The welfare cost of market incompleteness: optimal financial contracts with non-enforceability constraints

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Abstract

In this paper we develop a general equilibrium model in which firms finance investment by signing long-term contracts with a financial intermediary. Due to enforceability problems, financial contracts are constrained optimal, that is, they maximize the surplus of the contract subject to incentive compatibility constraints. By comparing this model with an alternative model in which contracts are fully enforceable, we evaluate the quantitative importance of non-enforceability for the aggregate allocation of the economy. We find that in the steady state the welfare level in the economy with enforceable contracts is 2.6 percent larger than in the economy with non-enforceable contracts.

VERY PRELIMINARY AND INCOMPLETE

1 Introduction

Several empirical studies of the investment behavior of firms find that financial factors play a significant role. Following the results of these studies, there has been an interest in studying, first, the kind of market imperfections that are able to account for the impact of financial factors on the investment decision of the firm and, second, the impact of these frictions on the aggregate allocation of the economy.

In this paper we study market incompleteness that derives from the lack of commitment of the firm. Therefore, as in Albuquerque-Hopenhayn-97, Hart-Moore-98 and Kehoe-Levine-93, we assume that contracts are not fully enforceable. After showing that this framework is able to capture some of the key empirical features of the investment behavior of the firm, we ask the following question: Are the distortions induced by the presence of non-enforceability constraints quantitatively important for the aggregate allocation of resources in the economy? What is the welfare reduction due to the presence of these constraints?

We answer this question by comparing the welfare level reached in the equilibrium with non-enforceable contracts with the welfare level reached in the economy in which contracts are fully enforceable. We find that the welfare level reached in the economy with enforceable contracts is 2.6 percent higher than the welfare level reached in the economy without enforceable contracts.

The plan of the paper is as follows. In Section 2 we describe the model economy in a partial equilibrium analysis and in Sections 3 and 4 we describe the problem to find the optimal constrained contract. Then in Section 5 the model is extended in a general equilibrium framework and in Section 6 we conduct the welfare comparison described above. Finally, section 7 concludes.

2 The model

The economy is populated by a continuum of entrepreneurs of total mass 1. A mass $1-\alpha$ of new entrepreneurs are born in each period and each entrepreneur faces a probability $1-\alpha$ of dying. In addition to entrepreneurs, there is a continuum of infinitely lived workers, employed by entrepreneurs.

Entrepreneurs search for a project characterized by the revenue function $F(k, \eta)$, where k is the input of capital and η an idiosyncratic shock. The revenue function is strictly increasing in k and η, and strictly concave in k. The shock η is independently and identically distributed in the interval $(-\infty, \bar{\eta})$, with distribution function $G(\eta)$. Moreover, we assume that $\lim_{\eta \to -\infty} F(k, \eta) = -\infty$ for all $k > 0$. As we will see, by assuming that the revenue function can take infinite negative numbers we make sure that there is always a non-zero probability that an active project will be liquidated. The capital input (investment) is chosen one period in advance and depreciates at rate δ. To simplify notations I will also use the function $f(k, \eta)$ to denote the revenue plus the non-depreciated capital, that is, $f(k, \eta) = F(k, \eta) + (1-\delta)k$.

Searching for a project requires entrepreneurial effort with disutility κ. By devoting effort to searching, the entrepreneur finds a profitable project with probability $q(z, e)$ where e is the number (mass) of searching entrepreneurs, and z is a shock that changes the ability with which searching entrepreneurs find profitable projects. The shock follows a two-state Markov process with transition probability $\Gamma(z'/z)$, that is, $z \in \{z^1, z^2\}$. We assume that $q(z, e)$ is decreasing and convex in e, and $q(z^1, e) \leq q(z^2, e)$ for all e.

An important assumption is that the probability of finding a project for each individual entrepreneur depends on the number of searching entrepreneurs with equal or higher entrepreneurial experience. Experienced entrepreneurs are those who are currently running a project and non-experienced entrepreneurs are those who are not currently running a project. If we denote by e^0 the number of searching entrepreneurs without experience and with e^1 the number of searching entrepreneurs with experience, the probability of finding a profitable project for experienced entrepreneurs is $q(z, e^1)$ and for non-experienced entrepreneurs is $q(z, e^0 + e^1)$. To simplify the analysis, we assume that entrepreneurial experience fully depreciates after one period.[1]

An entrepreneur who finds a profitable project, finances the input of capital by signing a long-term contract with an intermediary. The value for the entrepreneur of this contract is denoted by $V^0(\mathbf{s})$, where $\mathbf{s}$ denotes the aggregate states of the economy. The structure of the

[1] The assumption that experience affects the probability of starting a new business is consistent with the empirical findings of Quadrini-99a.

financial contract is complicated by enforceability problems. As in Albuquerque-Hopenhayn-97, Hart-Moore-98 and Kehoe-Levine-93, enforceability problems arise as the entrepreneur can appropriate the cash flow generated by the production process and default. By defaulting, the entrepreneur is always able to search for another project, in addition to appropriating (and consuming) the current cash flow. Denoting by $J(\mathbf{s}) = q(z, e^1)V^0(\mathbf{s}) - \kappa$ the value of searching for a project, the value of defaulting can be written as $D(\mathbf{s}, k, \eta) = \max\{0, F(k, \eta)\} + J(\mathbf{s})$.

The contract specifies, for each history realization of the individual shock and aggregate states, the payments to the intermediary τ, the payments to the firm d, and the next period capital input k'. In addition, it specifies the events under which the firm is liquidated. The liquidation of the firm takes place at the beginning of the period after the observation of the shocks, but before starting the production process. The liquidation value of the firm is a function of the installed capital and it is denoted by $L(k)$. The liquidation value of the firm is distributed in part to the intermediary (the quantity ℓ^b) and in part to the entrepreneur (the quantity ℓ^f). The liquidation value satisfies the condition $L(k) < (1-\delta)k$.

