# Bootstrap Unit Root Tests

Joon Y. Park[1]
School of Economics
Seoul National University
Seoul 151-742, Korea

## Abstract

We consider the bootstrap unit root tests based on autoregressive integrated models, with or without deterministic time trends. A general methodology is developed to approximate asymptotic distributions for the models driven by integrated time series, and used to obtain asymptotic expansions for the Dickey-Fuller unit root tests. The second-order terms in their expansions are of stochastic orders $O_p(n^{-1/4})$ and $O_p(n^{-1/2})$, and involve functionals of Brownian motions and normal random variates. The asymptotic expansions for the bootstrap tests are also derived and compared with those of the Dickey-Fuller tests. We show in particular that the usual nonparametric bootstrap offers asymptotic refinements for the Dickey-Fuller tests, i.e., it corrects their second-order errors. More precisely, it is shown that the critical values obtained by the bootstrap resampling are correct up to the second-order terms, and the errors in rejection probabilities are of order $o(n^{-1/2})$ if the tests are based upon the bootstrap critical values. Through simulation, we investigate how effective is the bootstrap correction in small samples.

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To the Reviewer:

This is a preliminary draft of my ongoing research. Some of the theorems and lemmas are stated without proofs. Also, there may be errors, both theoretical and typographical. I would be grateful if you could still consider the paper for presentation at the Eighth World Congress of the Econometric Society.

# 1. Introduction

It is now well perceived that the bootstrap, if applied appropriately, helps to compute the critical values of asymptotic tests more acurately in finite samples, and that the tests based on the bootstrap critical values generally have actual finite sample rejection probabilities closer to their asymptotic nominal sizes. See, e.g., Hall (1994) and Horowitz (1999). The bootstrap unit root tests, i.e., the unit root tests relying on the bootstrap critical values, seem particularly attractive in this respect. For most of the commonly used unit root tests, the finite sample size distortions are known to be large and often too large to make the tests any reliable. It has indeed been observed by various authors including Harris (1992), Ferretti and Romo (1996) and Nankervis and Savin (1996) that the bootstrap tests have actual sizes that are much closer to the nominal sizes, compared to the original tests, in the unit root models.

The main purpose of this paper is to provide the bootstrap theory for the unit root tests. The bootstrap theory for the unit root models has previously been studied by, among others, Basawa et al. (1991a, 1991b), Datta (1996), Park (1999) and Chang and Park (1999). However, they have all been restricted to the consistency (and inconsistency) of the bootstrap estimators and statistics from the unit root models. None of them considers the asymptotic refinement of the bootstrap. In this paper, we develop asymptotic expansions that are applicable for a wide class of unit root tests and their bootstrap versions, and provide a framework within which we investigate the bootstrap asymptotic refinement of various unit root tests. Our asymptotic expansions for the unit root models extend and generalize those developed recently by Park (2000), which yields the asymptotic expansions for the estimators and statistics derived from the random walk models.

In the paper, we consider more specifically the Dickey-Fuller unit root tests for the autoregressive unit root models possibly with the constant and the linear time terms. It can be clearly seen, however, that our methodology may also be used to analyze many other unit root tests as well. For the Dickey-Fuller unit root tests, the expansions have as the leading term the functionals of Brownian motion representing their asymptotic distributions. This is as expected. The second-order terms in the expansions are, however, quite different from the standard Edgeworth-type expansions for the stationary models. They are represented by functionals of Brownian motions and normal random variates, which are of stochastic orders $O_p(n^{-1/4})$ and $O_p(n^{-1/2})$. The second-order expansion terms involve various unknown model parameters. This is so for the tests in models with deterministic trends, as well as for the tests in purely stochastic models.

We show that the limiting distributions of bootstrap statistics have expansions that are analogous to the original statistics. The bootstrap statistics have the same leading expansion terms. This is well expected, since the statistics that we consider are asymptotically pivotal. More importantly, their second-order terms are also given exactly as the original statistics except that the unknown parameters included in the expansions of the original statistics are now replaced by their sample analogues, which strongly converge to the corresponding population parameters. Consequently, using the critical values obtained by the bootstrap is expected to reduce the order of discrepancy between the actual (finite sample) and nominal (asymptotic) sizes of the tests. The bootstrap thus provides an asymptotic refinement for the tests. Though our asymptotic expansions for the unit root models are quite different from the Edgeworth-type expansions for stationary models, the reason that the bootstrap offers a refinement of asymptotics is precisely the same.

Through simulation, we investigate how effective is the bootstrap correction in small samples. We consider both Gaussian and non-Gaussian models, and parametric and nonparametric bootstraps. The findings are generally consistent with the theory developed in the paper. For the Gaussian models, the bootstrap provides little improvements, if at all, unless the resampling is done parametrically using normal distribution. For the non-Gaussian models generated by uniform or chi-square distributions, the usual nonparametric bootstrap provides some obvious improvements. The tests based on the bootstrap critical values have rejection probabilities that are substantially closer to their nominal values. The parametric Gaussian bootstrap makes things worse for the non-Gaussian models, as expected from our theory.

The rest of the paper is organized as follows. Section 2 introduces the model, tests and bootstrap method. The test statistics are introduced together with the autoregressive unit root model and the moment condition, and how to obtain bootstrap samples from such a model is explained here. The asymptotic expansions are derived in Section 3. The section starts with the probabilistic embeddings that are essential for the development of our subsequent theory, and present the second-order asymptotic expansions for the original and bootstrap tests. Some of their implications are also discussed. Section 4 extends the theory to the models with deterministic trends. The asymptotic expansions for the tests in models with constant and linear time trend are presented and compared with the earlier results. The simulation results are reported in Section 5, and Section 6 concludes the paper. Mathematical proofs are given in Section 7.

# 2. The Model, Tests and Bootstrap Method

## 2.1 The Model and Test Statistics

We consider the test of the unit root hypothesis

$$H_0 : \alpha = 1 \tag{1}$$

in the AR(p) unit root model

$$y_t = \alpha y_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta y_{t-i} + \varepsilon_t \tag{2}$$

We define

$$\alpha(z) = 1 - \sum_{i=1}^{p} \alpha_i z^i$$

so that under the null hypothesis of the unit root (1) we may write $\alpha(L)\Delta y_t = \varepsilon_t$ using lag operator $L$. Assume

**2.1 Assumption** Let $(\varepsilon_t)$ be an iid sequence with $E\varepsilon_t = 0$ and $E|\varepsilon_t|^r < \infty$ for some $r > 6$. Also, we assume that $\alpha(z) \neq 0$ for all $|z| \leq 1$.

Under Assumption 2.1 and the unit root hypothesis (1), the time series $(\Delta y_t)$ becomes a stationary AR(p) process.

The unit root hypothesis is customarily tested using the t-statistic on $\alpha$ in regression (2). Denote by $\hat{\alpha}_n$ the OLS estimator for $\alpha$ in regression (2). If we let

$$x_{t-1} = (\Delta y_{t-1}, \ldots, \Delta y_{t-p})'$$

and define

$$y^p_{t-1} = y_{t-1} - \left(\sum_{t=1}^{n} y_{t-1} x'_{t-1}\right)\left(\sum_{t=1}^{n} x_{t-1} x'_{t-1}\right)^{-1} x_{t-1} \tag{3}$$

we may explicitly write the t statistic for the null hypothesis (1) as

$$t_n(1) = \frac{\hat{\alpha}_n - 1}{\hat{\sigma}_n \left(\sum_{t=1}^{n} y^{p\,2}_{t-1}\right)^{-1/2}} \tag{4}$$

where $\hat{\sigma}^2_n$ is the usual variance estimator for the regression errors. The test is first proposed and investigated by Dickey and Fuller (1979, 1981), and it is commonly referred to as the Dickey-Fuller test (if applied to the regressions with no lagged difference term) or the augmented Dickey-Fuller (ADF) test (if based on the regressions with augmented lagged difference terms).

We may also use the statistic

$$s_n(1) = \frac{n(\hat{\alpha}_n - 1)}{\alpha_n(1)} \tag{5}$$

to test the unit root hypothesis, where

$$\alpha_n(1) = 1 - \sum_{i=1}^{p} \alpha_{ni}$$

with the least squares estimators $\alpha_{ni}$ of $\alpha_i$ for $i = 1, \ldots, p$. The statistic $s_n(1)$ reduces to the normalized coefficient $n(\hat{\alpha}_n - 1)$ in the simple model with no lagged differences.

The asymptotic distributions of the statistics $t_n(1)$ and $s_n(1)$ are well known [see, e.g., Stock (1994)] and given by

$$t_n(1) \to_d t(1) = \frac{\int_0^1 W(t)\,dW(t)}{\left(\int_0^1 W(t)^2\,dt\right)^{1/2}} \tag{6}$$

$$s_n(1) \to_d s(1) = \frac{\int_0^1 W(t)\,dW(t)}{\int_0^1 W(t)^2\,dt} \tag{7}$$

where $W$ is standard Brownian motion. The statistics $t_n(1)$ and $s_n(1)$ have, in particular, asymptotic distributions which do not depend on any nuisance parameter, and hence, they are asymptotically pivotal. The distributions of $t(1)$ and $s(1)$ are of both mean and median zero, but are rather heavily skewed.

