Money, Intermediaries and Cash-in-Advance Constraints

Christian Hellwig[*]
London School of Economics[†]

First Draft: May 4, 1999
This Version: July 27, 2000

Abstract

I study a search economy in which intermediaries are the driving force co-ordinating the economy on the use of a unique, common medium of exchange for transactions. If search frictions delay trade, intermediaries offering immediate exchange opportunities can make arbitrage gains from a price spread. As these intermediaries take over transactions, they are confronted to the double coincidence problem of the search market. In the model presented here, intermediaries solve this problem best by imposing a common medium of exchange to other agents, such that a Cash-in-Advance constraint results: Agents trade twice in order to consume, once to exchange their production against the medium of exchange, and once to receive their consumption good. To select between multiple equilibria, I introduce a criterion of minimal coalition proofness, whereby arbitrarily small coalitions may induce a change from one equilibrium to another. I show that any minimally coalition-proof equilibrium is Pareto-efficient, and characterize the full set of minimally coalition-proof equilibria of this economy.

[*]Financial Markets Group G203, London School of Economics, Houghton Street, London WC2A 2AE; email: C.Hellwig@lse.ac.uk

[†]This is a revised version of the dissertation submitted for the MSc. in Econometrics and Mathematical Economics at the LSE, 1999. I thank Nobu Kiyotaki, who supervised this project, Martin Hellwig, Godfrey Keller, Thomas Mariotti and Michele Piccione, as well as audiences at the Financial Markets Group, the Sticerd lunchtime seminar and the Young Economists' Meeting 2000 in Oxford, for helpful comments, suggestions and discussion on various parts of this paper. I am also grateful for financial support from LSE. All remaining errors are my responsibility.

1 Introduction

This paper examines the joint role of intermediation and money in organizing exchanges in an economy where many agents decide independently, simultaneously and repeatedly on the transactions they want to carry out. I extend the model first studied by Kiyotaki and Wright (1989) and propose a simple form in which agents may forgo production opportunities and rather act as intermediaries.

Traditionally, economic theory views intermediation and the use of media of exchange as different and competing forms of dealing with the frictions that result from decentralized decision-making on production, consumption and exchange. Starting with Starr (1972), the literature on monetary exchange emphasizes the role of money in a decentralized economy, where exchange may be subject to search frictions, i.e. producers cannot immediately sell their production to a "market". In such an economy, money overcomes the problem of double coincidence of wants and reduces the time an agent has to wait until he finds a suitable trade partner. In more recent years, money has been endogenously explained as the result of a cost minimization process, where by agreeing on a common medium of exchange, traders can reduce search costs. In equilibrium, no one has an incentive not to accept money, if all others do.[1]

This approach can be criticized on several grounds. First, the monetary equilibrium is only one of many equilibria in this type of environment. It follows that hardly anything can be said about the characteristics of the resulting medium of exchange. While the Kiyotaki-Wright framework provides a useful framework in which to analyze trading patterns, and thereby, what kinds of goods end up being used as media of exchange, these search economies lack mechanisms which lead to explicit coordination of transactions and the selection of an equilibrium which promotes an efficient medium of exchange. I shall argue informally further below, that many of the qualitative conclusions drawn from the Kiyotaki-Wright framework appear not to be very robust, and under fairly general assumptions, any good could in principle be used as money (independent of inherent qualities).[2]

Furthermore, the search economies hardly replicate empirical observations about money. While they succeed in explaining why it may be socially

[1] see, for example, Kiyotaki and Wright (1989) or Aiyagari and Wallace (1991)

[2] A more formal analysis of this argument will be provided in two companion papers, see Hellwig (both forthcoming).

e¢cient and individually rational that all agents use a common medium of exchange, they cannot account for the fact that the vast majority of transactions involves the exchange of goods for money. One of the conclusions from the literature following Kiyotaki and Wright (1989) is that such a Cash-in-Advance constraint where "goods" are only traded for "money", fails to materialize, since in an environment characterized by search frictions, there is su¢cient incentive to accept "goods" for further exchange, instead of immediate consumption. In a sense, the same search frictions which motivate the use of a medium of exchange render the existence of a Cash-in-Advance constraint impossible.[3]

As a consequence, the existing search literature is in diametrical opposition to Walrasian equilibrium theory. Money has been introduced into Walrasian models through a liquidity or Cash-in-Advance constraint, or more generally by exogenously attributing it with some quality which other goods don't have.[4] In these models, however, money derives value only from this exogenous attribute, and by the very nature of Walrasian equilibrium theory, there is a failure to endogenously account for its use. The conceptual incompatibility of money with general equilibrium theory lies in the structure of frictionless, Walrasian markets, in which there is no clearly de…ned role for bilateral exchanges, which call for the use of money.

A similar critique applies to the role of intermediation, both in pure search and in perfectly frictionless economies: In search economies, there is no explicit role for intermediation as an economic activity, since the possibilities for trade are exogenously given by the search process. Perfect markets on the other hand cannot account for intermediation for the same reason as they cannot account for money: Both are essentially non-Walrasian features which arise through the bilateral nature of exchange. One can view intermediation as an economic activity which by centralizing exchanges reduces frictions and makes markets "look as if" they were Walrasian. But then, why should there be a need for money in a world in which intermediaries can deal with existing market frictions (or vice versa)?

Historically, one observes, in apparent contradiction with the previous

[3]While this result obviously clashes with the observation of Cash-in-Advance constraints in quasi-perfect markets, it may have some intuitive appeal with respect to the importance of barter trade in environments, in which markets are far from frictionless.

[4]See, for example, Lucas and Stokey (1987) for an example of a Cash-in-Advance economy. Hellwig (1993) provides a detailed, critical discussion of the recent and not so recent literature on monetary equilibrium theory, on which some of the ideas in this paper are based.

theoretical argument, that intermediaries have been a driving force in developing more efficient ways of exchanging goods, and particularly in introducing and using money in the economy. For instance, the development of new financial instruments by financial intermediaries during the Renaissance was essential to the promotion of trade in Europe. The unification of monetary standards simplified organized trade between very remote provinces of the Roman Empire. Many such examples throughout history seem to support the view that not only did the use of common monetary instruments promote the development of organized intermediation in goods and financial markets, but these instruments were often developed and introduced by the intermediaries themselves.

This paper gives a rationale for the complementarity of money and intermediation in organizing exchanges. I consider an inherently non-Walrasian economy, where exchange is subject to search frictions. I assume, however, that agents can modify the trading environment by acting as intermediaries. Trade with intermediaries is not subject to search frictions, and delays in trade only depend on the intermediaries' ability to accommodate the transactions demanded by other agents. I show that in this context, intermediaries can introduce a common medium of exchange to all other agents. They, in turn, are willing to use it, if it allows them to buy from the intermediary whatever good they want to consume. On the other hand, intermediaries can more easily respond to their task of centralizing exchanges, if a common medium of exchange is used by the agents with whom they trade.

Intermediaries have been introduced into search models in the past.[5] All these models focus on the exchange of a single good with a given number of buyers and sellers, who all want to make their transaction as quickly as possible, and at the best price they can obtain. In these models, intermediaries offer immediate exchange, and live off a price spread between bid and ask price. This intuition can easily be extended to economies with many commodities. The success of intermediaries then depends on their ability to match buyers and sellers. It is important to note that if intermediaries faced no limits to the extent to which they can perform exchanges, i.e. if they could trade with all goods of the economy at once, there would be no need for a medium of exchange. Consumers could simply trade their excess demand in all goods at once with an intermediary, at the prices set by the latter. If the intermediary fixes market-clearing prices, then no medium of exchange

[5] Rubinstein and Wolinsky (1987) explore this aspect in a search-theoretic model, in which one good is traded between buyers, sellers and intermediaries. Although different in its aims, the present analysis is closer in spirit to Gehrig (1993).

is needed to buy some goods from a di¤erent intermediary. If on the other hand, intermediaries are restricted to exchange only a limited amount of goods, either because some goods are not easily traded through an intermediary, or because there are decreasing returns to the scale of goods with which an intermediary trades, then the excess demand of goods by an individual from an intermediary may not have a value of 0 at market-clearing prices. In this case, a medium of exchange serves consumers to transfer wealth from exchanges with one intermediary to exchanges with another. Intermediaries thus face a double coincidence problem, which they can solve by introducing and promoting a common medium of exchange.[6]

In the search-theoretic environment considered here, the introduction of intermediaries alters the way trade decisions are made by other agents. Trade with intermediaries enables consumers and producers to direct their search towards a particular good, as opposed to the random search in economies without intermediation. By limiting their clients' choices to the use of a unique, common medium of exchange, intermediaries can introduce its use to the entire economy. In an equilibrium of the economy considered here, all agents trade twice to acquire what they want to consume: once to obtain the medium of exchange (sell their production), and once to buy their consumption good. E¤ectively, a Cash-in-Advance constraint for exchange with intermediaries is introduced. In addition, as the medium of exchange enables intermediaries to match buyers and sellers, agents face no waiting time to perform the exchange they want to perform. As a result, the search market empties, since most producers and consumers take advantage of the intermediaries' services. Equilibrium allocations bear the characteristics of Walrasian allocations. The resulting transaction patterns resemble trade in frictionless Walrasian markets: At any time, almost all agents are able to trade in every period, and consume in every other (unless they produce or consume the common medium of exchange, and only need one transaction).

More importantly, and in contrast with pure search economies, intermedi-

[6] From an empirical perspective, Radford's (1945) description of how exchange developed in a Prisoners of War camp is very close to the present model. He describes how economic institutions developed within the completely unorganized environment of a PoW-camp, driven mainly by the scope for trade arising from di¤erences in endowments (Red-Cross packages) and tastes. It is interesting to note that in the early days of the camp, some individuals who exploited the price margins between di¤erent parts of the camp ("intermediaries") promoted and established the cigarette as common money. This was fundamental for the later development of more sophisticated "market" institutions inside the camp, such as a store, and even the introduction of a paper money, backed by the store's inventories of goods.

ation provides a mechanism by which the economy can explicitly coordinate on the common use of an efficient medium of exchange. If an arbitrarily small set of agents coordinates their activities and offers some new organization of transactions, they may induce other agents, and eventually the entire economy, to adopt their innovation. In view of the previous comments, this can be assimilated to the historical role of intermediaries in developing more efficient means of exchange. It is shown here that small coordinated deviations can induce Pareto efficiency. In addition, while innovation by arbitrarily small coordinations does not entirely remove equilibrium indeterminacy in the context of this model, I show that in any Pareto efficient equilibrium, all producers trade twice in order to consume.[7]

The results presented here have several implications for the traditional equilibrium models of monetary exchange. The Cash-in-Advance constraints encountered here differ in important ways from those used in Walrasian models. In this context, Cash-in-Advance constraints result from equilibrium trading strategies, rather than from some exogenous assumption driving the model.

The efficiency of a Cash-in-Advance constraint in the present model responds to some of the critiques on the nature and use of Cash-in-Advance constraints (the reader is referred to Hellwig 1993 for further discussion). Viewing a constraint as efficiency enhancing seems contradictory. In fact, efficiency results from the strategic interaction of intermediaries. It turns out that the Cash-in-Advance constraint enables intermediaries to achieve Pareto-efficiency in the trade process, if they can enforce it on all agents who want to trade with them. The constraint is observed in all intermediated exchange, but is not binding for exchange outside intermediation. Formally, this model does not assume away the possibility that two agents exchange "goods" for "goods" outside intermediated transactions, but if all agents agree to exchange with an intermediary, they almost never incur a situation in which they can exchange "goods" for "goods".

At this point, I would like to relate this work to a series of recent papers, in order to clarify some of the objectives outlined in this introduction. Various recent working papers use a "trading-post" environment in order to analyze

[7] This does not imply that all agents use the most efficient medium of exchange in this context. Due to the specification of production here, small coalitions may not always be able to break out of every equilibrium with inefficient media of exchange. A companion paper (Hellwig, forthcoming) generalizes the approach chosen here and addresses this short-coming, giving conditions under which the most efficient good is the unique medium of exchange in equilibrium.

transaction patterns similar to ours in a monetary setting. Iwai (1988) uses this environment in a search-theoretical setting. In Howitt (2000) and Clower and Howitt (2000), these trading posts represent intermediaries similar to the ones encountered here. In their work, however, monetization results from increasing returns to scale in the transaction process, which leads to concentration on a small number of trading posts. Starr (1999), as well as Starr and Stinchcombe (1999) examine an environment similar to Howitt and Howitt and Clower, but do not formalize the activity of intermediation in the same way. Again, monetization of trade follows from increasing returns, but equilibrium trade patterns are derived differently: Instead of focusing on price competition among intermediaries in a fully dynamic setting, Starr (1999) uses a tatonnement approach similar in spirit to a "Walrasian auctioneer". He abstracts from the dynamics of repeated exchanges.

