A Model of Persuasion — With Implications for Financial Markets*

Peter M. DeMarzo
U.C. Berkeley

Dimitri Vayanos
MIT
and

Jeffrey Zwiebel
Stanford University

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Abstract

We propose a model of the phenomenon of persuasion. We argue that individual beliefs evolve in a way that overweights the opinions and information of individuals whom they "listen to" relative to other individuals. Such agents can be understood to be acting as though they believe they listen to a representative sample of the individuals with valuable information, even though they may not. We analyze dynamics and convergence of beliefs, characterizing when agents' beliefs converge over time to the same beliefs, and when they instead diverge. Convergent beliefs can be characterized as the weighted average of agents' initial beliefs, and these weights can be interpreted as a measure of "influence." We then explore implications in an asset trading setting. Here we demonstrate that agents profit from being influential as well as being accurate. When agents' choice of whom to listen to is endogenous, we show that an individual's influence can be persistent, even though the individual may be inaccurate.

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1 Introduction

It is well known that in a fully rational model, the beliefs of individuals must follow martingales.[1] An individual who anticipates a future change in her expected beliefs should incorporate this into beliefs immediately. Nonetheless, there seems to be a wide variety of settings where individuals' beliefs change in ex-ante predictable manners. Individuals frequently appear to believe statements such as: "If I associate with a certain group, I am likely to agree more with their beliefs"; "If I choose to attend school A, I will likely develop an A-school's style of thought"; and, "as I age, I am likely to become more conservative".

More generally, in a wide range of settings, the activity of persuasion appears to affect listeners' beliefs in a predictable manner, towards the direction of the persuader. Whether it is a speaker giving a lecture, an attorney arguing for a client, or a newspaper editorial, common observation suggests that, on average, such arguments are more likely to convince than to dissuade. In a fully rational model, however, once someone states their position, there should be, on average, no scope for persuasion, provided that the listener has an unbiased assessment of the accuracy of the persuader's views. Attempts at persuasion are just as likely to drive an opinion away from the persuader as towards her.

Of course, the fact that a persuader can affect a listener's opinion, by presenting unexpectedly good evidence for his viewpoint, is not itself a violation of rationality. Even if the listener has an unbiased assessment of how strong the persuader's evidence is likely to be, if the persuader has a "good draw" of convincing evidence, one would expect his arguments to have an influence. Thus, for example, if a persuader is better informed than expected, the persuader may be able to anticipate his information will move other's views towards his own. And in settings where it is in his interest for others to have similar opinions – be it for political, social, or market reasons – he will have an incentive to engage in persuasion. With full rationality, such activity would only be beneficial when one's information is better than expected. Nonetheless, if the lack of such activity was taken as a signal of low quality information, there could exist a standard fully revealing signaling equilibrium where everyone chooses to persuade, even though such activity on average has no effect, lest individuals otherwise infer that one's silence signals information of a very poor quality.

Such an explanation, however, seems to fall short of explaining much persuasive activity. While it may be true that one side of an argument – that with the better case than anticipated – would benefit from persuasive activity, often both sides appear to be effective in their persuasion. That is, both sides seem to affect in their favor the opinions of those with whom

[1] See for example, Bray and Kreps (1987).

they interact. For example, under complete rationality, the moderate reader of the liberal newspaper should be just as likely to reject the views he reads as to adopt them. He should realize that, even if he is not reading the conservative newspaper, it is likely to have an opposing position, that he, as a moderate, is likely to find equally compelling. But to the contrary, observation suggests that such a moderate reader is in expectation likely to become more liberal if she regularly reads the liberal newspaper, while a similar moderate reader of a conservative newspaper is more likely to adopt his paper's conservative views.[2] Likewise, it is hard to believe that activities such as going to college, joining the military, working in a profession, etc..., do not systematically affect the average individual's beliefs on many issues in predictable manners. In general, individuals' views typically seem to evolve towards standard views of groups with which they regularly interact.

