Dynamic Price Competition with Persistent Consumer Tastes

Toker DOGANOGLU*

Institute for Statistics and Econometrics,

Christian Albrechts University at Kiel,

D-24098 Kiel, GERMANY.

email: toker@stat-econ.uni-kiel.de

January 31, 2000

*I would like to thank Rabah Amir for helpful comments. All errors are mine.

Abstract

The dynamic price competition in a horizontally differentiated duopoly when consumers value previous market shares is analyzed. The conditions for the existence of stable Markov-Perfect Equilibrium(MPE) in linear strategies are established. When they exist, the optimal pricing policies suggest that a firm with a higher previous market share charges a higher price, all else equal. It is possible to observe pricing below cost for some periods. In the steady state, the MPE leads to a more competitive outcome (lower prices) than the case where there is no persistence in consumer tastes. The model can produce outcomes where the steady state is reached very slowly which provides an alternative explanation for slow emergence of competition when entrants face an established incumbent: It may be due to persistence in consumer tastes.

Keywords: Dynamic price competition, network externalities, fashion, markov-perfect equilibrium, product differentiation.

JEL Classification: C73, D21, D43, L13, L21.

1. Introduction

Consumers often consider the choice of the others when they face a decision problem. When buying a computer or a software package, one informally inquires about the choices of peers as well as consulting consumer reviews. People quite frequently choose the box office hits when going to a movie theater. When buying clothes, one tends to choose the fashionable items for that time period.

Another case which most people can relate to is the choice of restaurants. (See, for example, Becker 1991) When faced with the decision problem of choosing a dining location, people are inclined to patronize a restaurant which has waiting lines and high prices. Conceivably there are several factors at work in bringing about this situation. First of all, when consumers are not sure about the quality of food served in restaurants, the high rate of occupancy might serve as a signal that the food is quite good. Secondly, having shared an experience that others in one's social circle has already acquired might provide additional satisfaction. Therefore, it is plausible to conjecture that, when making a decision about a restaurant, a consumer considers her own taste for the particular kind of cuisine, the price of the meal that she has to pay for dining there, as well as the popularity of the restaurant.

In this paper, I will investigate the dynamics in a differentiated products duopoly where consumers value previous market shares. The origin of this valuation is not spelled out; it might be due to network externalities, uncertainty about the product quality or just fashion. The questions I want to address are, how would prices and market shares evolve if such an effect were present. Vettas (1997) develop a similar model where current demand is positively related to the previous sales levels. He derives the equilibrium entry path of perfectly competitive firms. The basic difference of my model with Vettas (1997) is the market structure. In my model firms behave strategically instead of taking prices as given.

The model in consideration consists of two symmetric firms which produce horizontally differentiated products with constant marginal cost. The consumers live for two periods and only the old generation makes purchases. A product differentiation model á la Hotelling is adopted; in addition, the consumers' valuation includes a term that reflects the effects of the previous market share of a particular brand which I refer to as persistence in consumer tastes. At the beginning of each period, firms simultaneously announce their prices and then the demands are resolved. Due to the persistence in the demand, a firm with a high market share faces a trade off between exploiting high valuation it receives from the consumer population and foregoing future market share advantage due to high prices this period.

I consider linear Markov strategies for both firms and require that the market is covered in each period. An outcome is defined to be stable when both firms have market shares between zero and one. I establish the conditions required for the existence of a stable Markov Perfect Equilibrium(MPE) in linear strategies. In equilibrium, a firm with a higher previous market share charges a higher price, all else being equal. In the steady state, the prices are below the prices which would have been quoted in the absence of persistence in the consumer tastes. That is, when firms consider intertemporal effects of their pricing policies, a more competitive outcome prevails. In addition, equilibrium may involve prices below cost at some periods.

An important point to note is the speed of convergence to the steady state(stable) equilibrium. Often in industries which are initially dominated by one firm, the entrants find it difficult to gain a foothold in the market. I show that this is possible when consumers tastes are persistent, that is when consumers value the previous market share of a firm, the penetration of a new firm's product in the market might be substantially slow. This outcome has important implications, since the slow penetration of the entrant's product is due to consumer preferences but not on the predatory strategies followed by incumbent firms.

The paper is organized as follows. The related literature is presented in section 2. The model is outlined in Section 3. The analysis of the model is presented in Section 4. I analyze the entry in a market which is initially dominated by an incumbent in Section 5. Section 6 concludes.

2. Related Literature

The feedback effect of what others do on the preferences of the consumers may lead to endogenously changing tastes. Formal analysis of such effects can be traced back to von Weiszäcker (1971). He develops a model where consumers' current utility is effected by the level of previous sales. He concludes that if indeed the consumer preferences evolve endogenously, policy decisions are better based on steady state preferences and derives conditions on the preferences so that a steady state is achieved. Becker (1991), also assuming positive feedback effects on consumer demand, provide an explanation for persistent excess demand and why prices do not increase to levels that might clear the market in the case of restaurants.

Economists have recently been trying to provide rational explanations of why people coordinate their actions with others. The conceptually easiest explanation is found in network industries. For example, in a telephone network one additional user substantially increases the number of potential connections. A software package which is used by a large number users induce others to adopt the same package as the possible exchanges of information are enhanced. The more customers a network has, the more demand is generated. Such positive feedback effects are called "network externalities". Brynjolfsson and Kemerer (1996) show a positive relationship between the installed base and the retail price of software products signaling existence of the feedback effects of the type in consideration.

Another avenue that has been explored in detail recently involves uncertainty and learning on consumers' and/or firms' side. Caminal and Vives (1995) develop a model showing that past sales might serve as a signal of product quality when previous prices are not observable. The consumers might view a good which is purchased by more people as of being better quality. Therefore, the market share of a firm may be valued positively by the consumers when making a purchase decision. In support of this view, Bergemann and Valimäki (1997) consider a model where consumers use market shares to update their beliefs about the quality of a new product.

Yet another strand of literature where people consider what others do and behave accordingly is related to fashion or herd behavior. Often it is observed that societies pass traditions, customs from one generation to the other even if such customs are undesirable from the viewpoint of social welfare. Sometimes people just take the same actions that others have taken regardless of their own preferences. (See for example, Banerjee 1992, Bikhchandani, Hirshleifer and Welch 1992)

3. The Model

Consider a duopoly which serves N customers who have different tastes along with a product differentiation model á la Hotelling. The two firms are located at each end of the unit interval. In the next subsection we define consumer preferences and derive demand functions for each product. The subsection after that will set up the firms profit functions.

3.1. Consumers

I consider a consumer population which lives only two periods. Each period N young consumers arrive to the market, and N old consumers purchase one unit of one of the products. It is assumed that the product of firm 1 is located at 0 and firm 2 is located at 1. Both young and old consumers are uniformly and independently distributed along the unit interval with their positions representing their ideal product. Every period the consuming segment (the old) of the population changes, therefore each period there is a different realization of consumer locations.[1]

The consumers' derive a utility of V for consuming their ideal product which would have been located at the same point with them on the unit interval. They incur a transportation cost for having to consume one of the available brands instead of their ideal brand. The per unit travel cost is denoted by $1/2s$.[2] They also incur a disutility due to the price paid for

[1] This assumptions on the consumer population aims to capture the fact that consumers tastes may change through time. One could make repeat purchases, but the needs and tastes of a consumer might change during the time between purchases.

[2] Observe that s here provides a measure of substitutability between both brands. When $s \to \infty$, the transportation cost approaches to zero, therefore both brands become close substitutes. On the other hand for $s \to 0$, the transportation cost approaches infinity,

purchasing.

In addition to this standard horizontal differentiation model, the customers have a perceived component in their valuation which represents the effect of fashion or a valuation of the market penetration of the product. Throughout the text, this component is referred as persistence in consumer tastes. This effect is assumed to be a linear function of the previous market share, i.e the valuation of the previous market share enters the consumers' utility function as am_j^{t-1}, $(j = 1, 2)$, where a is measure of the strength of this valuation. This could be viewed as a signal of the product's quality,[3] network externalities,[4] or some other attribute which leads to a given level of popularity.

Let us denote the price of firm j at period t by p_j^t and the previous market share by m_j^{t-1}, for $j = 1, 2$. Then, the valuation of the customer located at the point α for product j at period t is

$$U^t(\alpha, p_j^t, m_j^{t-1}) = V + am_j^{t-1} - \frac{|\alpha - F_j|}{2s} - p_j^t, \tag{1}$$

where $F_j \in \{0, 1\}$ is the location of firm j and $a \geq 0$ represents the strength

therefore consumers prefer the closest brand independent of their prices.

[3]For example, let the quality of a product be either V or $V + a$. People will buy a product either because it is of higher quality or it is substantially cheaper. Therefore, the consumers of firm 1, purchase the product either because they believe the product to be of quality $V + a$ or of quality V but it is much cheaper than the alternative. Therefore, the high quality product belongs to firm 1 with probability m_1^{t-1} or to firm 2, in which case firm 1 has a product of quality V, with probability $1 - m_1^{t-1}$. Therefore the expected quality of the product of firm 1 is given by $V + am_1^{t-1}$.

