from the theory of integration of correspondences.

Lyapunov's theorem. Given the correspondence $P: \Omega \to R^m$ and the measure space $(\Omega, \mathscr{F}, \lambda)$, let $\int_\Omega P\, d\lambda = \{\int_\Omega p\, d\lambda: p$ is integrable and $p(\omega) \in P(\omega)$ for a.e. $\omega \in \Omega\}$. If λ is atomless, $\int_\Omega P\, d\lambda$ is a convex set.

By Lyapunov's theorem, Range F, Range F_1, and Range F_2 are convex sets and therefore intervals. (For example, for the case of Range F_1, in the statement of Lyapunov's theorem the measure space is Lebesgue measure on C_1 and $P(t) = \{f(x_i(t)): x_i \in X\}$ for each $t \in C_1$.) Since $|\text{Range } F_1| > 1$ and $|\text{Range } F_2| > 1$, Range F_1 and Range F_2 have nonempty interiors.

If there exists an integrable $h: R^{n+1} \to R$ such that $G(F(x)) = \int_0^T h(x(t), t)\, d\mu(t)$ for all $x \in X$, we have $G(F(x)) = H_1(x_1) + H_2(x_2)$, for all x, where each $H_i(x_i) = \int_{C_i} h(x_i(t), t)\, d\mu(t)$. Apply Theorem 2.2 to conclude that G is affine. ▮

Theorem 4.2 implies that the Arrow/Koopmans theory imposes stricter measurability requirements on agents than does concavity. Thus, despite its preeminent position in preordinal utility theory, concavity is comparatively near to ordinalist standards of measurability.

5. Extended psychologies (*preliminary – proofs omitted*)

We expand the psychological model of section 2 by taking sets of utility functions defined on subsets of the space of consumption options as primitive. This generalization has two advantages. First, the preferences that cannot be induced by the model of section 2 – see Theorem 2.1 – can trivially be accommodated in the expanded setting. Extended psychologies are thus a more flexible tool than preference relations. Second, assumptions on utility functions defined on the entire set of consumption options are often counterintuitive, whereas assumptions defined on subsets can sometimes target genuine psychological primitives more precisely. As we will see, diminishing marginal utility is a pertinent case in point.

Let X again denote an arbitrary nonempty set of consumption possibilities. An *extended psychology* $\boldsymbol{U}$ on X is a family of functions such that $u \in \boldsymbol{U}$ if and only if $u \in \mathscr{F}_A$ for some $A \subset X$. For any $A \subset X$, $\mathscr{F}_A \cap \boldsymbol{U}$ should be interpreted as the set of utility functions on A that accurately depict the agent's psychological reactions to the consumption possibilities in A.

For any extended psychology $\boldsymbol{U}$ on X, let edom $\boldsymbol{U}$ (the extended domain of $\boldsymbol{U}$) denote $\{A \subset X: A = \text{Domain } u \text{ for some } u \in \boldsymbol{U}\}$. If edom $\boldsymbol{U} = \{X\}$, then the psychology is *simple*, i.e., a psychology in the sense of section 2. We assume that $\bigcup_{B \in \text{edom } \boldsymbol{U}} B = X$: each element of the set of consumption choices is in the domain of one of the utilities in $\boldsymbol{U}$.

We now define $A \subset X$ to be *decisive* for $\boldsymbol{U}$ if, for all $x, y \in A$ and all $u, v \in \boldsymbol{U}$ such that $\{x, y\} \subset \text{Domain } u \cap \text{Domain } v$, $u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$.

Preference relations are induced by extended psychologies in the same way they are induced by the (simple) psychologies of section 2: given an extended psychology $\boldsymbol{U}$, define $R_{\boldsymbol{U}}$ by $x \, R_{\boldsymbol{U}} \, y$ if and only if there exists a decisive $A \supset \{x, y\}$ and a $u \in \boldsymbol{U}$ such that $u(x) \geq u(y)$. It is easy to show that for any binary relation $\succeq$ on X, there exists an extended psychology $\boldsymbol{U}$ on X such that $R_{\boldsymbol{U}} = \succeq$. A suitable $\boldsymbol{U}$ can be assembled in many ways: for instance, for each $(x, y) \in \succeq$, let $u_{\{x,y\}}: \{x, y\} \to R$ be a function that satisfies $u(x) \geq u(y)$ if and only if $x \succeq y$, and let $\boldsymbol{U} = \bigcup_{\{x,y\} \in \succeq} \{u_{\{x,y\}}\}$. Extended psychologies therefore constitute a more general model of agents than ordinal preferences.