The present analysis differs from these papers in several ways: Most importantly, I emphasize the role of a double coincidence problem which motivates intermediaries to promote a common medium of exchange. Secondly, this paper provides a fully dynamic analysis of the development of the "trading-post" environment out of a search economy. The "trading posts" generate exchange opportunities only insofar as intermediaries become active - the choice of becoming intermediary is itself endogenized in this model. While much more intricate and complicated as a model, this environment provides a few simplifying insights: In equilibrium, the specialization between production and intermediation takes place in such a way that intermediaries maximize the number of transactions they carry out, and as a consequence, no transactions take place as a result of pure search. Thus, from a much less structured trading environment, we obtain the same transaction patterns as those in the previously mentioned papers. Finally, by using the search-theoretical framework as a background, we can give a strategic account as to how intermediation develops and induces improvements in the transaction process until at some point, transaction patterns and allocations closely resemble Walrasian equilibrium allocations, as previously discussed.

The rest of this paper is organized as follows: Section 2 describes the basic economic environment. I consider a search economy a la Kiyotaki-Wright (1989), in which agents can choose whether they want to produce goods or act as intermediaries. An intermediary is an agent who can offer immediate exchange between a pre-specified pair of goods, one of which is his own consumption good. Intermediaries agree to trade their consumption good against the other good at a price of one for one, but are only willing to pass on an amount less than one unit of their own consumption good for each unit of the other good they receive. The amount of trade intermediaries can carry out is limited by their initial inventory of goods. In section 3, I discuss strategy choices of all agents, and define equilibria for this economy. Section 4 considers one type of equilibrium, in which a particular good is used as a common medium of exchange. I contrast the findings of the economy with intermediation with the monetary equilibria resulting from pure search. In particular, it is shown that all exchange between goods and goods disappears, when a common medium of exchange is used for exchanges with intermediaries. Section 5 introduces an equilibrium refinement in the form of robustness with respect to minimal coalitions, and shows that any robust equilibrium must be Pareto efficient. I then go on to characterize the full set of robust equilibria. I conclude with some remarks on the general validity of the approach chosen in this model as compared with others.

2 The economic environment

I consider an atomless probability space $(\Omega; \mu)$ of infinitely lived agents. There are $N \geq 3$ different goods and N types of agents in the economy. An agent of type i only consumes good i, and never anything else. There is a measure of $\frac{1}{N}$ of each type.

In order to consume, an agent can engage in two different types of activities. He can either act as a consumer-producer or as an intermediary. A consumer-producer always holds one unit of a good, and tries to obtain, after a sequence of one-for-one exchanges, his own consumption good i. He then consumes and immediately thereafter, he is endowed with one unit of good $(i+1) \bmod N$. An intermediary does not produce, but can hold any arbitrarily large inventory of his own consumption good. He uses this inventory to offer one prespecified exchange of his consumption good i for some other good j or vice versa. With every pair of trades, the intermediary increases his inventory of good i by a fraction σ_{ij} that he retains when he sells a unit of good i against one of good j. He can reduce his inventory by consuming. In equilibrium, the proportion of agents engaging in each activity is determined by an open entry condition.

The time-path of this economy is described in figure 1. Time is discrete and infinite. Trade can take place in two ways in this economy. In every period, a consumer-producer can choose to visit an intermediary. Every intermediary is visible for all other agents in the economy, thus there is no search cost involved in finding a suitable intermediary. The intermediary initially holds some inventory k of his own consumption good, cut into units

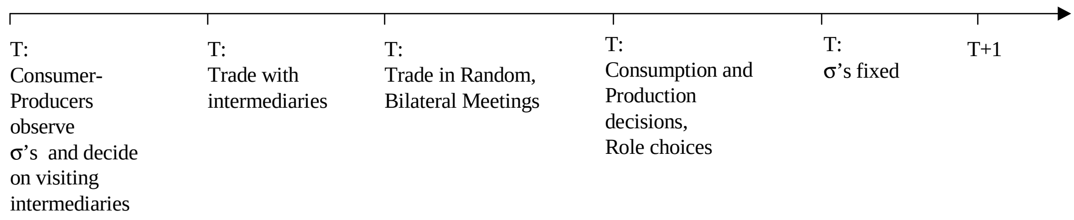

Figure 1: Time-path

of size $1 - \sigma_{ij}$. During the period, he offers these reduced units in exchange against integer units of good j. These units of good j are further exchanged against integer units of good i. The intermediary cannot store good j from one period to the next, and in order to transform the integer units of good i which he receives into reduced units which he can sell, he must store them for at least one period. Thus, the intermediary will make sure he performs the same number of exchanges in each direction, so that he doesn't hold good j at the end of a period. This number of two-way exchanges is bounded by the initial inventory of good i, as well as by the number of agents who want to complete a transaction with him.

If the intermediary's inventory is insufficient, or if there is a difference between the demand for exchanging good i for good j and the demand for the opposite exchange, some agents will be unable to perform the exchange they wished to carry out. In a second stage of the period, after all transactions with intermediaries are completed, all agents, who could not trade, as well as those who decided not to visit an intermediary during this period, are bilaterally matched and thus have a second opportunity for exchange. In such a random match, each agent observes what good the other agent holds. Both then decide whether or not to accept the other agent's good in exchange for one's own. Exchange takes place if both agents agree to it. At this point, I make the usual assumption that agents cannot observe their trade partners' trading histories, nor their types. As the matches are random, the probability of encountering an i-consumer who holds good j (henceforth called ij-agent) in such a meeting is given by the measure of ij-agents who enter the bilateral matching stage.

At the end of each period, all agents decide on consumption, and on their role during the following period. An intermediary can become consumer-producer simply by consuming his entire inventory and producing one unit of good $i+1$. An i-consumer can become an intermediary whenever he holds his own consumption good, simply by forgoing consumption and using his one unit as starting inventory for intermediation. There is no cost of setting up or abandoning intermediation. After consumption and role choices, each intermediary fixes his "bite" σ_{ij} for the following period. All consumer-producers observe these and choose which intermediary they want to visit during the next period.

Preferences of agents are assumed to be symmetric across types. Consumption utility is linear: An intermediary obtains an instantaneous utility cU from consuming c units of his inventory. A consumer-producer obtains utility $U(1-\sigma)$ from consuming a unit of his consumption good of size $1-\sigma$. Consuming any other good yields 0 utility. Time is discounted by a factor δ smaller than but close to 1. Whenever an agent trades, he incurs a direct transaction cost. Consumer-producers incur a cost of β_i, whenever they accept good i in a one-for-one exchange, or in a transaction with an intermediary. Intermediaries incur a cost of $\left(\beta_i+\beta_j\right)\gamma(k)$ from carrying out k two-way exchanges of good i for good j. For simplicity, I let good 1 be the good which has the lowest cost of acceptance. For expositional purposes, I let $\gamma(k)=k+\alpha k(k-1)$. Any convex, increasing transaction cost technology for intermediaries would yield the same results. Essentially, there are diminishing returns to scale in intermediation. In addition, $\gamma(0)=0$ and $\gamma(1)=1$, so that there is no fixed cost in setting up or maintaining intermediation.

Finally, some notation is needed: I denote by μ_{ij} the measure of ij-agents and by ν_{ij} the measure of ij-intermediaries in the economy. The $2N(N-1)$-dimensional vector of all μ_{ij} and all ν_{ij} is the distribution of inventories and role choices. The inventory of a generic ij-intermediary is denoted by k_{ij}, the "bite" by which i is reduced is denoted by σ_{ij}.

The main innovation here with respect to the original search-theoretical model of Kiyotaki and Wright (1989) is the formal introduction of intermediation. Like their framework, this paper aims to analyze transaction patterns and the emergence of media of exchange within a search economy. This implies considering an environment, in which goods are durable, and no commodity is predestined by its storability qualities to become a medium of exchange. In other words, we do not want to assume existence of a durable fiat money which dominates perishable goods for transaction purposes (as in Howitt 2000, for instance). Under this assumption, even for fairly special cases like the one considered here, characterization of equilibria becomes analytically very difficult, if not impossible. Second, the potential coexistence of intermediation with random search creates further notational and technical difficulties. Again, this choice follows from the motivations set out initially. It turns out that the way intermediation is introduced here leads to some surprising simplifications for the analysis of equilibrium strategies. Bertrand competition and open entry to (and exit from) intermediation ensure that in the equilibria on which our analysis focuses (for reasons explained later in section 5), search trade becomes inexistent so that all transactions will go through intermediaries. Overall, these insights lead to considerable technical simplifications and enable us to characterize equilibria simply in terms of their transaction patterns for intermediated exchange. Also, one can easily compare their welfare properties, and apply robustness analysis with respect to small coalitions, as is being done in part 5.

3 Strategy choice and equilibrium

In this section, I derive optimal strategies of agents and define the notion of stationary equilibrium for this economy. Given that the set-up is geared toward steady-state analysis, I consider the choice of inventories by intermediaries and the choice of trading strategies by consumer-producers separately, and as stationary over time. In equilibrium, an indifference condition between both roles must hold, such that no agent has an incentive to change his activity. I start by analyzing the behavior of intermediaries, taking as given the distribution of inventories and role choices and the trading strategies of consumer-producers.

3.1 Inventories of Intermediaries

Before the beginning of any period, an intermediary must choose his consumption, or equivalently the inventory he wishes to maintain, as well as the "bite" σ which he charges. For the moment, I assume that there exists σ_{ij}, such that the intermediary can perform any arbitrary large number of two-way exchanges, provided he charges $\sigma \leq \sigma_{ij}$, where as he will be unable to perform any two-way exchanges, if he charges $\sigma > \sigma_{ij}$. Under these conditions, the intermediary will always choose $\sigma = \sigma_{ij}$.

Let $W_{ij}(k_t)$ be the supremum of life-time utility for an ij-intermediary with an inventory of k_t units of good i at time t. According to its definition, $W(k_t)$ satisfies:

$$W_{ij}(k_t) = \sup_{\{c_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \delta^{s-t} \left(U c_s - \gamma \left(\overline{k_{s+1}} \right) \left(\beta_i + \beta_j \right) \delta \right) \quad (1)$$

subject to the constraints

$$k_{s+1} = k_s - c_s + \sigma_{ij}\overline{k_{s+1}} \tag{2}$$

$$\overline{k_{s+1}} = \left[\frac{k_s - c_s}{1 - \sigma_{ij}}\right] \tag{3}$$

for $s = t; t+1; \ldots$. $[a]$ denotes the integer part of a real valued number a. The per-period reward function in (1) is motivated as follows: Uc_s gives the utility of consuming c_s at time s. At time $s+1$, the intermediary can perform exactly $\overline{k_{s+1}}$ two-way trades, and he incurs a total transaction cost of $\gamma\left(\overline{k_{s+1}}\right)\left(\beta_i + \beta_j\right)$. (2) gives the evolution of inventories from one period to the next, while (3) gives the number of two-way trades during the following period. The discussion of this optimization problem is summarized in proposition 1 and in the subsequent lemma, the derivation and proof of which can be found in the appendix.

Proposition 1:

(i) For any initial inventory of k_t, there exists an optimal inventory- and consumption plan $\{k_s^*\}_{s=t}^{\infty}$.

(ii) There exists an integer κ_{ij}, such that:

(a) whenever $k_s^* \geq \kappa_{ij}$, $k_{s+1}^* = \kappa_{ij}$

(b) for any inventory $k_s^* < \kappa_{ij}$, an optimal inventory plan implies

$$k_{s+1}^* \geq \min\left\{\kappa_{ij}; \left[\frac{k_s^*}{1 - \sigma_{ij}}\right]\right\}$$

(iii) κ_{ij} is determined by

$$2\alpha\kappa_{ij}\left(\beta_i + \beta_j\right)\delta \geq (\sigma_{ij} + \delta - 1)U - \left(\beta_i + \beta_j\right)\delta \geq 2\alpha(\kappa_{ij} - 1)\left(\beta_i + \beta_j\right)\delta \tag{4}$$

Proposition 1 contains the main information about optimal inventory choice by intermediaries: There exists an optimal number of trades κ_{ij}, at which the benefit of one additional two-way exchange is outweighed by the increase in transaction costs. This optimal inventory increases, as σ_{ij} increases, and it goes to infinity, as α goes to 0, i.e. decreasing returns disappear. Whenever an intermediary's inventory exceeds this optimal number of trades, it is optimal to immediately consume any inventory that is not needed. Whenever an intermediary's inventory is insufficient to perform the optimal number of two-way trades, the number of trades performed does not decrease from one period to the next, and increases strictly, if the inventory is sufficient to accumulate one additional unit within one period. Lemma 1 considers the case where the gains from one period of trades are not sufficient to increase inventory by one additional unit.