[4]If this valuation is thought to be due to network externalities, this approach provides an adaptive expectations model of externalities. That is, consumers use the previous market share to estimate the expected network benefits, and in steady state these expectations are correct.

of the valuation of the previous market share and it is assumed to be time-invariant.[5]

Every period each customer consumes one unit of the product which provides the highest value. That is, the customer located at α chooses product j if and only if

$$U(\alpha, p_j^t, m_j^{t-1}) > U(\alpha, p_{-j}^t, m_{-j}^{t-1}),$$

where $-j$ represents the other product. Following the standard procedure, I look for the location of the indifferent customer, $\tilde{\alpha}$, to find the expected demand functions for each product. Observe that all the customers to the left of $\tilde{\alpha}$ consume product 1 while customers to the right prefer product 2. As every period the consuming segment of the population is changing,[6] and the actual location of each customer is private information, the firms can only calculate expected demands.

Depending on the values of parameters there are several possible market configurations, such as one firm cornering the market, or both firms producing but the market not being fully covered. The other possibility is both firms produce and the market is covered. I will concentrate on the latter case where $m_2^t = 1 - m_1^t$. This requires V to be large relative to $1/2s$. In this case, expected demand functions are given by

$$x_1^t(p_1^t, p_2^t, m_1^{t-1}) = N\left[\frac{1}{2} + as(2m_1^{t-1} - 1) + s(p_2^t - p_1^t)\right], \tag{2}$$

$$x_2^t(p_1^t, p_2^t, m_1^{t-1}) = N\left[\frac{1}{2} + as(1 - 2m_1^{t-1}) + s(p_1^t - p_2^t)\right]. \tag{3}$$

[5]Clearly, this is a strong assumption. Even though the consumers are different each period, the preference parameters are assumed to be constant through time. However, it is possible to imagine that young consumers inherit these preference parameters from the old.

[6]Recall that only the old consumers make purchases.

3.2. The firms

For simplicity, I assume firms incur zero marginal costs, however, this assumption can be relaxed.[7] Per period profit functions of the firms are given by

$$\pi_1(p_1^t, p_2^t, m_1^{t-1}) = p_1^t x_1^t(p_1^t, p_2^t, m_1^{t-1}) \quad (4)$$

$$\pi_2(p_1^t, p_2^t, m_1^{t-1}) = p_2^t x_2^t(p_1^t, p_2^t, m_1^{t-1}) \quad (5)$$

The competition has infinite horizon, and firms have a common discount factor β. Each firm maximizes the expected value of the profit streams. The objective function of each firm is given by

$$\Pi_j^t(m_1^{t-1}) = E^{t-1}\left[\sum_{i=0}^{\infty} \beta^i \pi_j^{t+i}(p_1^{t+i}, p_2^{t+i}, m_1^{t+i-1})\right], \qquad j = 1, 2. \quad (6)$$

The demand is resolved after the prices are announced, therefore the expectation is taken at the beginning of each period.

4. The Analysis

The underlying strategic interaction is dynamic, implying a multitude of possible outcomes in equilibrium. However, it is useful to briefly discuss the incentives a firm faces when choosing its price in the beginning of the period. Inspection of (2) and (3) reveals that a firm with a market share larger than a half faces a higher demand function due to the valuation of previous market shares by consumers. Therefore, a firm might be able to sustain a high price at a given period, but choosing a higher price may

[7] In fact, when the marginal costs of both firms are the same, our model will provide the same results if one views the strategic variable p_j^t as the markup above cost instead of prices.

decrease market share leading to a lower demand in the future. Hence, in equilibrium the contrasting incentives of higher profits this periods versus lower demand in the future must be balanced. An equilibrium concept that incorporates this intertemporal nature of the decision problem will be an appropriate choice.

I will, therefore, concentrate on one particular kind of equilibrium, namely, Markov Perfect Equilibrium(MPE) as introduced by Maskin and Tirole (1987). In a MPE, strategies are only functions of the payoff relevant information, that is, the strategies are only functions of the relevant physical state of the world, which, in this case is the previous market share of firm one. In the context of this paper, a MPE can formally be defined as below.

Definition 1. Markov Perfect Equilibrium

A Markov Perfect Equilibrium is a pair of functions from the state space $M = [0,1]$*, to the space of all possible prices,* $\mathbb{P} \subset \mathbb{R}$*. Formally, let* $p_1 : M \to \mathbb{P}$ *and* $p_2 : M \to \mathbb{P}$ *be two functions. Also let* $V_j(m_1^t)$*,(j=1,2), be the value of the game starting at period* $t+1$ *where both players play their optimal price policies.* p_1 *and* p_2 *constitute a MPE if and only if*

$$V_1(m_1^{t-1}) = \max_p E\left[\pi_1(p, p_2(m_1^{t-1}), (m_1^{t-1}) + \beta V_1(m_1^t(p, p_2(m_1^{t-1}), m^{t-1}))\right],$$

and

$$V_2(m_1^{t-1}) = \max_p E\left[\pi_2(p_1(m_1^{t-1}), p, m_1^{t-1}) + \beta V_2(m_1^t(p_1(m_1^{t-1}), p, m^{t-1}))\right].$$

The model posited above falls in to the category of linear-quadratic games which has been studied in detail. (See, for example, Basar and Oldser 1982) The strategic variable of each firm is the price and the physical state of the system is summarized by the previous market share. The per period profit function is quadratic-concave in the price(the strategic variable), and the state (market share) evolves as a linear function of the actions(prices) of the players. Therefore one can guess policy functions for p_1^t and p_2^t of the form

$$p_1(m_1^{t-1}) = l_1 + k_1 m_1^{t-1},$$

and

$$p_2(m_1^{t-1}) = l_2 + k_2(1 - m_1^{t-1}).$$

If one can find coefficients (l_1, l_2, k_1, k_2) which are consistent with maximization, leads to non-negative profits for both firms and, in addition, lead to a market structure where the market is covered, then the policy functions above will constitute a MPE. Given the linear-quadratic nature of the problem, the first order conditions(FOCs) will be necessary and sufficient. The solutions of the maximization problems are provided in the appendix.

Lemma 1. *The parameters (l_1, l_2, k_1, k_2) which satisfy the first order conditions are symmetric, that is $k_1 = k_2 = k$, $l_1 = l_2 = l$, and in addition k satisfies*

$$\frac{3k - 2a}{\beta} - 4s^2(k - a)^2(k - 2a) = 0, \tag{7}$$

and

$$l = \frac{1}{2s} - a\beta - \frac{1 - \beta}{2}k. \tag{8}$$

Proof. See appendix.

There are more than one possible (l, k) pair which satisfy the first order conditions. However, there are additional requirements for (l, k) to constitute a MPE of the price competition. It is needed that at every period the market is fully covered and both firms have non-negative market shares. If the pair (l, k) constitutes a MPE, the market share of firm 1 will evolve following

$$m^t(m_1^{t-1}) = \frac{1}{2} + s(2m_1^{t-1} - 1)(a - k). \tag{9}$$

Hence, for every previous market share the following condition has to hold:

$$0 \leq \frac{1}{2} + s(2m_1^{t-1} - 1)(a - k) \leq 1$$

There are two cases that has to be considered.

Case 1. $a \geq k$. In this case, the expected market share is increasing in the previous market share, therefore we need

$$a - k \quad \leq \quad \frac{1}{2s}, \tag{10}$$

Case 2. $a < k$ Since the expected market share is decreasing in the previous market share for this case, we need

$$-a + k \quad \leq \quad \frac{1}{2s}. \tag{11}$$

Combining (10) and (11)leads to

$$\gamma(k) :=\mid 2s(a - k) \mid\leq 1. \tag{12}$$

One can refer to (12) as the condition for stability, as it is required to hold for market shares to remain in $[0, 1]$ for all possible previous market share values and hence all possible prices. In the remainder of the text, $\gamma(k)$ is referred to as the stability factor. The steady state is attained when

$$m_1^{t+1}(m_1^t) = m_1^t(m_1^{t-1}) \equiv m_1^{ss}.$$

Solving for the steady state market share of firm 1 results in

$$m_1^{ss} = \frac{1}{2},$$

therefore, the steady state outcome is symmetric.

Another requirement is that each firm should prefer producing. This is possible when both firms have non-negative expected profits. Formally, it is necessary that $\Pi_j^t(m_1^{t-1}) \geq 0$ for $j = 1, 2$, since otherwise, by staying out of the market (for example by setting $p_j^t = \infty$), a firm may achieve zero profit.

Let $\mathcal{X} = \{(a, s, \beta) | a \in \mathbf{R}_+, s \in \mathbf{R}_+ \;\; \text{and} \;\; 0 \leq \beta \leq 1\}$ be the set of the parameters of the model, and denote a typical element of $\mathcal{X}$ by x. The following definition characterizes a stable MPE of dynamic price competition in linear strategies where the market is covered at every time period.