Similarly, the notion that persuasive activity makes a listener more sympathetic to an idea, independent of its merits, seems to underlie motivations for censorship and propaganda. If I believe my views are correct and someone else's are wrong – or even harmful, why shouldn't I afford my opponent every opportunity to speak, since this should in expectation serve to convince the listener that they had overrated this opposing view? In contrast, however, underlying the impulse to censor is clearly the notion that when individuals propound these views, they will be more likely to convince listeners than to dissuade them. Likewise, the use of propaganda seems based on the notion that repeated exposures to a view are likely to affect beliefs in a predictable manner. Such a view certainly seems to underlie much marketing activity. Along similar lines, a debate without equal time for each side, or a criminal trial where the defense was not allowed to present their case would generally be considered unfair. Yet if it was understood that such rules were imposed exogenously, they should not affect the average outcome in a setting where beliefs follow martingales. After hearing the prosecution, the jury should have a belief about the merits of the defendant's case, which is no more likely to be strengthened than weakened by hearing the defense.[3] In practice, however, such notions seem counterfactual. Rival viewpoints generally fight for "airtime," political spending does influence campaign outcomes, and attorneys do sway juries with their arguments.

What all these examples have in common is the notion that repeated exposure to an

[2] It is of course, not surprising that liberals and conservatives may choose to read newspapers similar to their political view – if for no other reason than they infer their respective newspaper will be more accurate. However, this does not explain why newspapers with opposite positions simultaneously move their respective readers' opinions in opposite directions.

[3] In fact, if the criterion for conviction is that the jury's posterior assessment of guilt lies in an exogenously determined lower tail of the distribution ("beyond a shadow of a doubt"), then hearing the defendant's case will raise the probability of conviction. This follows from noting that any new information leads to a mean-preserving spread of the assessment of guilt, raising the likelihood of a tail event.

opinion is likely to affect individuals' beliefs. If I explain the basis for my opinion to you, on average, you are more likely to adjust your opinion towards mine than away from it. And interacting regularly with a homogenous crowd is likely to lead one to move in the direction of this crowd's views. Evidence for such a salience bias is pervasive in the psychological literature.[4] One simple explanation for persuasive effects is that individuals give too much weight to the views that they hear relative to the ones they don't hear. Agents may simply not understand the full structure of their environment, and consequently may believe that those to whom they listen comprise a more representative sample of beliefs than in fact is the case. Intuitively, agents may not realize the extent to which the crowds in which they associate and the assumptions made by peers are not representative.[5]

As an introduction to the model of persuasion we will develop, consider the following extremely simple environment. Suppose the uncertain parameter that individuals are forming opinions about relates to the probability of heads from a particular type of coin. Individual i may choose to gather information by conducting some number of trials, n_i, and observing the frequency of heads, x_i.

Suppose next that individuals i and j meet. They each have different information, x_i and x_j, and thus will in general have different opinions. Individual j may try to "persuade" individual i by revealing his information. If individual i "listens to" individual j, then an obvious updating rule for i to use is

$$\hat{x}_i = \frac{n_i}{n_i + n_j} x_i + \frac{n_j}{n_i + n_j} x_j \qquad (1)$$

We will analyze updating rules of this form, and in fact show that if individuals do listen to representative samples from the entire population, such an updating rule will lead to the correct ultimate posterior beliefs.

On the other hand, note that if i and j's current information depends upon previous interactions with others, this updating rule is not necessarily Bayesian in the sense that there is no attempt to adjust for potential over-sampling of certain information. For instance, if i and j both previously listened to individual k, then the updating rule above will effectively "double count" this information. In such a circumstance, individual k's beliefs will have undue influence on the beliefs of individuals i and j.

[4] See, for example, Tversky and Kahneman (1973). Nisbett and Ross (1980) and Fiske and Taylor (1984) provide a review of this literature. Also closely related is the notion of confirmatory bias, whereby individuals interpret the evidence they observe in accord with their prior beliefs. See for example Lord, Ross and Lepper (1979). Rabin (1998) provides both an overview, and a discussion of applications in economic contexts.

[5] For an early discussion of this possibility in the psychology literature, see Ross, Greene and House (1977).

