A Generalised Model of Investment under Uncertainty: Aggregation and Estimation.

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Abstract

We propose a structural model of investment which is based on the aggregation of (S,s) investment projects within firms. This encompasses the findings that whilst firm level investment is smooth plant level investment is lumpy and frequently zero. We undertake stochastic aggregation and derive a structural firm level investment estimator.

The empirical performance and fit of this estimator on a panel of manufacturing firms is encouraging and provides an avenue for general policy simulation. This model also explains the rich non-linear dynamics of firm level investment data and the frequent simultaneity of firm level investment and disinvestment. This approach provides an alternative structural estimator to the standard convex adjustment cost models, such as Tobin's Q and the Euler equation. The is important because these estimators, which assume quadratic adjustment costs, appear to be misspecified and subject to a fallacy of composition between smooth firm level investment and lumpy plant level investment.

For completeness we also consider time aggregation as an alternative source of smoothing but statistically reject this as being insufficient to smooth investment alone. This test also rejects most plant level data, such as the USLRD and UKARD, as being generated from a single (S,s) process.

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1. Motivation & Related Literature

Following the seminal work of Eisner and Strotz (1963) empirical models of investment have typically assumed that the cost of adjustment is convex. Convex adjustment costs were introduced principally as a matter of analytical convenience without much underlying technical justification. Nevertheless models such Tobin's Q, Abel and Blanchard's (1986) direct forecasting approach, and the Euler equation, which have formed the mainstay of the investment literature, have typically been premised upon quadratic costs of adjustment[1]. However, these models are inconsistent with a number of stylized facts of investment and appear to match the smooth behaviour of firm level investment only because of aggregation across plants and projects.

Studies of the US Longitudinal Research Database (LRD) carried out by Doms and Dunne (1994) and Cooper et al. (1999) demonstrate that plant level investment is largely a lumpy activity with investment spikes accounting for a large proportion of total investment. Cooper and Haltiwanger (1999) report that 10.4% of observations entail near zero investment (less than 1%). Nilsen and Schiantarelli (1998) report in Norwegian micro data that 33% of observations display zero-investment, whilst Reduto dos Reis (1999) reports that in UK establishment data 2.4% of observations display zero-investment[2]. This compares starkly to Abel and Eberly's (1999) Compustat firm level results which displays no zero-investment episodes in their 12075 observations, and Bloom et al. (1999) who report only 1 zero-investment observation in their 5068 UK Datastream observations.

That plant level data displays a prevalence of investment zeros is not surprising. The predictions of convex adjustment cost models are not robust to the addition of even tiny non-convex adjustment costs. Dixit (1991) and Abel and Eberly (1996, 1997) report that fixed costs or partial irreversibilities leads firms to undertake lumpy Ss style investment behaviour, even for adjustment costs of less than 5% of the price of new capital goods. In reverse, the lumpy investment predictions of Ss models are robust to being supplemented by convex adjustment costs which does not eliminate the investment zeros[3]. This suggest that firm level investment only looks smooth because of aggregation across production plants and projects. This would also explain another stylized fact of investment, that simultaneous investment and disinvestment occurs in approximately half of all UK and US firm level observations.

[1] See Blundell et al. (1992), Chirinko (1993), and Bond and Van Reenen (2000) for reviews of the investment literature, and Abel and Eberly (1994) for an important exception.

[2] The wide difference in the number of zero-investment observations across data sets reflects national differences in the grouping of production units within reporting units, with the Norwegian data set apparently closest to single production plant data.

[3] See Abel and Eberly (1994).

Alternative Ss models of investment, which assume irreversibilities and/or fixed costs, are consistent with lumpy plant level investment but not smooth firm level investment. But by generalising the standard Ss models of investment to allow for aggregation within each firm it is possible to encompass both stylized results. In this paper we build a generalised model of aggregated investment under partial irreversibility. However, for empirical completeness, before we proceed to model aggregation across units we must consider the alternative source of investment smoothing, aggregation across time. Calculations of the impact of time aggregation on the frequency of zero-investment episodes demonstrate this is insufficient for explaining smooth firm level investment. Furthermore, the frequency of predicted investment zeros is so high for standard parameter values, around 50% per year, that even lumpy micro-level data appears to be too smooth to be characterised as a single investment project.

So in the absence of any truly single project data we must allow for an arbitrary degree of aggregation. Under a maintained hypothesis on the separability of marginal project level revenues, we undertake stochastic aggregation, and develop an estimation procedure for firm level panel data. The encouraging empirical fit of this estimator on firm panel data, the consistency with lumpy plant level data, and the consistency with simultaneous investment and disinvestment by firms suggests this provides a more reasonable structural model of investment than the convex adjustment cost approach.

Abel and Eberly (1999) also estimate aggregated firm level investment, but in a Tobin's Q framework[4], which assumes perfect competition and constant returns to scale. Since the firm's capital stock may have no finite size under these conditions without convex adjustment costs our estimator rules this out by assumption. Hence, our two approaches are founded on mutually exclusive maintained hypotheses, although as the conditions in our model approach perfect competition and constant returns to scale our estimator would approach theirs if augmented by convex adjustment costs. Caballero and Engel's (1999) estimator is also similar in spirit to ours. Their innovative approach assumes a discrete time distribution for fixed costs to derive an approximate maximum likelihood for investment. Their double aggregation over adjustment draws and investment projects requires larger investment "project" numbers for the convergence of expected and realised investment however, leading them to estimate on industry level data. Their approach is also based on fixed costs and discrete time and so less closely linked in terms of the notion of real options and the impact of uncertainty on investment.

In Section 2 we consider the investment problem for a single investment project, drawing on previous results in Abel and Eberly (1996). In Section 3 we examine the impact of time aggregation on the investment profile of a single project

[4]see Abel and Eberly (1994)

firm, solving for the evolution of the investment probability to calculate the sample mean and variance of the frequency of zero investment episodes that a single project firm would generate. This is (statistically) significantly different from the observed frequency of zero investment episodes in firm level data, and we reject the single project firm hypothesis. In Section 4 we discuss a model of stochastic aggregation similar to those considered in Bertola and Caballero (1990, 1994), and derive its implications for estimating investment and disinvestment. In Section 5 we discuss our data and in Section 6 we present the estimation results from a panel of 212 UK manufacturing firms. Section 7 concludes and extensive technical appendices follow.

2. The Single Project Firm:

We consider a firm which owns a single production project. For this project the operating revenues are a constant elasticity function of K, the installed capital stock, and Z, an index of business conditions;

$$Y(K;Z) = AK^{a}Z; \qquad 0 < a < 1 \tag{2.1}$$

This functional form nests a Cobb-Douglas production function with an iso-elastic demand curves, in which flexible inputs, such as labor, have been optimised out. This has been used, for example, by Dixit (1991) and Dixit and Pindyck (1994) in their analyses of irreversible investment. The business conditions Z are a proxy for the stochastic elements in the firms environment, such as factor prices, factor productivity and demand conditions.

Let $\{Z(t)\}$ follow the process

$$dZ(t) = Z(t)(\mu_Z dt + \sigma dW(t)) \tag{2.2}$$

where μ_Z and σ^2 represent the brownian mean and variance and $W(t)$ is a Wiener process. The assumption that business conditions evolve as a brownian motion is not a critical assumption for the (S,s) style of investment (only persistence in the stochastic process is required) but is made for analytical tractability. Random walks, which are the discrete time equivalent of Brownian motion, also have some empirical support from the economics literature. Gibrat's law of proportions, which can be states as the log of firm size approximately evolves as a random walk, has been hard to decisively reject[5]. Asset prices, commodity prices and exchange rates are also usually modelled as random walk processes. In the macro literature the hypothesis that productivity evolves as a random walk has had mixed empirical success, suggesting this process is at least highly autocorrelated[6].

[5] See Mansfield (1988), Sutton (1996) and Geroski et al. (1997).

[6] See for example King et al., (1991).

We assume that the firm is risk neutral and maximises the expected present value of its cash flow over an infinite horizon. Its cash flow at any point in time equals the operating profit $AK(t)^{\alpha}Z(t)$, minus the cost of purchasing capital at a constant price B, plus the proceeds received from selling capital at a constant price S. The optimal investment for each project is the solution to the dynamic programme

$$V(K(t);Z(t)) = \max_{\{I(s)\}} E_t \left\{ \int_t^{\infty} \exp^{-r(s-t)}(AZ(s)K(s)^{\alpha}ds - B\,dI^{+}(s) + S\,dI^{-}(s)) \right\} \tag{2.3}$$

$$\text{subject to} \quad dK(t) = -\delta K\,dt + I(t)$$

Abel and Eberly (1996) prove that there exists a unique (S,s) solution to this problem which can be fully characterized in terms of the marginal revenue product of capital $\alpha AK^{\alpha-1}Z$, an upper investment trigger (big S) and a lower disinvestment trigger (little s). These investment triggers represent the standard Jorgensonian user cost of capital terms supplemented by a positive real option term ϕ_U at the investment trigger and a negative real option term ϕ_L at the disinvestment trigger. Investment only takes place when the marginal revenue product of capital hits the upper trigger and disinvestment only when it hits the lower trigger. This investment policy is summarized in the table 1 below:

Table 1: The marginal revenue product investment triggers

Invest if:	$\alpha AK^{\alpha-1}Z = r + \alpha\delta - \mu_Z + \phi_U$
Innaction when	$r + \alpha\delta - \mu_Z + \phi_L < \alpha AK^{\alpha-1}Z < r + \alpha\delta - \mu_Z + \phi_U$
Dis–Investment if:	$\alpha AK^{\alpha-1}Z = r + \alpha\delta - \mu_Z + \phi_L$

Since these option terms are independent of the size of the project K, the difference between the logged upper and lower investment triggers $\log(Z_U) - \log(Z_L)$ is also independent of the size of the project. This is a critical property for aggregation. It allows a group of investment projects to be characterized by the distribution of their individual marginal productivities on a common support - the interval between the lower and upper investment triggers. To simplify notation for aggregation we label the marginal revenue product of capital $Y = \alpha AK^{\alpha-1}Z$. Using lower cases for logs we find that the logged marginal revenue product, $y = \log(\alpha AK^{\alpha-1}Z)$, has is a brownian motion process with drift $\mu_Y = \mu_Z + (1-\alpha)\delta - \frac{\sigma^2}{2}$ and variance σ^2. For notational simplicity we re-normalize the upper and lower investment triggers to be $\bar{y}$ and 0.