Definition 2. *For every $x \in \mathcal{X}$, a MPE in linear strategies where each period the market is covered has parameters (l, k) that satisfy the FOCs, the stability condition (12), and leads to $\Pi_j^t(m_1^{t-1}) \geq 0$, for $j = 1, 2$, and $0 \leq m_1^{t-1} \leq 1$,as defined in (6).*

Before proceeding with the main result, it is useful to partition the set of parameters as follows:

$E_1^1 := \{x \;\mid\; x \in \mathcal{X}, \text{ and, } as < \frac{1}{2}\frac{3-\beta}{1+\beta}, \text{ and, } 0 \leq \beta < \frac{1}{3}\}$,

$E_1^2 := \{x \;\mid\; x \in \mathcal{X}, as < \frac{1}{2} + \frac{1}{2}\frac{1-\beta}{\sqrt{\beta(1+\beta)}}, \text{ and, } \frac{1}{3} \leq \beta \leq 1\}$,

$E_1^3 := \{x \;\mid\; x \in \mathcal{X}, \frac{1}{2} + \frac{1}{2}\frac{1-\beta}{\sqrt{\beta(1+\beta)}} < as < \frac{1}{2}\frac{3-\beta}{1+\beta}, \text{ and, } \frac{1}{3} \leq \beta \leq 1\}$,

$E_1 := \{x \;\mid\; x \in E_1^1 \cup E_1^2 \cup E_1^3\}$,

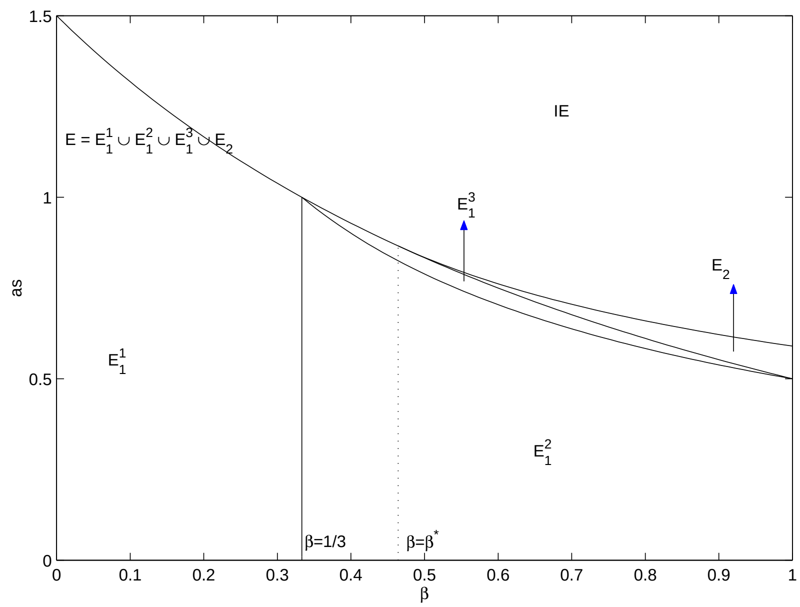
$E_2 := \{x \;\mid\; x \in \mathcal{X}, \frac{1}{2}\frac{3-\beta}{1+\beta} \leq as \leq \sqrt{\frac{\theta}{\beta}}, \text{ and, } \beta^* \leq \beta \leq 1\}$,

$E \;\; := \{x \;\mid\; x \in E_1 \cup E_2\}$

$IE := \{x \;\mid\; x \in \mathcal{X} \setminus E\}$,

where $\theta = -\frac{9}{4} + \frac{3}{2}\sqrt{3}$ and $\beta^* = 2\sqrt{3} - 3$. As stated in the following proposition, the sets E_1^1, E_1^2, E_1^3 and E_2 define the parameters where a stable MPE in linear strategies exists. In each of these sets the price policies exhibit different behavior. In figure 1, these sets are presented visually. Notice that both E_1^3 and E_2 are relatively smaller than all other sets.

Figure 1: The Partition of the Parameter Set



Proposition 1. Existence of stable Markov Perfect Equilibrium in linear strategies (LSMPE)

a. For all $x \in E$ $0 \leq \gamma(k) \leq 1$.

b. There exists a unique LSMPE for $x \in E_1$ *with* $k > 0$ *and*

i) $l \geq 0$*, if* $x \in E_1^1 \cup E_1^2$,

ii) $l < 0$*, if* $x \in E_1^3$.

c. There are two sets of (l, k) pairs which lead to a LSMPE for $x \in E_2$ with $k > 0$ and $l < 0$.

d. For $x \in IE$, there exists no LSMPE.

Proof. See Appendix.

The fact that $k \geq 0$ for $x \in E$ implies that a firm with a higher market share in the previous period will charge a higher price than a firm with a smaller market share would, everything else being equal. This result is consistent with the observation of higher prices charged by popular brands. In addition, for the cases where $l < 0$, it is possible to observe negative(below cost) prices when a firm has a lower market share in the previous period. Such practices, that is below cost pricing, are quite common in many industries. Often, an entrant offers introductory prices, free samples or even explicit subsidies to penetrate the market, for example, in software industry or mobile telecommunications industry.

For large values of the product of popularity and substitutability, as, a MPE in linear strategies where the market is covered cannot be sustained. The largest permissible value of as is attained when $\beta \to 0$, and even then it is required that $as < 3/2$. Moreover, for $\beta \to 1$, the set of possible parameters for which a LSMPE exists shrinks considerably. When $x \in IE$, the endogeneity of the demands leads to unstable outcomes. This unstability stems from the particular definition that I employed which requires both firms to have market shares in $[0, 1]$ each period. It might be possible to derive equilibrium strategies where the firms explicitly account for the constraint that their market shares should remain in between zero and one, but we do not extend the analysis in this direction in the present context. However, my conjecture is that, for $x \in IE$, the market will be cornered by one firm, that is de facto standardization will occur.

It is interesting to note that the existence results depends on the product of a, the magnitude of the valuation of previous market shares, and s the substitutability between the brands. As two brands become more and more close substitutes, it becomes more difficult to sustain an equilibrium outcome even with small magnitudes of persistence in demand.

Proposition 2. Steady State Prices

The steady state prices of both firms are equal and given by

$$p^{ss} = \frac{1}{2s} - a\beta - \frac{\beta k}{2}. \tag{13}$$

Moreover, $0 < p_1^{ss} \leq \frac{1}{2s}$.

Proof. See Appendix.

It is easy to see from (7) that $k \to 0$ when $a \to 0$. Therefore, in the absence of any persistence in the demand function the MPE of the dynamic price competition would entail charging prices that are equal to the transportation cost. If the firms were operating in an industry where there are no persistence in demand, the transportation cost would have been the equilibrium price of the dynamic interaction. Surprisingly, introduction of the valuation of the previous market shares by the consumers decreases the prices charged by the firms below the transportation cost. Therefore, the equilibrium, that is derived here, of the dynamic price competition in an environment with persistence in the demand involves more competitive behavior. This points out a very interesting phenomenon: Competition between two brands which exhibit externalities or fashion effects will be more intense than two brands which do not possess such effects. The mechanism leading to such a result is due to the incentives of both firms to cut prices to gain future market share advantage. Indeed, a firm with a market share slightly above one half does not enjoy any significant valuation due to persistence this period and have incentives to cut prices to face a higher demand next period. At the same time, the other firm has even stronger incentives to cut prices this period since it has a market share below one half leading to a lower demand. In equilibrium, when such incentives are balanced, the resulting prices are much lower. Similar results showing that the MPE might involve more competitive behavior is also shown by Jun and Vives (1998) in a different context.

This more competitive outcome is in sharp contrast with for example markets with switching costs where the intertemporal nature of the decision problem allows firms to sustain high(close to perfectly collusive) prices.(See for example, Beggs and Klemperer (1992), Padilla (1995)) The reason behind this difference is that in this model the feedback effects are eliminated when both firms serve close to half of the market and hence they have incentives to cut prices to enhance future demand. In models with switching costs, however, a firm always have captive consumers, therefore, the incentives to exploit these locked-in consumers are always present.

5. Entry

As discussed in the previous section, the competition in the presence of persistence in consumer tastes is rather intense. In this section, I will analyze the evolution of market shares when a firm dominates the market initially.

Let us, first, consider the time required to reach steady state. First assume that (l, k) leads to a LSMPE. Let $m_0 > 1/2$ be the initial market share of firm 1, the incumbent. Solving (9) recursively results in

$$m_1(t, m_0) = \frac{1}{2} + (\gamma(k))^t (m_0 - \frac{1}{2}), \tag{14}$$

where $\gamma(k) = 2s(a - k)$. And define an ϵ-neighborhood of the steady state as $[1/2 - \epsilon, 1/2 + \epsilon]$ recalling that $m_1^{ss} = 1/2$. The next proposition provides a lower bound for the expected number of periods that is required for the market share of the incumbent to decrease to an ϵ-neighborhood of $m_1^{ss} = 1/2$ where $\epsilon < m_0 - 1/2$.

Proposition 3. Rate of Convergence

For $\epsilon < m_0 - 1/2$, the expected number of time periods, τ, which is required for the incumbent's market share to be in an ϵ-neighborhood of $m_1^{ss} = 1/2$, satisfies

$$\tau > \frac{\log \frac{\epsilon}{m_0 - \frac{1}{2}}}{\log(\gamma(k))}.$$

Proof. See Appendix.