### 2.2 The Bootstrap Method

Implementation of the bootstrap method in our unit root model is pretty straightforward, once we fit regression (2) and obtain the coefficients estimates $(\hat\alpha_{ni})$ and the fitted residuals $(\hat\varepsilon_t)$. We will not use the estimate $\hat\alpha_n$ of $\alpha$ in any of resampling procedures, where we impose the unit root restriction. In what follows, we let $(y_0, \ldots, y_{-p})$ are known and make all our arguments conditional on them. Also, we define $u_t = \Delta y_t$ for $t = -p+1, \ldots, n$. Of course, we may equivalently assume $(y_0, (u_0, \ldots, u_{-p+1}))$, instead of $(y_0, \ldots, y_{-p})$, are known.

The first step is to get bootstrap samples for the innovations $(\varepsilon_t)$ after mean correction. As usual, we denote by $(\varepsilon_t^*)$ their bootstrap samples, i.e., $(\varepsilon_t^*)$ are samples from

$$\left(\hat\varepsilon_t - \frac{1}{n}\sum_{i=1}^{n}\hat\varepsilon_i\right)_{t=1}^{n}$$

which can be viewed as iid samples from the empirical distribution given by $(\hat\varepsilon_t - \sum_{i=1}^{n}\hat\varepsilon_i/n)$. Note that the mean adjustment is necessary, since otherwise the mean of bootstrap samples is nonzero.

Once the bootstrap samples $(\varepsilon_t^*)$ are obtained, we may reconstruct bootstrap samples $(u_t^*)$ for $(u_t)$ recursively from $(\varepsilon_t^*)$

$$u_t^* = \sum_{i=1}^{p}\hat\alpha_{ni}u_{t-i}^* + \varepsilon_t^*$$

conditional on $(u_0, \ldots, u_{-p+1})$. Finally, resamples $(y_t^*)$ for $(y_t)$ can be obtained just by integrating $(u_t^*)$, i.e.,

$$y_t^* = y_0 + \sum_{i=1}^{t}u_i^*$$

given $y_0$. The bootstrap versions of the statistics $t_n(1)$ and $s_n(1)$, which we denote by $t_n^*(1)$ and $s_n^*(1)$ respectively, can be obtained from $(y_t^*)$ similarly as in (4) and (5).

## 3. Asymptotic Expansions of Test Statistics

### 3.1 Probabilistic Embeddings

Our subsequent theoretical development relies heavily on the probabilistic embedding of the partial sum process constructed from the innovation sequence $(\varepsilon_i)$ into a Brownian motion in an expanded probability space. This will be given below. Throughout the paper, we denote by $E\varepsilon_i^2 = \sigma^2$, $E\varepsilon_i^3 = \mu^3$ and $E\varepsilon_i^4 = \kappa^4$.

**Lemma 3.1** Let Assumption 2.1 hold. Then there exist a standard Brownian motion $(W(t))_{t\ge 0}$ and a time change $(T_i)_{i\ge 0}$ such that $T_0 \equiv 0$ and for all $n \ge 1$,

$$W(T_i/n) =_d \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{i} \varepsilon_k \tag{8}$$

$i = 1, \ldots, n$, and if we let $\Delta_i = T_i - T_{i-1}$, then $\Delta_i$'s are iid with $E\Delta_i = 1$ and and $E|\Delta_i|^r \le K_r E|\varepsilon_t|^{2r}$ for all $r \ge 1$, where $K_r$ is an absolute constant depending only upon $r$.

The reader is referred to Hall and Heyde (1980) for the explicit construction of the time change $(T_i)_{i\ge 0}$. As shown in Park and Phillips (1999), we have under Assumption 2.1 that

$$\sup_{1\le i\le n} \left| \frac{T_i - i}{n^r} \right| \to_{a.s.} 0$$

for any $r > 1/2$. Therefore, if we define

$$W_n(t) = \sum_{i=1}^{n} W(T_{i-1}/n) 1\{(i-1)/n \le t \le i/n\} \tag{9}$$

for $t \in [0,1]$ then we have due to the Hölder continuity of the Brownian sample path

$$\sup_{t\in[0,1]} |W_n(t) - W(t)| \le \left| T_{[nt]}/n - t \right|^{1/2-\epsilon} = o(n^{-1/4+\epsilon}) \text{ a.s.}$$

for any $\epsilon > 0$. In particular, $W_n \to_{a.s.} W$ uniformly on $[0,1]$. We let $T_{ni} = T_i/n$, $i = 1, \ldots, n$, for notational brevity.

In what follows, we will assume that $(\varepsilon_i)$ and $(W, (T_i))$ are defined in the common probability space $(\Omega, \mathcal{F}, P)$. This causes no loss in generality since we are concerned only with distributional results of the test statistics defined in (4) and (5), yet it will greatly simplify and clarify our subsequent exposition. The convention will be made throughout the paper. From now on, we would thus interpret the distributional equality in (8) as the usual equality.

For the development of our asymptotic expansion, it is necessary to define additional sequences defined from the Brownian motion $W$ and the time change $(T_i)$ introduced in Lemma 3.1. We let

$$\delta_i = \Delta_i - 1$$

for $i = 1, \ldots, n$. Moreover, we define

$$\eta_i = n \int_{T_{n,i-1}}^{T_{ni}} [W(t) - W(T_{n,i-1})] dW(t)$$

for $i = 1, \ldots, n$. Note that $(\delta_i)$ and $(\eta_i)$ are iid sequences of random variables. We also need to consider the sequence $(\xi_i)$ given by

$$\xi_i = (1/\sigma^2) x_{i-1} \varepsilon_i$$

Clearly, $(\xi_i)$ is a martingale difference sequence. Under the null hypothesis of unit root, it has conditional covariance matrix whose expectation is given by $\Gamma$, where $\Gamma = Ex_i x_i'/\sigma^2$. Finally, we let $E\delta_i^2 = \tau^4/\sigma^4$, which is finite under Assumption 2.1. Note that $\delta_i \equiv 0$, when and only when $(\varepsilon_i)$ are normal. The parameter $\tau$ can therefore be regarded as the nonnormality parameter. Subsequently, we set $\tau = 0$ if and only if $(\varepsilon_i)$ are normal. The parameters $\Gamma$ and $\tau^4$ defined here, in addition to $\sigma^2, \mu^3$ and $\kappa^4$ introduced earlier, will appear frequently in the development of our asymptotic expansions.

Now we define

$$v_i = (\varepsilon_i, \delta_i, \eta_i, \xi_i')'$$

and let

$$B_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} v_i$$

Then invariance principle holds, and $B_n \to_d B$ for a properly defined vector Brownian motion $B$. We present this formally as a lemma.

**Lemma 3.2** Let Assumption 2.1 hold. Then $B_n \to_d B$, where $B$ is a vector Brownian motion with covariance matrix $\Sigma$ given by

$$\Sigma = \begin{pmatrix} \sigma^2 & \mu^3/3\sigma^2 & \mu^3/3\sigma^2 & 0 \\ & \tau^4/\sigma^4 & (\kappa^4 - 3\sigma^4 - 3\tau^4)/12\sigma^4 & 0 \\ & & \kappa^4/6\sigma^4 & 0 \\ & & & \Gamma \end{pmatrix}$$

where parameters are defined earlier.

Following our earlier convention, we subsequently assume that both $B_n$ and $B$ are defined in the probability space $(\Omega, \mathcal{F}, P)$, and that $B_n \to_{a.s.} B$. It is well known that any weakly convergent random sequence can be represented, up to the distributional equivalence, by a random sequence which converges a.s. [see, e.g., Pollard (1982)]. Moreover, we make a partition of the limit Brownian motion $B$ as

$$B = (\sigma W, V, U, Z')'$$

conformably with $v_i = (\varepsilon_i, \delta_i, \eta_i, \xi_i')'$. Assume that $(\varepsilon_i)$ are nonnormal and $\tau \neq 0$, and let $(\underline{W}, \ddot{W})$ be a bivariate standard Brownian motion independent of $W$. Then we may write

$$V = \omega W + \underline{\omega}\underline{W}$$

where $\omega$ and $\underline{\omega}$ are the constants given by

$$\omega = \frac{\mu^3}{3\sigma^3} \quad \text{and} \quad \underline{\omega} = \left(\frac{\tau^4}{\sigma^4} - \frac{\mu^6}{9\sigma^6}\right)^{1/2}$$

Also, we may represent $U$ as

$$U = \omega W + \underline{\omega}\underline{W} + \bar{\omega}\bar{W}$$

where the constants $\omega$, $\underline{\omega}$ and $\bar{\omega}$ are given by

$$\omega = \frac{\mu^3}{3\sigma^3}$$

$$\underline{\omega} = \left(\frac{\tau^4}{\sigma^4} - \frac{\mu^6}{9\sigma^6}\right)^{-1/2}\left(\frac{\kappa^4 - 3\sigma^4}{12\sigma^4} - \frac{\tau^4}{4\sigma^4} - \frac{\mu^6}{9\sigma^6}\right)$$

$$\bar{\omega} = \left[\frac{\kappa^4}{6\sigma^4} - \frac{\mu^6}{9\sigma^6} - \left(\frac{\tau^4}{\sigma^4} - \frac{\mu^6}{9\sigma^6}\right)^{-1}\left(\frac{\kappa^4 - 3\sigma^4}{12\sigma^4} - \frac{\tau^4}{4\sigma^4} - \frac{\mu^6}{9\sigma^6}\right)^2\right]^{1/2}$$

For the normal $(\varepsilon_i)$, we have $\tau = 0$ and therefore $V \equiv 0$. Moreover, we have $\mu^3 = 0$ and $\kappa^4 = 3\sigma^4$, and we may write

$$U = (1/\sqrt{2})\underline{W}$$

where $\underline{W}$ is a standard Brownian motion independent of $W$. For both the Gaussian and non-Gaussian models, we may represent $Z(1) = \Gamma^{1/2}S$ using a multivariate normal random vector $S$ with identity covariance matrix. Since $Z$ is independent of $(W, V, U)$, so is $S$. Finally, we let $M$ be an extended standard Brownian motion on $\mathbb{R}$ independent of $B$ (and therefore all the Brownian motions and normal random variates defined above), and let $N$ be another extended Brownian motion on $\mathbb{R}$ defined by $N(t) = W(1+t) - W(1)$. The notations defined here will be used throughout the paper without any further reference.