3. Time Aggregation

The time series of the investment behaviour that solves this control problem is characterised by bursts of investment at the upper trigger, disinvestment at the lower trigger and periods of inaction in between. Firm level investment appears too smooth to be consistent with this, lacking both the characteristic investment bursts and the intervening periods of zero-investment. One potential explanation is time aggregation. Because we only observe investment cumulated across discrete periods of time - generally across the accounting year - this may obscure this investment impulse behaviour. We derive a statistical test for time aggregation below.

Let $f(y;t,T)$ denote that probability that a project at time t and position y will undertake an investment impulse by time $T \geq t$. This probability distribution evolves according to the Kolmogorov backward equation[7]

$$\frac{\sigma^2}{2} f_{yy}(y;t,T) + \mu_Y f_y(y;t,T) + f_t(y;t,T) = 0 \tag{3.1}$$

with boundary conditions

$$\begin{aligned} f(0;t,T) &= 1 \quad \forall \; 0 \leq t \leq T & (3.2)\\ f(\bar{y};t,T) &= 1 \quad \forall \; 0 \leq t \leq T & (3.3)\\ f(y;T;T) &= 0 \quad \forall \; 0 < y < \bar{y} & (3.4)\\ \lim_{T \to \infty} f(y;t,T) &= 1 \quad \forall \; 0 < y < \bar{y}; \quad \forall \; t & (3.5) \end{aligned}$$

The first two boundary conditions in (3.2) and (3.3) state that investment takes place almost surely at the lower ($y = 0$) and upper ($y = \bar{y}$) investment triggers. The third boundary condition (3.4) states that with no time remaining the probability of hitting the investment trigger for an interior point is zero. The fourth boundary condition (3.5) states that as the time frame extends to infinity the probability of an investment impulse for any starting position approaches one. For the investment problem under complete irreversibility this probability function $f(y;t,T)$ has a solution in terms of the standard normal distribution[8]. For the more general partial irreversibility problem the solution has a power series form which is derived in Appendix B.

Figure 1 plots the probability of a firm starting at y hitting either investment trigger within one year, two years and five years for the parameter values

[7] see Cox and Miller (1996)

[8] see pages Harrison (1990)

$\mu_Y = 0.02; \sigma^2 = 0.5; a = 0.8; r = 0.08$, and buy/sell = 2. The investment probability increases as the starting position moves towards either barrier reflecting the underlying uncertainty in the Wiener process. The underlying drift towards the upper barrier (μ_Y) tends to increase the investment probability for higher starting values. Figure 2 plots the investment probability for a firm with the same parameters but a buy/sell ratio of 1.1. The increased reversibility of investment reduces the distance between the lower and upper triggers and increases the probability of an investment episode.

[FIGURES 1 AND 2 - SEE BACK OF PAPER]

To generate a statistical test on the observed frequency of investment we integrate the expected frequency of investment conditional on the initial value of y with respect to the ergodic density. The ergodic density is long run probability density for the marginal revenue product of capital, y, and would reflect the distribution one in a large sample of projects. For the thresholds $(0; \bar{y})$ the ergodic distribution has the form[9]

$$e(y) = \exp\left(\frac{2\mu_Y}{\sigma^2} y\right) \left(\frac{\frac{2\mu_Y}{\sigma^2}}{\exp\left(\frac{2\mu_Y}{\sigma^2}\bar{y}\right) - 1} \right) \tag{3.6}$$

so that the expected frequency of investment episodes at t until time T is equal to;

$$h(t,T) = \int_0^{\bar{y}} e(y) f_t(y; t, t+T) dy \tag{3.7}$$

Letting $T - t = 1$ (one year) enables us to calculate the mean probability of observing a zero investment episode[10]. For a large numbers of observations the binomial distribution approaches the normal distribution $N[p; p(1-p)/n]$ where n is the number of trials and p the probability of success. We can use to this to calculate the approximate likelihood of observing any share of zero investment episodes in a panel of projects. These results also apply to (S,s) models generated by fixed adjustment costs since the only relevant variable is the band width and the $(\mu; \sigma^2)$ of the brownian process.

[9] See Harrison (1990)

[10] Since investment and dis-investment are continuous valued functions the incidence of exactly offsetting positive and negative investment impulses within any period has measure zero. Hence, the probability of zero net investment is the same as the probability of zero gross investment. This is not true for models with fixed costs alone, which have a discrete investment function. So whilst our results would be correct for gross investment they would provide a lower bound on the predicted frequency for net investment. This is due to the small probability of exactly offsetting positive and negative investments within a period.

Table 2 displays the observed incidence of zero investment episodes in our firm level and plant level data sets. Table 3 displays the predicted incidence of zero investment episodes (and standard deviations) for four sets of standard parameter values based around those estimated in Section 5 below. Table A2 in the Appendix presents a reference table of investment probabilities for a much wider range of parameter values.

Table 2: ACTUAL Frequency of Zero Investment Episodes

	Obs.	Buildings & Land	Plant, Machinery	Vehicles	Total
Firm Level	2,434	5.9	0.1		0.0
Single Plant	20,907	53.0	4.3	23.6	2.4
Sources: UK Datastream and UK ARD (see Reduto dos Reis, 1999)					

Table 3: PREDICTED Frequency of Zero Investment Episodes

Parameter Values	(1)	(2)	(3)	(4)
Uncertainty (σ)	0.5	0.15	1.5	0.5
Buy/Sell ratio	2	2	2	1.05
Predicted Frequency (st.dev.)				
Firm Level (2,434 obs.)	58.2 (0.01)	30.3 (0.00)	37.2 (0.01)	34.9 (0.00)
Single Plant (20,907 obs.)	58.2 (0.00)	30.3 (0.00)	37.2 (0.00)	34.9 (0.00)
Other Parameter Values: $\mu_Y = 0.036$; $r = 0.08$; $a = 0.8$				
The standard deviations, $\sqrt{p(1-p)/N}$, depend on the number N of observations.				

In column (1) of Table 3 we see that the predicted frequency of zero-investment episodes is higher and significantly different at 58.2% from the observed frequency of zero investment episodes in Table 2 for firm data and plant level data. Columns (2) and (3) demonstrate that this conclusion is robust to ten fold changes to the level of uncertainty, whilst column (4) demonstrates this is even true for an irreversibility cost of only 5% of the price of new capital goods. In table A2 of the Appendices it is clear that the predicted level of investment zeros is significantly too high for firm and plant level data to be interpreted as a single investment project for a wide range of parameters.

One explanation for the lack of zero investment episodes in actual plant level data may be that investment actually takes two general forms: continuous maintenance investment which is not subject to adjustment costs, and project level investment as modelled above. However, it appears that only by relegating an unacceptable high level of investment into the unexplained residual category of maintenance investment does the frequency of zero episodes in US LRD and UK ARD data once again the become consistent with the single project model. For example, Cooper et al.'s. (1999) study of the LRD reports that even the frequency of low investment episodes (less than 4%) is 24%, which is still too low for most reasonable parameter values.

This suggests that even plant level investment processes are aggregated across investment projects within the plant so that estimation of Ss models on plant level data will encounter significant aggregation problems. A fortiori this suggests that firm level investment is aggregated across both plants and projects, obscuring the link between the theory and data. This has been confirmed by Hammermesh (1989) and Caballero et al. (1985) who find clear evidence of labour demand and investment smoothing across plants within firms.

4. Aggregation of Investment Projects up to the Firm Level

In the absence of any truly project level data a robust estimation strategy requires accounting for aggregation. We generalise the firm level production function to allow the firm to operate N of separate production projects. We could think of these representing individual projects, lines and vintages of capital, separate production plants providing intermediate inputs or regional production sites. These separate projects may even involve entirely distinct operations owned by a conglomerated parent. For brevity we shall continue to refer to these "units" as projects but with this more general interpretation in mind.