Lemma 1: Suppose an intermediary holds k units of inventory, and let $T \geq 2$ be the minimum number of periods needed to increase inventory by one unit. Let $\Delta(k) = (\sigma_{ij} + \delta - 1)U - \left(\beta_i + \beta_j\right)\delta(\gamma(k+1) - \gamma(k))$ be the per period increase in utility obtained from holding $k+1$ units of inventory rather than k. If

$$\Delta(k)\frac{\delta^{T-1}}{1-\delta} + Uk\sigma_{ij}\left(\delta^{T-1}T - \frac{1-\delta^{T}}{1-\delta}\right) \geq 0$$

then it is optimal to increase inventory from k to $k+1$.

Lemma 1 states that even if integer increases in inventories cannot be realized from one period to the next, intermediaries are happy to increase their inventories, as long as the long-term gain in utility outweighs the cost of forgoing short-term consumption. The second term in the expression is negative, but converges to 0, as $\delta \to 1$. Thus, in the no-discounting case, the intermediary's inventory will always be increased until the optimal level is reached. Unless the marginal benefit of the final two-way exchange is negligeable, the unique inventory level compatible with the notion of stationary equilibrium is κ_{ij} (determined in equilibrium by (4)). The indeterminacy that might occur if the last two-way exchange yields low additional utility is due to the non-convexity arising from the fact that the number of transactions comes as an integer. For simplicity, I will therefore assume that an intermediary always builds up his inventory to the optimal number of transactions κ_{ij}. This will be satisfied in most cases of interest here, particularly, when decreasing returns to scale set in slowly, and each intermediary is willing to serve a large number of consumer-producers.

3.2 Trading strategies of consumer-producers

I now turn to the description of optimal trading strategies by consumer-producers. A trading strategy for an ij-agent consists of two decision rules, one that relates the current inventory to the choice of visiting an intermediary, and one that indicates acceptance probabilities of exchanges in a bilateral meeting. An ij-agent can choose to visit any jl-intermediary in the hope of exchanging good j for an integer good l, or he can choose to visit an ij-intermediary and receive a reduced unit of his own consumption good. This completes the set of feasible trading strategies for intermediated exchange, as an ij-agent would not be willing to receive any reduced good l other than his own consumption good. Thus, for any inventory, the set of feasible strategies for intermediated exchange is fully described by the set $\{\emptyset, 0, 1, 2, ..., N\}$, where $\emptyset$ is associated with the strategy of not visiting an intermediary, 0 is associated with the choice of visiting an ij-intermediary, and any $l = 1, ..., N$ is assigned to visiting a jl-intermediary. Formally, a trading strategy for intermediated exchange is a function $\phi_i : \{1, 2, ..., N\} \times \{0, 1, 2, ..., N\} \to [0, 1]$, where $\phi_i(j, l)$ indicates the probability assigned to strategy l (visiting a jl-intermediary, or visiting an ij-intermediary, if $l = 0$) by an ij-intermediary. ϕ_i must satisfy $\sum_{l=0}^{N} \phi_i(j, l) \leq 1$, and the residual probability $1 - \sum_{k=0}^{N} \phi_i(j, k)$ is assigned to the strategy of not visiting an intermediary.

Similarly, trading rules for bilateral meetings are described by a function $\tau_i : \{1, 2, ..., N\} \times \{1, 2, ..., N\} \to [0, 1]$, where $\tau_i(j, k)$ indicates the probability that an ij-agent accepts good k for good j. When an ij-agent meets an lk-agent, trade occurs with probability $\tau_i(j, k)\tau_l(k, j)$. To complete the notation, $\pi_{ij}(k)$ denotes the probability that a consumer-producer receives good $k \in \{i, j\}$ when he visits an ij-intermediary. In a steady-state equilibrium, they are determined by the consumer-producers' trading strategies $\{\phi_i\}_{i=1}^{N}$ for trade with intermediaries, and by the distribution of inventories and role choices, but I shall take them as given here.

Taking as given the "bites" σ_{ij}, the trading probabilities $\pi_{ij}(k)$, trading strategies τ_i, and the distribution of inventories and role choices $\left\{\mu_{ij}, \nu_{ij}\right\}_{i,j=1}^{N}$, one can now derive an optimal trading strategy for a generic ij-agent as a maximizer for a set of Bellman equations describing the choice of trade strategies. Let $V_i(j)$ be the expected life-time utility of an ij-agent. $V_i(j)$ satisfies:[8]

[8] One observes that this set of Bellman equations, as much of the preceeding section, restricts attention to stationary trade strategy profiles. A routine argument shows, however, that non-stationary strategies are weakly dominated in a steady-state equilibrium with stationary strategies. In addition, standard results imply that under stationarity, the solution to this set of Bellman equations is equivalent to the corresponding sequential optimisation problem.

$$
\begin{aligned}
(1-\delta)V_i(j) = \; & \delta \max_{\phi_i;\tau_i} \Bigg\{ \sum_{l=1}^{N} (V_i(l) - \beta_l - V_i(j))\, \phi_i(j;l)\, \pi_{jl}(l) \\
& + (V_i(i) - \sigma_{ij} U - \beta_i - V_i(j))\, \phi_i(j;0)\, \pi_{ij}(i) \\
& + \left(1 - \sum_{l=1}^{N} \phi_i(j;l)\, \pi_{jl}(l) - \phi_i(j;0)\, \pi_{ij}(i)\right) \\
& \quad \sum_{k;l} \frac{\mu'_{kl}\, (V_i(l) - \beta_l - V_i(j))\, \tau_i(j;l)\, \tau_k(l;j)}{\sum_{k;l} \mu'_{kl}} \Bigg\}
\end{aligned} \tag{5}
$$

$$
V_i(i) = U + V_i((i+1) \bmod N)
$$

and

$$
\mu'_{ij} = \mu_{ij} \left(1 - \sum_{l=1}^{N} \phi_i(j;l)\, \pi_{jl}(l) - \phi_i(j;0)\, \pi_{ij}(i)\right)
$$

μ'_{ij} denotes the measure of ij-agents who did not visit an intermediary, and as a result enter a bilateral match. Before discussing the implications of (5) in more detail, it is worth noting that one can abstract from consumption choices: once an agent holds his consumption good, he can consume immediately and hold on to his production good forever thereafter, which yields a life-time utility of U. Trading good i against some other good is, obviously, strictly dominated, so that any agent who receives his own consumption good will consume immediately.

Several results follow from (5). One observes that for every consumer-producer, not visiting an intermediary is a weakly dominated strategy. The analysis of optimal strategies for trade with intermediaries and bilateral trade can be separated. The following proposition summarizes the findings and gives simple rules which optimal trading strategies have to satisfy:

Proposition 2: If $(\tau_i; \phi_i)$ is an optimal trade strategy for a consumer-producer of type i, then the following must be true:

(i) If $\phi_i(j;k) > 0$, then $k \in \arg\max_l (V_i(l) - \beta_l - V_i(j))\, \pi_{jl}(l)$, and

$$
\max_l (V_i(l) - \beta_l - V_i(j))\, \pi_{jl}(l) \geq (V_i(i) - \sigma_{ij} U - \beta_i - V_i(j))\, \pi_{ij}(i)
$$

If k is a unique maximizer, then $\chi_i(j;k) = 1$

(ii) If $\chi_i(j;0) > 0$, then

$$(V_i(i) - \sigma_{ij}U - \beta_i - V_i(j))\,\pi_{ij}(i) \geq \max_l (V_i(l) - \beta_l - V_i(j))\,\pi_{jl}(l)$$

where $\chi_i(j;0) = 1$, if the inequality is strict

(iii) If $\tau_i^*(j;k) > 0$, then $V_i(k) - \beta_k - V_i(j) \geq 0$;
and $V_i(k) - \beta_k - V_i(j) > 0$ implies $\tau_i^*(j;k) = 1$

While evident in its content, proposition 2 highlights the main difference between trade with intermediaries and random bilateral trade: Strategies for the latter amount to simple decision rules indicating whether one good is accepted in exchange for another, and agents might be willing to accept more than one good in exchange for their current inventory. As a result, trading patterns remain indeterminate, as there may be many possible sequences of exchanges which lead a consumer-producer from his current inventory to his consumption good. Trading with an intermediary enables him to follow a different strategy and direct himself towards the one trade where the expected surplus is maximized. The consumer-producer can follow a predetermined sequence of intermediated exchanges in order to eventually receive his consumption good, and if more than one such sequence occurs, this implies that they yield the same expected value. This is very similar to the deterministic trading zones in Iwai (1988), where agents need to visit an "ij-island" in order to trade good i for good j. However, there is a key difference: in Iwai (1988), frictions in trade depend on an exogenously given search externality, whereas here, trade frictions are endogenized. Any delay in trade results from the inability of intermediaries to accommodate all the exchanges demanded by consumer-producers.

3.3 Stationarity and Role choices

To complete the description of how this economy behaves, one needs to consider how the distribution of inventories and role choices evolves from one period to the next, and how competition among intermediaries determines the equilibrium measures of intermediation. I will first look at competition among intermediaries.

Given the choice of σ's by all other intermediaries, there exists a level σ_{ij} at which either ij-agents are indifferent between visiting an ij-intermediary and their best outside option, or the inventories of all intermediaries charging $\sigma \leq \sigma_{ij}$ is exactly sufficient to satisfy all possible two-way exchanges. In either case, the intermediary will not perform any trades when he charges $\sigma > \sigma_{ij}$, but is not constrained in his number of exchanges if he charges $\sigma \leq \sigma_{ij}$. As all intermediaries of the same type face this same constraint, they will all charge σ_{ij} in equilibrium. In addition, the possibility of undercutting implies that in equilibrium, σ_{ij} and κ_{ij} together satisfy condition (4): If at a given σ_{ij}, the optimal level of exchanges exceeds the number of possible two-way exchanges, every intermediary has an incentive to undercut, and Bertrand competition reduces σ_{ij} and κ_{ij} until every intermediary can carry out the optimal number of transactions. On the other hand, if inventories are too small to perform all possible two-way exchanges, intermediaries can increase their σ_{ij} and at the same time their inventory without loosing transactions, until either σ_{ij} reaches the threat point of consumer-producers, or all two-way exchanges can be accommodated. For further analysis, mainly the second possibility is of interest. As a summary, Bertrand competition implies

$$\nu_{ij}\kappa_{ij} \leq \min\left\{ \mu_{ij}\phi_i(j;0);\ \sum_l \mu_{li}\phi_l(i;j) \right\} \tag{6}$$

with equality, if

$$(V_i(i) - \sigma_{ij}U - \beta_i - V_i(j))\,\pi_{ij}(i) > \max_l (V_i(l) - \beta_l - V_i(j))\,\pi_{jl}(l)$$

In addition, an indifference condition between role choices is required. This implies that

$$W_{ij}(\kappa_{ij}) \geq \kappa_{ij}U + V_i([i+1] \bmod N) \tag{7}$$

whenever $\nu_{ij} > 0$, i.e. whenever ij-intermediaries are active. Also,

$$U + V_i([i+1] \bmod N) \geq \max_{j \neq i} W_{ij}(1) \tag{8}$$

must hold. (7) and (8) simply say that no intermediary should have an incentive to become a consumer-producer and no consumer-producer should want to become an intermediary. As the discount rate tends to 1, it can be shown that the two inequalities converge, and in the limit, they can be rewritten as

$$\widetilde{W}_{ij}(\kappa_{ij}) = \widehat{V}_i([i+1] \bmod N)$$

where $\widehat{W}_{ij}(\kappa_{ij})$ and $\widehat{V}_i([i+1] \bmod N)$ represent the long-run average values of intermediaries and consumer-producers, respectively.[9]