## 3.2 Asymptotic Expansions

We are now ready to obtain asymptotic expansions for the distributions of the statistics $t_n(1)$ and $s_n(1)$ introduced in (4) and (5). We define

$$P_n = \frac{1}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_t - \frac{1}{n}\left(\sum_{t=1}^{n} y_{t-1}x_{t-1}'\right)\left(\sum_{t=1}^{n} x_{t-1}x_{t-1}'\right)^{-1}\left(\sum_{t=1}^{n} x_{t-1}\varepsilon_t\right) \quad (10)$$

$$Q_n = \frac{1}{n^2}\sum_{t=1}^{n} y_{t-1}^2 - \frac{1}{n^2}\left(\sum_{t=1}^{n} y_{t-1}x_{t-1}'\right)\left(\sum_{t=1}^{n} x_{t-1}x_{t-1}'\right)^{-1}\left(\sum_{t=1}^{n} x_{t-1}y_{t-1}\right) \quad (11)$$

Also, we write the error variance estimate $\sigma_n^2$ as

$$\sigma_n^2 = \frac{1}{n}\sum_{t=1}^{n} \varepsilon_t^2 - \frac{1}{n}\left(\sum_{t=1}^{n} \varepsilon_t x_{t-1}'\right)\left(\sum_{t=1}^{n} x_{t-1}x_{t-1}'\right)^{-1}\left(\sum_{t=1}^{n} x_{t-1}\varepsilon_t\right) \quad (12)$$

and write

$$\alpha_n(1) = \alpha(1) - \iota\left(\sum_{t=1}^{n} x_{t-1}x_{t-1}'\right)^{-1}\left(\sum_{t=1}^{n} x_{t-1}\varepsilon_t\right) \quad (13)$$

Here and elsewhere in the paper, $\iota$ denotes the $p$-vector of ones. The statistics $t_n(1)$ and $s_n(1)$ can now be written respectively as

$$t_n(1) = \frac{P_n}{\sigma_n \sqrt{Q_n}} \quad \text{and} \quad s_n(1) = \frac{P_n}{\alpha_n(1) Q_n} \tag{14}$$

Here we assume that $\sigma^2$ and $\alpha(1)$ are estimated under the unit root restriction. This assumption is made purely for the expositional purpose. All of our subsequent results also hold for the unrestricted estimators of $\sigma^2$ and $\alpha(1)$.

To derive the asymptotic expansions for the statistics $t_n(1)$ and $s_n(1)$, we need to consider various sample product moments in (10) – (13). The asymptotics for some of them are presented in Lemma 3.3, which can be directly obtained from the probabilistic embeddings developed in the previous section. Proposition 3.4 is a direct consequence of Lemma 3.3. To simplify the subsequent exposition, we use $X$ to denote $X(1)$, as well as the process itself, for Brownian motion $X$. This should cause no confusion.

**Lemma 3.3** Let Assumption 2.1 hold. Then we have

(a) $\displaystyle \frac{1}{n\sigma^2}\sum_{t=1}^{n} \varepsilon_t^2 = 1 + n^{-1/2}(V + 2U) + o_p(n^{-1/2})$

(b) $\displaystyle \frac{1}{n^{1/2}\sigma^2}\sum_{t=1}^{n} x_{t-1}\varepsilon_t = Z + o_p(1)$

(c) $\displaystyle \frac{1}{n\sigma^2}\sum_{t=1}^{n} x_{t-1}x_{t-1}' = \Gamma + O_p(n^{-1/2})$

for large $n$.

**Proposition 3.4** Let Assumption 2.1 hold. Then we have

(a) $\sigma_n^2 = \sigma^2 \left[ 1 + n^{-1/2}(V + 2U) \right] + o_p(n^{-1/2})$

(b) $\alpha_n(1) = \alpha(1) - n^{-1/2}\iota'\Gamma^{-1}Z + o_p(n^{-1/2})$

for large $n$.

We now obtain the asymptotic expansions for the sample product moments $\sum y_{t-1}\varepsilon_t$, $\sum y_{t-1}^2$ and $\sum x_{t-1}y_{t-1}$. To effectively analyze these product moments, we define $w_t = \sum_{i=1}^{t} \varepsilon_i$ for $t \geq 1$ and $w_0 \equiv 0$ and first consider the asymptotic expansions for the sample product moments of $(w_t)$ and $(\varepsilon_t)$. We let $u_t = \Delta y_t$ as before, so that $\alpha(L)u_t = \varepsilon_t$ under the null hypothesis of the unit root. Under the unit root hypothesis, $(u_t)$ is just a linearly filtered process of $(\varepsilon_t)$, and $(y_t)$ becomes an integrated process generated by such a process. Our subsequent asymptotic expansions involve various functionals of Brownian motions. To ease the exposition, we let for Brownian motions $X$ and $Y$,

$$I(X) = \int_0^1 X(t)dt \quad \text{and} \quad J(X;Y) = \int_0^1 X(t)dY(t)$$

in the subsequent development of our theory. This shorthand notation, together with $X = X(1)$ introduced above, will be used for the rest of the paper.

**Lemma 3.5** *Let Assumption 2.1 hold. Then we have,*

(a) $\displaystyle \frac{1}{n^{1/2}\sigma}\sum_{t=1}^{n}\varepsilon_t = W + n^{-1/4}M(V) + n^{-1/2}N(V) + o_p(n^{-1/2})$

(b) $\displaystyle \frac{1}{n^{3/2}\sigma}\sum_{t=1}^{n} w_{t-1} = I(W) + n^{-1/2}[WV - J(W;V) - \omega] + o_p(n^{-1/2})$

(c) $\displaystyle \frac{1}{n^{2}\sigma^2}\sum_{t=1}^{n} w_{t-1}^2 = I(W^2) + n^{-1/2}\left[W^2V - J(W^2;V) - 2\omega I(W)\right] + o_p(n^{-1/2})$

(d) $\displaystyle \frac{1}{n\sigma^2}\sum_{t=1}^{n} w_{t-1}\varepsilon_t = J(W;W) + n^{-1/4}WM(V)$
$\displaystyle \qquad + n^{-1/2}\left[(1/2)M(V)^2 + WN(V) - (1/2)(V+2U)\right] + o_p(n^{-1/2})$

*for large n.*

The asymptotic expansions for $\sum y_{t-1}\varepsilon_t$, $\sum y_{t-1}^2$ and $\sum x_{t-1}y_{t-1}$ can now be obtained using the relationships between $(y_t)$ and $(w_t)$, and between $(u_t)$ and $(\varepsilon_t)$. To write down more explicitly their relationships, we need to define some new notation. We let

$$\pi = 1/\alpha(1) \quad \text{and} \quad \pi_i = \sum_{j=i}^{p}\alpha_j/\alpha(1)$$

for $i = 1,\ldots,p$, and let

$$\varpi = (\pi_1,\ldots,\pi_p)'$$

We also define

$$\nu = (1/\sigma)\left(y_0 + \sum_{i=1}^{p}\pi_i u_{1-i}\right) \tag{15}$$

Note that we assume $(y_0;(u_0,\ldots,u_{-p+1}))$ to be given. Therefore, we may and will regard $\nu$ as a parameter in our subsequent analysis.

With the notation introduced above, we may write after some algebra

$$u_t = \pi\varepsilon_t + \varpi'(x_{t-1} - x_t) \tag{16}$$

and subsequently get

$$y_t = \sigma\nu + \pi w_t - \varpi' x_t \tag{17}$$

It is now rather straightforward to deduce from Lemma 3.5 that

**Proposition 3.6** Let Assumption 2.1 hold. Then we have

(a) $\frac{1}{n^{3/2}¼¾} \sum_{t=1}^{n} y_{t-1} = I(W) + n^{-1/2}[W V ¡ J(W;V) + (º=¼ ¡ !)] + o_p(n^{-1/2})$

(b) $\frac{1}{n¼^2¾^2} \sum_{t=1}^{n} x_{t-1} y_{t-1} = ¶[1 + J(W;W)] ¡ ¡(\$=¼^2) + o_p(1)$

(c) $\frac{1}{n^2¼^2¾^2} \sum_{t=1}^{n} y_{t-1}^2 = I(W^2)$

$+ n^{-1/2} £ W^2 V ¡ J(W^2;V) + 2(º=¼ ¡ !) I(W) ¤ + o_p(n^{-1/2})$

(d) $\frac{1}{n¼¾^2} \sum_{t=1}^{n} y_{t-1} "_t = J(W;W) + n^{-1/4} W M (V)$

$+ n^{-1/2}[(1=2) M (V)^2 + W N (V) + (º=¼) W ¡ (1=2)(V + 2U) ¡ (\$=¼) Z] + o_p(n^{-1/2})$

for large n.