The firms total operating revenue is assumed to be linearly separable in the marginal operating revenue of each production project, and can be represented by the form.

$$Y(K_1;K_2;...;Z_1;Z_2;...;Z_f) = H(\chi_1;\chi_2;...) + Z_f \sum_{i=1}^{N} Z_{p;i} K_i^a \qquad 0 < a < 1 \quad (4.1)$$

where χ_i is the indicator function which takes the value 1 if project i exists and zero otherwise, $H(...)$ is some finite function, Z_f is a firm level shock to business conditions and Z_p are idiosyncratic project level shocks to business conditions. The project returns are subject to common shocks to the firm's conditions and idiosyncratic shocks to the project's conditions. Let $\{Z_f; Z_{p;i}\}$ follow the process

$$\begin{aligned} dZ_f &= Z_f(\mu_f dt + \sigma_f dW_f) & (4.2) \\ dZ_{p;i} &= Z_{p;i}(\mu_p dt + \sigma_p dW_p) & (4.3) \end{aligned}$$

where $\{W_f; \{W_{p;i}\}_{i=1}^{N}\}$ are all independent Wiener processes[11].

[11] The project and firm level shocks need not be independent but this assumption significantly simplifies the mathematics with little loss of generality. A necessary condition for our aggregation procedure is that plant and firm level shocks are not perfectly dependent.

Several points are worth noting about this production function. Firstly, by assuming the linear separability of the marginal rather than absolute revenue product of projects we can provide some rationale for the organisation of production into firms rather than atomistic projects. The action of $H(.)$ on the indicator functions allows firms to derive value from combining different types of production projects. For example, a mining firm could derive a positive revenue flow from combining the ownership of ore extraction and refining plants if $H(0,0,..\chi_N,\chi_{N+1},0,0..) > 0$ where χ_N and χ_{N+1} are the indicator functions for the ownership of ore extraction and refining plants.

Secondly, by being deliberately vague about the definition of a production project we can allow these to represent groups of investment units which are observed to be non-separable.. Hence, this approach is robust to the non-separability of some groups of production processes, with the group level Z_p shock representing the Hicksian index of the processes.

Finally, since the number of projects N is arbitrary (and estimatable) this nests our earlier model based on Abel and Eberly (1996) in which the firm operated one project. A one project specification also nests the model of Cobb-Douglas production and Iso-elastic demand in the standard Euler investment model[12].

So this linearly separable multi-project structure is a generalisation of the more standard single project firm, which as section (3) reports, is not supported by the data. Whilst is would be desirable to generalize still further to allow for arbitrary interactions between projects within each firm this is not analytically tractable. The difficulty is that the distribution of projects becomes a state variable, dramatically increasing the dimensionality of the optimisation problem[13]. This requires us to make the structural trade-off between the generality of the underlying model and the versatility of the estimation procedure, the success of which can be gauged by the performance of our estimator.

5. Estimation Techniques[14]

Since the marginal revenue function is linearly separable at the project level, the first order conditions can be considered separately across projects, and the opti-

[12] see for example, Bond and Meghir (1994)

[13] Some progress has been made in solving models of aggregation with interactions across agents by using approximations of the distribution (see Krussell and Smith, 1998) or the ignoring elements of the "echoes" of previous shocks (see Caplin and Leahy, 1998). The extension of these methods to our application is left for future reseach.

[14] We are happy to provide the full Gauss estimation codes and data (subject to Datastream copyright) on request at (e-mail) nick.bloom@ifs.org.uk. To facilitate accessability we are currently testing, editing and simplifying these codes and writing an accompanying manual for on-line availability.

mal investment rule is an aggregation of section (2). This is facilitated by two important results. Firstly, each project has the same lower and upper investment triggers and so the distribution of their marginal revenue product has a common support. Secondly, since the positioning of the marginal revenue product of projects between their lower and upper investment triggers is a bounded I(0) process and their capital stock is an unbounded I(1) process these must have zero limiting correlation. This permits a mapping from the cross sectional distribution of projects between the investment triggers to firm level investment and disinvestment.

This distribution of projects within the firm is shaped by two forces. Firstly firm level shocks act to shifts the entire distribution of projects between the investment triggers. Secondly, the project level shocks act as mean reverting force which smooths the effects of the firm level shocks. The probability distribution of projects, $p(y;t)$, satisfies the Kolmogorov forward equation[15]

$$\frac{1}{2}\sigma_p^2 \partial_{yy} p(y;t) - \mu_Y \partial_y p(y;t) - \partial_t p(y;t) = 0 \tag{5.1}$$

Since continuous information is not available as to the evolution of firm level shocks these are modelled as period specific rates of drift μ_Y, based of the growth of logged sales as a proxy for Z (see section (6)). This is only an approximation because investment is path dependent, but this can be tested using the Levy procedure discussed in Appendix C.

The expected rate of investment at the firm level is just the integral over the density of projects at either investment boundary during the period. So denoting the density of projects at the lower and upper triggers by $p(0;t)$ and $p(\bar{y};t)$, the expected investment and disinvestment over a period of length T is equal to

$$E[\frac{I^+}{K}] = \frac{1}{1-a}\int_0^T \frac{\sigma^2}{2} p(\bar{y};t)dt \tag{5.2}$$

$$E[\frac{I^-}{K}] = -\frac{1}{1-a}\int_0^T \frac{\sigma^2}{2} p(0;t)dt \tag{5.3}$$

This is only an identity for expected firm level investment. Previous estimations using stochastic aggregation, such and Caballero and Engel (1991), Caballero (1993), and Bertola and Caballero (1990,1994), modelled aggregate investment or consumption, and so assumed that the number of "projects" within the economy is infinite. This allows the application of a "strong law of large numbers" style of argument to demonstrate that the distribution of projects matches its probability distribution[16] .to demonstrate that the expectations of the process mean converges to the realised investment rate. Whilst this convergence still holds for our

[15] see Cox and Miller (1996)

[16] This involves an application of the Glivenko-Cantelli (see Billingsley, 1979).

estimation procedure, since we assume that firms operates only a finite number of projects, the estimation of actual investment will involve a forecast error, similar to the error term usually appended to standard linear. Finally we need to an initial distribution for the marginal revenue products of the investment projects, which we assume is the long run ergodic distribution. Since this also introduces a source of error, but which decays over time, we discard the first three observations for each firm when fitting our parameters.

Estimation is undertaken by minimising the sum of squared deviations between actual and forecasted net investment over a grid of parameter values. This estimation iteratively updates the estimated distribution of projects within the firm given the shock to sales (as a proxy for business conditions). The distribution of projects within each firm confers an estimate of investment, using (5.2) - (5.3), and a sum of squared residuals on a firm by firm basis. The sum of these firm level estimation errors provides the total squared error and associated R-Squared. Because of this firm by firm estimation approach, boostrapping standard errors by randomising over the weightings given to each firm in the summation, is computationally quick.

Figures 3 to 6 display the sequential evolution of this project level distribution for BASS Plc's first four years, the first firm in our dataset and a large UK Brewer, which we use as an example to illustrate this technique. In Figure 3 we plot the expected distribution of Bass's investment projects, in terms of their marginal revenue products of capital, between their lower and upper investment triggers. In this example these projects could represent some number of Bass's regional brewing facilities. If we knew the exact marginal revenue product of capital of every one of these units we would have a discrete distribution - a histogram - but since this estimator only hold in expectations this plot looks smooth.

[FIGURES 3, 4, 5 AND 6 - SEE BACK OF PAPER]

Every time Bass has a good firm level sales shock the distribution of these investment projects moves towards the upper barrier. In figure 3 we plot the first year distribution of projects and in figure 4 the first two years' distributions, with the second year distribution (the additional one to figure 3) representing the effect of the negative sales shock which hit Bass in year 2. As a result, there is a lower density of projects at the investment trigger (the right trigger) and a higher density of projects are now at the lower trigger (the left trigger). Figure 5 displays (in addition) the distribution of projects after another bad sales shock in year 3 and figure 6 the distribution of projects after Bass received a good sales shock in year 4.

As equations (5.3) and (5.2), investment and disinvestment are just the time integral of projects at the upper and lower triggers over the year. We present the full 19 year series for sales shocks, actual investment and predicted investment for Bass in figure 7. It is clear that both the predicted and actual investment series for Bass are much smoother than the sales series, the latter being a result of the aggregation. The poor realisations of sales (solid line) which led to the downward shift in the project distributions in figures 4 and 5, can be seen in years 2 and 3 for Bass. This poor sales realisations leads to a reduction in predicted investment (dashed & doted line) which matched the fall in actual investment (dashed line). In year 4 the good shock to sales is also visible in figure 7 and leads to a prediction of a pick-up in investment, although this is at odds with actual investment which continued to fall. Figures 8, 9 and 10 represent the next three firms in our data set (APV, Croda International, and Senior Engineering Group) with their sales, investment and predicted investment figures, as a further illustration of this estimator.

[FIGURES 7, 8, 9 AND 10 - SEE BACK OF PAPER]

Our first set of empirical results fits estimated net investment to observed net investment. Since our estimator yields separate predictions on gross investment, (5.2), and disinvestment, (5.3), we jointly fit these processes by combining the squared residuals from both series using an equal weight. More generally, company accounts also contains good information on the hiring and firing of labor. Our Cobb-Douglas production function implicitly involves a predicted labor input, which although maximised out of the concentrated profit function in section (2), is easily recoverable from the (implicit) first order conditions. This allowed us to jointly estimate the firms investment and labor demand strategy in a single coherent model and estimation procedure. This confers a significant advantage over the standard models based on Tobin's Q or Abel and Blanchard (1986) which lack the structural foundations to estimate using the joint information in these processes.