Clearly, as $\gamma(k) \to 1$, the time required to reach an ϵ-neighborhood of the steady state market share increases indefinitely. This provides an alternative explanation of the hardships that entrants face in order to penetrate a market. In this case, contrary to the common belief that the difficulties faced by the entrants are caused by predatory strategies of incumbent firms, the slow penetration is due to the interaction between consumers' preferences and the discount rate.

In the model, consumers' taste parameters(locations) are changing from period to period, since the consumer population changes every period. This fact, combined with a stability factor near one implies that the market leader might change in a particularly biased realization of consumer tastes. If in one period, for some external reason which is not modeled here, a larger population prefers a particular brand, then the dominance of the producer of this brand might prevail for quite a long time. This fact can explain why one observes fashion cycles where different firms are dominant in a periodic manner.

It is useful to simulate the equilibrium outcomes for some parameter values to get a feeling for the dynamics of the market shares and prices. When $\gamma(k)$ is close to zero, prices and market shares converge to their steady state values rather quickly and exhibit oscillations with a small magnitude, therefore the results are not too interesting. However, when $\gamma(k) \to 1$, the model produces striking results as the shocks have long lived effects.

I have chosen $s = 1$, $N = 1000$, $m_0 = 1$ and $\epsilon = 0.05$. Every period the consuming population is randomly located by means of a uniform random number generator. Then, given the prices and the previous market shares of the firms, each consumer is assigned to the firm which provides the higher value. The realized market share of a firm can be calculated as a ratio of the customers of a particular firm to the total number of consumers.

Case 1. $a = 1.099$ and $\beta = 0.25$

The parameters of the model are in the set E_1^1. The parameters of the equilibrium strategies in this case are given by $l = 0.0002$ and $k = 0.6001$ and the stability factor becomes $\gamma(0.6001) = 0.9978$. The expected time to reach the steady state in this case is 1701 periods. The expected profits of the firms are given by $\Pi_1^0(1) = 798.45$ and $\Pi_2^0(1) = 0.00052$. Clearly, the early advantage of firm 1 translates in to much higher expected profits. The market share and price dynamics are presented in Figures 2 and 3, respectively. An occasional shock to consumer tastes, that biases the population towards one brand, has long lived effects. One such shocks can be observed around $t = 700$ and, resulting in ,what one might call, a fashion cycle. For about a thousand periods the entrant remains as the market leader and around $t = 1800$ the incumbent again captures of half the market.

Case 2. $a = 0.555$ and $\beta = 0.9$

The parameters, in this case, are in the set E_2. Therefore there are two possible (l, k) pairs which can be sustained in equilibrium. The parameters of the equilibrium strategies in this case are given by $l = -0.0024(-0.0153)$ and $k = 0.0577(0.3151)$ and the stability factors turn out to be $\gamma(0.0577) = 0.9946$ and $\gamma(0.3151) = 0.4798$. For $k = 0.3151$, only six periods are needed to get in a 0.05-neighborhood of the steady state, while for $k = 0.0577$ the number of periods becomes 687. The expected profits of the firms are given by $\Pi_1^0(1) = 526.20(957.88)$ and $\Pi_2^0(1) = 0.0381(560.43)$. With the larger value of k the steady state is reached very quickly and the steady state price is $p^{ss} = 0.1423$ which is still much lower than $\frac{1}{2s} = 0.5$. I will present the dynamics for $k = 0.0577$ in Figures 4 and 5, respectively. Observe the below cost pricing by the entrant in the beginning of competition. Also in this case, the market leader changes by a strong shock on the consumer tastes; two such instances are observed at around $t = 600$ and $t = 900$. It is interesting to note that the price process resembles an autoregressive stochastic process. And since the coefficient $\gamma(k)$ is closed to one in this case, a shock drastically biasing the tastes of the consumer population has long lived effects.

Figure 2: Market Share Dynamics ($a = 1.099$, $\beta = 0.25$)

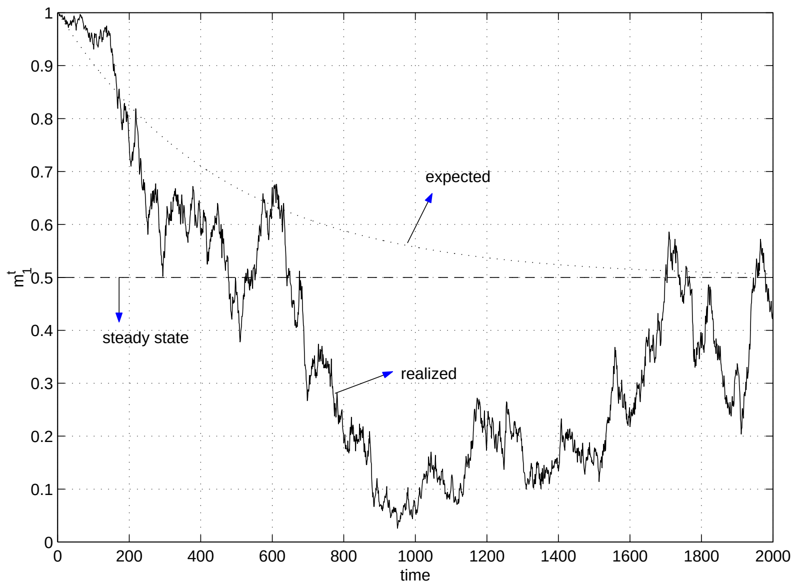


Figure 3: Price Dynamics ($a = 1.099$, $\beta = 0.25$)

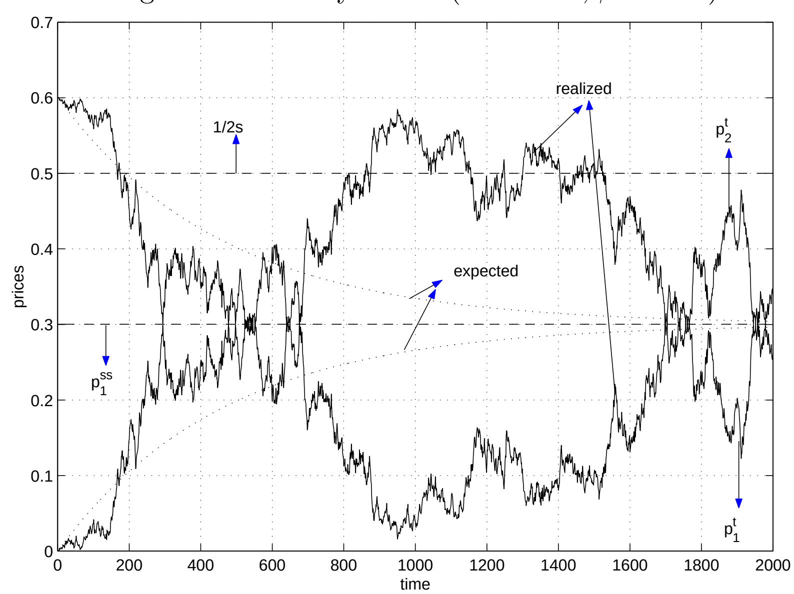


Figure 4: Market Share Dynamics ($a = 0.555$ and $\beta = 0.9$)

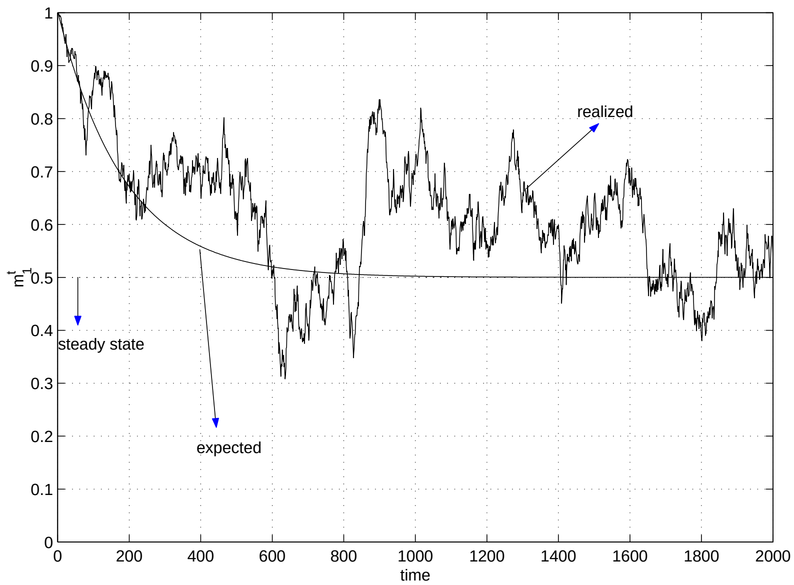
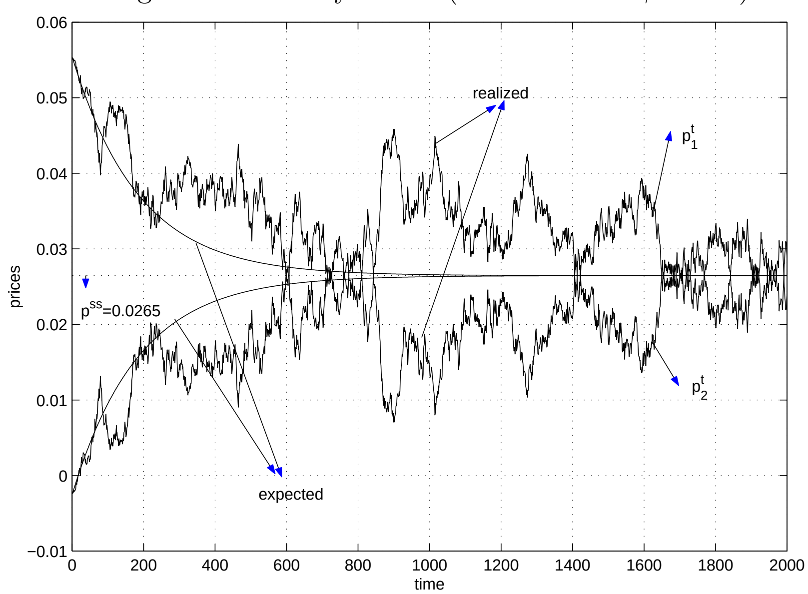


Figure 5: Price Dynamics ($a = 0.555$ and $\beta = 0.9$)



6. Conclusion

I have analyzed price competition in a duopoly when customers value the previous market share. The conditions for the existence of a stable MPE in linear strategies where the market is covered at each period are established. One feature of equilibrium price policies is that a firm with a high market share will choose higher prices. This prediction well fits with empirical observations of the higher prices charged by popular firms.