It is in this sense that we attempt to model the phenomenon of persuasion. We assume that individuals use a heuristic updating rule such as (1) above, without properly adjusting for the fact that the sample of individuals whom they listen to may be non-representative of the population as a whole. Thus, repeated sampling of information from the same individual or group of individuals will result in that individual or group having influence over the beliefs of others. In the context of the examples described above, listening repeatedly to like-minded colleagues or the same conservative authors (or individuals who read those authors) will tend to sway one's opinions towards that group.

Thus, persuasion bias in our model results from the fact that not all individuals communicate with a representative sample of the population, and that they do not adjust for this in their updating rule. While this is not fully rational, we claim it is "near-rational" in the sense that this updating rule leads to correct information aggregation under the simpler model of the world that samples are representative. Moreover, it seems to fit well with average behavior. For example, when listening to the opinions of others, most individuals do not appear to put less weight on the opinions of those who read the same newspaper as they do – though they should since otherwise that common component of the information will be over-represented.

Again, one justification for our model is that individuals believe they communicate with a random sample of the population when they do not. An alternative interpretation of our updating rule is that individuals believe that only a subset of the population possesses useful information worth listening to, and that the beliefs of the remainder of the population consist of useless noise. Thus, even though individuals may realize that they do not listen to a representative sample of the entire population, they may believe that they listen to a representative sample of the "right" people. For example, the economist might dismiss out of hand the market analyst who begins his argument with chartist notions, while the chartist might in turn dismiss the views of anyone subscribing to "crazy" academic notions of market efficiency. Insofar as all individuals' assessment of who does not possess useful information is not completely correct, our persuasion bias will follow. In this case, individuals know that the individuals that they listen to do not correspond to a representative sample, yet this is justified given their model of the world.

We take such a "persuasion bias" as a primitive in our model, and examine its implications for the dynamics of beliefs in general and an asset trading market in particular. In Section 2 of the paper, we develop a formal model of this environment, and show that if all individuals listen to unbiased samples of individuals from the entire population, then information is correctly aggregated. If, however, there are biases in the patterns of communication, information will be aggregated in a way which gives the information held by certain groups of individuals more weight than that held by others. In particular, we characterize the dynamics of beliefs in the population, and show that under a simple condition on the pattern of communication, beliefs of all agents converge to the same beliefs, given by a weighted average of initial beliefs. The weight associated with any type, in turn, can readily be interpreted as that type's "influence," and will be seen to depend on both the number and the influence of the types who listen to this given type.

In Section 3, we incorporate our model of persuasion into a model of asset markets. Standard models of asset markets generate trading volume through risk-sharing and liquidity motives, generally induced by the presence of noise traders. In actuality, however, much trading volume seems to be generated because agents believe that they can process the same, public information better than others. We model this by assuming agents believe that only a subset of the population possesses useful information worth listening to, and that the beliefs of the remainder of the population consist of useless noise.[6] That is, agents believe that they are better than others in identifying the useful information in the population.

In our asset market model, agents listen to the information of others, update their beliefs, and then trade on their updated beliefs. We demonstrate that in this setting, there is not only a role for using information accurately, but also a role for influence. Intuitively, influential agents may be able to persuade others of their beliefs, and in the process affect price dynamics in a manner that benefits them, even if their information is not particularly accurate.

One obvious question that this suggests is whether others will listen to such a poorly informed but influential agent in the future. That is, if we repeat such a game and allow agents to choose whom to listen to based on some measure of past performance, will they choose to keep listening to such an agent? We take up this question at the end of Section 3. We show that influence can be self-perpetuating. In particular, it can be in the interest of agents to listen to agents who are influential, even if they are inaccurate. Furthermore, influence can generally lead to higher profits. We characterize in settings in which influence can be "stable," in a manner to be defined.