The asymptotic expansions for the statistics $t_n(1)$ and $s_n(1)$ can now be easily obtained from (14), using the results in Lemma 3.3 and Propositions 3.4 and 3.6.

**Theorem 3.7** Let Assumption 2.1 hold. Then we have

$$t_n(1) = t(1) + n^{-1/4} F_1(\mu) + n^{-1/2} F_2(\mu) + o_p(n^{-1/2})$$
$$s_n(1) = s(1) + n^{-1/4} G_1(\mu) + n^{-1/2} G_2(\mu) + o_p(n^{-1/2})$$

for large n, where

$$F_1(\mu) = \frac{W M (V)}{I(W^2)^{1/2}}$$

$$F_2(\mu) = \frac{(1=2) M (V)^2 + W N (V) + (º=¼)}{I(W^2)^{1/2}} ¡ \frac{[1 + J(W;W)][(V + 2U)=2 + ¼¶ ¡^{-1} Z]}{I(W^2)^{1/2}} ¡ \frac{J(W;W)[W^2 V ¡ J(W^2;V) + 2(º=¼ ¡ !) I(W)]}{2 I(W^2)^{3/2}}$$

and

$$G_1(\mu) = \frac{W M (V)}{I(W^2)}$$

$$G_2(\mu) = \frac{(1=2) M (V)^2 + W N (V) + (º=¼)}{I(W^2)} ¡ \frac{(V + 2U)=2 + ¼¶ ¡^{-1} Z}{I(W^2)} ¡ \frac{J(W;W)[W^2 V ¡ J(W^2;V) + 2(º=¼ ¡ !) I(W)]}{I(W^2)^2}$$

in notation defined earlier.

Naturally, the asymptotic expansions for the statistics $t_n(1)$ and $s_n(1)$ have the leading terms $t(1)$ and $s(1)$ representing their asymptotic distributions. For both $t_n(1)$ and $s_n(1)$, the second terms in our expansions are of stochastic order $O_p(n^{-1/4})$. Their effects are, however, distributionally of order $O(n^{-1/2})$. More precisely, we have

$$P\left\{t(1) + n^{-1/4}F_1(\theta) \le x\right\} = P\{t(1) \le x\} + O(n^{-1/2})$$
$$P\left\{s(1) + n^{-1/4}G_1(\theta) \le x\right\} = P\{s(1) \le x\} + O(n^{-1/2})$$

uniformly in $x$. This is because the process $M$ included in $F_1$ and $G_1$ is independent of $(W, V, U)$ and

$$\varphi(W)M(V) \sim MN\left[0, \varphi(W)^2|V|\right)$$

for any functional $\varphi$ of $W$, where $MN$ stands for mixed normal distribution. Therefore, we let

$$F_n(\theta) = n^{-1/4}F_1(\theta) + n^{-1/2}F_2(\theta) \quad \text{and} \quad G_n(\theta) = n^{-1/4}G_1(\theta) + n^{-1/2}G_2(\theta)$$

and call from now on $F_n(\theta)$ and $G_n(\theta)$ the second-order terms in our asymptotic expansions of $t_n(1)$ and $s_n(1)$.

The results in Theorem 3.7 suggest that our second-order asymptotic expansions of the statistics $t_n(1)$ and $s_n(1)$ provide refinements of their asymptotic distributions up to order $o(n^{-1/2})$, which we present more formally as

**Corollary 3.8** Under Assumption 2.1, we have

$$P\{t_n(1) \le x\} = P\{t(1) + F_n(\theta) \le x\} + o(n^{-1/2})$$
$$P\{s_n(1) \le x\} = P\{s(1) + G_n(\theta) \le x\} + o(n^{-1/2})$$

uniformly in $x \in \mathbb{R}$.

It is thus expected in general that the actual finite sample rejection probabilities of the tests $t_n(1)$ and $s_n(1)$ disagree with their nominal size only by order $o(n^{-1/2})$, if the second-order corrected critical values are used, i.e., $a_\lambda$ and $b_\lambda$ such that $P\{t(1) + F_n(\theta) \le a_\lambda\} = \alpha$ and $P\{s(1) + G_n(\theta) \le b_\lambda\} = \alpha$ for size $\alpha$ tests.

For both statistics, the second-order terms $F_n(\theta)$ and $G_n(\theta)$ involve various functionals of Brownian motions and normal random variates. The functionals are dependent upon various model parameters, not only those included explicitly, but also those are implicitly given by the variances and covariances of $(W, V, U, Z)$ which we introduce below Lemma 3.2. Of course, we may make more transparent the dependence of $F_n(\theta)$ and $G_n(\theta)$ on these implicit parameters by representing $V$ and $U$ in terms of linear combinations of standard Brownian motions $\underline{W}$ and $\overline{W}$ which are independent of $W$, and $Z$ by a linear transform of standard normal random vector $S$, as suggested there.

Some of the distributional properties and roles of the second-order terms are obvious. For instance, it is clear that the second $O_p(n^{-1/4})$ terms in the expansions of $t_n(1)$ and $s_n(1)$ contribute nothing to the finite sample bias, since they represent mixed normal distributions with mean zero. It can also be easily seen that the initial values have effects on their finite sample distributions, which are distributionally of order $O(n^{-1/2})$. We may further analyze the second-order terms in the expansions of $t_n(1)$ and $s_n(1)$ in some special cases. If $\mu_3 = 0$ and the distribution of $(\varepsilon_i)$ is unskewed, then $V$ and $U$ become independent of $W$. In this case, we may indeed show that the third $O_p(n^{-1/2})$ terms of their expansions are of zero expectations and do not affect the locations of their distributions. The finite sample bias is thus of order $o(n^{-1/2})$ for both $t_n(1)$ and $s_n(1)$.[2]

It is rather straightforward to obtain the second-order asymptotic expansions for other unit root tests using our results here. For the tests considered in Stock (1994, pp2772–2773), it is indeed not difficult to see that the tests classified as $\alpha$-class, $\rho$-class and $P_T$ all have the asymptotic expansions that are obtainable from the results in Lemmas 3.3 and 3.5 and Propositions 3.4 and 3.6. Moreover, our approach developed here can also be used to analyze the models with the local-to-unity formulation of the unit root hypothesis. The asymptotics for such models are quite similar to those for the unit root models, except that they involve Ornstein-Uhlenbeck diffusion process in place of Brownian motion. Their asymptotic expansions can be obtained exactly in the same manner using the probabilistic embedding of Ornstein-Uhlenbeck process. Finally, the asymptotic expansions for the tests of stationarity based on the cumulated stationary processes, such as the one proposed by Kwiatkowski, Phillips, Schmidt and Shin (1992), are also possible using our methodology. However, to conserve space, we do not report their details.

### 3.3 Bootstrap Asymptotic Expansions

To develop the asymptotic expansions for the bootstrapped statistics $t_n^*(1)$ and $s_n^*(1)$ comparable to those for $t_n(1)$ and $s_n(1)$ presented in the previous section, we first need a probabilistic embedding of the standardized partial sum of the bootstrap samples $(\varepsilon_i^*)$ into a Brownian motion defined on an extended probability space. Once this embedding is done in an appropriately extended probability space, the rest of the procedure to obtain the asymptotic expansions for $t_n^*(1)$ and $s_n^*(1)$ is essentially identical to that for $t_n(1)$ and $s_n(1)$.

Let $W$ be a standard Brownian motion independent of $(\varepsilon_i)_{i=1}^\infty$, and assume that they are defined on the common probability space $(\Omega, F, P)$. Of course, there exists a probability space rich enough to support $W$ together with $(\varepsilon_i)_{i=1}^\infty$, since we assume they are independent. We then let $(T_i^*)_{i\geq 0}$ be a time change defined in $(\Omega, F, P)$ such that

$$W(T_i^*/n) =_{d^*} \frac{1}{\sigma_n \sqrt{n}} \sum_{k=1}^{i} \varepsilon_k^* \qquad (18)$$

[2] Abadir (1993) shows that the bias of $\hat{\alpha}_n$ is of order $O(n^{-1})$ for the simple first-order Gaussian autoregression.

where $=_{d^*}$ denotes the equivalence of distribution conditional on a realization of $(\varepsilon_i)_{i=1}^{\infty}$. Note that, for each n and any possible realization of $(\varepsilon_i)_{i=1}^{n}$, we may find a time change $(T_i^*)_{i=1}^{n}$ for which (18) holds with the same Brownian motion $W$. The Brownian motion $W$ therefore is not dependent upon the realizations of $(\varepsilon_i)_{i=1}^{\infty}$.