6. Data

The company data is drawn from the datastream on-line service and consists of all manufacturing companies quoted on the UK stock market in 1973, with financial holding companies dropped. We deleted firms with less than ten consecutive observations, broke the series for firms whose accounting period fell outside 300 to 400 days due to changes in year end timing, and excluded the observations for firms where there are 100% plus jumps in any of the explanatory variables. Our capital stock measure is derived from the book value of the firm's stock of net fixed assets, using the investment data in a standard perpetual inventory formula.

Sales figures have been adjusted for changes in inventories and cash flow represents pre-tax earnings with depreciation allowances added back in (see Appendix D). Table 5 reports some summary statistics on the firms in our sample.

Table 5: Unbalanced Panel of 212 Manufacturing firms, 3070 observations

	mean	median	stand. dev.			min.	max.
			total	within	betwn		
investment (I_t/K_{t-1})	0.113	0.095	0.10	0.05	0.09	-0.10	1.14
Δ logged real sales	0.026	0.025	0.14	0.05	0.13	-0.61	0.65
σ share returns	1.56	1.63	0.71	0.55	0.49	0.01	11.03
cash flow (C_t/K_{t-1})	0.18	0.140	0.16	0.09	0.17	-0.09	1.96
observatns per firm	15	14	2.8	0	2.8	11	20

The underlying driving variable for investment - the change in logged business conditions, $\Delta\log(Z)$, - is unobservable and must be proxied by logged real sales, $\Delta\log(S)$, where

$$\Delta\log(S) = \Delta\log(Z) + a\Delta\log(K) \qquad (6.1)$$

It is clear from the identity (6.1) that we could use subtract $a\Delta\log(\hat{K})$ from $\Delta\log(S)$ to obtain an estimate of $\widehat{\Delta\log}(Z)$. However, because the construction of the capital stock through the perpetual inventory method can be subject to measurement errors, we avoid using this estimated $\widehat{\Delta\log}(Z)$ to prevent introducing the same source of error into our dependent and explanatory variables. Furthermore, as we show in Appendix D, changes in log sales provides a proxy for $\frac{1}{1-a}\widehat{\Delta\log}(Z)$, with the degree of accuracy increasing with the time interval between observations. Hence, we estimate investment using $(1-a)\Delta\log(S)$ to proxy $\Delta\log(Z)$.

7. Results [work in progress]

We estimate over a three dimensional grid of parameters, with flexible values for the standard deviation of project level shocks [σ_P] the degree of reversibility in the purchase and resale value of capital [buy/sell] and the returns to scale parameter [a]. The remaining parameters were determined as follows: the real discount rate r was assumed to be 8%; the drift rate of demand μ was the 2.1% average from our firm data, and the standard deviation of the firm level shock σ_F was the 13.5% average from our data. The depreciation rate δ was a firm specific parameter set to meet the long run capital accumulation constraint $\overline{\frac{I}{K}} = \delta + \overline{d\log(S)}$ where S is firm sales. This long run relationship on the cointegration of capital and sales holds empirically and is implied by our model. The flexibility in the depreciations rates across firms reflects the diversity of asset mixes with this parameter ranging from 1.6% to 19%[17].

Our estimated values and bootstrapped standard errors from minimising the residual sum of squares over net investment are presented in Table 6.

Table 6: Estimation Results - Net Investment: $R^2 = 0.276$

	$\widehat{buy/sell}$	$\hat{\sigma}_P$	$\hat{a}$
Parameter estimate (standard error)	2 (0.32)	0.6 (0.13)	0.8 (0.12)
Parameter grid $buy/sell = \{1.1; 1.3; 1.5; 2; 2.5; 3\}$;			
$\sigma_P = \{0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8\}$; $a = \{0.4; 0.5; 0.6; 0.7; 0.8; 0.9; 0.95\}$			

The $\widehat{buy/sell}$ parameter estimate suggest that the degree of irreversibility is significant with a 50% resale loss incurred. This accords with the low level of capital disinvestment observed in our data, where the mean disinvestment/investment ratio is 0.181. Without an important reversibility constraint this ratio would be closer to one given the high empirical standard deviation (0.135) and low drift (0.026) of firm level demand. This ratio of observed total disinvestment/investment also approximately matches the predicted ratio of 0.26 from our estimator.

With $\hat{\sigma}_P = 0.6$, the marginal revenue product of capital at the individual project level should vary by as much as 60% in about two thirds of cases, on a year-to-year basis. This seems implausibly high and we have two comments in relation to this. Firstly, we have ignored the issue of cross-sectional heterogeneity in the project level parameters, which as Caballero and Engel (1991) demonstrate, also generates the type of smoothing properties we attribute to these idiosyncratic project level shocks. Hence, it is likely that the omission of cross sectional heterogeneity biases this project level shock parameter upwards[18]. Secondly, to maintain tractability of approach we ignore the possibility of firm level convex adjustment costs. This will also bias $\hat{\sigma}_P$ upwards in attempting to generate the additional firm level investment smoothness.

The estimated value of $\hat{a}$, the degree of homogeneity on capital, is 0.8. This is not significantly different from unity which is the value for a under perfect competition and constant returns to scale (CRS). However, since the firm's capital stock may have no finite size under these conditions our estimator rules this out by assumption. The Abel and Eberly (1994) model of non-linear Q, on the contrary, requires perfect competition and CRS for empirical application, and so

[17] There is no necessity for depreciation rates to be positive if the capital stock includes a large share of property.

[18] We could include cross sectional heterogeneity in this estimator but at considerable computational expense as we would need to estimate the process for each (plants, type) combination.

is founded on a mutually exclusive maintained hypothesis. This is an important issue although the economic evidence for perfect competition is not strong [ref: Hall(1987?)]

The R-Squared for this estimation was 0.276 which appears to be a satisfactory fit for a structural estimator but poor for a general reduced form estimator. As a rough guide we present the R-Squared for some OLS and within groups estimators on the same data set below. It is clear that our estimator out performs the simple OLS estimators in rows (1) (2) and (3) in terms of goodness of fit. The within-groups estimators in rows (5), (6) and (7), which allow for 210 extra parameters $\{\alpha_2 ... \alpha_{211}\}$ in the form of firm specific intercepts, have a superior goodness of fit to our structural estimator[19]. Columns (7) and (8) present results from the OLS and within groups Euler equation, an alternative structural estimator[20]. The explanatory power of the OLS Euler equation, which has 4 free parameters $\{\alpha_1; \beta_1; \beta_2; \beta_3\}$, is much lower than our estimator, whilst the within groups Euler equation, which has 214 free parameters $\{\alpha_1 ... \alpha_{211}; \beta_1; \beta_2; \beta_3\}$ is approximately equal.

	OLS Estimators- Reduced Form	R-Squared
(1)	$\frac{I}{K}_{i,t} = \alpha_1 + \beta_1 \Delta \log y_{i,t}$	0.156
(2)	$\frac{I}{K}_{i,t} = \alpha_1 + \beta_1 \Delta \log y_{i,t} + \beta_2 \Delta \log Y_{i,t-1} + \beta_3 \Delta \log Y_{i,t-2}$	0.201
(3)	$\frac{I}{K}_{i,t} = \alpha_1 + \beta_1 \Delta \log Y_{i,t} + \beta_2 \Delta \log Y_{i,t-1} + \beta_3 \Delta \log Y_{i,t-2} + \beta_4 \frac{I}{K}_{i,t-1} + \beta_5 \frac{I}{K}_{i,t-2}$	0.259
	Within Groups Estimators-Reduced Form	
(4)	$\frac{I}{K}_{i,t} = \alpha_i + \beta_1 \Delta \log Y_{i,t} + \beta_2 \Delta \log Y_{i,t-1} + \beta_3 \Delta \log Y_{i,t-2}$	0.333
(5)	$\frac{I}{K}_{i,t} = \alpha_i + \beta_1 \Delta \log Y_{i,t} + \beta_2 \Delta \log Y_{i,t-1} + \beta_3 \Delta \log Y_{i,t-2} + \beta_4 \frac{I}{K}_{i,t-1} + \beta_5 \frac{I}{K}_{i,t-2}$	0.353
(6)	$\frac{I}{K}_{i,t} = \alpha_i + \beta_1 \Delta \log Y_{i,t} + \beta_2 \Delta \log Y_{i,t-1} + \beta_3 \Delta \log Y_{i,t-2} + \beta_4 (\Delta \log Y_{i,t})^2$	0.418
	$\beta_5 (\Delta \log Y_{i,t})^{1/2} + \beta_6 \frac{I}{K}_{i,t-1} + \beta_7 \frac{I}{K}_{i,t-2} + \beta_8 (\frac{I}{K}_{i,t-1})^2 + \beta_9 (\frac{I}{K}_{i,t-1})^{1/2}$	
	OLS and Within Groups - Euler Specification	
(7)	$\frac{I}{K}_{i,t} = \alpha_1 + \beta_2 \Delta \log Y_{i,t-1} + \beta_4 \frac{I}{K}_{i,t-1} + \beta_5 \left(\frac{I}{K}_{i,t-1}\right)^2$	0.138
(8)	$\frac{I}{K}_{i,t} = \alpha_i + \beta_2 \Delta \log Y_{i,t-1} + \beta_4 \frac{I}{K}_{i,t-1} + \beta_5 \left(\frac{I}{K}_{i,t-1}\right)^2$	0.237

These comparisons are very preliminary. They are estimated with OLS and within groups rather than consistent GMM, exclusive of financial variables, and based on the Euler equation alone, but nevertheless encouraging. As a structural estimator the sample fit of our estimator is satisfactory. Furthermore, a more

[19] This confirms a common result in empirical investment that there is a trade-off between structural estimators suitable for general policy simmulation and reduced form estimators with superior forecasting and fit (for example see Oliner et al., 1995).