3 The optimal financial contract

In this section we characterize the optimal financial contract signed by the entrepreneur and the intermediary using the methodology developed by Marcet-Marimon-92,Marcet-Marimon-97 to characterize recursive dynamic contracts. This methodology consists of solving a planner problem that maximizes a weighted sum of the value of the contract for the entrepreneur and the intermediary, subject to the incentive-compatibility and resource constraint. For the moment we assume that the weights attributed to the entrepreneur and the intermediary by the planner are given. Later we will specify the condition that pins down these weights.

Assume that the entrepreneur and the intermediary discount flows at the common factor β and denote with $\phi_t \in \{0, 1\}$ the choice to liquidate the firm at time t. When $\phi_t = 1$ the firm continues to operate; when $\phi_t = 0$ the firm is liquidated. Then, define $\tilde{\beta}_{t,n}$ as:

$$\tilde{\beta}_{t,j} = \begin{cases} 1 & \text{for } j = t \\ \prod_{i=t}^{j-1} \phi_i \alpha\beta & \text{for } j > t \end{cases} \tag{1}$$

The variable $\tilde{\beta}_{t,j}$ is the $j-t$ periods discount factor at time t when the contract can be permanently discontinued at any date between period t and period $j-1$ due to voluntary liquidation or entrepreneur's death.

Define λ to be the weight assigned to the entrepreneur by the planner and ζ the weight assigned to the intermediary. Without loss of generality we normalize $\zeta = 1$ and the planner problem can be written as:

$$\max_{\{\phi_t, d_t, \tau_t, k_{t+1}, \ell_t^f, \ell_t^b\}_{t=0}^{\infty}} \quad E_0 \sum_{t=0}^{\infty} \tilde{\beta}_{0,t} \Big\{ (1-\phi_t) \cdot \Big[\lambda(\ell_t^f + J(\mathbf{s}_t)) + \ell_t^b\Big] + \phi_t \cdot \Big[\lambda d_t + \tau_t + (1-\alpha)\beta L(k_{t+1})\Big] \Big\} \tag{2}$$

subject to

$$E_t \sum_{j=t}^{\infty} \tilde{\beta}_{t,j} \Big\{ (1-\phi_j)(\ell_j^f + J(\mathbf{s}_j)) + \phi_j d_j \Big\} \geq \phi_t D(\mathbf{s}_t, k_t, \eta_t) \tag{3}$$

$$k_{t+1} = f(k_t, \eta_t) - d_t - \tau_t \tag{4}$$

$$\ell_t^f + \ell_t^b = L(k_t) \tag{5}$$

$$d_t \geq 0, \quad \ell_t^f \geq 0, \quad k_0 = \text{given} \tag{6}$$

The objective (2) defines the surplus of the contract as the expected discounted value of per-period flows. The per-period flow is defined as the weighted sum of the return for the lender and the entrepreneur with weights given by λ and $\zeta = 1$. The dummy variable ϕ_t separates the different returns following the endogenous liquidation choice. Notice that, for analytical convenience, we have added to the current return of the intermediary the expected discounted next period value of the firm's assets in case of exogenous liquidation (entrepreneurial death), that is, the term $(1-\alpha)\beta L(k_{t+1})$.[2]

Equation (3) defines the incentive compatibility constraint: the value of continuing the contract for the entrepreneur ($\phi_t = 1$) must always be non-smaller than the default value. As specified above, the default value is equal to the cash flow generated by the firm (if positive) plus the expected value of searching for a new project, that is, $D(\mathbf{s}, k, \eta) = \max\{0, F(k, \eta)\} + J(\mathbf{s})$. If the cash flow is negative, implying that the firm realizes losses, the entrepreneur can always default and leave the non-depreciated value of capital to compensate for these losses.

After writing this problem in lagrangian form, with γ_t the Lagrange multiplier at time t associated with the incentive compatibility constraint (3), the planner problem can be transformed in the following saddle-point formulation:

$$\min_{\{\mu_{t+1}\}_{t=0}^{\infty}} \quad \max_{\{\phi_t, d_t, \tau_t, k_{t+1}, \ell_t^f, \ell_t^b\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \tilde{\beta}_{0,t} \cdot \Big[(1-\phi_t)\omega_0(\mathbf{x}_t) + \phi_t \omega(\mathbf{x}_t) \Big] \tag{7}$$

subject to

$$\omega_0(\mathbf{x}_t) = (\lambda + \mu_{t+1})(\ell_t^f + J(\mathbf{s}_t)) + \ell_t^b \tag{8}$$

$$\omega(\mathbf{x}_t) = (\lambda + \mu_{t+1}) d_t + \tau_t + (1-\alpha)\beta L(k_{t+1}) - (\mu_{t+1} - \mu_t) D(\mathbf{s}_t, k_t, \eta_t) \tag{9}$$

$$k_{t+1} = f(k_t, \eta_t) - d_t - \tau_t \tag{10}$$

$$L(k_t) = \ell_t^f + \ell_t^b \tag{11}$$

$$d_t \geq 0, \quad \ell_t^f \geq 0, \quad \mu_{t+1} \geq \mu_t, \quad \mu_0 = 0, \quad k_0 = \text{given} \tag{12}$$

where $\omega_0(\mathbf{x}_t)$ and $\omega(\mathbf{x}_t)$, defined in (8) and (9), can be interpreted as the returns at time t for the planner conditional on the choice to liquidate or operate the firm. These functions depend on the vector of variables $\mathbf{x}_t = (\mathbf{s}_t, k_t, \eta_t, \mu_t, d_t, \tau_t, k_{t+1}, \mu_{t+1})$, where we have denoted with $\mathbf{s}_t$ the vector of aggregate state variables at time t. Of particular interest is the new variable μ

[2] Remember that $(1-\alpha)$ is the probability with which the entrepreneur dies. In this case the liquidation value, $L_(k_{t+1})$, is taken by the intermediary. Because the liquidation arises with probability $(1-\alpha)$ at the beginning of the next period, this value has to be discounted at rate $(1-\alpha)\beta$.

that evolves according to $\mu_{t+1} = \mu_t + \gamma_t$. Therefore, this variable increases when the lagrange multiplier γ_t is positive, that is, when the incentive-compatibility constraint (3) is binding. A brief inspection of the objectives in problems (2) and (7) reveals that the objective function in the second problem is equivalent to the objective function in the first problem, but with variable weights assigned by the planner to the entrepreneur and the intermediary (in addition to the term $(\mu_{t+1} - \mu_t)D(\mathbf{s}_t, k_t, \eta_t)$ that also enters the period return function in the saddle-point formulation). More specifically, the weight assigned to the entrepreneur is now $\lambda + \mu_{t+1}$. Because μ increases when the incentive compatibility constraint is binding, the effect of this constraint is to increase the weight assigned to the entrepreneur by the planner. This property has a very simple intuition: The weight used by the planner determines the value of the contract for the entrepreneur, and larger is this weight, higher is the value of the contract for the entrepreneur. When the default condition is binding, the value of the contract for the entrepreneur is smaller than the default value. In this circumstance, to prevent the entrepreneur from defaulting, his contract value must be increased. The way to increase this value is by attributing a larger weight to the entrepreneur, that is, by increasing $\lambda + \mu_{t+1}$.