Obviously, extended psychologies are also more general than simple psychologies. Unfortunately, defining an ordering of extended psychologies that is sufficiently discriminating is less straightforward. Consider the following extension of the ordering of psychologies defined in section 2. Recall that for a simple psychology $\boldsymbol{U}$ and a $A \subset X$, $\boldsymbol{U}|A$ (the restriction of $\boldsymbol{U}$ to A) is the set $\{w \in \mathscr{F}_A : w = u|A$ for some $u \in \boldsymbol{U}\}$. The same definition holds without change for extended psychologies. Define $\boldsymbol{U} \preceq_E \boldsymbol{V}$ if (1) $\boldsymbol{U}$ and $\boldsymbol{V}$ have the same decisive sets, and (2) for any decisive A, $\boldsymbol{U}|A \supset \boldsymbol{V}|A$. The difficulty is that $\preceq_E$ can allow too few comparisons. For instance, even if A is decisive for $\boldsymbol{U}$ and $\boldsymbol{V}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ may not have extended domains containing a set that contains A; ranking according to $\preceq_E$ is then impossible. It is helpful, therefore, to define simple psychologies that summarize extended psychologies; the problem of limited extended domains then disappears and there will be greater scope for ranking.

Definition 5.1. $\boldsymbol{S} \subset \mathscr{F}_X$ is a *summary* of the extended psychology $\boldsymbol{U}$ if (1) for each decisive A, $\boldsymbol{S}|A = \boldsymbol{U}|A$, and (2) there does not exist a $\boldsymbol{S}' \supset \boldsymbol{S}$, where $\boldsymbol{S}' \subset \mathscr{F}_X$, that satisfies (1).

Clearly, an extended psychology can have at most one summary. Also, the preferences induced by a summary of an extended psychology $\boldsymbol{U}$ coincide with $R^{\boldsymbol{U}}$.

We may define an extended psychology $\boldsymbol{U}$ to be *weaker than* extended psychology $\boldsymbol{V}$ if either $\boldsymbol{U} \preceq_E \boldsymbol{V}$ or both $\boldsymbol{U}$ and $\boldsymbol{V}$ have summaries and the summary of $\boldsymbol{U}$ is weaker than the summary of $\boldsymbol{V}$ (in the sense of Definition 2.1). It is easy to confirm that if $\boldsymbol{U} \preceq_E \boldsymbol{V}$ and $\boldsymbol{U}$ and $\boldsymbol{V}$ have summaries, then the summary of $\boldsymbol{U}$ is weaker than the summary of $\boldsymbol{V}$ and that the "weaker than" relation on extended psychologies is transitive.

Of course, summaries of extended psychologies need not exist. In particular, summaries do not exist when $R_{\boldsymbol{U}}$ is intransitive. But it is straightforward to show that summaries do exist when edom $\boldsymbol{U}$ is a partition of X or when $\boldsymbol{U}$ is a simple psychology. For our purposes, one key value of summaries is that they shed light on the link between diminishing marginal utility and concavity.

Let $X = R_+^n$. The set $L \subset R_+^n$ is a *line segment in* R_+^n if there exist $a, b \in R_+^n$ such that, for all $x \in L$, $x = \lambda a + (1-\lambda)b$ for some $\lambda \in [0, 1]$. An extended psychology $\boldsymbol{U}$ satisfies *diminishing marginal utility* if (1) edom $\boldsymbol{U}$ is the set of all line segments in R_+^n, (2) each line segment L is decisive and $\mathscr{F}_L \cap \boldsymbol{U}$ contains only concave functions, and (3) there does not exist a $\boldsymbol{U}' \supsetneq \boldsymbol{U}$ that satisfies (2).

Theorem 5.1. If $\boldsymbol{U}$ satisfies diminishing marginal utility, $\boldsymbol{U}$ has a summary $\boldsymbol{S}$. If A is decisive for $\boldsymbol{U}$, then $\boldsymbol{S}|A$ consists of all concave utility representations of $R_{\boldsymbol{U}} \cap (A \times A)$.

Proof. [Not included in this version.]

Theorem 5.1 is motivated by the idea that the psychologically plausible kernel of concavity is the assumption of diminishing marginal utility along line segments. That concave utility functions imply DMU on lines is obvious. The additional message of Theorem 5.1 is that if a psychology contains all ordinally equivalent utility functions that are concave on an agent's decisive sets, those concave utilities retain all of the information contained in the DMU functions on lines – which is important when the latter are the genuine psychological primitives. That is, when we project the concave utilities onto their decisive line segments, we recover the entire set of original DMU functions.

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