[20] see for example Bond and Meghir (1994)

general metric of fit which included the ability to explain other facets of investment data, such as the lumpy nature of plant level investment, the smooth nature of firm level investment and the frequency of simultaneous firm level investment and disinvestment observations, would favour our approach.

Extensions: These results are preliminary and current extensions involve

1. Extending the estimators to use additional disinvestment and labour demand data
2. Using our variance of stock market returns to proxy for firm level uncertainty to test the underlying modelling assumptions (see Bloom et al., 1999).
3. Extending our data set and grid search
4. Making the Gauss estimation codes available on-line with some accompanying explanatory "manual"

8. Conclusions

The standard approach to estimating investment, followed by models such as Tobin's Q, Abel and Blanchard, and the Euler equation, is based on quadratic adjustment costs. Whilst this approach is satisfactory for explaining smooth firm level investment it is inconsistent with two important stylized facts: that plant level investment is lumpy and frequently zero and that firms are typically observed investing and dis-investing simultaneously. In fact, it should not be surprising that plant level investment does not conform to the smooth predictions of quadratic adjustment costs. Quadratic adjustment cost models are not robust to the inclusion of even tiny irreversibilities and fixed costs, displaying (S,s) style lumpy investment for irreversibilities and fixed costs as little as 5% of the price of new capital. In fact the surprising result is that firm level data is so smooth.

We investigate whether this smoothness of firm investment is due to time aggregation alone, but by calculating the statistical distribution for yearly zero-investment observations, we are able to statistically reject this hypothesis. More importantly, we find that the predicted frequency of zero investment episodes is also too high for plant level investment to be generated by a single (S,s) process. This result and the disappearance of zeros when moving from small plants to large plants and from plants to firms leads us to conclude that: aggregation across investment projects within each firm is responsible for generating smooth investment and simultaneous investment and disinvestment.

Given these findings we propose a structural model of investment based on the aggregation of (S,s) investment projects within firms. Under a maintained hypothesis on the separability of marginal revenue product of capital across projects, we undertake stochastic aggregation, develop an estimation procedure for firm level panel data, and estimate a structural investment equation on our panel of firm data. Our preliminary results are promising: the empirical performance and fit of this on our panel is good for a structural model, and appears to provide an avenue for general policy simulation. The estimated parameter values suggest that: irreversibilities are important with capital resale incurring a 50% loss; that project level idiosyncrasies are large; and that the homogeneity of returns to capital is 0.8 suggesting imperfect competition and/or decreasing returns to scale.

This approach also yields predictions on the comovement of labour and capital in a single model and estimation procedure. This confers a significant advantage over the standard models of investment based on Tobin's Q or Abel and Blanchard (1986) which lack the structural foundations to undertake estimation using the joint information in the labour and capital processes. This should help to address the question over the relative adjustment of labour and capital and the appropriate modelling strategy between alternative assumptions of costly capital adjustment or costly labour adjustment. More generally this estimation procedure is also appropriate for the other aggregated (S,s) processes which have been considered in the literature, such as household demand for consumer durables.

9. Appendix A

10. Appendix B: The solution to the time aggregation problem.

Let $f(x;t,T)$ be the probability of a project in (position, time) space $(x;t)$ hitting either investment barrier by time T. To simplify the use of the boundary conditions further on we solve for $g(x;t,T)$, which is the probability of not investing, so that $g(x;t,T) = 1 - f(x;t,T)$. Then taking binomial approximations we can derive the Kolmogorov backward equation for $g(x;t,T)$

$$\frac{\sigma^2}{2} g_{xx}(x;t,T) + \mu g_x(x;t,T) + g_t(x;t,T) = 0 \tag{10.1}$$

with boundary conditions

$$\begin{aligned}
g(0;t,T) &= 0 \quad \forall\ 0 \le t \le T && (10.2)\\
g(\bar{x};t,T) &= 0 \quad \forall\ 0 \le t \le T && (10.3)\\
g(x;T;T) &= 1 \quad \forall\ 0 < x < \bar{x} && (10.4)\\
\lim_{T \to \infty} g(x;t,T) &= 0 \quad \forall\ 0 < x < \bar{x}; \quad \forall\ t && (10.5)
\end{aligned}$$

The first two boundary conditions in 10.2 and 10.3 state that investment takes place almost surely at the lower ($x = 0$) and upper ($x = \bar{x}$) boundaries. The third boundary condition 10.4 states that at the end of the time period the probability of hitting the investment barrier for an interior point is zero. The fourth boundary condition 10.5 states that as the time frame extends to infinity the probability of an investment impulse for any starting position approaches one in finite time.

Proceeding to solve 10.1 by separation of variables $g(x;t,T) = u(t)v(x)$, where we temporarily suppress the dependence on (the parameter) T until later, we derive two sets of ordinary differential equations and an unknown connecting term k

$$u_t(t) - ku(t) = 0 \tag{10.6}$$

$$v_{xx}(x) + \frac{2\mu}{\sigma^2} v_x(x) + \frac{2k}{\sigma^2} v(x) = 0 \tag{10.7}$$

The first ordinary differential equation 10.6 has an exponential solution

$$u(t) = A \exp(kt) \tag{10.8}$$

The second ordinary differential equation 10.7 and boundary conditions 10.2 and 10.3 define a Sturm-Liouville problem with integrating factor $e^{\frac{2\mu}{\sigma^2}x}$ (see e.g.

Churchill and Brown, 1993) with characteristic equation

$$\beta^2 + \frac{2\mu}{\sigma^2}\beta + \frac{2k}{\sigma^2} = 0 \tag{10.9}$$

If $k \le \frac{2\mu^2}{\sigma^2}$ then the roots of are real and the solutions take the general form

$$\begin{aligned} v(x) &= A_1 e^{\beta_1 x} + A_2 e^{\beta_2 x} \qquad k < \frac{\mu}{2\sigma^2}, \text{ distinct roots } (\beta_1 \neq \beta_2) && (10.10) \\ v(x) &= A_1 e^{\frac{\mu}{\sigma^2}x} + xA_2 e^{\frac{\mu}{\sigma^2}x} \qquad k = \frac{2\mu^2}{\sigma^2}, \text{ the repeated root case} && (10.11) \end{aligned}$$

These solutions cannot meet the boundary conditions so we consider $k > \frac{2\mu^2}{\sigma^2}$, which delivers complex roots;

$$\begin{aligned} \beta &= -\frac{\mu}{\sigma^2} \pm i\left(\frac{2k}{\sigma^2} - \frac{\mu^2}{\sigma^4}\right)^{\frac{1}{2}} && (10.12) \\ &= -\frac{\mu}{\sigma^2} \pm i\lambda(K) && (10.13) \end{aligned}$$

where the solutions have the form

$$v(x) = e^{-\frac{\mu}{\sigma^2}s}\left(A_1 \cos(\lambda(k)x) + A_2 \sin(\lambda(k)x)\right) \tag{10.14}$$

Imposing 10.2 we obtain $A_1 = 0$, and imposing 10.3 we obtain

$$k_n = \frac{\mu}{2\sigma^2} + \frac{n^2\pi^2\sigma^2}{2\bar{X}^2} \qquad n = 1, 2, 3 \ldots \tag{10.15}$$

from which we can derive a general solution to 10.1 in terms of its Fourier series;

$$\begin{aligned} g(x;t) &= \sum_{n=1}^{\infty} A_n \exp(k_n t)\exp\left(-\frac{\mu}{\sigma^2}x\right)\sin\left(\frac{n\pi}{\bar{X}}x\right) && (10.16) \\ g(x;t;T) &= \sum_{n=1}^{\infty} B_n \exp(k_n(t-T))\exp\left(-\frac{\mu}{\sigma^2}x\right)\sin\left(\frac{n\pi}{\bar{X}}x\right) && (10.17) \end{aligned}$$

where 10.17 comes from redefining the coefficients $B_n = A_n \exp(k_n T)$. The terminal condition

$$g(x;T;T) = 1 \tag{10.18}$$

determines the constants $\{A_n\}_{n=0}^{\infty}$. Setting $t = T$ in 10.17 using 10.18, multiplying both sides by $e^{\frac{\mu}{\sigma^2}x}\sin(\frac{n\pi}{\bar{X}}x)$, integrating between 0 and $\bar{X}$, and exploiting the orthogonality of the Fourier terms we obtain

$$B_n = \frac{\int_0^{\bar{x}} e^{\frac{\mu}{\sigma^2}x} \sin(\frac{n\pi}{\bar{x}}x)dx}{\int_0^{\bar{x}} \sin(\frac{n\pi}{\bar{x}}x)^2 dx}; \qquad n = 1, 2, \ldots \tag{10.19}$$

$$= \frac{2n\pi\sigma^4(1 + \exp(\frac{\mu}{\sigma^2}\bar{x})(-1)^{n+1})}{\bar{x}(\mu^2\bar{x}^2 + n^2\pi^2\sigma^4)}$$

Using the ergodic distribution $e(x)$ which can be solved from the stationary Kolmogorov equation

$$e(x) = \exp(\frac{2\mu}{\sigma^2}x)\left(\frac{\frac{2\mu}{\sigma^2}}{\exp(\frac{2\mu}{\sigma^2}\bar{x}) - 1}\right) \tag{10.20}$$

we can integrate across the initial states to derive the probability that an individual x with an ergodic initial distribution will hit either barrier between t and T.