This saddle-point formulation allows us to rewrite the planner's problem recursively as follows:

$$W(\mathbf{s}, k, \eta, \mu) = \max\left\{ \min_{\mu'} \max_{\ell^f, \ell^b} \left\{ \omega_0(\mathbf{x}) \right\},\ \min_{\mu'} \max_{d, \tau, k'} \left\{ \omega(\mathbf{x}) + \alpha\beta E W(\mathbf{s}', k', \eta', \mu') \right\} \right\} \quad (13)$$

subject to

$$\begin{aligned}
\omega_0(\mathbf{x}) &= (\lambda + \mu')(\ell^f + J(\mathbf{s})) + \ell^b && (14)\\
\omega(\mathbf{x}) &= (\lambda + \mu')d + \tau + (1-\alpha)\beta L(k') - (\mu' - \mu)D(\mathbf{s}, k, \eta) && (15)\\
k' &= f(k, \eta) - d - \tau && (16)\\
L(k) &= \ell^f + \ell^b && (17)\\
\mathbf{s}' &= H(\mathbf{s}) && (18)\\
d &\geq 0, \quad \ell^f \geq 0, \quad \mu' \geq \mu && (19)
\end{aligned}$$

where the prime denotes the next period variable and H is the law of motion for the aggregate states.

Before proceeding, let's observe that for $\lambda > \zeta = 1$, the problem is not well defined. This is because the planner would attribute more weight to the firm and he always prefers to shift resources from the intermediary to the entrepreneur (remember that τ is unbounded below). So it will be optimal to ask for an infinite amount of transfers from the intermediary to the entrepreneur. This also implies that $\lambda + \mu'$ cannot be bigger than 1. Because $\lambda + \mu' \leq 1$, the optimal solution for the distribution of the liquidation value of the firm is $\ell^f = 0$ and $\ell^b = L(k)$.[3] Notice also that the solution of the minimization problem with respect to μ', in the case of liquidation, is simply given by $\mu' = \mu$. Therefore, the problem can be rewritten as:

$$W(\mathbf{s}, k, \eta, \mu) = \max\left\{ (\lambda + \mu)J(\mathbf{s}) + L(k),\ \min_{\mu'} \max_{d, \tau, k'} \left\{ \omega(\mathbf{x}) + \alpha\beta E W(\mathbf{s}', k', \eta', \mu') \right\} \right\} \quad (20)$$

[3] The only exception is when $\lambda + \mu' = 1$. In that case the solution is not determined.

subject to (15), (16), (18), (19)

3.1 First order and exit conditions

When the contract is not discontinued, the optimization solution is characterized by the following first order conditions:

$$\lambda + \mu' - 1 \le 0 \qquad (= \text{if } d > 0) \tag{21}$$

$$D(\mathbf{s}, k, \eta) - \alpha\beta E\Big[(1-\phi')J(\mathbf{s}') + \phi' D(\mathbf{s}', k', \eta')\Big] - d \le 0 \qquad (= \text{if } \mu' > \mu) \tag{22}$$

$$\beta\Big[1 - \alpha + \alpha E(1-\phi')\Big]L_{k'} + \alpha\beta E\phi'\left[f_{k'} - (\mu'' - \mu')D_{k'}\right] - 1 = 0 \tag{23}$$

Condition (21) tells us that if some dividend is distributed, then μ' must be set to 1. Condition (22) imposes a limit to the firm's growth. When the default condition is binding (that is, $D(\mathbf{s}, k, \eta)$ is greater than the expected discounted value of dividends), then $\mu' > \mu$ and the next period stock of capital grows at the rate that satisfies equation (22) with equality. When the default condition is not binding, $\mu' = \mu$ and (22) can be satisfied with the inequality sign. Because the default value is increasing in the value of the idiosyncratic shock, the default condition is binding only for values of the shock above a certain threshold. This threshold depends on the aggregate and individual states and we denote it by $\hat{\eta}(\mathbf{s}, k, \mu)$. Therefore, for $\eta \le \hat{\eta}(\mathbf{s}, k, \mu)$ the default condition is not binding and $\mu' = \mu$.

If the firm survives enough periods, $\lambda + \mu'$ converges to 1. The limiting value of μ is equal to 1 and it is denoted by μ^*. At this point the input of capital is always kept at the optimal level and any scheme through which the production surplus is distributed is a solution to the problem. The optimal input of capital is denoted by $k^*(\mathbf{s})$ and it depends only on the aggregate states of the economy.

We can solve for the value of the total surplus as a function of initial capital k, without solving for the distribution of dividends. This can be easily seen from problem (20). Eliminating τ in equation (15) using equation (16), the function $\omega(\mathbf{x})$ becomes:

$$\omega(\mathbf{x}) = (\lambda + \mu')d + f(k, \eta) - d - k' + (1-\alpha)\beta L(k') - (\mu' - \mu)D(\mathbf{s}, k, \eta) \tag{24}$$

Independently of whether $\lambda + \mu'$ is smaller or equal to 1, $\omega(\mathbf{x}) = f(k, \eta) - k' + (1-\alpha)\beta L(k') - (\mu' - \mu)D(\mathbf{s}, k, \eta)$. This is because when $\lambda + \mu' < 1$, $d = 0$ and when $\lambda + \mu' = 1$, the variable d cancels out. Therefore, we can solve for the optimal investment without considering the dividend policy that the contract recommends.