The existence of equilibrium depends on the product of the magnitude of feedback effects and the substitutability of the available brands. A stable MPE exists only when this product is sufficiently low, and for high values of this product a stable equilibrium does not exist. When this product is closed to its boundary value and the discount factor is sufficiently large, a firm with a very low market share chooses a price below cost to gain market share next period.

It is also interesting to note that, in the steady state, firms choose prices which are below the prices that might have been chosen in the absence of persistent consumer tastes. These effects make the market share a very important strategical variable for a firm. Therefore, the presence incentives of gaining future market share for both firms, leads to fiercer price competition driving the prices down in the steady state.

I have introduced a concept of stability which relies on a shared market structure. The evolution of markets shares are governed by a parameter which is referred to as the coefficient of stability in the text. The convergence to the steady state levels of market shares (prices) may be substantially slow after a shock to the consumer preferences if this coefficient has a value closer to one. Therefore, the emergence of a competitive market structure may take a long time in markets where there is persistence on the demand side.

There are several possible extensions of the model. Different magnitudes of the persistence effects for each product may lead to asymmetric steady state outcomes. It would be interesting to analyze a model where firms control the popularity of their brands through costly advertising. Another possible extension is negative persistence effects, that is the case where the wide spread consumption of a product decreases its value. This may provide a framework for the analysis of competition in the markets for luxury goods.

References

Banerjee, A.V. (1992): "A Simple Model of Herd Behavior", *The Quarterly Journal of Economics*, 107, pp. 797-817.

Basar, T. and G. Oldser (1982) : *Dynamic Noncooperative Game Theory*, Academic Press:New York.

Becker, G. S. (1991) : "A Note on Restaurant Pricing and Other Examples of Social Influences on Price", *Journal of Political Economy*, 99, pp. 1109-1116.

Beggs, A. and Klemperer,P.D. (1992): " Multiperiod Competition with Switching Costs", *Econometrica*, 60, pp. 651-666.

Bergemann, D. and J. Välimaki (1997): "Market Diffusion with Two-Sided Learning", *The Rand Journal of Economics*, 28, pp. 773-795.

Bikhchandani, S., D. Hirshleifer, and I. Welch (1992) : "A theory of Fads, Fashion, Custom and Cultural Change as Informational Cascades", *Journal of Political Economy*, 100, pp. 992-1026.

Brynjolfsson, E. and C.F. Kemerer, (1996) : "Network Externalities in Microcomputer Software: An Econometric Analysis of the Spreadsheet Market", *Management Science*, 42, pp. 1627-1647.

Caminal, R. and X. Vives (1996): "Why Market Shares Matter? An Information-Based Theory", *Rand Journal of Economics*, 27, pp. 221-239

Jun, B. and X. Vives (1999) : "Strategic Incentives in Dynamic Duopoly", mimeo.

Maskin E. and J. Tirole (1987): "A Theory of Dynamic Oligopoly, III: Cournot Competition", *European Economics Review*, 31, pp. 947-968.

Padilla, A.J. (1995): " Dynamic Duopoly with Consumer Switching Costs", *Journal of Economic Theory*, 67, pp. 520-530.

Vettas, N. (1997): "Entry to New Markets with Endogenous Demand", Fuqua School of Business, Duke University, working paper.

von Weizsäcker, C. C. (1971): "Notes on Endogenous Change of Tastes", *Journal of Economic Theory*, 3, pp. 345-372.

Appendix

Derivation of the Markov Perfect Equilibrium

Both firms solve the maximization problems

$$V_1(m_1^{t-1}) = \max_p E\Big[\pi_1(p, p_2(m_1^{t-1}), m_1^{t-1}) + \beta V_1(m_1^t(p, p_2(m_1^{t-1}), m^{t-1}))\Big],$$

and

$$V_2(m_1^{t-1}) = \max_p E\Big[\pi_2(p_1(m_1^{t-1}), p, m_1^{t-1}) + \beta V_2(m_1^t(p_1(m_1^{t-1}), p, m^{t-1}))\Big].$$

Since the firms consider the expected demands at the beginning of each period, the expectation is dropped by using the expressions for the expected profits each period. In equilibrium, the value functions satisfy

$$\begin{aligned} V_j(m_1^{t-1}) &= \pi_j(p_1(m_1^{t-1}), p_2(m_1^{t-1}), m_1^{t-1}) \\ &\quad + \beta V_j(m_1^t(p_1(m_1^{t-1}), p_2(m_1^{t-1}), m_1^{t-1})), \end{aligned} \tag{15}$$

for $j = 1, 2$. The first order conditions are obtained as

$$V_j'(m_1^t) = -\frac{\frac{\partial \pi_j}{\partial p_j}(p_j, p_{-j}(m_1^{t-1}), m_1^{t-1})}{\beta \frac{\partial m_1^t}{\partial p_j}(p_j, p_{-j}(m_1^{t-1}), m_1^{t-1})}, \tag{16}$$

for $j = 1, 2$, and $V_j'(x) = (\partial/\partial x)V_j(x)$. In addition, by using the equilibrium condition (15), one can relate $V'(m_1^t)$ and $V'(m_1^{t+1})$, that is

$$\begin{aligned} V_j'(m_1^t) &= \frac{\partial \pi_j}{\partial m_1^t}(p_1(m_1^t), p_2(m_1^t), m_1^t) \\ &\quad + \frac{\partial p_{-j}}{\partial m_1^t}(m_1^t)\frac{\partial \pi_j}{\partial p_{-j}}(p_1(m_1^t), p_2(m_1^t), m_1^t) \\ &\quad + \beta V_j'(m_1^{t+1})\Big[\frac{\partial m_1^{t+1}}{\partial m_1^t}(p_1(m_1^t), p_2(m_1^t), m_1^t) \\ &\quad + \frac{\partial p_{-j}}{\partial m_1^t}(m_1^t)\frac{\partial m_1^{t+1}}{\partial p_{-j}}(p_1(m_1^t), p_2(m_1^t), m_1^t)\Big], \end{aligned} \tag{17}$$

for $j = 1, 2$. $V_j'(m_1^{t+1})$ can be calculated by moving the time index one period further in (16), and substitute this in to (17). This provides us with the euler equations for both firms. Both equations in (17) can be reduced to a function of m_1^{t-1}, by first substituting state variable at $t+2$ as $m_1^{t+1} = m_1^{t+1}(p_1(m_1^t), p_2(m_1^t), m_1^t)$, the optimal policy functions of the form $p_j(m_1^t) = l_j + k_j m_1^t$, then the state variable at time $t+1$ as $m_1^t = m_1^t(p_1(m_1^{t-1}), p_2(m_1^{t-1}), m_1^{t-1})$, and $p_j(m_1^{t-1}) = l_j + k_j m_1^{t-1}$. It is easy to verify that both function are linear in m_1^{t-1}, therefore by equating the coefficients of m_1^{t-1}, and the constants and after rearranging, the following four equations in (l_1, l_2, k_1, k_2) are obtained:

$$\frac{(k_2 + 2k_1 - 2a)}{\beta} - s^2(k_2 - 2a)(k_2 + k_1 - 2a)^2 = 0 \tag{18}$$

$$\frac{(2k_2 + k_1 - 2a)}{\beta} - s^2(k_1 - 2a)(k_2 + k_1 - 2a)^2 = 0 \tag{19}$$

$$\begin{aligned} &-\frac{sl_2 + sk_2 - 2sl_1 - as + .5}{\beta s} \\ &-s(k_2 - 2a)(sk_2 + sk_1 - 2as - 1)(l_2 - l_1) \\ &+s^2(k_2 - a)(k_2 - 2a)^2 + s^2(k_2 - a)(k_2 - 2a)k_1 \\ &+.5(sk_1 - sk_2 - 1)(k_2 - 2a) = 0 \end{aligned} \tag{20}$$

$$\begin{aligned} &-\frac{2sl_2 + 2sk_2 - sl_1 - as - .5}{\beta s} \\ &-s(k_1 - 2a)(sk_2 + sk_1 - 2as - 1)(l_2 - l_1) \\ &+k_1 s^2(k_2 - a)(k_2 - 4a) - 2as^2(k_2 - a)(k_2 - 2a) \\ &+s^2 k_1^2(k_2 - a) + .5(sk_1 - sk_2 + 1)(k_1 - 2a) = 0 \end{aligned} \tag{21}$$