$$P(t - T) = \int_0^{\bar{x}} e(x)f(x; t, T)dx \tag{10.21}$$

$$= \int_0^{\bar{x}} e(x)(1 - g(x; t, T))dx \tag{10.22}$$

$$= 1 - \sum_{n=1}^{\infty} B_n \int_0^{\bar{x}} e^{\frac{\mu}{\sigma^2}x} \sin(\frac{n\pi}{\bar{x}}x)dx \tag{10.23}$$

$$= 1 - \left(\frac{\frac{2\mu}{\sigma^2}}{\exp(\frac{2\mu}{\sigma^2}\bar{x}) - 1}\right) \sum_{n=1}^{\infty} \frac{n\pi\sigma^4(1 + \exp(\frac{\mu}{\sigma^2}\bar{x})(-1)^{n+1})}{(4\mu^2\bar{x}^2 + n^2\pi^2\sigma^4)} B_n \exp(k_n(t - T)) \tag{10.24}$$

Table A2: Predicted Zero-Investment Frequencies (%)

¾^{2}	Buy=Sell	¹$_{Y}$	r	a	frequency (100 £p)
0.2	1.1	0.02	0.08	0.8	44.0%
0.2	1.5	0.02	0.08	0.8	60.2%
0.2	2	0.02	0.08	0.8	69.4%
0.2	4	0.02	0.08	0.8	73.1%
0.5	1.1	0.02	0.08	0.8	25.4% .
0.5	1.5	0.02	0.08	0.8	46.7%
0.5	2	0.02	0.08	0.8	62.4%
0.5	4	0.02	0.08	0.8	72.5%
1	1.1	0.02	0.08	0.8	11.4%
1	1.5	0.02	0.08	0.8	30.3%
1	2	0.02	0.08	0.8	48.4%
1	4	0.02	0.08	0.8	61.0%
1.5	1.1	0.02	0.08	0.8	5.6%
1.5	1.5	0.02	0.08	0.8	20.0%
1.5	2	0.02	0.08	0.8	37.3%
1.5	4	0.02	0.08	0.8	51.0%
0.5	2	0.1	0.08	0.8	57.4%
0.5	2	-0.1	0.08	0.8	57.4%
0.5	2	0.02	0.2	0.8	56.8%
0.5	2	0.02	0.02	0.8	66.7%
0.5	2	0.02	0.08	0.95	62.9%
0.5	2	0.02	0.08	0.5	61.5%
0.5	2	0.02	0.08	0.25	60.4%

The standard deviations for N obs. is p(1-p)/N

11. Appendix C: The Evolution of the Project Level Distribution

Since continuous information is not available as to aggregate developments, we assume the realizations of firm level uncertainty to be evenly spread within each observation period. This is only an approximation of course. Irreversible investment is path dependent, and so the variability of the capital stock upper bound at higher frequencies is in principle relevant for the observed path of installed capital. However, we believe any empirical importance of these issues is overshadowed by the substantial simplification of the analytical and estimation problems: it reduces an intractable stochastic partial differential equation (SDE) to a sequence of deterministic linear partial differential equations (PDEs) whose solution is presented below. At the end of this section we explain the Levy test for discerning the estimated error in approximating a SDE with a PDE in this manner

Let $f(s;t)$ denote the probability density of a process $s(t)$ with stochastic differential

$$ds(t) = \mu dt + \sigma dz \tag{11.1}$$

where dz is a standard Wiener process, $\mu < 0$, and let $\{s\}$ be reflected at 0. The function can be derived by solving the Kolmogorov forward equation

$$\partial_t f(s;t) = \frac{1}{2}\sigma^2 \partial_{ss} f(s;t) - \mu \partial_s f(s;t) \tag{11.2}$$

with the boundary conditions

$$\frac{1}{2}\sigma^2 \partial_s f(0;t) = \mu f(s;t) \qquad \forall t \tag{11.3}$$

$$\frac{1}{2}\sigma^2 \partial_s f(S;t) = \mu f(S;t) \qquad \forall t \tag{11.4}$$

the integration constraint on the distribution

$$\int f(s;t) = 1 \qquad \forall t \tag{11.5}$$

and the initial condition

$$f(s;0) = j(s) \tag{11.6}$$

This can be solved via a separation of variables to obtain a couple of ordinary differential equations. Denoting $f(s;t) = g(s)h(t)$, we can write

$$\frac{h_t(t)}{h(t)} = \frac{1}{2}\sigma^2 \frac{g_{ss}(s)}{g(s)} - \mu \frac{g_s(s)}{g(s)} = \lambda \tag{11.7}$$

for some constant λ: In the t direction,

$$h_t(t) - \lambda h(t) = 0 \tag{11.8}$$

has the general solution $h(t) = Ae^{\lambda t}$. In the s direction,

$$g_{ss}(s) - \frac{2\mu}{\sigma^2} g_s(s) - \frac{2\lambda}{\sigma^2} g(s) = 0 \tag{11.9}$$

$$g_s(0) = \frac{2\mu}{\sigma^2} g(0) \qquad \forall t \tag{11.10}$$

$$g_s(S) = \frac{2\mu}{\sigma^2} g(S) \qquad \forall t \tag{11.11}$$

Equations 11.9 to 11.11 define a Sturm-Liouville problem with integrating factor $e^{-\frac{2\mu}{\sigma^2} s}$ (see e.g. Churchill and Brown, 1993) with characteristic equation

$$x^2 - \frac{2\mu}{\sigma^2} x - \frac{2\lambda}{\sigma^2} = 0 \tag{11.12}$$

If $\lambda \geq \frac{\mu}{2\sigma^2}$ then the roots of are real and the solutions take the general form

$$g(S;\lambda) = A_1 e^{x_1 S} + A_2 e^{x_2 S} \qquad \lambda > -\frac{\mu}{2\sigma^2}, \text{ distinct roots } (x_1 \neq x_2) \tag{11.13}$$

$$g(S;\lambda) = A_1 e^{\frac{\mu}{\sigma^2} S} + sA_2 e^{\frac{\mu}{\sigma^2} S} \qquad \lambda = -\frac{\mu}{2\sigma^2}, \text{ the repeated root case} \tag{11.14}$$

Putting the distinct roots solutions into the boundary conditions

$$x_1 A_1 + x_2 A_2 = \frac{2\mu}{\sigma^2}(A_1 + A_2) \qquad \forall t \tag{11.15}$$

$$x_1 A_1 e^{x_1 S} + x_2 A_2 e^{x_2 S} = \frac{2\mu}{\sigma^2}(A_1 e^{x_1 S} + A_2 e^{x_2 S}) \qquad \forall t \qquad \forall t \tag{11.16}$$

and rearranging yields

$$(x_2 - \frac{2\mu}{\sigma^2})A_2(e^{(x_1 - x_2)S} - 1) = 0; \qquad (x_1 - \frac{2\mu}{\sigma^2})A_1(e^{(x_2 - x_1)S} - 1) = 0; \qquad \forall t \tag{11.17}$$

For the repeated roots putting the solutions into the boundary conditions and rearranging yields

$$\frac{2\mu}{\sigma^2} SA_2 = 0; \qquad A_1 = -\frac{\sigma^2}{\mu} A_2; \qquad \forall t \tag{11.18}$$

Hence, solutions of this form with real roots can only satisfy the boundary conditions if $\lambda = 0; A_2 = 0$, where

$$g(S; \lambda = 0) = A_1 e^{\frac{2\mu}{\sigma^2} S} \tag{11.19}$$

We then consider the solutions obtained for complex roots of the characteristic equation, $\lambda < -\frac{\mu^2}{2\sigma^2}$;

$$x = \frac{\mu}{\sigma^2} \pm i \left(-\frac{\mu^2}{\sigma^4} - \frac{2\lambda}{\sigma^2} \right)^{\frac{1}{2}} \quad (11.20)$$

$$= \frac{\mu}{\sigma^2} \pm \beta(\lambda) \quad (11.21)$$

where the solutions have the form

$$g(S;\lambda) = e^{\frac{\mu}{\sigma^2}s}\left(A_1 \cos(\beta(\lambda)s) + A_2 \sin(\beta(\lambda)s)\right) \quad (11.22)$$

Imposing 11.9 to 11.11 we obtain

$$\beta(\lambda)A_2 = \frac{\mu}{\sigma^2}A_1 \qquad \forall t \quad (11.23)$$

$$-\beta(\lambda)A_1 \sin(\beta(\lambda)S) + \beta(\lambda)A_2 \cos(\beta(\lambda)S) = \frac{\mu}{\sigma^2}\left(A_1 \cos(\beta(\lambda)S) + A_2 \sin(\beta(\lambda)S)\right) \qquad \forall t \quad (11.24)$$

which after re-arranging yields,

$$\sin(\beta(\lambda)s) = 0 \quad (11.25)$$

so that the eigenvalues for $g(S;\lambda)$ are

$$\lambda_n = \frac{\sigma^2}{2}\left(\left(\frac{n\pi}{S}\right)^2 - \frac{\mu^2}{\sigma^4}\right) \quad (11.26)$$