Here and elsewhere in the paper we follow the usual convention in the bootstrap literature and put superscript $*$ to the quantities and relationships depending upon the realizations of $(\varepsilon_i)_{i=1}^{\infty}$. In particular, $P^*$ and $E^*$ refer respectively to the probability and expectation operators given a realization of $(\varepsilon_i)_{i=1}^{\infty}$. They can be more formally defined as the conditional probability and expectation operators $P(\cdot|(\varepsilon_i)_{i=1}^{\infty})$ and $E(\cdot|(\varepsilon_i)_{i=1}^{\infty})$ on the probability $(\Omega, F, P)$ introduced above. For the functionals of $W$, however, $P^*$ and $E^*$ agree with $P$ and $E$ respectively, since they are independent of $(\varepsilon_i)$ by construction.

Just as the convention made in Section 3.1, we identify $(\varepsilon_i^*)$ only up to their distributional equivalences so that we may assume $(\varepsilon_i^*)$ are also defined in the probability space $(\Omega, F, P)$. We define the bootstrap sequences $(\delta_i^*)$, $(\eta_i^*)$ and $(\xi_i^*)$ analogously as $(\delta_i)$, $(\eta_i)$ and $(\xi_i)$, and $v_i^* = (\varepsilon_i^*, \delta_i^*, \eta_i^*, \xi_i^{*\prime})'$ as in Section 3.1. Also, we let

$$B_n^*(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} v_i^*$$

Then it can be readily established that

**Lemma 3.9** Let Assumption 2.1 hold. Then $B_n^* \to_{d^*} B^*$ a.s., where $B^*$ is a vector Brownian motion with covariance matrix $\Sigma_n$ given by the sample analogue estimator of $\Sigma$ defined in Lemma 3.2.

As before, we may assume that $B^*$ is defined in the probability space $(\Omega, F, P)$. Furthermore, we may let

$$B^* = (\sigma_n W, V^*, U^*, Z^{*\prime})'$$

and further represent $V^*, U^*$ and $Z^*$ in terms of independent standard Brownian motions $W$, $\underline{W}$ and $\overline{W}$, as in Section 3.1. The coefficients in the representations are, of course, now given by the sample analogue estimators $\omega_n$, $\underline{\omega}_n$ and $\overline{\omega}_n$, say, of $\omega$, $\underline{\omega}$ and $\overline{\omega}$.

We now introduce the bootstrap stochastic order symbols. For a sequence of random sequences $(X_n)$ on the probability space $(\Omega, F, P)$, we let $X_n = o_p^*(1)$ if $P^*\{|X_n| > \epsilon\} \to 0$ a.s. for any $\epsilon > 0$. Likewise, we denote by $Y_n = O_p^*(1)$ for $(Y_n)$ on $(\Omega, F, P)$ if a.s. for any $\epsilon > 0$ there exists a constant K such that $P^*\{|Y_n| > K\} \leq \epsilon$. The symbols $o_p^*(1)$ and $O_p^*(1)$ are the bootstrap versions of the stochastic order symbols $o_p(1)$ and $O_p(1)$. It is easy to check that $o_p^*(1) = o_p(1)$ and $O_p^*(1) = O_p(1)$. For the random sequences whose distributions do not depend upon the realizations of $(\varepsilon_i)$, the converse is also true, i.e., $o_p(1) = o_p^*(1)$ and $O_p(1) = O_p^*(1)$. Moreover, we may easily see that $o_p^*(1)$ and $O_p^*(1)$ satisfy the usual product rules that apply to $o_p(1)$ and $O_p(1)$. Needless to say, the definitions of $o_p^*(1)$ and $O_p^*(1)$ naturally extend to $o_p^*(c_n)$ and $O_p^*(c_n)$ for some numerical sequence $(c_n)$.

**Theorem 3.10** *Let Assumption 2.1 hold. We have*

$$t_n^*(1) = t(1) + F_n(\theta_n) + o_p^*(n^{-1/2})$$
$$s_n^*(1) = s(1) + G_n(\theta_n) + o_p^*(n^{-1/2})$$

*where $F_n$ and $G_n$ are defined earlier and $\theta_n$ denotes the sample moment estimator of the parameter $\theta$. Moreover, it follows that*

$$P^*\{t_n^*(1) \le x\} = P\{t_n(1) \le x\} + o_p(n^{-1/2})$$
$$P^*\{s_n^*(1) \le x\} = P\{s_n(1) \le x\} + o_p(n^{-1/2})$$

*uniformly in $x \in \mathbb{R}$.*

The asymptotics for the bootstrap statistics $t_n^*(1)$ and $s_n^*(1)$ are completely analogous to those for the corresponding statistics $t_n(1)$ and $s_n(1)$. The parameters appeared in the asymptotic expansions of the original statistics are replaced by their sample analogue estimators, as in the bootstrap Edgeworth expansions for the standard stationary models. Also, the residual terms that are of order $o_p(n^{-1/2})$ are now majorized by $o_p^*(n^{-1/2})$.

Now we define

$$P^*\{t_n^*(1) \le a_\lambda^*\} = P^*\{s_n^*(1) \le b_\lambda^*\} = \lambda$$

The values $a_\lambda^*$ and $b_\lambda^*$ are thus the bootstrap critical values for the $\lambda$-level tests based on the statistics $t_n(1)$ and $s_n(1)$. It follows directly from Theorem 3.10 that

$$P\{t_n(1) \le a_\lambda^*\},\ P\{s_n(1) \le b_\lambda^*\} = \lambda + o(n^{-1/2})$$

as $n \to \infty$. The tests using the bootstrap critical values $a_\lambda^*$ and $b_\lambda^*$ thus have rejection probabilities with errors of order $o(n^{-1/2})$.

## 4. Tests in Models with Deterministic Trends

In this section, we investigate the unit root tests in the model

$$y_t = D_t + \alpha y_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta y_{t-i} + \varepsilon_t \tag{19}$$

where $D_t$ is the deterministic trend which is further specified as

$$D_t = \sum_{i=0}^{q} \beta_i t^i \tag{20}$$

with parameters $\beta_i$, $i = 0, \ldots, q$. In the paper, we consider only the simplest (but most frequently used) cases $q = 0$ and $q = 1$. Higher order cases pose no fundamental difficulty, but would not be attempted here to save space.

We need to consider model (19), instead of (2), when it is believed that the observed time series $(y_t)$ includes deterministic trend specified as $D_t$ and can be appropriately modelled as

$$y_t = D_t + y_t^{\circ} \tag{21}$$

where the stochastic component $(y_t^{\circ})$ is assumed to follow (2). As an alternative to testing for the unit root in regression (19), we may detrend $(y_t)$ directly from the regression given by (21) with (20) to obtain the fitted residuals $(\hat{y}_t^{\circ})$, and base the unit root tests on regression (2) using $(\hat{y}_t^{\circ})$. It turns out that they are asymptotically equivalent not only in the first order, but also in the second order. All our subsequent results are therefore applicable for both procedures.[3]

To obtain the asymptotic expansions for the Dickey-Fuller tests in the presence of linear time trends, we need the following lemma and the subsequent proposition. We denote by $\iota$ the identity function $\iota(x) = x$ in what follows.

**Lemma 4.1** Let Assumption 2.1 hold. Then we have

(a) $$\frac{1}{n^{1/2}\sigma}\sum_{t=1}^{n}\frac{t}{n}\varepsilon_t = J(\iota;W) + n^{-1/4}M(V) - n^{-1/2}\left[WV - J(W;V) - N(V) - \omega\right] + o_p(n^{-1/2})$$

(b) $$\frac{1}{n^{3/2}\sigma}\sum_{t=1}^{n}\frac{t}{n}w_{t-1} = I(\iota W) + n^{-1/2}\left[WV - I(WV) - J(\iota W;V) - \omega/2\right] + o_p(n^{-1/2})$$

for large $n$.

**Proposition 4.2** Let Assumption 2.1 hold. Then we have

$$\frac{1}{n^{3/2}\pi\sigma}\sum_{t=1}^{n}\frac{t}{n}y_{t-1} = I(\iota W) + n^{-1/2}\left[WV - I(WV) - J(\iota W;V) + (\nu/\pi - \omega/2)\right] + o_p(n^{-1/2})$$

for large $n$

We now present the asymptotic expansions of the Dickey-Fuller tests for the models with constant, $q = 0$, and for the models with linear time trend, $q = 1$. They are quite similar, and we present them together in a single framework. For both cases $q = 0$ and $q = 1$, we denote by $\tilde{t}_n(1)$ and $\tilde{s}_n(1)$ the Dickey-Fuller statistics based on regression (19), or equivalently, the ones defined as in (4) and (5) from regression (2) run with the demeaned or detrended $(y_t)$. We denote by $\tilde{W}$ the demeaned or detrended Brownian motion, for each of the cases $q = 0$ and $q = 1$. Moreover, we let $\tilde{t}(1)$ and $\tilde{s}(1)$ respectively be the functionals of Brownian motions defined similarly as $t(1)$ and $s(1)$ with $W$ replaced by $\tilde{W}$. It is well known that $\tilde{t}_n(1)$ and $\tilde{s}_n(1)$ have the limiting distributions given by $\tilde{t}(1)$ and $\tilde{s}(1)$ respectively.