Lemma 1. *The parameters* (l_1, l_2, k_1, k_2) *which satisfy the first order conditions are symmetric, that is* $k_1 = k_2 = k$*,* $l_1 = l_2 = l$*, and in addition* k *satisfies*

$$f(k) = \frac{3k - 2a}{\beta} - 4s^2(k - a)^2(k - 2a) = 0, \tag{22}$$

and

$$l = \frac{1}{2s} - a\beta - \frac{1 - \beta}{2}k. \tag{23}$$

Proof. Subtracting (19) from (18), amounts to

$$\frac{(k_2 - k_1)}{\beta} = -s^2(k_2 - k_1)(k_2 + k_1 - 2a)^2 \tag{24}$$

Therefore k_1 and k_2 must be equal for (18) and (19) to hold for all parameter values, that is $k_1 = k_2 = k$. One can find the value of k by solving (22) which is derived by substituting $k_1 = k_2 = k$ in (18) or (19).

Now, adding (20) and (21), then substituting $k_1 = k_2 = k$ and simplifying, one obtains

$$\begin{aligned} \frac{4s^2\beta(k - 2a)(k - a) - 2\beta s(k - 2a) - 3}{\beta}(l_1 - l_2) & \\ + \frac{3k - 2a}{\beta} - 4s^2(k - a)^2(k - 2a) &= 0. \end{aligned} \tag{25}$$

it is easy to check that second line of (25) is equal to (22), hence is zero. Therefore, (25) holds for all parameter values if and only if $l_1 = l_2$. After substituting $k_1 = k_2 = k$ and $l_1 = l_2 = l$ in to the difference of (20) and (21), it is easy to check that l is given by (23). ■

It is useful to rewrite (22) as

$$f(k) = f_1(k) + f_2(k),$$

with

$$f_1(k) = \frac{3k - 2a}{\beta},$$

$$f_2(k) \quad = \quad -4s^2(k-a)^2(k-2a).$$

Moreover,

$$f'(k) = \frac{3}{\beta} - 4s^2(3k-5a)(k-a),$$

and

$$f''(k) = -8s^2(3k-4a).$$

Lemma 2. *The only possible solutions of (22) which satisfy* $\mid \gamma(k) \mid \leq 1$ *are such that* $k \in I_1 = [a - 1/2s, 2a/3]$*, and* I_1 *is nonempty only for* $as \leq 3/2$*. Moreover,* $k \in I_1 \Rightarrow 0 \leq \gamma(k) \leq 1$*.*

Proof. For the market shares to remain in $[0,1]$ for every previous market share, it is needed that $\mid 2s(a-k) \mid \leq 1$ or equivalently $k \in [a - 1/2/s, a + 1/2/s]$. Observe first of all that $f_2(k) \geq 0$ for $k \leq 2a$, and $f_1(k) \geq 0$ for $k \geq 2a/3$. Therefore, the only candidates for solutions of (22) lies in either $I_1 = [a - 1/2/s, 2a/3]$ or $I_2 = [2a, a + 1/2/s]$. For, $as > 1/2$, I_2 is empty, hence a root of $f(k)$ can be in I_2 only if $as \leq 1/2$. It is easy to verify $f'(2a) > 0$ and $f''(k) < 0$ for $k \in I_2$. For there to be a root in I_2, it is needed that $f(a + 1/2/s) < 0$. However,

$$f(a + 1/2/s) = \frac{1}{2}\frac{2as(1-\beta) + 3 - \beta}{s\beta} > 0,$$

for all $0 \leq \beta \leq 1$, therefore there is no root of $f(k)$ in I_2.

The only possible roots which leads to $\gamma(k) \leq 1$, then, must be in I_1 which is nonempty only when $as \leq 3/2$. The largest possible value of k in this case is $2a/3$, which implies $0 \leq \gamma(k) = 2s(a-k)$ and since $\gamma(k)$ decreases with k, the smallest value of k leads to $\gamma(a - \frac{1}{2s}) = 1$. Therefore $0 \leq \gamma(k) \leq 1$, for $k \in I_1$. ■

Lemma 3. *For* $as < \frac{1}{2}\frac{3-\beta}{1+\beta} \le \frac{3}{2}$*, there is a unique root of* $f(k)$ *in* I_1*.*

Proof. Observe first of all that $f(2a/3) > 0$, and $f''(k) > 0$ when $k \in I_1$. Therefore, there can only be one root of $f(k)$ in I_1, if $f(a - 1/2/s) < 0$. It is easy to check that this inequality is satisfied whenever

$$as < \frac{1}{2}\frac{3-\beta}{1+\beta}. \tag{26}$$

The right hand side of (26) is always less than 3/2 for $0 \le \beta < 1$. ■

Lemma 4. *For*

$$\frac{1}{2}\frac{3-\beta}{1+\beta} \le as \le \sqrt{\frac{\theta}{\beta}},$$

and $\beta > \beta^* = 2\sqrt(3) - 3$*,* $f(k)$ *has two roots in* I_1*, where* $\theta = -\frac{9}{4} + \frac{3}{2}\sqrt{3}$*. For* $\min(\frac{3}{2}, \sqrt{\frac{\theta}{\beta}}) > as$*,* $f(k)$ *has no roots in* I_1 *for every* $0 \le \beta \le 1$*.*

Proof. Since $as \ge \frac{1}{2}\frac{3-\beta}{1+\beta}$, $f(a - 1/2/s) \ge 0$. If for $k < 2a/3$ the minimum of $f(k) > 0$, then $f(k)$ has no roots when $k < 2a/3$; observe that $f''(k) > 0$ in this region so there exists a unique minimum. It is easy to verify that $f(k)$ is minimized at

$$k_{min} = \frac{1}{6}\frac{8as\beta - \sqrt{4a^2s^2\beta^2 + 9\beta}}{s\beta},$$

and

$$f(k_{min}) = \frac{4as\beta(4a^2s^2\beta + 27) - \sqrt{\beta(4a^2s^2\beta + 9)}(18 + 8a^2s^2\beta)}{54s\beta^2}.$$

Then $f(k_{min}) > 0$, whenever

$$4as\beta(4a^2s^2\beta + 27) > \sqrt{\beta(4a^2s^2\beta + 9)}(18 + 8a^2s^2\beta). \tag{27}$$

By squaring both both sides of (27), and simplifying the condition for $f(k_{min}) > 0$ becomes

$$16a^4s^4\beta^2 + 72a^2s^2\beta - 27 > 0,$$

which holds whenever $a^2s^2\beta > -\frac{9}{4} + \frac{3}{2}\sqrt{3} = \theta = .34808$. Hence, for $as > \sqrt{\theta/\beta}$, there are no roots of $f(k)$ when $k < 2a/3$. Also for $as > 3/2$, I_1 is empty, therefore for $\min(3/2, \sqrt{\theta/\beta}) > as$, $f(k)$ has no roots in I_1.

Observe that $k_{min} < 2a/3$, for $as < \sqrt{3/(4\beta)}$, and since $\sqrt{\theta/\beta} < \sqrt{3/4\beta}$, $f(k)$ has two roots to the left of $k = 2a/3$ for $as < \sqrt{\theta/\beta}$. It is also shown previously in lemma (3) that one of these roots is in I_1 whenever $as < \frac{1}{2}\frac{3-\beta}{1+\beta}$. For $as > \frac{1}{2}\frac{3-\beta}{1+\beta}$, there are two possibilities. Since, $f(a - \frac{1}{2s}) > 0$ in this case, either both roots are in I_1, or both of them are smaller than $a - \frac{1}{2s}$. Both roots fall in I_1 only if the minimum is attained in I_1, which would be the case when $k_{min} > a - 1/2s$ since it is already shown that $k_{min} < 2a/3$. It easy to verify that $k_{min} > a - 1/2s$, whenever

$$as < \frac{3}{4}\frac{1-\beta}{\beta}.$$

Therefore, $f(k)$ has two roots in I_1 for

$$\max(\frac{1}{2}\frac{3-\beta}{1+\beta}, \frac{3}{4}\frac{1-\beta}{\beta}) \leq as \leq \min(\frac{3}{2}, \sqrt{\frac{\theta}{\beta}}).$$