The exit condition (decision to liquidate the firm) is given by:

$$(\lambda + \mu)J(\mathbf{s}) + L(k) = \min_{\mu'} \max_{d,\tau,k'} \Big\{\omega(\mathbf{s}, k, \underline{\eta}, \mu, k', \mu') + \alpha\beta E W(\mathbf{s}', k', \eta', \mu')\Big\} \tag{25}$$

By assuming that the shock can take unbounded negative numbers we make sure that there is always a positive probability that the firm is liquidated. The value of the idiosyncratic shock below which the firm is liquidated depends on the aggregate states $\mathbf{s}$ and individual states k, μ, and it is denoted by $\underline{\eta}(\mathbf{s}, k, \mu)$.

Conditions (21)-(23) along with the exit condition (25) characterize an optimal contract.

4 More on the properties of the optimal contract

We now characterize the properties of an optimal contract under two special cases: in the steady state equilibrium and in the case in which the external value of the entrepreneur J only depends on the aggregate shock z. We will see that under particular assumptions, this property of the function J will hold in the general equilibrium of this economy as described in section 5.

4.1 Steady State equilibrium

In a steady state equilibrium the outside value $J(\mathbf{s})$ is constant, and the contractual problem can be written as:

$$W(k,\eta,\mu) \;=\; \max\left\{(\lambda+\mu)J + L(k)\ ,\ \min_{\mu'}\max_{d,\tau,k'}\Big\{\omega(\mathbf{x}) + \alpha\beta EW(k',\eta',\mu')\Big\}\right\} \tag{26}$$

subject to

$$\omega(\mathbf{x}) = (\lambda+\mu')d + \tau + (1-\alpha)\beta L(k') - (\mu'-\mu)D(k,\eta) \tag{27}$$

$$k' = f(k,\eta) - d - \tau \tag{28}$$

$$d \geq 0, \quad \mu' \geq \mu, \quad \mu_0 = 0, \quad k_0 = \text{given} \tag{29}$$

Define $\underline{\underline{\eta}}(k)$ the value of the shock for which, given a certain input of capital k, the firm's revenue is zero. Therefore $\underline{\underline{\eta}}(k)$ satisfies the condition $F(k, \underline{\underline{\eta}}) = 0$. The first order and exit conditions are:

$$\lambda + \mu' - 1 \,\leq\, 0 \tag{30}$$

$$\max\{0, F(k,\eta)\} + J - \alpha\beta\int_{\max\{\underline{\underline{\eta}}(k'),\underline{\eta}(k',\mu')\}}^{\bar{\eta}} F(k',\eta')G(d\eta') - \alpha\beta J - d \,\leq\, 0 \tag{31}$$

$$\beta\Big[1-\alpha+\alpha G(\underline{\eta}(k',\mu'))\Big]L_{k'} + \alpha\beta\int_{\underline{\eta}(k',\mu')}^{\bar{\eta}}\left[f_{k'} - (\mu''-\mu')D_{k'}\right]dG(\eta') = 1 \tag{32}$$

$$(\lambda+\mu)J + L(k) \;=\; \min_{\mu'}\max_{d,\tau,k'}\Big\{\omega(k,\underline{\eta},\mu,k',\mu') + \alpha\beta EW(k',\eta',\mu')\Big\} \tag{33}$$

In the above first order conditions we have used the explicit formula for the default value, that is, $D(k,\eta) = \max\{0, F(k,\eta)\}$ and $D_{k'} = \max\{0, F_{k'}\}$. As for the first order conditions for the original problem, condition (30) is satisfied with equality if $d > 0$ and condition (31) is satisfied with equality if $\mu' > \mu$.

Lemma 4.1 *If the sequence of functions for $\underline{\eta}, k', \mu'$ satisfying conditions (30)-(33) is unique, then along the solution path of the contract there is a unique correspondence between k and μ, and we can write $\mu = m(k)$.*

Proof 4.1 *The stochastic properties of μ'' depend on k' and μ'. Therefore, we can write $\mu'' \sim \Gamma(k', \mu')$, where Γ is the density function of μ''. Given this, equation (32) can be written as $h(k', \mu') = 0$. Now consider the case in which the default condition is binding. In this case equation (31) is satisfied with equality. Given the lower bound function $\underline{\eta}(k', \mu')$, equations (31) and the condition $h(k', \mu') = 0$ (that is equation (32)) uniquely determine the two variables k' and μ'. Notice that the variables k' and μ' are determined independently of the current value of μ, as long as the default condition is binding. Therefore, the current stock of capital is a sufficient statistics for the determination of k' and μ'. Because k' and μ' are both uniquely determined by k, we can express μ' as a function of k', or $\mu' = m(k')$. Along the solution path then, a similar expression must hold in the current period, that is $\mu = m(k)$. A similar result holds if the default condition is not binding. In this case $\mu' = \mu$ and $k' = k$. To see this observe that if equation (32) was satisfied in the previous period, that is, $h(k, \mu) = 0$, and $\mu' = \mu$, then $h(k', \mu' = \mu) = 0$ if and only if $k' = k$.*

The above lemma states that the capital stock is a sufficient statistic for the characterization of the contract and there exists a function relating μ and k, that is, $\mu = m(k)$. Consequently, we can write the exit shock and the threshold shock for the default condition as a function of k, that is, $\underline{\eta}(k)$ and $\hat{\eta}(k)$.

Corollary 4.1 *The stock of capital changes only if the default condition is binding.*

Proof 4.1 *The prof is implicitly provided by the proof of lemma 4.1.*

This property of the optimal contract has a very simple intuition that derives from the fact that the increase in the stock of capital has redistributional consequences from the intermediary to the entrepreneur. This is because the increase in output induced by the increase in the stock of capital, translates in one-to-one increase in the value of the firm (through the increase in the value of defaulting). Therefore, any output gains induced by the increase in the stock of capital, are totally appropriated by the entrepreneur. Because capital depreciates and the discount rate is positive, the intermediary faces a cost for expanding the stock of capital, which in turn reduces the value of the contract for the intermediary. Now because the planner is giving a larger weight to the intermediary than to the entrepreneur, it prefers not to expand the stock of capital, unless he must do so to prevent default.