In addition, in a steady state, the distribution of inventories and role choices remains constant over time. Taking $\{\gamma_{ij}, \phi_i, \kappa_{ij}, \sigma_{ij}, \tau_i\}_{i,j=1}^N$ as given, $\left\{\mu_{ij}\right\}_{i,j=1}^N$ must satisfy:

$$\mu_{ij} = \mu'_{ij}\left(1 - \frac{1}{\sum_{k,l}\mu'_{kl}} \sum_{k,l} \mu'_{kl}\tau_i(j,l)\tau_k(l,j)\right) + \sum_{l=1}^{N} \phi_i(l,j)\pi_{lj}(j)\mu_{il} \tag{9}$$

$$+ \frac{1}{\sum_{k,l}\mu'_{kl}} \sum_{l} \mu'_{il} \sum_{k} \mu'_{kj}\tau_i(l,j)\tau_k(j,l)$$

whenever $j \neq i, (i+1) \bmod N$, and

$$\mu_{i,i+1} = \mu'_{i,i+1}\left(1 - \frac{1}{\sum_{k,l}\mu'_{kl}} \sum_{k,l} \mu'_{kl}\tau_i(j,l)\tau_k(l,j)\right) \tag{10}$$

$$+ \sum_{l=1}^{N} \left(\phi_i(l,i)\pi_{li}(i) + \phi_i(l,0)\pi_{il}(i)\right)\mu_{il} + \frac{1}{\sum_{k,l}\mu'_{kl}} \sum_{l} \mu'_{il} \sum_{k} \mu'_{ki}\tau_i(l,i)\tau_k(i,l)$$

Here, an i-agent's production good has to be treated separately from all other goods he may hold as an inventory. Condition (9) can be explained as follows: μ'_{ij} is the set of ij-agents who are unsuccessful in trading with an intermediary. A fraction $1 - \frac{1}{\sum_{k,l}\mu'_{kl}} \sum_{k,l} \mu'_{kl}\tau_i(j,l)\tau_k(l,j)$ of these is unsuccessful in bilateral exchange as well. $\sum_{l=1}^{N} \phi_i(l,j)\pi_{lj}(j)\mu_{il}$ is the measure of i-agents who acquire good j from an intermediary, and a measure of $\frac{1}{\sum_{k,l}\mu'_{kl}} \sum_{l} \mu'_{il} \sum_{k} \mu'_{kj}\tau_i(l,j)\tau_k(j,l)$ acquires good j through a bilateral

[9] For simplicity, I will later use the no-discounting version of this condition.

match. Similarly, $\mu_{i,i+1}$ can be decomposed into those agents who were unsuccessful in trading, and those who were able to consume after visiting an intermediary, or after a successful bilateral meeting. Since holding one's own consumption good stands at the beginning of any sequence of trades, no agent will trade in his inventory for good $i+1$.

Trading probabilities for trade with intermediaries can be derived from the distribution of inventories and role choices as

$$\pi_{ij}(i) = \frac{\min\left\{\nu_{ij}\kappa_{ij}, \mu_{ij}\phi_i(j,0), \sum_l \mu_{li}\phi_l(i,j)\right\}}{\mu_{ij}\phi_i(j,0)}$$

$$\text{and } \pi_{ij}(j) = \frac{\min\left\{\nu_{ij}\kappa_{ij}, \mu_{ij}\phi_i(j,0), \sum_l \mu_{li}\phi_l(i,j)\right\}}{\sum_l \mu_{li}\phi_l(i,j)}$$

respectively, simply the maximum possible measure of two-way transactions divided by the measure of agents wishing to perform the same transaction. Since $\nu_{ij}\kappa_{ij} \leq \min\left\{\mu_{ij}\phi_i(j,0), \sum_l \mu_{li}\phi_l(i,j)\right\}$, this is reduced to $\pi_{ij}(i) = \frac{\nu_{ij}\kappa_{ij}}{\mu_{ij}\phi_i(j,0)}$ and $\pi_{ij}(j) = \frac{\nu_{ij}\kappa_{ij}}{\sum_l \mu_{li}\phi_l(i,j)}$. If the total inventory of intermediaries $\nu_{ij}\kappa_{ij}$ is exactly sufficient to carry out the maximum number of possible two-way transactions, then these probabilities are reduced to

$$\pi_{ij}(i) = \min\left\{1, \frac{\sum_l \mu_{li}\phi_l(i,j)}{\mu_{ij}\phi_i(j,0)}\right\} \text{ and } \pi_{ij}(j) = \min\left\{1, \frac{\mu_{ij}\phi_i(j,0)}{\sum_l \mu_{li}\phi_l(i,j)}\right\}$$

.

3.4 Steady state equilibrium

Building on the previous sections, a stationary equilibrium of this economy is defined as follows:

Definition: A stationary equilibrium consists of

$$\left\{\mu_{ij}, \nu_{ij}, \phi_i, \kappa_{ij}, \sigma_{ij}, \tau_i\right\}_{i,j=1}^{N}$$

satisfying, for all i, j:

(i) Optimality and Stationarity of intermediaries' inventories κ_{ij} and σ_{ij}: Condition (4) must hold.

(ii) Optimality of ϕ_i and τ_i: ϕ_i and τ_i maximize the set of Bellman equations (5), taking as given $\{\mu_{ij}, \nu_{ij}, \phi_{-i}, \kappa_{ij}, \sigma_{ij}, \tau_{-i}\}_{i,j=1}^{N}$

(iii) Bertrand competition of intermediaries: Condition (6) must hold

(iv) Optimality of role choice: Conditions (7) and (8) must hold.

(v) Stationarity of $\{\mu_{ij}\}_{i,j=1}^{N}$: Conditions (9) and (10) must hold.

The existence of an equilibrium can be proved by a fixed point argument along the lines of Aiyagari and Wallace (1991). In the next section, I will derive and characterize some types of equilibria. The following proposition states some useful characteristics which any stationary equilibrium satisfies:

Proposition 3: (i) In a stationary equilibrium, the measure of consumer-producers who consume in any given period is constant across types and is given by

$$\sum_{l=1}^{N} \left(\phi_i(l,i)\,\pi_{li}(i) + \phi_i(l,0)\,\pi_{il}(i)\right)\mu_{il} + \frac{1}{\sum_{k,l}\mu'_{kl}} \sum_{l} \mu'_{il} \sum_{k} \mu'_{ki}\,\tau_i(l,i)\,\tau_k(i,l)$$

(ii) as $\alpha \to 0$: $\kappa_{ij} \to \infty$, $\nu_{ij} \to 0$, and $\sigma_{ij} \to 1 - \delta + \frac{(\beta_i+\beta_j)\delta}{U}$.

(i) follows from the definition of stationarity: The measure of consumption of good i must equal the measure of production of good i. (ii) is less immediate, but follows from conditions (4) and (6), as well as from the indifference of role choices. As the decreasing returns disappear, the optimal inventory becomes infinite, competition lets the sets of intermediaries tend to 0, and the "bite" charged tends to a minimum sustainable level.

4 Commodity Money

In this section, I discuss the development of media of exchange as an equilibrium property of the economy outlined above. The concept of money referred to in this context is commodity money, i.e. a good which is used by all consumer-producers for indirect exchange. It can be noted that introducing intermediation in this way widens the possible set of equilibria of this economy from the one originally studied in Kiyotaki and Wright. If no agent acts as an intermediary, it is weakly optimal for consumer-producers not to visit an intermediary. But then, no agent has an incentive for becoming intermediary and trade will only take place in bilateral meetings. Thus, any steady-state equilibrium of the original Kiyotaki-Wright economy can be supported as an equilibrium of this economy with intermediation, setting $\nu_{ij} = \kappa_{ij} = \sigma_{ij} = 0$ and $\phi_i(j;k) = 0$, for all $i;j;k$. This reduces the equilibrium definition to the distribution of inventories and to the search strategy profile $\left\{\mu_{ij};\tau_i\right\}_{i;j=1}^N$.

As the properties of monetary equilibria in the pure search economy have been studied extensively by Kiyotaki and Wright (1989) and Aiyagari and Wallace (1991), their main results will only briefly be reviewed here. The purpose of this section is to contrast the definitions and equilibrium characteristics of commodity money of the intermediated economy with those resulting from pure search. In a similar way, I will compare implications for Cash-in-Advance constraints in the two cases. A commodity money or a Cash-in-Advance constraint is defined as a property of an equilibrium strategy profile, and as such they must satisfy the optimality condition. For the non-intermediated economy, a commodity money is a good that is always accepted by all consumer-producers of the economy. A strategy profile entails a Cash-in-Advance constraint, if in every exchange, the commodity money is exchanged against some other good.

Definition: An equilibrium $\left\{\mu_{ij};\tau_i\right\}_{i;j=1}^N$ has good m as a universally accepted medium of exchange, if, whenever $\mu_{ij} > 0$, $\tau_i(j;m) = 1$. An equilibrium $\left\{\mu_{ij};\tau_i\right\}_{i;j=1}^N$ results in a "Cash-in-Advance" constraint for some good m, if m is a universally accepted medium of exchange, and if, for all $i;j;l;k$: $\mu_{ij}\mu_{kl}\tau_i^*(j;l)\tau_k^*(l;j) = 0$ whenever $j \neq m$ or $l \neq m$.

As opposed to the storage cost economy studied in the afore mentioned papers, no general conclusions can be drawn about the existence of equilibria with commodity money in the transaction cost economy. Here, an equilibrium without intermediaries, with a medium of exchange may fail to exist: for this purpose, it suffices to note that if there is a good i, which is never accepted for the purpose of indirect exchange, only i-producers will hold it, and an i-consumer will be indifferent between holding the medium of exchange and holding $i-1$, the good consumed by i-producers. But then, i-consumers will not accept the medium of exchange in exchange for good $i-1$ (a simple 3-good economy as studied by Kiyotaki and Wright provides an example for this case). On the other hand, if every good is produced at least by two types of consumers, then there is always a benefit from accepting the common medium of exchange, and, in principle, any good could be supported as a universally accepted medium of exchange.[10]

Along similar lines, one can show that an equilibrium with a Cash-in-Advance constraint cannot exist for this economy, unless direct transaction costs are sufficiently high to prevent agents from any indirect trade of "goods" for "goods": A Cash-in-Advance constraint implies that any agent must first acquire the medium of exchange, before he can acquire his own consumption good. There will be an advantage in terms of expected waiting time for holding one good rather than the other, giving agents an incentive to trade "goods" for "goods", which only high transaction costs may prevent. The Cash-in-Advance constraint breaks down because the existing search frictions give consumer-producers an incentive to trade "goods" for "goods" in an attempt to reduce expected waiting time.

I want to contrast these results with monetary and Cash-in-Advance equilibria of the intermediated economy. It is straight-forward to consider equilibria of the following type: For a medium of exchange m, there exist $N - 1$ sets of im-intermediaries exchanging good m each against their consumption good i. Consumer-producers always decide to obtain good m first, before they exchange m against their consumption good. Intermediated exchange in this case leads to the use of a generalized medium of exchange, and results in a Cash-in-Advance constraint for good m: In order to obtain their consumption good through an intermediary, consumer-producers must first obtain the medium of exchange.

Definition: $\{\phi_i\}_{i=1}^N$ exhibits a Cash-in-Advance constraint for some good m, iff $\phi_i(j; m) = 1$, and $\phi_i(m; 0) = 1$, for all $i; j$.

Proposition 4: If transaction costs are small enough, then for any good m, there exists a stationary equilibrium $\langle \mu_{ij}; \nu_{ij}; \phi_i; \eta_{ij}; \sigma_{ij}; \tau_i \rangle_{i;j=1}^N$ in which

[10] A formal justification of this statement can be found by analyzing the 3-good economy studied in Kiyotaki and Wright. In their "speculative equilibrium", the good that is least costly to store circulates as common medium of exchange only because it reduces storage costs for some agents, not because it has a higher liquidity for exchanges. The storage cost effect falls away in this environment, and can be replicated, only if using the medium of exchange reduces overall discounted transaction costs - a condition which isn't satisfied a priori. On the other hand, in an environment where every good is produced by more than one type, accepting a common medium of exchange always increases the probability of further trade. If storage or transaction costs are sufficiently small, this effect dominates, and in principle, any good could thus be supported as a common medium of exchange.

$\{\phi_i\}_{i=1}^N$ exhibits a Cash-in-Advance constraint for m.

In this equilibrium, the medium of exchange is the result of the intermediaries' strategies: Their coordination favors one good for the use in indirect exchange. Intermediaries can deliver this good much quicker than the search market. If transaction costs are small enough, Bertrand competition among intermediaries guarantees that the benefits of intermediation exceed its costs, so that consumer-producers have no incentive to deviate from the proposed trading sequence. The next result discusses how the commodity money for exchange with intermediaries affects bilateral trade:

Proposition 5: (i) The measure of bilateral trades involving the commodity money is 0.