A number of literatures are relevant to this paper. Most closely related, some of the fundamental elements of our general model of persuasion have been previously analyzed, though under a significantly different context, motivation and interpretation. In particular, French (1956) and Harary (1959) consider a setup similar to our abstract model in Section 2,

[6] Hence, while there are no actual noise traders in the model, the fact that each agent believes there are is sufficient to generate trade.

to examine "social influence." Thus, our Theorem 2 is generalization of the results of Harary.[7] And whereas neither French or Harary recognize the characterization for the convergent point in beliefs that we find in Theorem 3, this result follows from a combination of their Theorems with the characterizations of a stationary distribution of a Markov chain in Freidlin and Wentzell (1984), applied to the specific case of our model. We depart from French and Harary in motivating the problem as one of persuasion, in our ability to characterize influence in accord with the graphical structure of who listens to whom, in developing a setting under which the mechanics of beliefs follow from the behavior of near-rational agents, and in thinking about the implications of such a setting in a finance market with maximizing agents.

The paper whose motivation is perhaps most similar to our finance application is Harris and Raviv (1993). Like us, they also consider a setting where investors employ different models to interpret the same public information, and are willing to trade because they believe they have a superior model.[8] We differ from this work in that we explicitly model the nature of traders' different models of the world, and consequently how information will be differentially interpreted. More specifically, our agents' have models of the world which yield an important role for communication and updating of beliefs. Agents in our setting do understand that alternative interpretations of public information made by other traders is of value. This focus on the communication structure allows us to derive implications for the role of "influence" and to explore naturally the possibility of traders altering their models. In contrast, Harris and Raviv and other similar papers are instead primarily concerned with a very different set of issues relating to the cross-sectional and time-series relationship between price and volume.

A number of recent papers consider the implications for financial markets of biases in traders' inference processes. One particular manifestation, common to Daniel, Hirshleifer and Subrahmanyam (1998), Gervais and Odean (1997) and Benos (1996), considers traders who are overconfident in their assessment of their information. These papers are similar to ours in considering the consequences of psychologically motivated biases for asset markets. They differ in that traders make mistakes directly about the value of relevant information, and there is no role for communication. In contrast, in our setting, traders understand their

[7] Additionally, DeGroot (1974) examines a model virtually identical to that of French and Harary. Neither DeGroot, nor the literature that follows DeGroot (see for example Press (1978) and Berger (1981)) seem to be aware of this previous work. As with French and Harary, none of this work considers the economic context of maximizing behavior nor the application to financial markets which we pursue.

[8] See also Kandel and Pearson (1992) and Hindy (1994) for models where traders have different models of the world. Similarly, Beltratti and Kurz (1996) and Garmaise (1997) consider trade in a setting where agents must obey rational beliefs rather than the stronger notion of rational expectations, effectively allowing agents to believe any model not contradictable by the data.

own information is far from complete, and seek out the information of others. Our traders can be interpreted to be mistaken about the representativeness of the information that they obtain from others. This focus on the consequences of communication and updating more naturally allows us to explore influence and the evolution of mistakes over time. In much of the related literature, mistakes clearly cost traders wealth or utility, and hence are unlikely to be evolutionarily persistent.[9]

One recent literature which does focus on the evolution of beliefs given repeated interactions between agents is the large literature on social learning. Work in this area assumes that agents choose actions over time, and base their choice on information received from other agents. The choice of action can be in a non-strategic context, where an agent's action does not directly affect other agents (for instance, agents are consumers who decide which product to buy), or in a strategic context, where agents play a game with other agents. Agents may observe information from the whole population, or there may be "local interactions", where agents only observe their neighbors. See, for example, Ellison (1993) and Fudenberg and Ellison (1995) for local interactions in a strategic and non-strategic context, respectively, and Bala and Goyal (1998) for a general neighborhood structure. Although there is a formal similarity between this paper and the literature on social learning, the questions asked are quite different. The social learning literature studies whether agents adopt the optimal action (in a non-strategic context) or converge to a Nash equilibrium (in a strategic context). By contrast, this paper studies how influential each agent is in affecting the long-run limit of others' beliefs, and how influence matters in a financial market setting.