[3] We do not consider in the paper the GLS detrending proposed by Elliot, Rothenberg and Stock (1992) based on the local-to-unity formulation of the unit root hypothesis. Such detrending in general yields asymptotics distinct from those for the usual OLS detrending considered here.

**Theorem 4.3** *Let Assumption 2.1 hold, and let $\tilde F_n$ and $\tilde G_n$ be defined similarly as $F_n$ and $G_n$ with $\tilde W$ in place of $W$. We have*

$$\tilde t_n(1) = \tilde t(1) + n^{-1/4}\tilde F_1 + n^{-1/2}\tilde F_2 + o_p(n^{-1/2})$$
$$\tilde s_n(1) = \tilde s(1) + n^{-1/4}\tilde G_1 + n^{-1/2}\tilde G_2 + o_p(n^{-1/2})$$

*for large n, where*

$$\tilde F_1 = \frac{\tilde W M(V)}{I(\tilde W^2)^{1/2}}$$
$$\tilde F_2 = \frac{(1/2)M(V)^2 + \tilde W N(V) - [1 + J(\tilde W,\tilde W)](V + 2U)/2 + \pi\iota^{-1}Z}{I(\tilde W^2)^{1/2}} - \frac{J(\tilde W,\tilde W)[\tilde W^2 V - J(\tilde W^2,V) - 2\omega I(\tilde W)]}{2I(\tilde W^2)^{3/2}}$$

*and*

$$\tilde G_1 = \frac{\tilde W M(V)}{I(\tilde W^2)}$$
$$\tilde G_2 = \frac{(1/2)M(V)^2 + \tilde W N(V) - (V + 2U)/2 - \pi\iota^{-1}Z}{I(\tilde W^2)} - \frac{J(\tilde W,\tilde W)[\tilde W^2 V - J(\tilde W^2,V) - 2\omega I(\tilde W)]}{I(\tilde W^2)^2}$$

*Moreover, if we let $\tilde F_n = n^{-1/4}\tilde F_1 + n^{-1/2}\tilde F_2$ and $\tilde G_n = n^{-1/4}\tilde G_1 + n^{-1/2}\tilde G_2$, then it follows that*

$$P\left\{\tilde t_n(1) \le x\right\} = P\left\{\tilde t(1) + \tilde F_n \le x\right\} + o(n^{-1/2})$$
$$P\left\{\tilde s_n(1) \le x\right\} = P\left\{\tilde s(1) + \tilde G_n \le x\right\} + o(n^{-1/2})$$

*for large n, uniformly in $x \in \mathbb{R}$.*

The asymptotic expansions for $\tilde t_n(1)$ and $\tilde s_n(1)$ in Theorem 4.2 are quite similar to those for $t_n(1)$ and $s_n(1)$ in Theorem 3.7. We only have two differences. First, all the terms in the expansions for $t_n(1)$ and $s_n(1)$ representing the dependency on the initial value $\nu$ disappear, and are not present in the expansions of $\tilde t_n(1)$ and $\tilde s_n(1)$. This is naturally expected, since the demeaning or detrending makes the statistics $\tilde t_n(1)$ and $\tilde s_n(1)$ invariant with respect to the initial values. Second, the Brownian motion $W$ is replaced by the demeaned or detrending Brownian motion $\tilde W$ in all the expansion terms. The demeaning or detrending thus affects not only the first-order asymptotics, but also the lower order asymptotics.

## 5. Monte Carlo Simulations

Table 1. Rejection Probabilities: Normal Innovations with $n = 100$

| $\alpha$ | $\beta$ | Asymptotic Tests t Test | Asymptotic Tests s Test | Bootstrap Tests t Test | Bootstrap Tests s Test |
|---|---|---|---|---|---|
| 1.0 | 0.4 | 0.048 | 0.053 | 0.049 | 0.051 |
| | 0.0 | 0.048 | 0.051 | 0.050 | 0.052 |
| | –0.4 | 0.049 | 0.051 | 0.054 | 0.054 |
| 0.9 | 0.4 | 0.957 | 0.963 | 0.956 | 0.959 |
| | 0.0 | 0.727 | 0.738 | 0.737 | 0.734 |
| | –0.4 | 0.498 | 0.500 | 0.509 | 0.507 |

Table 2. Rejection Probabilities: Normal Innovations with $n = 50$

| $\alpha$ | $\beta$ | Asymptotic Tests t Test | Asymptotic Tests s Test | Bootstrap Tests t Test | Bootstrap Tests s Test |
|---|---|---|---|---|---|
| 1.0 | 0.4 | 0.051 | 0.057 | 0.053 | 0.053 |
| | 0.0 | 0.051 | 0.056 | 0.056 | 0.057 |
| | –0.4 | 0.050 | 0.052 | 0.058 | 0.059 |
| 0.9 | 0.4 | 0.559 | 0.584 | 0.568 | 0.562 |
| | 0.0 | 0.306 | 0.317 | 0.328 | 0.329 |
| | –0.4 | 0.202 | 0.206 | 0.225 | 0.223 |

## 6. Conclusion

## 7. Mathematical Proofs

Proof of Lemma 3.1 See Hall and Heyde (1980, Theorem A.1, p269). ¤

Proof of Lemma 3.2 That $B_n \to_d B$ follows from an invariance principle for martingale difference sequences [see Hall and Heyde (1980, p99)]. The covariance matrix of $B$ can be obtained from Park (2000, Remarks 2.4 and 2.5) and that $(\xi_i)$ are uncorrelated with $(\varepsilon_i, \delta_i, \eta_i)$. ¤

Proof of Lemma 3.3 Part (a) follows from Park (2000). Part (b) is immediate from the probabilistic embedding introduced in Section 3.1. Part (c) is well known [see, e.g., Berk (1974, p491)]. ¤

Proof of Lemma 3.4 Given (12) and (13), both Parts (a) and (b) readily follow from Lemma 3.3. ¤

**Proof of Lemma 3.5** See Lemma 3.2 of Park (2000). ■

**Proof of Proposition 3.6** It follows from (17) that

$$\sum_{t=1}^{n} y_{t-1} = n\sigma\nu + \pi \sum_{t=1}^{n} w_{t-1} - \varpi' \sum_{t=1}^{n} x_{t-1} \tag{22}$$

$$\sum_{t=1}^{n} x_{t-1} y_{t-1} = \pi \sum_{t=1}^{n} x_{t-1} w_{t-1} - \sum_{t=1}^{n} x_{t-1} x_{t-1}' \varpi + \sigma\nu \sum_{t=1}^{n} x_{t-1} \tag{23}$$

$$\begin{aligned}\sum_{t=1}^{n} y_{t-1}^2 = {} & \pi^2 \sum_{t=1}^{n} w_{t-1}^2 + 2\pi\sigma\nu \sum_{t=1}^{n} w_{t-1} + n\sigma^2\nu^2 + \varpi' \sum_{t=1}^{n} x_{t-1} x_{t-1}' \varpi \\ & - 2\pi\varpi' \sum_{t=1}^{n} x_{t-1} w_{t-1} - 2\sigma\nu\varpi' \sum_{t=1}^{n} x_{t-1}\end{aligned} \tag{24}$$

$$\sum_{t=1}^{n} y_{t-1}\varepsilon_t = \pi \sum_{t=1}^{n} w_{t-1}\varepsilon_t + \sigma\nu \sum_{t=1}^{n} \varepsilon_t - \varpi' \sum_{t=1}^{n} x_{t-1}\varepsilon_t \tag{25}$$

Consequently, we have from (22) – (25)

$$\frac{1}{n^{3/2}} \sum_{t=1}^{n} y_{t-1} = \pi \frac{1}{n^{3/2}} \sum_{t=1}^{n} w_{t-1} + n^{-1/2}\sigma\nu + O_p(n^{-1}) \tag{26}$$

$$\frac{1}{n} \sum_{t=1}^{n} x_{t-1} y_{t-1} = \pi \frac{1}{n} \sum_{t=1}^{n} x_{t-1} w_{t-1} - \frac{1}{n} \sum_{t=1}^{n} x_{t-1} x_{t-1}' \varpi + O_p(n^{-1/2}) \tag{27}$$

$$\frac{1}{n^2} \sum_{t=1}^{n} y_{t-1}^2 = \pi^2 \frac{1}{n^2} \sum_{t=1}^{n} w_{t-1}^2 + n^{-1/2} \left( 2\pi\sigma\nu \frac{1}{n^{3/2}} \sum_{t=1}^{n} w_{t-1} \right) + O_p(n^{-1}) \tag{28}$$

$$\frac{1}{n} \sum_{t=1}^{n} y_{t-1}\varepsilon_t = \pi \frac{1}{n} \sum_{t=1}^{n} w_{t-1}\varepsilon_t + n^{-1/2} \left( \sigma\nu \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \varepsilon_t - \varpi' \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_{t-1}\varepsilon_t \right) \tag{29}$$

We may now easily deduce Parts (a), (c) and (d) from Lemma 3.5, using (26), (28) and (29).