(ii) If the probabilities of trading with an intermediary are sufficiently close to 1 for an agent holding his production good, he will never accept to trade in a bilateral meeting.

(i) is motivated as follows: If an agent holding the medium of exchange meets some other agent in a random bilateral meeting, he will only accept to exchange for his own consumption good, say i. But any agent holding good i would have visited an im-intermediary as well (i.e. the same intermediary as the im-agent would have visited). Since consumer-producers strictly prefer the equilibrium trading sequence to any other possible trading sequence, it must be that the im-intermediaries had sufficient inventory so that with probability 1, either the im-agent or the i-holding agent could have traded with the intermediary. It follows that they cannot meet in a bilateral random match.

(ii) follows from the fact that, if $\pi_{jm}(m)$ is close to 1, $V_i(j)$ is close to $(V_i(m) - \beta_m)\frac{\delta\pi_{jm}(m)}{1-\delta+\delta\pi_{jm}(m)}$, and the advantage of holding one good over another for exchange with intermediaries disappears. As soon as the cost of the additional transaction outweighs this gain, no good other than the medium of exchange will be accepted for indirect trade, and trade in bilateral meetings disappears.

Summing up, the medium of exchange resulting in this type of equilibrium is used only in transactions with intermediaries, and it is the result of strategic interaction of the latter. Outside intermediated exchange, only "goods" for "goods" exchange occurs, but this exchange disappears when intermediation is sufficiently efficient in exchanging the production good for the medium of exchange. In this case, the Cash-in-Advance constraint holds universally for all exchanges in the economy. From proposition 3(ii), we know that as α goes to 0, all measures of intermediaries converge to 0. Proposition 5 raises the question, whether, as α goes to 0, it is possible that an equilibrium exhibits approximate "market-clearing", in the sense that the probabilities of succeeding to trade with an intermediary converge to 1 for all exchanges? This would imply that all agents are always able to carry out their most preferred intermediated exchange.

Proposition 6: As $\alpha \to 0$, no Cash-in-Advance equilibrium exhibits approximate market-clearing.

This result follows directly from proposition 3(i): The measure of consumer-producers consuming at any given time must be constant across types and is bounded above by $\frac{1}{2}\left(\frac{1}{N} - \max_j \gamma_{jm}\right) < \frac{1}{2N}$. In an equilibrium with a Cash-in-Advance constraint, there exists no set of intermediaries consuming the medium of exchange m. In addition, all m-agents only pursue one exchange. In contradiction, market-clearing would imply that the measure of m-consumer-producers consuming at any given time converges to $\frac{1}{N}$.

The impossibility of complete "market-clearing" is the result of a disequilibrium in trade sequences: In an equilibrium with a Cash-in-Advance constraint, the producers of the commodity money and the consumers of it only trade once in order to consume, while all other agents trade at least twice between production and consumption. Thus, for goods m and $m + 1$, the commodity money equilibrium creates an imbalance in the underlying equality of demand and supply which has been assumed for this model if markets were Walrasian. Obviously, this result would not be robust, if the specification of pricing were altered in such a way that intermediaries could change prices to equate the aggregate quantities demanded and supplied for each transaction. Nevertheless, proposition 6 provides an important insight: the liquidity demand for the medium of exchange distorts such market-clearing prices away from underlying Walrasian prices.

While complete market-clearing is impossible in equilibrium, further characterization of Cash-in-Advance equilibria with intermediaries shows that as α goes to 0, the probabilities of trading with an intermediary converge to 1 for all jm- and mj-exchanges, except where $m + 1$ is exchanged against m (m-agents selling their production good), and where m is exchanged against $m - 1$ ($m - 1$ agents selling their production good). In other words, in all markets, where the commodity money does not perturb the underlying equality between supply and demand, approximate market-clearing occurs.

For the full characterization of the Cash-in-Advance equilibrium with intermediaries, it is useful to assume first that no trade will take place in bilateral meetings. As previously argued, this will be satisfied in equilibrium, if trading probabilities go to 1. Taking an arbitrary good m as the medium of exchange, and taking the measures of intermediaries γ_{jm} (sufficiently small) as given for all $j \neq m$, and $\gamma_{lm} = \max_j \gamma_{jm}$, one can conjecture the following about the probabilities of trade with intermediaries:

Type: i=	*m*	*m+1*	…	*l-1*	*l*	*l+1*	…	*m-2*	*m-1*
$\pi_{i+1,m}(m)$	*<1*	*<1*	…	*<1*	*1*	*1*	…	*1*	
$\pi_{im}(i)$		*1*	…	*1*	*1*	*<1*	…	*<1*	*<1*

Figure 2: Transaction probabilities

One first conjectures $\pi_{m+1;m}(m) < 1$ and $\pi_{m-1;m}(m) < 1$. This implies $\pi_{m+1;m}(m+1) = 1$ and $\pi_{m-1;m}(m-1) < 1$. Now, by proposition 3, either $\pi_{i+1;m}(m) < 1$ or $\pi_{im}(i) < 1$ for any i, such that $\gamma_{im} < \gamma_{lm}$. Given the structure of the table so far, at least for one type of agents i, one must have $\pi_{i+1;m}(m) = \pi_{im}(i) = 1$, and by proposition 3, this can only be at l.[11] By consequence, for $i = m; m+1; ...l-1$, $\pi_{i+1;m}(m) < 1$ and for $i = l+1; ...; (m+N-1) \bmod N$, $\pi_{im}(i) < 1$.

Using this conjecture, one can easily find a stationary inventory distribution. Stationarity for type l implies that $\mu_{l;l+1} = \mu_{lm} = \frac{1}{2}\left(\frac{1}{N} - \gamma_{lm}\right)$, and it follows again from proposition 3 that for $i = m+1; ...l-1$, $\mu_{im} = \frac{1}{2}\left(\frac{1}{N} - \gamma_{lm}\right)$, whereas for $i = l+1; ...; (m+N-1) \bmod N$, $\mu_{i;i+1} = \frac{1}{2}\left(\frac{1}{N} - \gamma_{lm}\right)$. It follows that for $i = m+1; ...l-1$, $\mu_{i;i+1} = \frac{1}{N} - \gamma_{im} - \frac{1}{2}\left(\frac{1}{N} - \gamma_{lm}\right)$ and for $i = l+1; ...; (m+N-1) \bmod N$, $\mu_{im} = \frac{1}{N} - \gamma_{im} - \frac{1}{2}\left(\frac{1}{N} - \gamma_{lm}\right)$. Finally, one has $\mu_{m;m+1} = \frac{1}{N}$ and $\mu_{m-1;m} = \frac{1}{N} - \gamma_{m-1;m}$. This completely describes the stationary inventory distribution.

To complete the characterization of the Cash-in-Advance equilibrium, one must consider the indifference condition between role choices (9), the

[11] If $\max_j \gamma_{jm}$ as attained at more than one type, say j and l, then $\pi_{j+1;m}(m) = \pi_{jm}(j) = 1$ and $\pi_{l+1;m}(m) = \pi_{lm}(l) = 1$. This adds some indeterminacy to the trading probabilities and consequently, the resulting stationary inventory distributions for types $i = j+1; ...; l-1$, but as the measures of agents acquiring their consumption good are constant across types and are determined by $\max_j \gamma_{jm}$, the overall characterization of the equilibrium remains unchanged

optimality condition on inventories (4), and the condition (8) on ν_{jm} that the aggregate inventory of intermediaries is sufficient to carry out the maximal possible number of trades. For the indifference condition on role choices, it is advisable to consider the case of no discounting, where long-run averages are compared. The inequalities then collapse to an equality that the long-run average consumption of consumer-producers must be equal in value to the per period pay-off of an intermediary. In the case of the Cash-in-Advance equilibrium, this yields the following conditions in σ_{im}, ν_{im}, and κ_{im}, for each $i \neq m$:

$$2\alpha\kappa_{im}(\beta_i+\beta_m) \geq \sigma_{im}U - (\beta_i+\beta_m) \geq 2\alpha(\kappa_{im}-1)(\beta_i+\beta_m) \tag{11}$$

$$(U(1-\sigma_{im}) - (\beta_i+\beta_m))\frac{1}{2}\frac{\frac{1}{N}-\nu_{lm}}{\frac{1}{N}-\nu_{im}} \tag{12}$$
$$= \kappa_{im}(\sigma_{im}U - (\beta_i+\beta_m) - \alpha(\kappa_{im}-1)(\beta_i+\beta_m))$$

$$\nu_{im}\kappa_{im} = \frac{1}{2}\left(\frac{1}{N}-\nu_{lm}\right) \tag{13}$$

For $i = m-1$, (12) is replaced by

$$\left(U(1-\sigma_{m-1;m}) - \beta_{m-1}\right)\frac{1}{2}\frac{\frac{1}{N}-\nu_{lm}}{\frac{1}{N}-\nu_{m-1;m}}$$
$$= \kappa_{m-1;m}\left(\sigma_{m-1;m}U - \left(\beta_{m-1}+\beta_m\right) - \alpha(\kappa_{m-1;m}-1)\left(\beta_{m-1}+\beta_m\right)\right)$$

One can now substitute (13) into (12) to find that

$$(U(1-\sigma_{im}) - (\beta_i+\beta_m))\frac{\nu_{im}}{\frac{1}{N}-\nu_{im}} = \sigma_{im}U-(\beta_i+\beta_m)-\alpha(\kappa_{im}-1)(\beta_i+\beta_m)$$

It follows from further simple manipulation that $\nu_{lm} = \max_j \nu_{jm}$ if $\beta_l = \max_{j\neq m;m-1}\beta_j$, i.e. the good with the highest transaction costs will require the largest set of intermediaries. In addition, one can easily show that there exists a unique solution $\{\sigma_{im};\nu_{im};\kappa_{im}\}_{i\neq m}$, such that (11)-(13) are satisfied. Moreover, since $\nu_{im} \to 0$ as $\alpha \to 0$, it follows that $\pi_{im}(m) \to 1$ for all $i \neq m+1$ and $\pi_{im}(i) \to 1$ for all $i \neq m-1$. Thus, for sufficiently low α, no trade will occur in bilateral meetings. Proposition 7 summarizes the characterization of the Cash-in-Advance equilibria with intermediaries:

Proposition 7: For sufficiently low α, there exists a Cash-in-Advance equilibrium $\left\{\mu_{ij}, \nu_{ij}, \phi_i, \kappa_{ij}, \sigma_{ij}, \tau_i\right\}_{i,j=1}^{N}$ for any good m, such that the following are true:

(i) $\{\sigma_{im}, \nu_{im}, \kappa_{im}\}_{i \neq m}$ is the unique solution to (11)-(13)

(ii) The inventory distribution is given as follows: for $i = m+1, \ldots, l-1$, $\mu_{im} = \frac{1}{2}\left(\frac{1}{N} - \nu_{lm}\right)$ and $\mu_{i,i+1} = \frac{1}{N} - \nu_{im} - \frac{1}{2}\left(\frac{1}{N} - \nu_{lm}\right)$

for $i = l+1, \ldots, (m+N-1) \bmod N$, $\mu_{im} = \frac{1}{N} - \nu_{im} - \frac{1}{2}\left(\frac{1}{N} - \nu_{lm}\right)$ and $\mu_{i,i+1} = \frac{1}{2}\left(\frac{1}{N} - \nu_{lm}\right)$, where $\nu_{lm} = \max_j \nu_{jm}$

$\mu_{m,m+1} = \frac{1}{N}$ and $\mu_{m-1,m} = \frac{1}{N} - \nu_{m-1,m}$

(iii) Trading probabilities for exchanges with intermediaries are close to 1 for all except $\pi_{m+1,m}(m)$ and $\pi_{m-1,m}(m-1)$, and as a consequence, all exchange goes through intermediaries.

In the pure search economy without intermediation, the Cash-in-Advance constraint failed to capitalize because goods were endogenously characterized by their qualities for indirect exchange. There was a considerable probability that the most preferred transaction could not be carried out immediately, and agents were unable to direct their search towards a predetermined sequence of transactions. In the intermediated economy, the medium of exchange results from the strategic interaction of intermediaries. Consumer-producers can direct their strategy towards a predetermined sequence of trades, in this case the one imposed by intermediaries. As the intermediaries become more and more efficient in carrying out exchanges, consumer-producers are able to almost immediately carry out the exchange proposed by the trade sequence. Holding a particular good at time t becomes equivalent in value to exchanging it against the next good of the trading sequence at time $t+1$. In the Cash-in-Advance equilibrium, any good can almost directly be exchanged against the commodity money, so that there is no incentive to reduce search frictions by goods-for-goods trade, as in the pure search model.