Combining results we find that the general solution to 11.2 to 11.4 can be written

$$f(s;t) = f_0(s) + \sum_{n=1}^{\infty} e^{\lambda_n t} f_n(s) \quad (11.27)$$

$$f_0(s) = A_0 e^{\frac{2\mu}{\sigma^2}s}; \qquad f_n(s) = e^{\frac{\mu}{\sigma^2}s} A_n \left(\cos\left(\frac{n\pi}{S}s\right) + \frac{S}{n\pi}\frac{\mu}{\sigma^2}\sin\left(\frac{n\pi}{S}s\right)\right) \quad (11.28)$$

The initial condition

$$\sum_n f_n(s;t) = j(s) \quad (11.29)$$

determines the constants $\{A_n\}_{n=0}^{\infty}$. Multiplying both sides of 11.29 by $e^{-\frac{2\mu}{\sigma^2}s} f_n(s;t)$, integrating between 0 and S, and exploiting the orthogonality of the eigenfunctions so that $\int_0^S f_i(s;t) f_j(s;t) = 0 \quad \forall \; i \neq j$, we obtain

$$A_0 = -\frac{2\mu}{\sigma^2}\frac{1}{1-e^{\frac{2\mu}{\sigma^2}S}} \tag{11.30}$$

$$A_n = \frac{\int_0^S j(s)\left(\cos(\frac{n\pi}{S}s) + \frac{S}{n\pi}\frac{\mu}{\sigma^2}\sin(\frac{n\pi}{S}s)\right)ds}{\int_0^S \left(\cos(\frac{n\pi}{S}s) + \frac{S}{n\pi}\frac{\mu}{\sigma^2}\sin(\frac{n\pi}{S}s)\right)^2 ds}; \qquad n = 1, 2, \ldots \tag{11.31}$$

The integral can be computed numerically for general $j(.)$ and the numerator has the closed form

$$\int_0^S \left(\cos(\frac{n\pi}{S}s) + \frac{S}{n\pi}\frac{\mu}{\sigma^2}\sin(\frac{n\pi}{S}s)\right)^2 = \frac{S}{2}\left((\frac{n\pi}{S}s)^2 + 1\right) \tag{11.32}$$

11.1. An approach for testing the importance of the approximation of the SDE by a PDE

The problem of approximating continuous time stochastic differential equations (SDE) using discrete time data is common in the physical sciences and a number of methods have been developed to deal with this. The Levy approach starts from the proof that conditional on the $(\mu;\sigma)$ brownian motion process being observed at $(x_0; t)$ and $(x_1; t+T)$ in (event, time) space its intervening probability distribution is normal. For example, at time $t+\frac{T}{2}$ the unknown position of the brownian process has a probability distribution $N(\frac{x_1 - x_0}{2}; \sigma^2\frac{T}{2})$. The evolution of the SDE over the interval $(t; t+T)$ can then be approximated by integrating with respect to this normal distribution over every pair of half period partial differential equations (PDEs) across $(t; t+\frac{T}{2})$ and $(t+\frac{T}{2}; t+T)$ which join at $t+\frac{T}{2}$. By splitting this half period into four quarter periods, eight $\frac{1}{8}$ periods, sixteen $\frac{1}{16}$ periods etc. any degree of accuracy can be achieved to the exact probabilistic solution to the SDE conditional on the available discrete information. This provides a computationally intensive test of the importance of assuming the smooth evolution of the firm level shock between periods of observation which we perform at $\frac{1}{16}$ splits for the parameter estimates - [insert results here].

12. Appendix D: Data

The UK data is taken from the accounts of firms listed on the UK stock market with financial holding companies excluded. This data is contained in the Datastream on-line service.

Investment (I). The basic variable used is total new fixed assets less revenue raised from fixed asset sales: DS435-DS423

Capital Stock (K): Was computed by adjusting the historic cost values for inflation and applying a perpetual inventory procedure with a depreciation rate of 8% per annum for all years following the first year for which historic cost data were available. Ideally we would use the depreciation rates from the firm level long run investment identity, $\overline{\frac{I}{K}} = \delta + \overline{\Delta \log(s)}$, but the average rate of 8% has been used to facilitate comparability with other estimators and to avoid additional estimation loops.

$$P^I_t K_t = (1-\delta)P^I_{t-1}K_{t-1}\left(\frac{P^I_t}{P^I_{t-1}}\right) + P^I_t I_t \qquad (12.1)$$

$$\text{where: } \left\{\begin{array}{ll} K_t & : \text{Capital Stock} \\ P^I_t & : \text{Price of Investment Goods} \\ I_t & : \text{Real Investment Goods} \\ \delta & : \text{Depreciation Rate} \end{array}\right\}$$

The starting value was based on the net book value of tangible fixed capital assets in the first observation within our sample period, adjusted for previous years inflation. Subsequent values were obtained using accounts data on investment and disposals, national price indices for investment goods prices.

Output (Y): Sales, DS104, deflated by the aggregate GDP deflator

Cash Flow (C): For the purposes of the regressions, cash flow is computed as funds available for investment, i.e. as net income plus depreciation.

12.1. Substitution of Sales Data as a Proxy for Business Conditions

To use sales data, $\log(S) = \log(AK^{\alpha}Z)$, as a proxy for business conditions, $\log(Z)$, we need to argue that these processes are closely related. This proof involves us first that the change in logged sales is a some weighted combination w of $(w \ast \Delta\log(Z) + (1-w)\ast\frac{1}{1-\alpha}\Delta\log(Z))$. We then show that for any set of initial conditions this weighting function w tends to 0, so that the relationship between logged sales and business conditions tends to $\Delta\log(S) = \frac{1}{1-\alpha}\Delta\log(Z)$, the proxy relationship adopted for estimation.

Taking first differences of logged sales we obtain

$$\Delta\log(S) = \Delta\log(Z) + \alpha\Delta\log(K) \qquad (12.2)$$

From section (2) it is clear that at either investment trigger

$$\Delta\log(K) = \frac{1}{1-\alpha}\Delta\log(Z) \qquad (12.3)$$

so that by combining (12.2) and (12.3) we obtain that at either trigger $\Delta \log(S) = \frac{1}{1-a}\Delta \log(Z)$. In between either investment trigger $\Delta \log(K) = 0$ so that from (12.2) $\Delta \log(S) = \Delta \log(Z)$. Hence, over any period of time $\Delta \log(S) = \phi(t)\Delta \log(Z) + (1-\phi(t))\frac{1}{1-a}\Delta \log(Z)$, where $\phi(t) \in [0;1]$ is a weighting function which represents the time proportion of projects hitting the investment triggers. For example, for a single project $\phi(t)$ would equal 0 if the project was at the investment trigger and 1 otherwise. To complete the assertion that $\Delta \log(S) \approx \frac{1}{1-a}\Delta \log(Z)$ we prove that $\lim_{T\to\infty} \phi(t) = 0$.

The results from Abel and Eberly (1996) can be generalised to demonstrate that for a model of the type presented in section (2) the gap between the logged lower and upper triggers is constant, and so I(0). The driving process for investment, the change in business conditions $\Delta \log(Z)$, is I(1). By cointegration it is clear that the long run growth rate is unaffected by irreversibility.

Formally, define $K_r(Z)$ to the be the level of capital stock that would prevail under complete reversibility conditional on business conditions. This can be derived from the first order conditions on the equality between the marginal product of capital and the user cost of capital

$$aAK^{a-1}Z = (r + a\delta - \mu_Z) \quad (12.4)$$

so that

$$\log K_r(Z) = \frac{\log aA - \log Z - \log(r + a\delta - \mu_Z)}{1-a} \quad (12.5)$$

Define $K_I(Z)$ and $K_D(Z)$ to be the capital stocks which would induce investment and disinvestment at the upper and lower investment triggers under irreversibility. By revealed preference theory the complete reversibility level of capital $K_r(Z)$ must lie strictly between $K_I(Z)$ and $K_D(Z)$[21]. We can use the definitions of the lower and upper investment triggers in section (2) to obtain

$$\log K_D = \log K_I + \frac{(\phi_U - \phi_L)}{1-a} \quad (12.6)$$

which is positive since $\phi_L < 0$ and $\phi_U > 0$. This provides bounds for the actual capital stock under partial reversibility

$$\begin{aligned} \frac{\log aA - \log Z - \log(r + a\delta - \mu_p) - (\phi_U - \phi_L)}{1-a} &< \log K \quad (12.7) \\ &< \frac{\log aA - \log Z - \log(r + a\delta - \mu_p) + (\phi_U - \phi_L)}{1-a} \end{aligned}$$

[21] If a (dis)invesment is optimal under partial reversibility it must be optimal or value adding under complete reversibility since no adjustment costs are incurred.

where Since the capital stock is bounded within a constant window between two Brownian processes, $\log K_D$ and $\log K_I$, which have a common drift, variance and Wiener term, $\log K$ must also have the same drift. Thus,

$$\lim_{T\to\infty}\left[\left(\frac{1}{T}\log(K_{T+t}) - \log(K_t)\right) - \frac{\mu}{1-\alpha}\right] \xrightarrow{a.s.} 0 \qquad (12.8)$$

on the filtered probability space $(\Omega; F_t; P)$ on which brownian process $\log(Z_t)$ is adapted, and so

$$\lim_{T\to\infty}\left[\frac{1}{T}\left(\log(K_{T+t}) - \log(K_t)\right)\right] \xrightarrow{a.s.} \lim_{T\to\infty}\left[\frac{1}{T}\frac{1}{1-\alpha}\left(\log(Z_{T+t}) - \log(Z_t)\right)\right] \qquad (12.9)$$

It is interesting to note that this also proves that the long run growth rate of capital is independent of any of the parameters of irreversibility or the variance of the business conditions. That is, uncertainty has no long run impact of the accumulation of capital. This has also been demonstrated by Hartman and Hendrickson (1999) using an alternative method of proof.