One can verify that for $\beta < \beta^* = 2\sqrt(3) - 3 \approx 0.4641$,

$$\frac{1}{2}\frac{3-\beta}{1+\beta} < \sqrt{\frac{\theta}{\beta}} < \frac{3}{4}\frac{1-\beta}{\beta},$$

hence both of the roots of $f(k)$ are to the left of I_1. However for $\beta \geq \beta^*$,

$$\frac{3}{2} > \sqrt{\frac{\theta}{\beta}} \geq \frac{1}{2}\frac{3-\beta}{1+\beta} \geq \frac{3}{4}\frac{1-\beta}{\beta},$$

therefore there are two roots of f(k) in I_1, for

$$\frac{1}{2}\frac{3-\beta}{1+\beta} \leq as \leq \sqrt{\frac{\theta}{\beta}}.$$

■

Lemma 5. *All the possible solutions of (22) which leads to* $0 \leq \gamma(k) \leq 1$ *are positive.*

Proof. First recall that $f(2a/3) > 0$. In addition, k can be negative only when $a - 1/2s < 0$, or equivalently $as < 1/2$. It is easy to verify that

$$\frac{1}{2} \leq \frac{1}{2}\frac{3-\beta}{1+\beta}.$$

Therefore, for $as < 1/2$ there is only one possible solution of k. Then if $f(0) < 0$, this root must be in $[0, 2a/3]$, since $f(2a/3) > 0$. Evaluating $f(k)$ at zero results in

$$f(0) = \frac{2a(4a^2s^2\beta - 1)}{\beta},$$

which can be shown to be negative whenever

$$as \leq \sqrt{\frac{1}{4\beta}}.$$

Since the right hand side of the above inequality decreases in β, the smallest value it can take, $1/2$, is obtained when $\beta = 1$, and hence $f(0) < 0$ holds for $as \leq 1/2$. Therefore all the possible solutions of $f(k)$ in I_1 are such that $k \geq 0$.■

Lemma 6. *For*

$$\frac{1}{2}\frac{3-\beta}{1+\beta} \leq as \leq \sqrt{\frac{\theta}{\beta}},$$

$l < 0$. *Moreover, for* $\beta \geq \frac{1}{3}$ *and*

$$\frac{1}{2} + \frac{1}{2}\frac{1-\beta}{\sqrt{\beta(1+\beta)}} \leq as \leq \frac{1}{2}\frac{3-\beta}{1+\beta},$$

$l \leq 0$. *Otherwise,* $l > 0$.

Proof. One can easily check $l < 0$ whenever

$$k_{l=0} = \frac{1 - 2as\beta}{s(1-\beta)} < k.$$

When $as > \frac{1}{2}\frac{3-\beta}{1+\beta}$, $k_{l=0} < a - \frac{1}{2s}$, therefore when $f(k)$ has two roots in I_1, the corresponding values of l are negative. However,

$$a - \frac{1}{2s} \leq k_{l=0} \leq \frac{2}{3}a,$$

whenever

$$\frac{3}{2}\frac{1}{1+2\beta} \leq as \leq \frac{1}{2}\frac{3-\beta}{1+\beta}.$$

Therefore, $f(k_{l=0}) < 0$ will imply that the solution of the root of $f(k)$ is larger than $k_{l=0}$ and hence $l < 0$. Observe that

$$f(k_{l=0}) = \frac{(2as(1+\beta)+\beta-3)(4as(as-1)\beta(1+\beta)+3\beta-1)}{s\beta(1-\beta)^3}.$$

Since $(2as(1+\beta)+\beta-3) < 0$ for $as < \frac{1}{2}\frac{3-\beta}{1+\beta}$, it is needed that $(4as(as-1)\beta(1+\beta)+3\beta-1) > 0$ for $f(k_{l=0})$ to be negative. It is easy to verify that for

$$as \geq \frac{1}{2} + \frac{1}{2}\frac{1-\beta}{\sqrt{\beta(1+\beta)}},$$

$(4as(as-1)\beta(1+\beta)+3\beta-1) > 0$, and hence $f(k_{l=0}) < 0$. However, it is also needed that

$$\frac{1}{2} + \frac{1}{2}\frac{1-\beta}{\sqrt{\beta(1+\beta)}} \leq \frac{1}{2}\frac{3-\beta}{1+\beta},$$

which holds whenever $\beta \geq \frac{1}{3}$. ■

The fact that $l < 0$ may be true raises the possibility of non-positive prices at some periods when the previous market share is closed to zero. Consequently, this may cause firms to have non-positive expected profits, therefore it is needed to check whether firms obtain non-negative profits in equilibrium. When the firms play their affine price policies with the parameters (l, k), the expected market share of firm evolves as

$$m_1(t) = \frac{1}{2} + \gamma(k)(m_1(t-1) - \frac{1}{2}), \tag{28}$$

and starting from an initial market share m_0 solving (28) recursively leads to

$$m_1(t, m_0) = \frac{1}{2} + \gamma(k)^t (m_0 - \frac{1}{2}), \tag{29}$$

Observe that $1 - m_1(t, m_0) = m_1(t, 1 - m_0)$. Then, the value functions of both firms with m_0 being the initial value for the state can be written as

$$V_1(m_0) = \sum_{i=1}^{\infty} \beta^{i-1}(l + k m_1(i-1, m_0)) m_1(i, m_0),$$

and

$$V_2(m_0) = \sum_{i=1}^{\infty} \beta^{i-1}(l + k(1 - m_1(i-1, m_0)))(1 - m_1(i, m_0)),$$

therefore $V_2(m_0) = V_1(1 - m_0)$. Hence, it is sufficient to analyze $V_1(m_0)$ as a function of the initial state. Calculating $V_1(m_0)$ results in

$$\begin{aligned} V_1(m_0) &= \frac{k\gamma(k)}{1 - \beta\gamma(k)^2} m_0^2 \\ &+ \frac{1}{2} \frac{k(1 + \beta\gamma(k)^2)(1 - \gamma(k)) + 2l\gamma(1 - \beta\gamma(k)^2)}{(1 - \beta\gamma(k)^2)(1 - \beta\gamma(k)} m_0 \\ &+ \frac{1}{4} \frac{(1 - \gamma(k))(k\beta(1 - \gamma(k)^2) + 2l(1 - \beta\gamma(k)^2))}{(1 - \beta)(1 - \beta\gamma(k))(1 - \beta\gamma(k)^2)}. \end{aligned} \tag{30}$$

Lemma 7. *For $m_0 \geq 1/2$, $V_1(m_0) \geq V_2(m_0)$.*

Proof. It is easy to check that

$$V_1(m_0) - V_2(m_0) = \frac{1}{2} \frac{(2m_0 - 1)((2l + k)\gamma(k) + k)}{1 - \beta\gamma(k)}. \tag{31}$$

For $l \geq 0$, (31) is clearly positive. For $l < 0$, it is needed that $(2l + k)\gamma(k) + k > 0$, or after substituting the values of $\gamma(k)$ and l in terms of k

$$(2a - k)(1 - 2s\beta(a - k)) > 0,$$

which holds for $k < 2a/3$ and $2s(a - k) < 1$. ■

Lemma 8. $V_1(0) \geq 0$.

Proof. $V_1(0)$ is equal to the constant term in (30), that is

$$V_1(0) = \frac{1}{4}\frac{(1-\gamma(k))(k\beta(1-\gamma(k)^2) + 2l(1-\beta\gamma(k)^2))}{(1-\beta)(1-\beta\gamma(k))(1-\beta\gamma(k)^2)}. \tag{32}$$

Hence it is needed that $h_1(k) = k\beta(1-\gamma(k)^2) + 2l(1-\beta\gamma(k)^2) \geq 0$, which trivially holds for $l \geq 0$. Substituting the values of l and $\gamma(k)$ in terms of k in $h_1(k)$, one obtains

$$h_1(k) = \frac{1-2s(a-k)\beta}{s}h_2(k),$$

where $h_2(k) = -2s^2\beta(a-k)(2a-k) + sk - 1$, which is quadratic concave in k. If $h_2(0) \geq 0$ and $h_2(2a/3) \geq 0$, then for any $k \in [0, 2a/3]$, $h_2(k) \geq 0$. Clearly, $h_1(k) > 0$ whenever $h_2(k) > 0$, since $\frac{1-2s(a-k)\beta}{s} > 0$ holds always for $0 \leq 2s(a-k) \leq 1$. It is also only necessary to consider k, such that $f(k) = 0$, $\beta > 1/3$ and $as < \sqrt{\frac{\theta}{\beta}}$, since otherwise $l > 0$ and therefore $V_1(0) \geq 0$.

It is easy to verify that $h_2(0) = 1 - 4a^2s^2\beta$ which is positive whenever

$$as < \sqrt{\frac{1}{4\beta}},$$

and, since $\sqrt{\frac{\theta}{\beta}} \leq \sqrt{\frac{1}{4\beta}}$ for all $0 \leq \beta \leq 1$, it holds always. Therefore, $h_2(0) > 0$.

On the other hand, $h_2(2a/3)$ is given by

$$h_2(2a/3) = -\frac{8}{9}a^2s^2\beta - \frac{2}{3}as + 1,$$

and is positive whenever

$$as < \frac{3}{8}\frac{-1+\sqrt{1+8\beta}}{\beta}.$$

It can be easily verified that for $\beta > 1/3$,

$$\sqrt{\frac{\theta}{\beta}} < \frac{3}{8}\frac{-1+\sqrt{1+8\beta}}{\beta},$$

and hence $h_2(2a/3) > 0$ for all parameters that might lead to k in I_1 and $l < 0$. Since $h_2(k)$ is quadratic concave in I_1, $h_2(0) > 0$ and $h_2(2a/3) > 0$, $h_2(k) > 0$ for all $k \in [0, 2a/3]$ such that $l < 0$ and hence $V_1(0) \geq 0$. ■

Lemma 9. $\frac{\partial V_1}{\partial m_0}(0) > 0$.