Figure 3 provides a graphical representation of the transaction pattern in a Cash-in-Advance equilibrium. In addition to the Cash-in-Advance equilibrium, other equilibria with intermediation exist. Any network of intermediaries that gives every consumer-producer exactly one trading sequence by which he can acquire his consumption good, can be supported as an equilibrium (as long as transaction costs are sufficiently small). It follows that there must be at least $N-1$ sets of intermediaries operating in equilibrium. For

[12] Equilibrium representation: An arrow from i to j represents the activity of ij-intermediaries. Trading strategies can always follow an arrow, but move against it, only if the good received is immediately consumed.

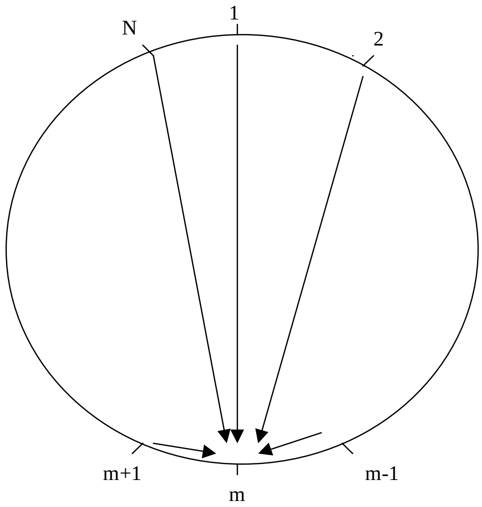

Figure 3: Cash-in-Advance equilibrium[12]

example, one can consider the following few alternative examples of intermediation networks. I shall not proceed at a full equilibrium characterization, since most of them can be analyzed using similar techniques as above:

(i) for $i = 1; \ldots; N - 1$, there exist $i; i+1$-intermediaries. In this case, all agents trade their production good directly against their consumption good, except for type N, who trade good 1 for good 2, then 3, etc. until they receive good N.

(ii) for $i = l + 1; \ldots; m - 1$, there exist im-intermediaries, and for $i = m; \ldots; l - 1$, there exist il-intermediaries. In this case, both good l and good m are locally used as medium of exchange. Good l is used by agents of types m to $l - 1$, good m is used by types l to $m - 1$. Type l acts as a middleman, who exchanges his production good $l + 1$ for the medium of exchange m, and then exchanges m against his consumption good l, which acts also as a medium of exchange.

(iii) for $i = 1; \ldots; m - 1$, there exist $i; i+1$-intermediaries, and for $i = m + 1; \ldots; N$, there exist im-intermediaries. This is a mixture of case (i) and the Cash-in-Advance equilibrium. Types 1 to $m - 1$ trade their production good directly against their consumption good, type N trades good 1 for good 2, then good 3 and so on, until he receives the medium of exchange m, which is used as in the Cash-in-Advance equilibrium by types m to $N - 1$.

(iv) as the Cash-in-Advance equilibrium, with the difference that $m; m+1$-intermediaries replace the $m+1; m$-intermediaries. Trading strategies re-

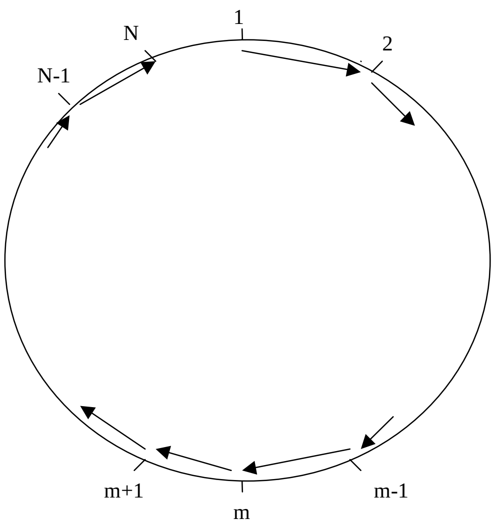

Figure 4: "Trade-one-up" equilibrium, case (i)

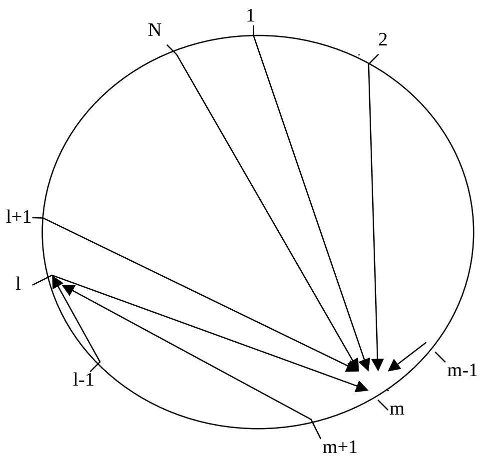

Figure 5: Two-money equilibrium, case (ii)

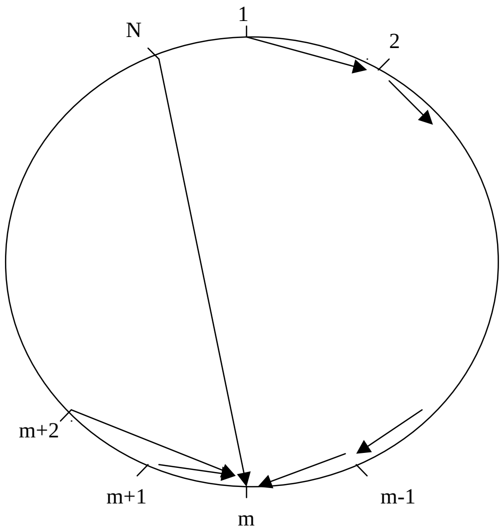

Figure 6: Case (iii) combines a Cash-in-Advance constraint with case (i)

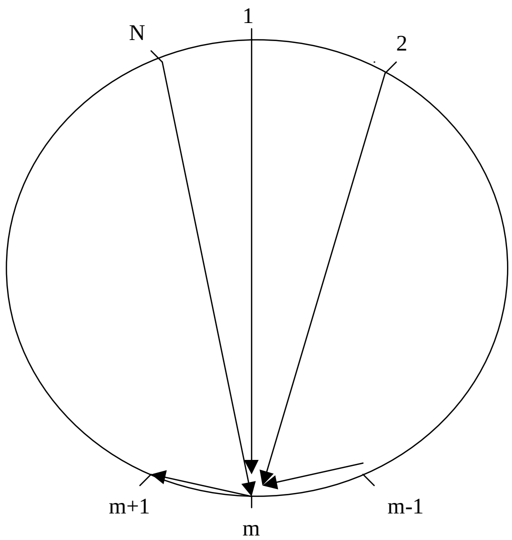

Figure 7: Alternative Cash-in-Advance equilibrium

main una¤ected, since all agents trading m for $m+1$ immediately consume thereafter. This represents an alternative version of the Cash-in-Advance equilibrium.

Obviously it is possible to consider equilibria of type (ii) with more than two local media of exchange, or to consider other combinations of these networks.

5 E¢ciency and Minimal Coalition Proofness

The previous discussion of Cash-in-Advance equilibria with and without intermediaries raises the problem of multiple equilibria, selection of a medium of exchange, and coordination of strategies. In this context, I shall also consider the welfare properties of the various equilibria. In related models without intermediaries, resulting Nash equilibria may be ine¢cient because of a coordination failure: agents cannot explicitly coordinate their actions to agree on a Pareto-e¢cient equilibrium. A second form of ine¢ciency arises from when some coordinated strategy pro…le Pareto-dominates the equilibrium, but is not individually rational (see, for example, the discussion in Aiyagari and Wallace, 1991). In this section, I show that the coordination problem and the discussion of welfare properties can be considered as related problems in an intermediated economy. For this purpose, I introduce a re…nement of Nash equilibrium strategies in the form of minimal coalition proofness.

In pure search economies, a Pareto-improving change in the equilibrium strategy pro…le can only be induced by a measure of agents that is strictly positive and bounded away from 0. I will argue here that, as long as a Pareto-improvement results from a change in the decisions $\acute{A}_i$ on what intermediaries to visit, this change can be enforced by an arbitrarily small set of agents who coordinate their actions as intermediaries. For instance, a small set of agents may become intermediaries and coordinate their actions with some small set of consumer-producers. By doing so, the intermediaries o¤ers a higher life-time utility to both their clients and to themselves. This gives the opportunity to all other agents to deviate from their initial strategy pro…le to take advantage of the higher life-time utility o¤ered by the new intermediaries. The old equilibrium is then no longer stable and will be replaced by a new one. This type of coordination is more explicit than the one resulting from Nash equilibrium strategies, however it only requires coordination of an arbitrarily small, but positive measure of agents.

More formally, the notion of minimal coalition proofness states that the minimum set of agents which can increase their life-time utility by coordination, taking as given the behavior of all other agents of the economy, must have a measure strictly greater than 0. This leads to the following:

Definition: A stationary equilibrium $\left\langle \mu_{ij}; \nu_{ij}; \phi_i; \kappa_{ij}; \sigma_{ij}; \tau_i \right\rangle_{i;j=1}^{N}$ has a subset $S \subseteq \Omega$ with a coordinated deviation (at time τ), if for each $s \in S$, there exists a plan $\{\phi_s(t); \kappa_s(t); \sigma_s(t); \tau_s(t)\}_{t=\tau}^{1}$, such that:

(i) given strategies $\{\phi_i; \kappa_{ij}; \sigma_{ij}; \tau_i\}_{i;j=1}^{N}$ followed by agents in $\Omega - S$, strategies $\{\phi_s(t); \kappa_s(t); \sigma_s(t); \tau_s(t)\}_{t=\tau}^{1}$ for each agent $s \in S$, and the resulting inventory distribution $\left\langle \hat{\mu}_{ij}(t); \hat{\nu}_{ij}(t) \right\rangle_{i;j=1}^{N}$, $\{\phi_s(t); \kappa_s(t); \sigma_s(t); \tau_s(t)\}_{t=\tau}^{1}$ is weakly preferred to the equilibrium strategies $\{\phi_i; \kappa_{ij}; \sigma_{ij}; \tau_i\}_{i;j=1}^{N}$ for each s, and strictly preferred for some non-zero measure of agents in S.

(ii) given the deviation strategies, and given the resulting inventory distribution $\left\langle \hat{\mu}_{ij}(t); \hat{\nu}_{ij}(t) \right\rangle_{i;j=1}^{N}$, for some "switching" subset S' of agents in $\Omega - S$, the equilibrium strategies $\{\phi_i; \kappa_{ij}; \sigma_{ij}; \tau_i\}_{i;j=1}^{N}$ are no longer optimal.

Definition: A stationary equilibrium $\left\langle \mu_{ij}; \nu_{ij}; \phi_i; \kappa_{ij}; \sigma_{ij}; \tau_i \right\rangle_{i;j=1}^{N}$ is minimally coalition proof, if for every sequence of subsets of agents S_n with a coordinated deviation, and such that $\mu(S_n) \to 0$, the sequence of measures of the corresponding "switching" subsets also converges to 0.

The previous definition of minimal coalition proofness combines elements of general coalition-proofness with evolutionary stability. As in evolutionary stability, we consider whether changes in strategies by an arbitrarily small set of agents can eventually lead to changes of strategies of the entire population. We restrict ourselves, however, in the set of strategy changes that we consider, by considering coalitions of agents who "coordinate" their actions such that given the status quo, all deviators can benefit. From the perspective of general coalition-proofness, we restrict our attention to coalitions of arbitrarily small size. Clearly, minimal coalition-proofness is weaker as a selection mechanism than either evolutionary stability or general coalition-proofness. According to this definition, the deviation plan need not be stationary. This is essential for considering the transition from one stationary equilibrium to another.