The threshold value of the shock, $\hat{\eta}(k)$, is determined by equation (31) after imposing $d = 0$ and $k' = k$, that is by the condition:

$$F(k, \hat{\eta}) = \alpha\beta \int_{\max\{\underline{\underline{\eta}}(k), \underline{\eta}(k)\}}^{\bar{\eta}} F(k, \eta) dG(\eta) + (\alpha\beta - 1)J \tag{34}$$

Notice that, if we assume $F(0, \eta) = 0$, then for values of k sufficiently small, the default condition is always binding. More specifically, the default condition is always binding for values of capital that are smaller than $\tilde{k}$, where $\tilde{k}$ satisfies the condition:

$$0 = \alpha\beta \int_{\max\{\underline{\underline{\eta}}(\tilde{k}), \underline{\eta}(\tilde{k})\}}^{\bar{\eta}} F(\tilde{k}, \eta) dG(\eta) + (\alpha\beta - 1)J \tag{35}$$

4.2 Equilibrium with aggregate shocks

We now consider the case in which the economy is affected by aggregate shocks. In this section we make the simplifying assumption that the outside value for the firm depends only on the aggregate shock z. Therefore, we will write $J(z)$. Under this assumption, the contractual problem can be written as:

$$W(z, k, \eta, \mu) = \max\left\{(\lambda + \mu)J(z) + L(k)\ ,\ \min_{\mu'} \max_{d, \tau, k'} \left\{\omega(\mathbf{x}) + \alpha\beta EW(z', k', \eta', \mu')\right\}\right\} \tag{36}$$

subject to

$$\omega(\mathbf{x}) = (\lambda + \mu')d + \tau + (1 - \alpha)\beta L(k') - (\mu' - \mu)D(z, k, \eta) \tag{37}$$

$$k' = f(k, \eta) - d - \tau \tag{38}$$

$$d \geq 0, \quad \mu' \geq \mu, \quad \mu_0 = 0, \quad k_0 = \text{given} \tag{39}$$

Also in the case of aggregate shocks, there is a unique correspondence between k' and μ', which however now depends on the current aggregate shock z. This can be shown using the same argument used in the previous section. Therefore, we can express μ as a function of k, that is, $\mu = m(z_{-1}, k)$, where z_{-1} is the realization of the aggregate shock in the previous period. This implies that, before the realization of the idiosyncratic shock, the contract can be characterized by three state variables: z_{-1}, z, and k. Given this result, the lower bound shock $\underline{\eta}$ can be expressed as a function of the previous and current aggregate shocks and the current value of k, that is, $\underline{\eta}(z_{-1}, z, k)$. This lower bound is determined by the condition:

$$(\lambda + \mu)J(z) + L(k) = \omega(z, k, \underline{\eta}, \mu, k', \mu') + \alpha\beta EW(z, z', k', \eta') \tag{40}$$

The analysis of the contract will be simplified by making the following assumption.

Assumption 4.1 *The properties of the aggregate shock z are such that the contract is never discontinued if $\eta > \underline{\underline{\eta}}(k)$.*

The above assumption excludes the possibility that the liquidation of the firm is optimal if the revenue is non-negative. This property is always satisfied (at least along the equilibrium path) in the steady state equilibrium. However, when the default value is not constant but depends on the aggregate shock z, then this property is not necessarily satisfied. Some restrictions have to be imposed on the properties of the shock z.

Under assumption 4.1, the first order and exit conditions are:

$$\lambda + \mu' - 1 \;\leq\; 0 \tag{41}$$

$$\max\{0, F(k,\eta)\} + J(z) - \alpha\beta \int_{\underline{\underline{\eta}}(k')}^{\bar{\eta}} F(k',\eta')G(d\eta') - \alpha\beta \sum_{z'} J(z')\Gamma(z'/z) - d \;\leq\; 0 \tag{42}$$

$$\beta\Big[1 - \alpha + \alpha \sum_{z'} G(\underline{\eta}(z,z',k'))\Gamma(z'/z)\Big] L_{k'} + \alpha\beta \sum_{z'} \int_{\underline{\eta}(z,z',k')}^{\bar{\eta}} f_{k'} G(d\eta')\Gamma(z'/z) - \\ \alpha\beta \sum_{z'} \int_{\max\{\underline{\underline{\eta}}(k'),\hat{\eta}(z,z',k')\}}^{\bar{\eta}} \left[(\mu'' - \mu')F_{k'}\right] G(d\eta')\Gamma(z'/z) - 1 \;=\; 0 \tag{43}$$

$$(\lambda + \mu)J(z) + L(k) \;=\; \min_{\mu'} \max_{d,\tau,k'} \Big\{\omega(\mathbf{x}) + \alpha\beta E W(z',k',\eta',\mu')\Big\} \tag{44}$$

First notice that, according to equation (42), the next period stock of capital depends on the previous and current realization of the aggregate shock, the current stock of capital, and the current idiosyncratic shock.[4] Therefore, we can write $k' = g(z_{-1}, z, k, \eta)$. While in absence of aggregate shocks the stock of capital never decreases, with aggregate shocks k' may be lower than k.

Using the function $\mu = m(z_{-1}, k)$, define $k = m^{-1}(z_{-1}, \mu)$ to be the inverse of the function m conditional on the previous realization of the shock. Then, the next period capital stock when the default condition is not binding ($\mu' = \mu$) is given by $k' = m^{-1}(z, \mu)$. When $z = z_{-1}$, then $k' = k$. However, when $z \neq z_{-1}$, the next period capital stock differs from the current one. The threshold shock is determined by the condition:

$$F(k,\hat{\eta}) \;=\; \alpha\beta \int_{\underline{\underline{\eta}}(m^{-1}(z,\mu))}^{\bar{\eta}} F(m^{-1}(z,\mu),\eta')G(d\eta') + \alpha\beta \sum_{z'} J(z')\Gamma(z'/z) - J(z) \tag{45}$$

If the value of η that satisfies this condition is greater than $\bar{\eta}$, then the default condition is not binding for any realization of the shock. The threshold value can be written as $\hat{\eta}(z_{-1}, z, k)$. On the other hand, if the value of $\hat{\eta}$ that satisfies this condition is negative, then the default condition is always binding.