Proof. The derivative of $V_1(m_0)$ at $m_0 = 0$ is given by the coefficient of m_0 in (30), that is

$$\frac{\partial V_1}{\partial m_0}(0) = \frac{1}{2} \frac{k(1 + \beta\gamma(k)^2)(1 - \gamma(k)) + 2l\gamma(1 - \beta\gamma(k)^2)}{(1 - \beta\gamma(k)^2)(1 - \beta\gamma(k)},$$

which again is trivially positive whenever $l \geq 0$. It is, therefore, only necessary to check for the parameter values which may lead to $l < 0$. Let us introduce

$$g_1(k) = k(1 + \beta\gamma(k)^2)(1 - \gamma(k)) + 2l\gamma(k)(1 - \beta\gamma(k)^2).$$

Clearly $\frac{\partial V_1}{\partial m_0}(0) > 0$, whenever $g_1(k) > 0$. It is easy to verify, after substituting the values of l and $\gamma(k)$, that $g_1(k) = (1 - 2s(a - k)\beta)g_2(k)$, with $g_2(k) = 4s^2\beta(k-a)^2(k-2a)+4ks(k-a)-(k-2a)$. It is clear that $g_1(k) > 0$ whenever $g_2(k) > 0$. Observe that $4s^2\beta(k-a)^2(k-2a) = -\beta f_2(k)$ and since only values of k such that $f_2(k) = -f_1(k)$ are of interest, one can rewrite $g_2(k)$ as

$$g_2(k) = \beta f_1(k) + 4ks(k - a) - (k - 2a) = 2k(1 - 2s(a - k)).$$

Therefore $g_2(k) > 0$ for all $k > 0$ and $2s(a - k) < 1$. ■

Lemma 10. $V_j(m_0) \geq 0$, *for all* $0 \leq m_0 \leq 1$, *and* $j = 1, 2$.

Proof. By lemma 8, $V_1(0)$ is positive, and by lemma 9, $V_1(m_0)$ is increasing at $m_0 = 0$. Observe in (30) that V_1 is a quadratic convex function of m_0, therefore $V_1(m_0) > V_1(0) > 0$ for $0 \le m_0 \le 1$. Since $V_2(m_0) = V_1(1 - m_0)$, $V_2(m_0)$ is also positive for all $0 \le m_0 \le 1$. ■

Let us define the following sets:

$E_1^1 := \{x \mid x \in \mathcal{X},\ \text{and},\ as < \frac{1}{2}\frac{3-\beta}{1+\beta},\ \text{and},\ 0 \le \beta < \frac{1}{3}\}$

$E_1^2 := \{x \mid x \in \mathcal{X}, as < \frac{1}{2} + \frac{1}{2}\frac{1-\beta}{\sqrt{\beta(1+\beta)}},\ \text{and},\ \frac{1}{3} \le \beta \le 1\}$

$E_1^3 := \{x \mid x \in \mathcal{X}, \frac{1}{2} + \frac{1}{2}\frac{1-\beta}{\sqrt{\beta(1+\beta)}} < as < \frac{1}{2}\frac{3-\beta}{1+\beta},\ \text{and},\ \frac{1}{3} \le \beta \le 1\}$

$E_1 := \{x \mid x \in E_1^1 \cup E_1^2 \cup E_1^3\}$

$E_2 := \{x \mid x \in \mathcal{X}, \frac{1}{2}\frac{3-\beta}{1+\beta} \le as \le \sqrt{\frac{\theta}{\beta}},\ \text{and},\ \beta^* \le \beta \le 1\}$

$E\ := \{x \mid x \in E_1 \cup E_2\}$

$IE := \{x \mid x \in \mathcal{X} \setminus E\}$

Proposition 1. Existence of stable Markov Perfect Equilibrium in linear strategies (LSMPE)

a. For all $x \in E$ $0 \le \gamma(k) \le 1$.

b. There exists a unique LSMPE for $x \in E_1$ with $k > 0$ and

i) $l \ge 0$, if $x \in E_1^1 \cup E_1^2$,

ii) $l < 0$, if $x \in E_1^3$.

c. There are two sets of (l, k) pairs which lead to a LSMPE for $x \in E_2$ with $k > 0$ and $l < 0$.

d. For $x \in IE$, there exists no LSMPE.

Proof.

a) This follows from lemma (2).

b)The uniqueness of k follows from lemma (3) and while positivity is proved in (5). In lemma (6), it is shown that for $x \in E_1^1 \cup E_1^2$, l is also positive, while for $x \in E_1^3$, l is negative. Both firms achieve nonnegative expected profits by lemma 10. Therefore there is a unique pair (l, k) which leads to a LSMPE.

c) It is shown in lemma 4 there are two possible values of k that satisfy the FOCs and stability requirement when $x \in E_2$. The positivity of k again follows from lemma 5 and negativity of l is shown in lemma 6. Both firms have non-negative expected profits by lemma 10, hence both (l, k) pairs can be sustained as a LSMPE.

d) When $x \in IE$, there are no possible solutions of the FOCs which satisfy stability requirement, and therefore there exists no LSMPE. ■

Proposition 2. Steady State Prices

The steady state prices of both firms are equal and given by

$$p^{ss} = \frac{1}{2s} - a\beta - \frac{\beta k}{2}. \tag{33}$$

Moreover, $0 < p_1^{ss} \leq \frac{1}{2s}$.

Proof. Recall that the steady state market share of each firm is 1/2 and since prices of both firms are given by

$$p_j^t = l + km_j^{t-1} = l + \frac{k}{2} = \frac{1}{2s} - a\beta - \frac{1-\beta}{2}k + \frac{k}{2},$$

which after simplifications leads to

$$p^{ss} = \frac{1}{2s} - a\beta - \frac{\beta k}{2} = \frac{1}{2s}(1 - \beta\frac{\gamma(k)}{2}) - \frac{a\beta}{2}.$$

Since $0 \leq (1 - \beta\frac{\gamma(k)}{2}) < 1$, it is clear that $p^{ss} \leq \frac{1}{2s}$. To show that $p^{ss} > 0$, let us first note that $p^{ss} < 0$, if

$$k < 2a - \frac{1}{s\beta} = k_{pn}.$$

One needs to consider only the case where $k_{pn} > a - \frac{1}{2s} > 0$ which is possible only if

$$as > \frac{1}{2}\frac{2-\beta}{\beta} \geq \frac{1}{2}\frac{3-\beta}{1+\beta}.$$

Therefore, p^{ss} can be negative only when $f(k)$ has two roots in I_1. If one can show that $f(k_{pn}) > 0$ and further $f'(k_{pn}) < 0$, then both roots of $f(k)$ must be larger than k_{pn}. It easy to verify that

$$\begin{aligned} f(k_{pn}) &= \frac{4as\beta^2 - 3\beta + 4(as\beta-1)^2}{s\beta^3} \\ &= \frac{4(as\beta-1)^2 + 4\beta(as\beta-1) + \beta}{s\beta^3} \\ &\geq \frac{4(as\beta-1)^2 + 4\beta(as\beta-1) + \beta^2}{s\beta^3} \\ &= \frac{(2(as\beta-1)+\beta)^2}{s\beta^3} \geq 0. \end{aligned}$$

Calculating $f'(k_{pn})$ results in

$$f'(k_{pn}) = \frac{-4a^2s^2\beta^2 + 16as\beta + 3\beta - 12}{\beta^2}.$$

$f'(k_{pn})$ is quadratic-concave in as, hence it can be shown that $f'(k_{pn}) < 0$ for

$$as < \frac{1}{2}\frac{4-\sqrt{3\beta+4}}{\beta},$$

and since for $0 \leq \beta \leq 1$,

$$\sqrt{\frac{\theta}{\beta}} < \frac{1}{2}\frac{4-\sqrt{3\beta+4}}{\beta},$$

the desired results obtains. ■

Proposition 3. Rate of Convergence

For $\epsilon < m_0 - 1/2$, the expected number of time periods, τ, which is required *for the incumbent's market share to be in an* ϵ*-neighborhood of* $m_1^{ss} = 1/2$*, satisfies*

$$\tau > \frac{\log \frac{\epsilon}{m^0 - \frac{1}{2}}}{\log(\gamma(k))}.$$

Proof. $m_1(t, m_0^1)$ is in an ϵ-neighborhood of the steady state when

$$\mid m_1(t, m_0^1 - 1/2 \mid < \epsilon,$$

or, equivalently, using the expression for $m_1(t, m_0)$ from (29)

$$\gamma(k)^\tau < \frac{\epsilon}{m_0 - \frac{1}{2}}.$$

Taking logarithms of both sides and then dividing both sides by

$$\log \gamma(k) < 0$$

yields

$$\tau > \frac{\log \frac{\epsilon}{m_0 - \frac{1}{2}}}{\log(\gamma(k))}.$$

■