Due to (27), Part (b) follows if we establish

$$\frac{1}{n} \sum_{t=1}^{n} x_{t-1} w_{t-1} = \pi\sigma^2\iota[1 + J(W;W)] + o_p(1)$$

or equivalently,

$$\frac{1}{n} \sum_{t=1}^{n} w_{t-1} u_{t-i} = \pi\sigma^2[1 + J(W;W)] + o_p(1) \tag{30}$$

for all $i = 1, \ldots, p$. This is what we set out to do. We can show after some algebra that

$$\sum_{t=1}^{n} w_{t-1} u_{t-i} = \sum_{t=1}^{n} w_t u_t + \sum_{j=1}^{i-1} \sum_{t=1}^{n} \varepsilon_t u_{t-j} - \sum_{t=1}^{n} \varepsilon_t \sum_{j=0}^{i-1} u_{n-j}$$

Moreover, it can be deduced that

$$\sum_{t=1}^{n} w_t u_t = \pi \sum_{t=1}^{n} w_t \varepsilon_t + \sum_{j=1}^{\infty} \pi_j \sum_{t=1}^{n} w_t (u_{t-j-1} - u_{t-j})$$

and that

$$\sum_{t=1}^{n} w_t (u_{t-j-1} - u_{t-j}) = \sum_{t=1}^{n} \varepsilon_{t+1} u_{t-j} - u_{n-j} \sum_{t=1}^{n+1} \varepsilon_t$$

Consequently, we have

$$\frac{1}{n} \sum_{t=1}^{n} w_{t-1} u_{t-i} = \pi \left( \frac{1}{n} \sum_{t=1}^{n} w_{t-1} \varepsilon_t + \frac{1}{n} \sum_{t=1}^{n} \varepsilon_t^2 \right) + R_n$$

where

$$\begin{aligned} R_n &= \frac{1}{n} \sum_{j=1}^{\infty} \pi_j \sum_{t=1}^{n} \varepsilon_{t+1} u_{t-j} + \frac{1}{n} \sum_{j=1}^{n-1} \sum_{t=1}^{n} \varepsilon_t u_{t-j} - \frac{1}{n} \sum_{t=1}^{n+1} \varepsilon_t \sum_{j=1}^{\infty} \pi_j u_{n-j} - \frac{1}{n} \sum_{t=1}^{n} \varepsilon_t \sum_{j=0}^{n-1} u_{n-j} \\ &= O_p(n^{-1/2}) \end{aligned}$$

The result in (30) now follows directly from Lemma 3.3(a) and Lemma 3.5(a). The proof is therefore complete. $\square$

**Proof of Theorem 3.7** We may deduce from Lemma 3.3 and Proposition 3.6 that

$$\begin{aligned} \frac{P_n}{\pi\sigma^2} &= \frac{1}{n\pi\sigma^2} \sum_{t=1}^{n} y_{t-1} \varepsilon_t \\ &\quad - \frac{\pi}{\sqrt{n}} \left( \frac{1}{n\pi^2\sigma^2} \sum_{t=1}^{n} y_{t-1} x'_{t-1} \right) \left( \frac{1}{n\sigma^2} \sum_{t=1}^{n} x_{t-1} x'_{t-1} \right)^{-1} \left( \frac{1}{\sqrt{n}\sigma^2} \sum_{t=1}^{n} x_{t-1} \varepsilon_t \right) \\ &= J(W;W) + n^{-1/4} W M(V) \\ &\quad + n^{-1/2} \left[ \frac{M(V)^2}{2} + W M(V) + (\nu/\pi) W - \frac{V + 2U}{2} - \pi (1 + J(W;W)) \Gamma^{-1} Z \right] \\ &\quad + o_p(n^{-1/2}) \end{aligned}$$

and that

$$\begin{aligned} \frac{Q_n}{\pi^2\sigma^2} &= \frac{1}{n^2\pi^2\sigma^2} \sum_{t=1}^{n} y_{t-1}^2 + O_p(n^{-1}) \\ &= I(W^2) + n^{-1/2} \big( W^2 V - J(W^2;V) + 2(\nu/\pi - \omega) I(W) \big) + o_p(n^{-1/2}) \\ &= I(W^2) \left[ 1 + n^{-1/2} \frac{W^2 V - J(W^2;V) + 2(\nu/\pi - \omega) I(W)}{I(W^2)} \right] + o_p(n^{-1/2}) \end{aligned}$$

Consequently, it follows that

$$Q_n^{-1} = \frac{1}{\pi^2\sigma^2 I(W^2)}\left[1 - n^{-1/2}\frac{W^2V - J(W^2, V) + 2(\nu/\pi - \omega)I(W)}{I(W^2)}\right] + o_p(n^{-1/2})$$

$$Q_n^{-1/2} = \frac{1}{\pi\sigma\sqrt{I(W^2)}}\left[1 - n^{-1/2}\frac{W^2V - J(W^2, V) + 2(\nu/\pi - \omega)I(W)}{2I(W^2)}\right] + o_p(n^{-1/2})$$

Moreover, we have from Proposition 3.4 that

$$\sigma_n^{-1} = \sigma^{-1}\left(1 - n^{-1/2}(V + 2U)/2\right) + o_p(n^{-1/2})$$

$$\alpha_n(1)^{-1} = \alpha(1)^{-1}\left(1 + n^{-1/2}\alpha(1)^{-1}\iota'\Gamma^{-1}Z\right) + o_p(n^{-1/2})$$

Now the stated results follow easily after some tedious algebra. □

**Proof of Lemma 4.1** For Part (a), we simply note that

$$\frac{1}{n^{1/2}\sigma}\sum_{t=1}^{n}\frac{t}{n}\varepsilon_t = -\frac{1}{n^{3/2}\sigma}\sum_{t=1}^{n}w_{t-1} + \frac{1}{n^{1/2}\sigma}\sum_{t=1}^{n}\varepsilon_t$$

The stated result then follows directly from Lemma 3.5 and the fact that $W - I(W) = J(\imath, W)$, which can easily be deduced using integration by parts formula.

Let $r_i = i/n$ for $i = 1, \ldots, n$. To prove Part (a), we first note that

$$\frac{1}{n^{3/2}\sigma}\sum_{t=1}^{n}\frac{t}{n}w_{t-1} = \frac{1}{n^{3/2}\sigma}\sum_{t=1}^{n}\frac{t-1}{n}w_{t-1} + O_p(n^{-1})$$

and write

$$\frac{1}{n^{3/2}\sigma}\sum_{t=1}^{n}\frac{t-1}{n}w_{t-1} = \frac{1}{n}\sum_{i=1}^{n}r_{i-1}W(T_{n,i-1}) = -A_n + B_n$$

where

$$A_n = \frac{1}{n}\sum_{i=1}^{n}(T_{n,i-1} - r_{i-1})W(T_{n,i-1}) \quad \text{and} \quad B_n = \frac{1}{n}\sum_{i=1}^{n}T_{n,i-1}W(T_{n,i-1})$$

each of which will be analyzed below.

It is straightforward to deduce that

$$A_n = n^{-1/2}\frac{1}{n^2}\sum_{i=1}^{n}V_n(r_{i-1})W(T_{n,i-1}) = n^{-1/2}I(WV) + o_p(n^{-1/2}) \tag{31}$$

Furthermore, we may write $B_n$ as

$$B_n = I(\imath W) + n^{-1/2}[WV - J(\imath W, V)] - C_n + o_p(n^{-1/2}) \tag{32}$$

where

$$C_n = \sum_{i=1}^{n} \int_{T_{n,i-1}}^{T_{ni}} [tW(t) - T_{n,i-1}W(T_{n,i-1})]dt$$

To deduce (32), note that

$$B_n = \sum_{i=1}^{n} T_{n,i-1}W(T_{n,i-1})(T_{ni} - T_{n,i-1}) - n^{-1/2}\sum_{i=1}^{n} T_{n,i-1}W(T_{n,i-1})[(V_n(n_i) - V_n(n_{i-1})]$$

and

$$\sum_{i=1}^{n} T_{n,i-1}W(T_{n,i-1})(T_{ni} - T_{n,i-1}) = I(W) + \int_1^{T_{nn}} tW(t)dt - \sum_{i=1}^{n}\int_{T_{n,i-1}}^{T_{ni}} [tW(t) - T_{n,i-1}W(T_{n,i-1})]dt$$

Moreover, observe that

$$n^{1/2}\int_1^{T_{nn}} tW(t)dt = WV + o_p(1)$$

and that

$$\sum_{i=1}^{n} T_{n,i-1}W(T_{n,i-1})[(V_n(n_i) - V_n(n_{i-1})] = J(W;V) + o_p(1)$$

due to Kurz and Protter (1992).

Now we write

$$C_n = \sum_{i=1}^{n} T_{n,i-1}\int_{T_{n,i-1}}^{T_{ni}} [W(t) - W(T_{n,i-1})]dt + \sum_{i=1}^{n} W(T_{n,i-1})\int_{T_{n,i-1}}^{T_{ni}} (t - T_{n,i-1})dt + \sum_{i=1}^{n}\int_{T_{n,i-1}}^{T_{ni}} (t - T_{n,i-1})[W(t) - W(T_{n,i-1})]dt$$

and show that

$$C_n = n^{-1/2}\frac{\mu_3}{6\sigma^3} + o_p(n^{-1/2}) \qquad (33)$$

Note that

$$n^{1/2}\sum_{i=1}^{n} T_{n,i-1}\int_{T_{n,i-1}}^{T_{ni}} [W(t) - W(T_{n,i-1})]dt = \frac{\mu_3}{3\sigma^3}\frac{1}{n}\sum_{i=1}^{n} T_{n,i-1} + o_p(1) = \frac{\mu_3}{6\sigma^3} + o_p(1)$$

which becomes the leading term in $C_n$. The rest terms are negligible as we show below.