Given the assumption that the aggregate shock takes only two values, we can identify two functions relating the value of μ to the value of the capital stock. Graphically we can plot the locus of points for μ and k for each value of the aggregate shock realized in the previous period. If at a particular moment in time the beginning of period combination of k and μ is not located

[4]To be more precise, k' depends on (z, μ, k, η). However, because μ can be identified by z_{-1} and k, we can use (z_{-1}, z, k, η) as the sufficient set of state variables.

on one of these two locus functions, then the values of k and μ chosen for the next period will be located in one of these functions, depending on the current realization of z. As noted above, this allows us to express μ as a function of the previous aggregate shock and the current stock of capital, that is, $\mu = m(z_{-1}, k)$. Given this result, and the fact that $\mu' \geq \mu$, it is possible to find a numerical solution of the steady state contract through a backward procedure. The details of this procedure are described in Appendix A.

4.3 Value of searching for a new project and the entrance of new firms

The analysis of the previous section assumes that the external value of the firm only depends on the aggregate shock z. By imposing certain restrictions on the searching function $q(z, s)$, this property is in fact an equilibrium outcome of the economy we are studying. Let's assume that the searching function is of the following form:

$$q(z, e) = z \cdot \Big(a + \max\{0, e - \bar{e}\}\Big)^{\theta} \qquad a > 0, \quad \theta < 0 \tag{46}$$

This simply says that the probability of finding a new project is constant when the number of searching entrepreneurs is smaller than $\bar{e}$, and decreases monotonically as the number of searchers increases above $\bar{e}$. Given the assumption that experienced entrepreneurs compete only with experienced entrepreneurs in searching for a new project, if in equilibrium the number of experienced entrepreneurs searching for a new project is smaller than $\bar{e}$, then for each value of the shock z the probability with which they find a new project is constant. Notice that this property will hold for relatively small values of $\bar{e}$ because in equilibrium only experienced entrepreneurs with terminated projects will effectively search for a new project. Let's denote with $\bar{q}(z) = za^{\theta}$ this constant probability. In the remaining sections of this paper we assume that in equilibrium the number of experienced entrepreneurs searching for a new project is smaller than $\bar{s}$.

The constancy of the searching probability for experienced entrepreneurs is a useful property in our model. It implies that the value of a contract for a firm is only a function of the shock z, that is, $V^0(z)$. This in turn implies that the value of searching for a new project for an experienced entrepreneur is given by $J(z) = \bar{q}(z)V^0(z) - \kappa$, and it only depends on the shock z.[5]

In order to determine $V^0(z)$ we have to find the expected value of a contract for the entrepreneur and the intermediary, for each initial value of capital. Denote by $\bar{V}^F(z, k')$ and $\bar{V}^B(z, k')$ the next period expected value of the contract for the entrepreneur and the intermediary, respectively, given the current aggregate shock z and the initial capital input k', and conditional on the survival of the entrepreneur to the next period. The initial cost of the contract for the intermediary is given by the initial stock of capital k', while the present value of the contract is $\alpha\beta\bar{V}^B(z, k') + (1-\alpha)\beta L(k')$. The term $\beta\bar{V}^B(z, k')$ is the expected discounted value for the intermediary conditional on the entrepreneur surviving to the next period; the term $\beta L(k')$ is instead the value if the entrepreneur dies. By assuming that the financial sector is competitive (zero profit condition for the intermediary), the value of starting a new project for the entrepreneur is:

[5] If the probability of finding a new project also depends on the number of experienced entrepreneurs searching for a project, then the outside value of a firm depends on the number of contracts that are endogenously discontinued (those are the experienced searchers). This value, in turn, affects the number of contracts that will be discontinued, and the problem would become quite complex.

$$
\begin{aligned}
V^0(z) \;&=\; \max_{k'} \alpha\beta\bar{V}^F(z,k') && (47)\\
&\text{subject to}\\
\alpha\beta\bar{V}^B(z,k') + (1-\alpha)\beta L(k') - k' \;&\geq\; 0 && (48)
\end{aligned}
$$

Because $\alpha\beta\bar{V}^B(z,k') + (1-\alpha)\beta L(k') - k'$ is decreasing in k', while $\alpha\beta\bar{V}^F(z,k')$ is increasing in k', the solution to the above problem is unique and implies zero expected profits for the intermediary (constraint (48) is binding). The function $\bar{V}^F(z,k')$ is derived by integrating the value of the firm after the observation of the next period aggregate and idiosyncratic shocks, denoted by $V^F(z,z',k',\eta')$. More specifically, $\bar{V}^F(z,k') = \sum_{z'}\int_{\eta'} V^F(z,z',k',\eta')G(d\eta')\Gamma(z'/z)$. Similarly, the value of the contract for the intermediary, $\bar{V}^B(z,k')$, is derived by integrating the value of the contract for the intermediary after the observation of the next period aggregate and idiosyncratic shocks, that is, $\bar{V}^B(z,k') = \sum_{z'}\int_{\eta'} V^B(z,z',k',\eta')G(d\eta')\Gamma(z'/z)$.

The value of the firm is equal to the default value $D(z',k',\eta') = J(z') + \max\{0, F(\eta',k')\}$, when the default condition is binding, and it is equal to $D(z',k',\hat{\eta})$, when the default condition is not binding. Here $\hat{\eta}$ is the value of the shock below which the default condition is not binding. This allows us to derive the firm's value function V^F. After finding V^F, we can derive $V^B(z,z',k',\eta')$ using the function $W(z,z',k',\eta')$ as follows:

$$
V^B(z,z',k',\eta') \;=\; W(z,z',k',\eta') - (\lambda + \mu')V^F(z,z',k',\eta') \tag{49}
$$

The arbitrage condition to searching for a project for inexperienced entrepreneurs is given by:

$$
\kappa = q(z,e)V^0(z) \tag{50}
$$

Because $V^0(z)$ depends only on z, the above equation uniquely determines the mass of searching entrepreneurs $e = e^0 + e^1$.