I now turn to the definition of constrained Pareto-efficiency for the purpose of this economy. Intuitively, arbitrarily small deviations from an inefficient equilibrium can only enforce a Pareto improvement, if they include some changes in the intermediation network. This excludes possible inefficiencies resulting from the transactions in bilateral search meetings. However, in an equilibrium, in which all agents trade with intermediaries with very high probability, the resulting strategies τ_i are prescribed by the network of intermediaries, and can have only minor welfare implications. Thus, I restrict attention to Pareto inefficiencies resulting from the intermediation network:

Definition: A stationary equilibrium $\left(\mu_{ij}; \nu_{ij}; \phi_i; \kappa_{ij}; \sigma_{ij}; \tau_i\right)_{i;j=1}^{N}$ is constrained Pareto-efficient, if there does not exist $\left(\mu_{ij}^{\delta}; \nu_{ij}^{\delta}; \phi_i^{\delta}; \kappa_{ij}^{\delta}; \sigma_{ij}^{\delta}; \tau_i^{\delta}\right)_{i;j=1}^{N}$, such that

(i) $\phi_i \neq \phi_i^{\delta}$ for at least one i.

(ii) $\left(\mu_{ij}^{\delta}; \nu_{ij}^{\delta}; \phi_i^{\delta}; \kappa_{ij}^{\delta}; \sigma_{ij}^{\delta}; \tau_i^{\delta}\right)_{i;j=1}^{N}$ is a Pareto improvement over $\left(\mu_{ij}; \nu_{ij}; \phi_i; \kappa_{ij}; \sigma_{ij}; \tau_i\right)_{i;j=1}^{N}$.

Proposition 8: Any minimally coalition-proof equilibrium is constrained Pareto-efficient.

This result is quite natural: If some Pareto improvement can be implemented by a change in the intermediation network, then groups of intermediaries and consumer-producers can implement this change on a small scale, and increase their personal welfare. Everyone else now individually has an interest in changing to the new strategies.

This result diverges from the main results on search economies without intermediaries, where the continuity of objective functions with respect to strategies meant that small deviations change overall utility only marginally. Changes in the intermediation network may lead to discontinuous changes in pay-offs, and thus to strategy changes by large parts of the population. From a historical perspective, small deviating coalitions can be viewed as an innovation mechanism: Someone proposes a new system for organizing his transactions. If others find that this arrangement is efficient, they will also start using it. As media of exchange, and more generally trading strategies are complementary across agents, everyone will start using the new system, if it leads to a Pareto-improvement. It becomes clear from the previous discussion, that intermediation is essential in promoting an innovation in the system of exchanges.

The converse of proposition 7 does not hold generally: An equilibrium may be Pareto-efficient, but not minimally coalition-proof, if the implemented changes lead to welfare losses for agents who don't participate in the change. In the context of intermediated, monetary exchanges, some agents may strictly prefer the old equilibrium over the innovation, but once the innovation is introduced, they will change, because their trade partners also start using the new medium of exchange. Loosely speaking, different media of exchange are substitutes, but there are complementarities in using a medium of exchange.

What are the implications for the Cash-in-Advance equilibria considered in the previous section? Clearly, for sufficiently low transaction costs, any equilibrium where some agents decide not see an intermediary is not minimally coalition-proof, since not visiting an intermediary is weakly dominated, and since there is a possibility to offer profitable intermediation. But what can be said about Cash-in-Advance equilibria with intermediation, or about the other intermediation networks considered at the end of the preceding section?

If transaction costs are sufficiently small, any minimally coalition-proof equilibrium must enable all agents to trade at most twice in order to consume, and as decreasing returns disappear, probabilities of trade with intermediaries must converge to 1 for all agents trading twice. Otherwise, the equilibrium would be Pareto-dominated by a Cash-in-Advance equilibrium. Also, a mixed strategy equilibrium cannot be minimally coalition-proof: A mixed strategy equilibrium leaves some agents indifferent between two transaction sequences with generically different expected transaction costs and delays of consumption. For the trading sequence with lower transaction costs, some trading probability is smaller than 1. Slightly increasing the total amount of intermediation for this exchange is feasible for a small group of agents and will lead to a higher trading probability, thus breaking the indifference and inducing a large measure of agents to switch.

It follows that an intermediation network of a minimally coalition-proof equilibrium consists of exactly $N - 1$ sets of intermediaries.[13] $N - 2$ types trade twice, while the remaining two types trade once. If type i and $i + 1$ both trade twice, they use the same good as a medium of exchange, and in equilibrium, at most two goods are used for indirect exchange. The remaining potential candidates are the Cash-in-Advance equilibrium, and the equilibrium with two commodity moneys (case (ii) of the previous section), since calculations along the lines of proposition 7 show that in the latter case, trading probabilities also converge to 1.

To check these equilibria for minimal coalition-proofness, we shall consider alternative intermediation networks in which (i) the same type of agents

[13] $N - 1$ is actually the minimum to sustain a complete intermediation network.

consumes in integer units, and (ii) all agents participating in the deviation incur (weakly) lower transaction costs. If (i) and (ii) are satisfied, the alternative intermediation network can be implemented by an arbitrarily small deviation, since reducing transaction costs will also reduce γ_{ij}'s and lead to a reduction in the delay of consumption, thus necessarily to a Pareto-improvement. (i) and (ii) are also necessary, since otherwise, either the type initially consuming in integer units would not be willing to participate in a deviation, or some type would incur higher transaction costs by following the deviation.

Proposition 9a: (i) A Cash-in-Advance equilibrium of either type with good 1 (the good with the lowest transaction costs) as medium of exchange is minimally coalition-proof.

(ii) A Cash-in-Advance equilibrium for any other good m as medium of exchange is coalition-proof, only if α is sufficiently low.

(iii) A Cash-in-Advance equilibrium of the type considered in case (iv) of the previous section is coalition-proof, only if good 1 is used as a medium of exchange.

(i) is straight-forward, since no agent would want to use a good other than the one with the lowest transaction cost as a medium of exchange. (ii) and (iii) follow from considering a deviation of $i1$-intermediaries, for $i = m + 1, \ldots, N$ and $1m$-intermediaries. The deviation triggered leads to an equilibrium with two media of exchange, one of which is good 1, the other good m.[14] One conclusion from the proof of (ii) is that if $\beta_1 = 0$, only the Cash-in-Advance equilibrium for good 1 is minimally coalition-proof.

The conclusions for two-money equilibria are similar. As in the previous description of the two-money equilibrium, let l be the type consuming in integer units. Then the following holds:

Proposition 9b: A two-money equilibrium is coalition-proof, if and only if $\beta_l < \beta_m$ and there exists no type k using good l as a medium of exchange, such that $\beta_k < \beta_l$.

Thus, a minimally coalition-proof equilibrium either has a Cash-in-Advance constraint, or has two media of exchange. In either case, trading probabilities converge to 1. What importance can be attached to the minimally coalition-proof equilibria where good 1 is not a universally accepted medium

[14] In (iii), the deviation considered is slightly different, and leads to a two-money equilibrium with 1 and $m + 1$ being used as moneys.

of exchange? In all minimally coalition-proof equilibria, one type i of agents does not o¤er any intermediation, and as a consequence, consumes his consumption good in integer units. Due to the special assumptions about consumption and production in this model, a deviating coalition can impose a good as universal medium of exchange, only if a full Pareto improvement is implemented by the deviation. Type i agents will never accept to participate in a deviation in which they o¤er some intermediation, since this strictly reduces their utility. This explains why some of the two-money equilibria cannot be broken.

The result is reversed when a strict subset of types can coordinate a deviation, without including the type i who doesn't o¤er intermediation. As an example, this could occur in a more generalized version of this economy where i-consumers are subdivided into di¤erent types of producers, such that each good is produced by at least two types of consumers. Under certain conditions, this implies a unique minimally coalition-proof equilibrium with a Cash-in-Advance constraint for good 1.

6 Conclusion

This paper has developed a modi…ed version of the traditional search economy model of monetary exchange. The introduction of intermediaries leads to two main results. First, intermediaries can induce the use of a common medium of exchange. As such, intermediation and money are complementary phenomena. Strategic interaction of intermediaries may lead to Cash-in-Advance constraints, such that trade sequences with intermediaries follow the well-known pattern that "goods buy money and money buys goods, but goods don't buy goods" (Clower 1965). As opposed to many other models of monetary exchange, this pattern is a result and not an assumption of the model. The second central result is that the characteristics of a monetary equilibrium with intermediaries di¤er fundamentally from those of equilibrium models without intermediaries. By forming coalitions, intermediaries can coordinate and lead the economy out of an ine¢ cient equilibrium. Resulting exchanges are such that every type of agent trades at most twice in order to consume, once to acquire the medium of exchange and once to acquire his consumption good.

A series of questions cannot be properly addressed within the framework of this model. The very special assumptions about production and consumption that are used here lend themselves for easy analysis, but also have some drawbacks. As previously discussed, the intuitively most appealing result that small, coordinated deviations necessarily induce the use of the most e¢cient medium of exchange throughout the entire economy can only be derived in a setting in which a successful deviation need not depend on the participation of consumers of all types.

Most importantly, the choice of production and consumption activities remains unaddressed. As in many related models, I have simply assumed the existence of an underlying Walrasian equilibrium, which in the absence of search frictions also represents an optimum. Production and Consumption choices are exogenously given in such a way that in a frictionless economy, all markets would clear at the relative prices of 1. Assuming the existence of such an equilibrium is not trivial per se. It is far from evident to assume that prices will correspond to the market clearing prices of a frictionless economy, as they would be the result of some bilateral bargaining process. As we have seen in proposition 6, the liquidity demand for the medium of exchange distorts market-clearing prices away from the Walrasian equilibrium. It is even more problematic to assume that consumption and production decisions do not depend on decisions about trade. It seems appealing to think that decision-makers take into consideration their opportunities for trade when they decide what goods to produce or to consume. Agents may decide to produce one good because it is easy to trade, even though they are more e¢cient at producing a di¤erent, less marketable good. This problem does not appear, however, in discussions on exchange in decentralized economies.

Finally, it should be noted that the model presented here relies on some fairly ad hoc assumptions about intermediation. The peculiar assumptions about intermediaries' inventories and trade can be motivated by the attempt to implement a sequential service constraint[15] in this simple discrete-time framework, and by the necessity of avoiding the problems of price theory in decentralized markets. The idea that a limit to intermediation generates a need for intermediaries to introduce a common medium of exchange requires more re‡ection. Precisely which technical restrictions a¤ect the behavior of intermediaries, and how can they alter an exogenously given environment? While such constraints are taken as given in this context, further thought is needed in order to assess the validity of the way intermediation is introduced

[15] Such a constraint seems essential to understanding the inventory and turn-over of goods in an economy with decentralized exchange with or without intermediation, and in the latter case, to examine the intermediary's ability to providing liquidity in the form of immediate exchange.

into the search economy here, and the robustness of the results that follow from it.

In spite of these technical short-comings, the results presented here provide some more general perspectives on intermediation. The complementarity of the medium of exchange and intermediation, the efficiency result, and the non-stability result for non-intermediated economies all follow from three basic assumptions about the nature of the economy:

(i) A Pareto-optimal, market-clearing allocation, which would result from a competitive equilibrium in perfect markets, cannot be attained because of a form of market imperfection,

(ii) some agents have a technology to alleviate the imperfection by offering intermediation, and by offering this technology to the economy, they can make arbitrage profits from a price spread, and

(iii) the success of intermediaries depends crucially on how they can deal with their own constraints.

In general, we know many reasons for frictions in a competitive economy, and the many facets of intermediation all respond to these imperfections. In this paper, I have considered search frictions as the reason for imperfection. Similarly, credit market imperfections are considered in the literature on financial intermediation. When these forms of market imperfections arise, intermediation performs a screening activity between both sides of the market, for which a price spread is charged. The success of intermediaries depends mostly on appropriating a large volume of transactions, and on establishing a repeated, credible interaction with their customers. This transfers the problems of price-setting and market allocation to the intermediation sector. Many features traditionally attributed to competitive markets, such as market clearing, the use of money and Cash-in-Advance constraints, can thus be explained as being in the interest of intermediaries who organize market exchange to alleviate an imperfection and take arbitrage gains from it.

Beyond these implications for the theory of intermediation, the results developed here also have some implications for existing Walrasian macroeconomic and monetary theory. The intermediation model combines frictionless market transactions a la Walras with an explicit, bilateral structure of exchanges. In addition, intermediation provides a channel, by which price-setting and information transmission can plausibly be discussed (although this exceeds the limits of this paper). It is hoped that extensions and simplifications of the intermediation model may prove useful to analyze questions in monetary and macroeconomic theory for which the existing theory has come to its limits due to the ad hoc structure of monetary exchange.