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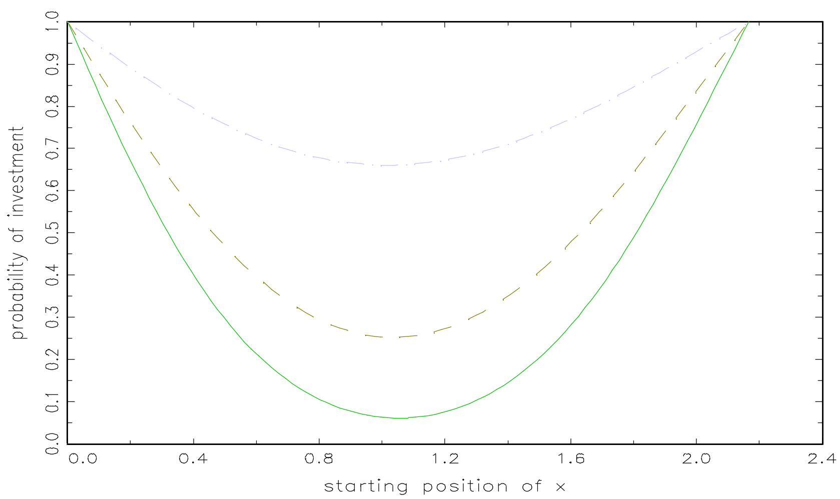

Figure 13.1: Investment Probabilities: One, Two & Five Years, buy=sell = 2

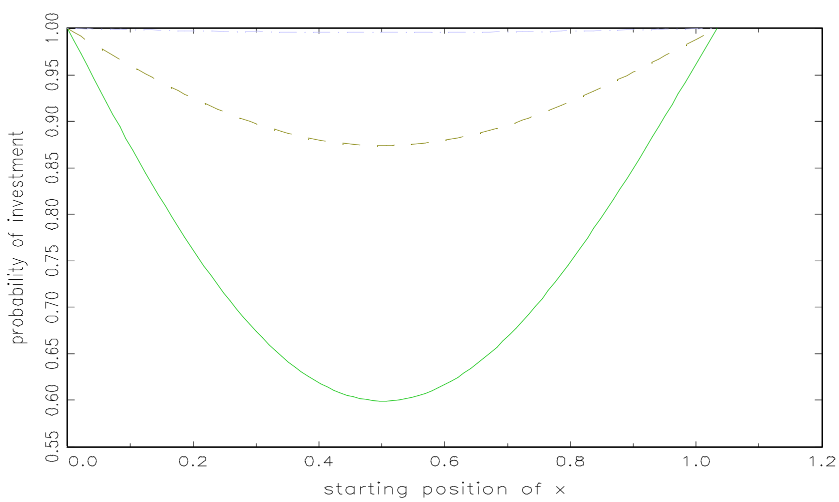

Figure 13.2: Investment Probabilities: One, Two & Five Years, buy=sell = 11

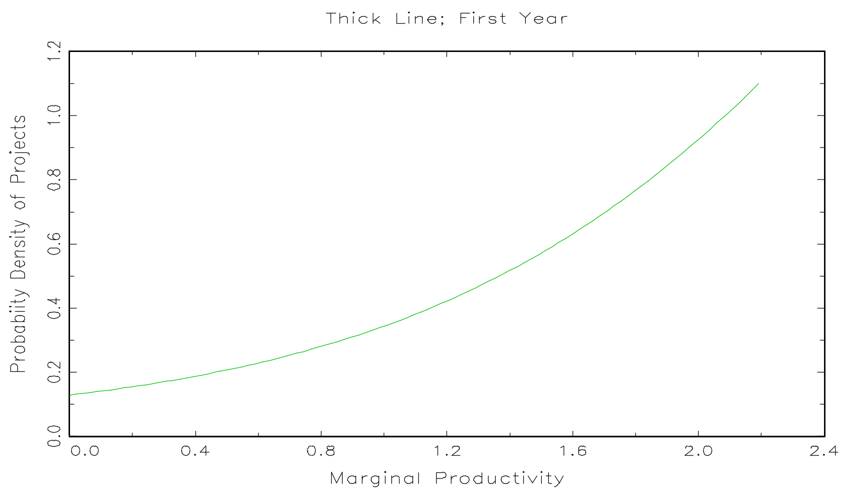

Figure 13.3: Project Level Distribution: Year 1: BASS

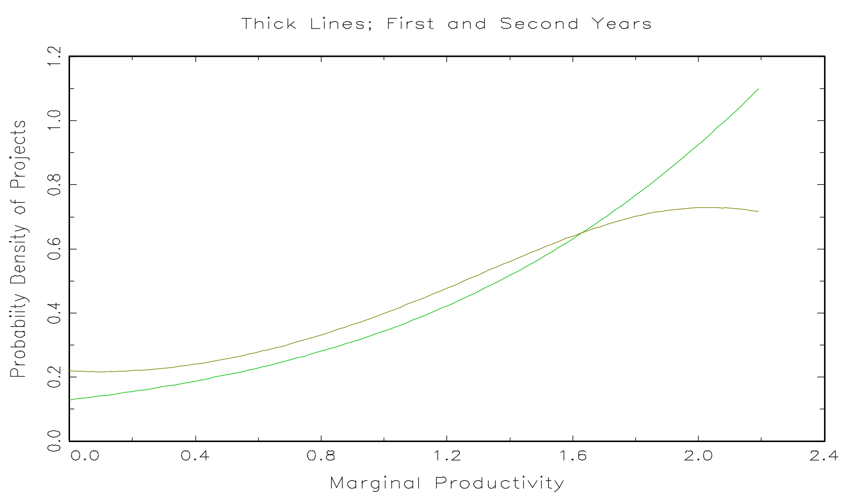

Figure 13.4: Project Level Distribution: Years 1 & 2: BASS

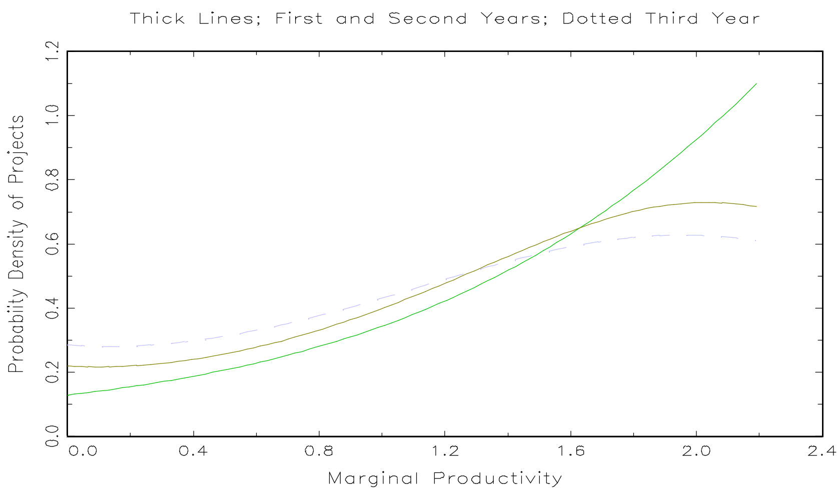

Figure 13.5: Project Level Distribution: Years 1,2 & 3: BASS

Figure 13.6 Project Level Distribution: Years 1,2, 3, & 4: BASS

Figure 13.7: Changes In Sales, Investment and Predicted Investment: BASS

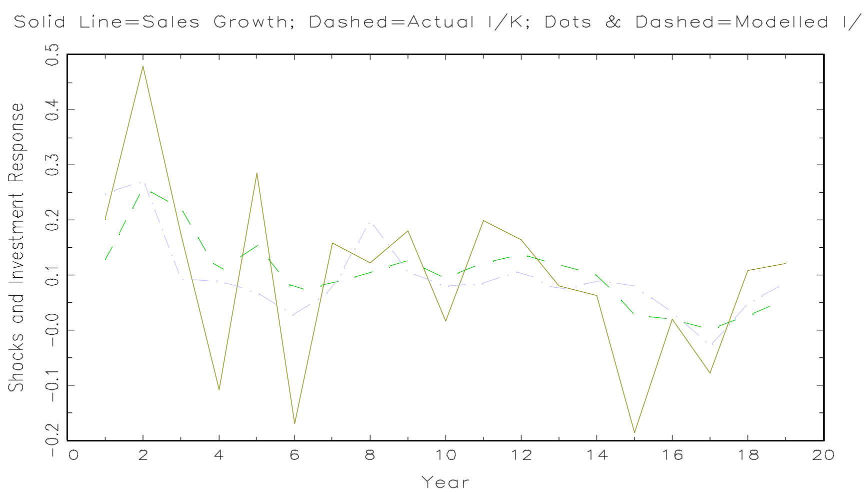

Figure 13.8: Changes In Sales, Investment and Predicted Investment: APV

Figure 13.9: Changes In Sales, Investment and Predicted Investment: Croda International

Figure 13.10: Changes In Sales, Investment and Predicted Investment: Senior Engineering Group