5 General equilibrium and first best allocation

In this section we compare the competitive allocation with the efficient allocation chosen by a benevolent social planner. The analysis is limited to the case in which there are not aggregate shocks.

In order to analyze the planner allocation, we need to consider a general equilibrium framework. To close the economy we assume that the production function also depends on the input of labor. However, capital and labor are perfect complements, so that the capital-labor ratio employed in production is always constant. More specifically, the production function is of the following form:

$$
y = F\Big(\min\{k, \xi\, l\}, \eta\Big) \tag{51}
$$

where l is the input of labor, and as before, k is the input of capital and η an idiosyncratic technology shock. In equilibrium the capital-labor ratio is equal to the parameter ξ.

Workers are endowed with one unit of time that is allocated between working activities and leisure. They choose sequences of consumption and leisure to maximize the expected discounted value of lifetime utility given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t (c_t - \underline{w} l_t) \tag{52}$$

where c_t is consumption, l_t is time spent working and $\underline{w}$ is a constant parameter that quantifies the disutility from working. We restrict the parameters of the economy to be in the range of values for which the average input of labor in the economy is smaller than 1, for all possible equilibria in the neighborhood of the steady state. This implies that the competitive wage, denoted by w, is constant and it is equal to $\underline{w}$. Given the constancy of w and the complementarity of labor and capital, we can redefine $F(k, \eta)$ to be the surplus produced by the firm, that is, final output net of the labor cost.[6]

Denote by $\bar{K}$ the total stock of capital given full employment. This is derived by multiplying the optimal capital-labor ratio (ξ) by the aggregate hours available to households. Also denote by n the number of entrepreneurs that at the beginning of the period (before death realization) are running a project. Finally denote by e^1 the number of searching entrepreneurs with experience and with e^0 the number of searching entrepreneurs without experience. Notice that, as long as e^0 chosen by the planner is positive, all the entrepreneurs with discontinued projects (experienced entrepreneurs) will search for a new project. Therefore, the number of searching entrepreneurs with experience is $e^1 = \alpha n G(\underline{\eta})$. The planner problem can be written recursively as:[7]

$$\Omega(n, k) = \max_{\underline{\eta}, e^0} \left\{ n\pi(k, \underline{\eta}) - (\alpha n G(\underline{\eta}) + s^0)\kappa - n'k' + \beta\Omega(n', k') \right\} \tag{53}$$

subject to

$$n' = \alpha n(1 - G(\underline{\eta})) + \alpha n G(\underline{\eta}) q(0) + e^0 q(\alpha n G(\underline{\eta}) + e^0) \leq \bar{K}/k' \tag{54}$$

$$\pi(k, \underline{\eta}) = [1 - \alpha(1 - G(\underline{\eta}))]L(k) + \alpha(1 - G(\underline{\eta}))(1 - \delta)k + \alpha \int_{\underline{\eta}}^{\bar{\eta}} F(k, \eta) G(d\eta) \tag{55}$$

[6] Why do we need labor? The reason is that we need to impose some bound on the size of the economy. Without labor, and elastic savings, there is no bound to the aggregate value of capital. In this case, the efficient size of a firm is always the size that maximizes the value of each individual firm, which is equivalent to the size of unconstrained firms in the competitive equilibrium. This implies that, if we compare the efficient allocation with the inefficient one, the aggregate stock of capital in the efficient allocation is much higher than the stock of capital in the competitive allocation. Not only because the average size of firms is larger, but also because the number of firms is bigger. With the introduction of labor, however, the aggregate stock of capital cannot be expanded by a large amount, due to the restrictions imposed by the availability of labor. Let's say, for example, that in the competitive equilibrium $l = 0.95$ (5 percent unemployment). Then the planner cannot increase total capital more than 5 percent). However, although the aggregate stock of capital cannot be increased by a large amount, the size distribution of firms will change dramatically. As we will show, in the efficient allocation there is an optimal size of firms that is smaller than the optimal size in the efficient allocation of the economy without labor. The reason we get these different results is because the planner solves two different problems in the economy with unbounded and bounded labor. With unbounded labor and elastic savings, the planner simple solves the problem of maximizing the output of each firm, in addition to choosing the optimal number of searching entrepreneurs. With bounded labor, instead, the planner will seek to maximize aggregate output, given the available labor and the optimal number of searchers.

[7] Notice that the planner will distribute capital equally to all active entrepreneurs and will choose the same lower shock to liquidate the firm.

The function $\Omega(n,k)$ is the planner's value function which depends on the number of projects n started in the economy with the input of capital k. The function $\pi(k,\underline{\eta})$ is the return function for the planner, which is given by the current expected amount of resources generated by the n activated projects.

6 The macroeconomic cost of market incompleteness

This section compares the welfare levels obtained under three different allocations. The first allocation is the market allocation in which firms' investment is financed with optimal constrained contracts signed between entrepreneurs and intermediaries. The second allocation is the market allocation when contracts are fully enforceable. This is not the first best allocation due to the presence of externalities associated with the process of searching for new projects. Finally, the third allocation is the first best allocation, that is, the allocation that would be chosen by a benevolent social planner. In what follows we will refer to the three allocations as "allocation with non-enforceable contracts", "allocation with enforceable contracts", and "planner allocation". The analysis will be limited to steady state equilibria.

In the allocation with enforceable contracts the optimal financial contract determines the input of capital that maximizes the surplus of the firm, independently of the distribution of this surplus between the entrepreneur and the intermediary. This is equal to the input of capital that is employed in the contract with non-enforceable contracts when the firm reaches the unconstrained status, that is, $\mu = 1 - \lambda$. Compared to the equilibrium with non-enforceable contracts, this implies a larger quantity of capital, and therefore, labor. Given the limited availability of labor, this introduces some pressure on the labor market which raises the wage rate, until the demand for labor is equal to the supply. In equilibrium, the size of firms is smaller than the size of unconstrained firms in the economy with non-enforceable contracts.