References

[1] Aiyagari, S. R. and N. Wallace (1991): "Existence of Steady-States with Positive Consumption in the Kiyotaki-Wright Model", Review of Economic Studies, vol. 58, p. 901-916

[2] Clower, R. W. (1965): "A Reconsideration of the Microfoundations of Monetary Theory", Western Economic Journal, vol. 6, p. 1-9

[3] Gehrig, T. (1993): "Intermediation in Search Markets", Journal of Economics and Management Strategy, vol.2, p. 97-120

[4] Howitt, P. (2000): "Beyond search: Fiat Money in Organized Exchange", Ohio State University

[5] Howitt, P. and R. W. Clower (2000): "The Emergence of Economic Organization", Journal of Economic Behavior and Organization, vol. 41, p.55-84

[6] Hellwig, C. (forthcoming): "Fiat Money in an Intermediated Economy", LSE, forthcoming

[7] Hellwig, C. (forthcoming): "Production and Exchange in a Search Economy", LSE, forthcoming

[8] Hellwig, M. F. (1993): "The Challenge of Monetary Theory", European Economic Review, vol. 37, p. 215-242

[9] Iwai, K. (1988): "The Evolution of Money: A Search-Theoretic Foundation of Monetary Economics", working paper no. 88-03, CARESS, University of Pennsylvania

[10] Kiyotaki, N. and Wright, R. (1989): "On Money as a Medium of Exchange", Journal of Political Economy vol. 97, p. 927-954

[11] Lucas, R. E. and Stokey, N (1987): "Money and Interest in a Cash-in-Advance Economy", Econometrica, vol. 53, p. 491-514

[12] Rubinstein, A. and Wolinsky, A (1987): "Middlemen", Quarterly Journal of Economics, vol. 102, p. 581-593

[13] Radford, R. A. (1945): "On the Economic Organisation of a P.O.W. Camp", Economica, vol. 12, p. 189-201

[14] Starr, R. M. (1972): "The Structure of Exchange in Money and Barter Economics", Quarterly Journal of Economics, vol. 88, p. 290-302

[15] Starr, R. M. (1999): "Why is there money? Convergence to a monetary equilibrium in a general equilibrium model with transaction costs", Department of Economics discussion paper 99-23, University of California, San Diego

[16] Starr, R. M. and M. B. Stinchcombe (1999): "Exchange in a Network of Trading Posts", in Markets, Information and Uncertainty: Essays in Economic Theory in Honor of Kenneth Arrow, G. Chichilnisky, ed., Cambridge University Press

7 Appendix: Selected Proofs

Proof of proposition 1: I first show that (1)-(3) can be equivalently reformulated as a Bellman equation. Existence of an optimal strategy are then easily obtained, as well as the fact that an optimal inventory plan must be non-decreasing. Because of the discontinuity in (3), the first step of this proof becomes possible only after the following short lemma:

Lemma: For any $k_s; k_{s+1}; c_s$ satisfying $\cdot > \left\lceil \frac{k_s - c_s}{1-\sigma} \right\rceil$, $[k_{s+1}] = \left\lceil \frac{k_s - c_s}{1-\sigma} \right\rceil$:

Proof: $[k_{s+1}] = \left\lceil k_s - c_s + \sigma_{ij} \left\lceil \frac{k_s - c_s}{1-\sigma} \right\rceil \right\rceil$

$$= \left(k_s - c_s - (1-\sigma_{ij}) \left(\frac{k_s - c_s}{1-\sigma_{ij}} \right) \right) + \left(\frac{k_s - c_s}{1-\sigma_{ij}} \right) = \left(\frac{k_s - c_s}{1-\sigma_{ij}} \right):$$

Now, define $\mathbb{R}_+$ as the state space of inventories. The feasibility correspondence $\Gamma_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ is given by:

$$\Gamma_{ij}(x) = \left\{ y \in \mathbb{R}_+ : 0 \leq y \leq x + \sigma_{ij} \left(\frac{x}{1-\sigma_{ij}} \right) \right\}$$

The one-period reward function $F_{ij} : \mathbb{R}^2_+ \to \mathbb{R}_+$ is given by:

$$F_{ij}(x; y) = U(x - y + \sigma_{ij}) - \delta \left(\beta_i + \beta_j \right) \gamma([y])$$

(1) can then be rewritten as:

$$W_{ij}(k_t) = \sup_{\{k_s\}_{s=t+1}^{\infty}} \sum_{s=t} \delta^{s-t} \left\{ U\left(k_s - k_{s+1} + \sigma_{ij}[k_{s+1}]\right) - \delta\left(\beta_i + \beta_j\right)\gamma([k_{s+1}]) \right\}$$

such that, for $s = t; t+1 :::$

$$k_{s+1} \in \Gamma_{ij}(k_s)$$

$\Gamma_{ij}(\cdot)$ is non-empty and compact-valued, and $F_{ij}(\cdot;\cdot)$ is bounded below on the set of all $(x;y)$ satisfying $y \in \Gamma_{ij}(x)$. It follows that $W_{ij}(\cdot)$ satisfies the following Bellman equation:

$$W_{ij}(x) = \sup_{y \in \Gamma_{ij}(x)} \left\{ U\left(x - y + \sigma_{ij}[y]\right) - \delta\left(\beta_i + \beta_j\right)\gamma([y]) + \delta W_{ij}(y) \right\}$$

The conventional solution techniques can now be applied to this Bellman equation. $\Gamma_{ij}(\cdot)$ is monotonic, convex-valued and upper hemi-continuous, but not continuous, while $F_{ij}(x;\cdot)$ is right-continuous with finite left limits, for all x. The following two lemmas complete the conditions necessary for characterizing a solution:

Lemma: Let κ_{ij} be given by (4). If $\left[\frac{k_s}{1-\sigma_{ij}}\right] \geq \kappa_{ij}$, an optimal continuation strategy is given by $k_{s+1} = k_{s+2} = ::: = \kappa_{ij}$.

Proof: (4) implies that κ_{ij} is the maximum number of trades an intermediary would be willing to carry out in any given period. It follows from the linearity of U that it is optimal to consume any excess inventory immediately.

Lemma: For any $k \geq k'$, $W_{ij}(k) - W_{ij}\left(k'\right) \geq \left(k - k'\right)U$

Proof: Any optimal plan starting from k' is feasible from k, and yields an additional utility of $\left(k - k'\right)U$.

It follows that intermediaries will never accumulate or keep an inventory higher than what is necessary to satisfy their most preferred number of trades. $[0;\kappa_{ij}]$ can be used as the state space, and is obviously compact. $W_{ij}(\cdot)$ is strictly increasing.

To establish existence, and to characterize $W_{ij}(\cdot)$, one can now use the Bellman operator. Standard results imply that the Bellman operator has

$W_{ij}(\cdot)$ as a unique fixed point in the space $B([0;\kappa_{ij}])$ of increasing, bounded functions from $[0;\kappa_{ij}]$ into $\Re_+$. Rewriting $W_{ij}(x)$ yields:

$$\begin{aligned}W_{ij}(x) &= \sup_{y\in\Gamma_{ij}(x)}\left\{U(x-y+\sigma_{ij}[y])-\delta\left(\beta_i+\beta_j\right)\gamma([y])+\delta W_{ij}(y)\right\}\\ &= Ux+\sup_{y\in\Gamma_{ij}(x)}\left\{-U(y-\sigma_{ij}[y])-\delta\left(\beta_i+\beta_j\right)\gamma([y])+\delta W_{ij}(y)\right\}\end{aligned}$$

Define $f^\star_{ij}(\cdot)$ by: $f^\star_{ij}(x)=W_{ij}(x)-Ux$, so that

$$\begin{aligned}f^\star_{ij}(x) = \sup_{y\in\Gamma_{ij}(x)}\Big\{&-U(y-\sigma_{ij}[y])-\delta\left(\beta_i+\beta_j\right)\gamma([y])+\delta Uy\\ &+\sup_{z\in\Gamma_{ij}(y)}\left\{\left(-U(z-\sigma_{ij}[z])-\delta\left(\beta_i+\beta_j\right)\gamma([z])+\delta W_{ij}(z)\right)\right\}\Big\}\end{aligned}$$

$$\begin{aligned}= \sup_{y\in\Gamma_{ij}(x)}\Big\{&-U(1-\delta)(y-[y])\\ &+\left((\delta+\sigma-1)U-\delta\left(\beta_i+\beta_j\right)\frac{\gamma([y])}{[y]}\right)[y]+\delta f^\star_{ij}(y)\Big\}\end{aligned}$$

From its definition, $f^\star_{ij}(x)$ is a non-decreasing step function. For given x, the supremum in $f^\star_{ij}(x)$ and $W_{ij}(x)$ exists and, upon inspection, must be at one of at most countably many discontinuities, since $F_{ij}(x;y)$ is decreasing in y. Furthermore, for any optimal y, $y\geq\left[\frac{x}{1-\sigma}\right]$, as long as $[y]\leq\kappa_{ij}$, where κ_{ij} is given by (4). □

Proof of lemma 1: Consider the following recursively defined plan of inventories: at T, choose $k+1$, at $T-1$, choose $(k+1)(1-\sigma_{ij})$, and for $T-1-t$, choose $(k+1)(1-\sigma_{ij})-tk\sigma_{ij}$. In words, this inventory plan consumes any unneeded inventory in the first period, and then lets the inventory increase until $(k+1)$ is reached. This plan yields life-time utility

$$\begin{aligned}\sigma_{ij}U+(Tk\sigma_{ij}-1)U-\delta\left(\beta_i+\beta_j\right)\gamma(k)\frac{1-\delta^T}{1-\delta}-\delta\left(\beta_i+\beta_j\right)(\gamma(k+1)-\gamma(k))\\ +\frac{\delta^T}{1-\delta}U(k+1)\sigma_{ij}-\frac{\delta^T}{1-\delta}\delta\left(\beta_i+\beta_j\right)\gamma(k+1)\end{aligned}$$

which must be larger than the utility of holding the inventory constant:

$$\frac{1}{1-\delta}Uk\sigma_{ij} - \frac{1}{1-\delta}\delta\left(\beta_i+\beta_j\right)\gamma(k)$$

Subtracting the second from the first yields, after some manipulation, that

$$\Delta(k)\frac{\delta^{T-1}}{1-\delta} + U\sigma_{ij} + (Tk\sigma_{ij}-1)U - U\sigma_{ij}\delta^{T-1} - \frac{1-\delta^{T}}{1-\delta}Uk\sigma_{ij} + \delta^{T-1}U > 0$$

Now, since T is the minimum number of periods needed to increase inventories, one has that $\sigma_{ij} \geq 1 - Tk\sigma_{ij}$. Substituting for σ_{ij} completes the proof.

Proof of proposition 9a: (i) and (ii) are straight-forward. A Cash-in-Advance constraint for good m is destabilized as proposed in the text, by a deviation of $i1$-intermediaries, for $i = m+1, \ldots, N$ and $1m$-intermediaries. For this deviation to be feasible, it is necessary that the additional transaction cost β_1 is more than offset by the increase in the frequency of consumption from $\frac{1}{2}(1-\max_l \gamma_{lk})$ to $\frac{1}{2}$. Using the no-discounting case, it must be the case that

$$\frac{1}{2}(U-\beta_m-\beta_1) > \frac{1}{2}\left(1-\max_l \gamma_{lk}\right)(U-\beta_m)$$

(iii) follows from the same deviation for type $m+1$. In this case,

$$\frac{1}{2}\left(U-\beta_{m+1}-\beta_1\right) > \frac{1}{2}\left(1-\max_l \gamma_{lk}\right)\left(U-\beta_{m+1}-\beta_m\right)$$

so that again, $m+1$-agents will have an incentive to follow the deviation. $\square$

Proof of proposition 9b: Clearly, a deviation is possible only towards a two-money equilibrium with good l and some other medium of exchange. If $\beta_m > \beta_l$, a deviation towards a Cash-in-Advance equilibrium for good l (or, a fortiori, to a two-money equilibrium with good 1 and good l) is feasible. If $\beta_m \leq \beta_l$, a deviation towards a two-money equilibrium is possible only if the second medium of exchange is some good k, where initially type k uses good l as medium of exchange, and $\beta_k < \beta_l$. If type k used good m as medium of exchange, the deviation would imply types "between" k and m to switch to a medium of exchange with higher transaction costs (from m to l). If $\beta_k \geq \beta_l$, the deviation would imply higher transaction costs for those switching from the use of l to k. $\square$

Note: Proofs for the remaining propositions (except prop. 4) follow directly from the arguments given in the text. The proof of proposition 4 can be established by a fixed point argument similar to the one in Aiyagari and Wallace (1991).