We have

$$\sum_{i=1}^{n} W(T_{n,i-1})\int_{T_{n,i-1}}^{T_{ni}}(t-T_{n,i-1})dt = \frac{1}{2}\sum_{i=1}^{n} W(T_{n,i-1})(T_{ni}-T_{n,i-1})^2$$
$$= \frac{1}{2n^2}\sum_{i=1}^{n} W(T_{n,i-1})\Delta_i^2 = O_p(n^{-1})$$

Moreover, we have

$$E\left|\int_{T_{n,i-1}}^{T_{ni}}(t-T_{n,i-1})[W(t)-W(T_{n,i-1})]dt\right|$$
$$\le E\left(\int_{T_{n,i-1}}^{T_{ni}}(t-T_{n,i-1})^2dt\right)^{1/2}\left(\int_{T_{n,i-1}}^{T_{ni}}[W(t)-w(T_{n,i-1})]^2dt\right)^{1/2}$$
$$\le \left(E\int_{T_{n,i-1}}^{T_{ni}}(t-T_{n,i-1})^2dt\right)^{1/2}\left(E\int_{T_{n,i-1}}^{T_{ni}}[W(t)-w(T_{n,i-1})]^2dt\right)^{1/2}$$
$$= O_p(n^{-5/2})$$

since

$$E\int_{T_{n,i-1}}^{T_{ni}}(t-T_{n,i-1})^2dt = O(n^{-3})$$
$$E\int_{T_{n,i-1}}^{T_{ni}}[W(t)-W(T_{n,i-1})]^2dt = O(n^{-2})$$

and therefore,

$$\sum_{i=1}^{n}\int_{T_{n,i-1}}^{T_{ni}}(t-T_{n,i-1})[W(t)-W(T_{n,i-1})]dt = O_p(n^{-3/2})$$

We thus have established (33). The stated result in Part (a) now follows immediately from (31), (32) and (33). The proof is therefore complete. ∎

**Proof of Proposition 4.2** The stated result is immediate from Lemma 4.1 and (17). ∎

**Proof of Theorem 4.3** For time series $(z_t)$, we let $\bar{z}_t = z_t - \sum_{t=1}^{n} z_t/n$ for the case $q = 0$, and let

$$\bar{z}_t = z_t - \frac{1}{n}\sum_{t=1}^{n} z_t - \left(\sum_{t=1}^{n}(t-c_n)z_t \Big/ \sum_{t=1}^{n}(t-c_n)^2\right)(t-c_n)$$

with $c_n = (n+1)/2$ for the case $q = 1$. Define $\tilde{P}_n$ and $\tilde{Q}_n$ by

$$\tilde{P}_n = \frac{1}{n}\sum_{t=1}^{n}\tilde{y}_{t-1}\tilde{\varepsilon}_t - \frac{1}{n}\left(\sum_{t=1}^{n}\tilde{y}_{t-1}\tilde{x}_{t-1}'\right)\left(\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{x}_{t-1}'\right)^{-1}\left(\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{\varepsilon}_t\right)$$

$$\tilde{Q}_n = \frac{1}{n^2}\sum_{t=1}^{n}\tilde{y}_{t-1}^2 - \frac{1}{n^2}\left(\sum_{t=1}^{n}\tilde{y}_{t-1}\tilde{x}_{t-1}'\right)\left(\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{x}_{t-1}'\right)^{-1}\left(\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{y}_{t-1}\right)$$

similarly as $P_n$ and $Q_n$ in (10) and (11). Also, we let

$$\tilde{\sigma}_n^2 = \frac{1}{n}\sum_{t=1}^{n}\tilde{\varepsilon}_t^2 - \frac{1}{n}\left(\sum_{t=1}^{n}\tilde{\varepsilon}_t\tilde{x}_{t-1}'\right)\left(\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{x}_{t-1}'\right)^{-1}\left(\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{\varepsilon}_t\right)$$

and define

$$\tilde{\alpha}_n(1) = \alpha(1) - \pi'\left(\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{x}_{t-1}'\right)^{-1}\left(\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{\varepsilon}_t\right)$$

which correspond to $\sigma_n^2$ and $\alpha_n(1)$ in (12) and (13). Then we may write

$$\tilde{t}_n(1) = \frac{\tilde{P}_n}{\tilde{\sigma}_n\sqrt{\tilde{Q}_n}} \quad \text{and} \quad \tilde{s}_n(1) = \frac{\tilde{P}_n}{\tilde{\alpha}_n(1)\tilde{Q}_n}$$

correspondingly as $t_n(1)$ and $s_n(1)$ in (14).

For both the cases $q = 0$ and $q = 1$, it can be easily deduced that

$$\frac{1}{n}\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{x}_{t-1}' = \frac{1}{n}\sum_{t=1}^{n}x_{t-1}x_{t-1}' + O_p(n^{-1})$$

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\tilde{x}_{t-1}\tilde{\varepsilon}_t = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}x_{t-1}\varepsilon_t + O_p(n^{-1/2})$$

$$\frac{1}{n}\sum_{t=1}^{n}\tilde{\varepsilon}_t^2 = \frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^2 + O_p(n^{-1})$$

Note in particular that

$$\frac{1}{n^3}\sum_{t=1}^{n}(t - c_n)^2 = \frac{1}{3} + O(n^{-1})$$

and

$$\frac{1}{n^{3/2}}\sum_{t=1}^{n}(t - c_n)z_t = \frac{1}{n^{1/2}}\sum_{t=1}^{n}\frac{t}{n}z_t + O_p(n^{-1}) \qquad (34)$$

for both $z_t = x_{t-1}$ and $\varepsilon_t$.

Moreover, we have for the case $q = 0$

$$\frac{1}{n}\sum_{t=1}^{n}\tilde y_{t-1}\tilde\varepsilon_t = \frac{1}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_t - \left(\frac{1}{n^{3/2}}\sum_{t=1}^{n} y_{t-1}\right)\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_t\right)$$

$$\frac{1}{n^2}\sum_{t=1}^{n}\tilde y_{t-1}^2 = \frac{1}{n^2}\sum_{t=1}^{n} y_{t-1}^2 - \left(\frac{1}{n^{3/2}}\sum_{t=1}^{n} y_{t-1}\right)^2$$

$$\frac{1}{n}\sum_{t=1}^{n}\tilde x_{t-1}\tilde y_{t-1} = \frac{1}{n}\sum_{t=1}^{n} x_{t-1}y_{t-1} - \iota\left(\frac{1}{n^{3/2}}\sum_{t=1}^{n} y_{t-1}\right)\left(\pi\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_t\right) + o_p(1)$$

and for the case $q = 1$

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{n}\tilde y_{t-1}\tilde\varepsilon_t = {} & \frac{1}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_t - \left(\frac{1}{n^{3/2}}\sum_{t=1}^{n} y_{t-1}\right)\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_t\right) \\ & - 3\left(\frac{1}{n^{3/2}}\sum_{t=1}^{n}\frac{t}{n}y_{t-1} - \frac{1}{2n^{3/2}}\sum_{t=1}^{n} y_{t-1}\right)\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{t}{n}\varepsilon_t\right) + O_p(n^{-1})\end{aligned}$$

$$\begin{aligned}\frac{1}{n^2}\sum_{t=1}^{n}\tilde y_{t-1}^2 = {} & \frac{1}{n^2}\sum_{t=1}^{n} y_{t-1}^2 - \left(\frac{1}{n^{3/2}}\sum_{t=1}^{n} y_{t-1}\right)^2 + O_p(n^{-1}) \\ & - 3\left(\frac{1}{n^{3/2}}\sum_{t=1}^{n}\frac{t}{n}y_{t-1} - \frac{1}{2n^{3/2}}\sum_{t=1}^{n} y_{t-1}\right)^2\end{aligned}$$

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{n}\tilde x_{t-1}\tilde y_{t-1} = {} & \frac{1}{n}\sum_{t=1}^{n} x_{t-1}y_{t-1} - \iota\left(\frac{1}{n^{3/2}}\sum_{t=1}^{n} y_{t-1}\right)\left(\pi\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_t\right) \\ & - 3\iota\left(\frac{1}{n^{3/2}}\sum_{t=1}^{n}\frac{t}{n}y_{t-1} - \frac{1}{2n^{3/2}}\sum_{t=1}^{n} y_{t-1}\right)\left(\pi\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{t}{n}\varepsilon_t\right) + o_p(1)\end{aligned}$$

which follows from (34) and

$$\frac{1}{n^{5/2}}\sum_{t=1}^{n}(t - c_n)y_{t-1} = \frac{1}{n^{3/2}}\sum_{t=1}^{n}\frac{t}{n}y_{t-1} - \frac{1}{2n^{3/2}}\sum_{t=1}^{n} y_{t-1} + O_p(n^{-1})$$

The stated results now follow easily. ¤

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