In the experiment, we specify the production technology as $F(k,\eta) = \eta k^{\nu}$. The technology shock takes values in $(-\infty, \overline{\eta})$ and the density function is $\frac{e^{\frac{\eta-\overline{\eta}}{\psi}}}{\psi}$. The probability density function of η is similar to an inverted exponential. The choice of this function is only due to its analytical simplicity. The liquidation value is assumed to be constant, that is, $L(k) = \bar{L}$. The probability of finding a new project for non-experienced entrepreneurs is given by the function $q(\bar{z}, s) = \bar{z}(a+s)^{\theta}$. The period in the economy is one year and the full set of parameter values, are reported in table 1.

Table 2 reports some key statistics of the competitive allocation with non-enforceable and enforceable contracts. In this computation we assume that in the equilibrium of the economy with non-enforceable contracts there is full employment. As reported in the last row of table 2, the welfare level reached in the steady state equilibrium of the economy with enforceable contracts is 2.6 percent higher than in the economy with non-enforceable contracts.

7 Conclusion

In this paper we have studied the welfare consequences of market incompleteness deriving from the lack of commitment of the firm in Pareto optimal contracts. This lack of commitment implies that contracts are second best as they have to satisfy incentive compatibility constraints. Our findings are that the welfare reduction induced by these constraints is in the order of 2-3 percent.

Table 1: Parameter values.

Discount factor	β	0.960
Entrepreneur's survival probability	α	0.990
Disutility from working	$\underline{w}$	0.150
Production technology ηk^{ν}	ν	0.980
Capital-labor ratio	ξ	0.200
Parameter of the density function	ψ	0.156
Maximum value of the shock	$\bar{\eta}$	0.156
Depreciation rate	δ	0.070
Liquidation value $L(k)$	$\bar{L}$	0.005
Searching technology $\bar{z}(a+s)^{\theta}$	$\bar{z}$	0.200
	a	0.200
	$\bar{s}$	0.200

Table 2: Properties of the economy with non-enforceable and enforceable contracts.

	Non-enforceable Contracts	Enforceable Contracts
Total mass of firms	2.495	2.716
Average size (capital)	0.780	0.716
Exit probability	0.036	0.037
Aggregate consumption	2.337	2.397

Given the quantitative importance of this welfare reduction, it is important to analyze what type of policies might be able to reduce these inefficiencies. This will be the object of future research.

A Numerical procedure

The numerical procedure starts with a guess for the values of $J(z) = q(z,n)V^0 - \kappa$, with $z \in \{z^1, z^2\}$. Given these values, the contract is first solved for $\mu = \mu^*$. This allows us to find the capital inputs $k^*(z_{-1})$, the lower bound shocks $\underline{\eta}(z, k^*(z_{-1}))$, and the value of the function $\bar{W}(z_{-1}, k^*(z_{-1})) = \sum_z \int W(z_{-1}, z, k^*(z_{-1}), \eta) dG(\eta)\Gamma(z/z_{-1})$. Then we form a grid for μ, with each grid point indexed by i, and we solve the model at each of these grid points sequentially, starting from the immediate adjacent point to μ^*. In solving the model in each of the grid point, we use the solutions found in the previous points and we approximate the function $\bar{W}(z_{-1}, k) = \sum_z \int W(z_{-1}, z, k, \eta) dG(\eta)\Gamma(z/z_{-1})$, for each z_{-1}, with piece-wise linear functions, that is, by connecting the values of $\bar{W}$ at two adjacent points with a linear segment. The steps followed to solve the contract at each particular grid point i are as follows.

1. In each grid point i, we have to solve for the values of $k_i(z_{-1}) = m^{-1}(z_{-1}, \mu_i)$, $\underline{\eta}_i(z_{-1}, z)$, and $\bar{W}(z_{-1}, k_i(z_{-1}))$, for $z_{-1}, z \in \{z^1, z^2\}$, where the latter is the expectation of the value function $W(z_{-1}, z, k_i(z_{-1}), \eta)$, and it is defined as:

$$\bar{W}(z_{-1}, k_i(z_{-1})) = \sum_z \Bigg\{ \Big[(\lambda + \mu_i)J(z) + L(k_i(z_{-1}))\Big] G(\underline{\eta}_i(z_{-1}, z)) + \int_{\underline{\eta}_i(z_{-1},z)}^{\bar{\eta}} \Big[\omega(\mathbf{x}_i) + \alpha\beta\bar{W}(z, g_i(z_{-1}, z, \eta))\Big] G(d\eta) \Bigg\} \Gamma(z/z_{-1}) \tag{56}$$

   Grid point values of $\bar{W}(z_{-1}, k_i(z_{-1}))$ are connected with linear segments and the decision rule for the next period value of capital, $g_i(z_{-1}, z, \eta)$, is provided by the first order condition (42). Using the functions $m(z, k')$ and $g_i(z_{-1}, z, \eta)$, we are able to determine the next period μ. To determine μ', for each $z \in \{z^1, z^2\}$, we connect grid points of the function $\mu' = m(z, k')$ with linear segments.

2. To find the values of $k_i(z_{-1})$, $\underline{\eta}_i(z_{-1}, z)$, and $\bar{W}(z_{-1}, k_i(z_{-1}))$, we solve the system of equations consisting of (43), (44), and (56), for $z_{-1}, z \in \{z^1, z^2\}$. This is a system of 8 equations in 8 unknowns. To evaluate these equations, we use the functions $g_i(z_{-1}, z, \eta)$ and $m(z_{-1}, k)$, and the approximations described above.

By repeating these steps for each grid point of μ backward, we solve the whole contract. After solving the contract, we are able to determine $J(z)$, for $z \in \{z^1, z^2\}$. We then restart the procedure using these new values of $J(z)$ until convergence.[8]

[8] The computational procedure described above starts by guessing outside values of searching for a new project $J(z) = q(z, n^1)V^0(z) - \kappa$. Then, after finding the value of V^0, the procedure is repeated until we find $J(z)$. However, a simpler way to proceed—particularly useful in the calibration stage—is to start the procedure by imposing certain values of $J(z)$, for $z \in \{z^1, z^2\}$. After finding the values of $V^0(z)$ (which are conditional on the assumed values of $J(z)$) we can parameterize the function $q(z, 0)$ so that the equation $J(z) = q(z, 0)V^0(z) - \kappa$ is satisfied.