2 The Dynamics of Beliefs Under Persuasion

2.1 Representing Beliefs

In developing a formal model of persuasion, one difficulty involves choosing a convenient representation for an agent's beliefs. In general, if there is some unknown parameter θ of interest, an agent i's beliefs at time t correspond to a probability distribution F_i^t for this parameter.

The difficulty with this most general formulation is two-fold. First, when modeling the communication between agents, it is unrealistically complicated for the agents to commu-

[9] An exception is De Long, Shleifer, Summers and Waldman (1990), which presents a model where traders making a mistake do make money over time. In this model, some traders underestimate the variance of a risky asset, and therefore over-invest in these assets. But since risky assets have higher expected returns, this leads to higher expected wealth, even though actual utility is lower.

nicate entire infinite-dimensional distribution functions. Any natural communication must involve the reduction to a low dimensional vector. The second difficulty comes from the aggregation of beliefs. Given agent i's own beliefs F_i^t and the information from agent j, F_j^t, how should agent i aggregate this information when forming his new beliefs F_i^{t+1}? Even if we assume agents are Bayesian, this updating is complicated and non-linear in general.

We introduce two different frameworks that allow us to simplify the problem of representing and updating beliefs. The first and most straightforward approach is to assume that the underlying parameter θ and all available information sources have a joint normal distribution. Under this parameterization, it is reasonable to think that agents communicate their beliefs using a simple sufficient statistic of the information they have acquired. Moreover, these sufficient statistics can be combined via a simple weighted average. This normal/linear framework is convenient, and we shall pursue it more detail in Section 3 when we imbed our model of persuasion within a standard noisy rational expectations model of asset markets.

The normal/linear framework, while simple, is of course a very special case. We would like to build a model of persuasion that does not rely on such strong distributional assumptions in order not to restrict its applicability. This is possible if instead of representing the beliefs directly through posterior distribution functions, we represent them indirectly through the data which generates those beliefs.

In particular, suppose that individuals learn about the unknown parameter θ by gathering "evidence" or "experimental" data. That is, each agent observes the outcomes from n identical experiments. Each observation can take on k possible realizations, with the likelihood of observing any particular outcome in the set depends upon the parameter θ. Furthermore, assume the joint distribution of the experimental outcomes (given θ) does not depend on the order that the outcomes are observed. Then the data from the experiments can be summarized by a vector

$$x \in \Re_+^k; \quad \sum_i x_i = 1; \tag{2}$$

where x_i represents the observed frequency of each outcome. Given this empirical distribution, agents then use their priors and Bayes rule to determine their posterior distribution for the unknown parameter θ.

This formulation has the advantage that the empirical frequency x acts as a sufficient statistic for an agent's posterior beliefs, and is much easier to communicate. It fits directly the situation of scientists sharing experimental data (or individuals sharing anecdotes about, for example, the efficacy of some home cold remedy). It also has the advantage of a simple, linear aggregation rule. For example, if individuals i and j have each observed (directly or indirectly) the results of n^t experiments at time t, and these experiments have outcome frequencies x_i^t and x_j^t, their information can be aggregated via $n^{t+1} = 2n^t$ and

$$x_i^{t+1} = \frac{1}{2}x_i^t + \frac{1}{2}x_j^t. \tag{3}$$

In what follows, we develop a model of communication and influence consistent with these two alternative representations of beliefs. That is, we model the beliefs held by an agent by a finite-dimensional vector, and suppose agents aggregate beliefs using a linear updating rule. While justifying this representation by presuming joint normality is convenient for many economic applications, the second approach based on communicating data makes clear that this representation is not a distributional restriction.

2.2 The Basic Model

Consider a setting with N "types" of agents, $\mathcal{N} = \{1, 2, \ldots, N\}$. For each type, there is a continuum of individuals uniformly distributed over the interval $[0, 1]$. We will denote an agent m of type i by i_m, and the "position" of this agent at time t by $x_{i_m}^t$.[10] An agent's position can be thought to represent his "beliefs" at a given time and is given by a point in $\mathbb{R}^p$, where p is some constant. An agent begins at time 0 with beliefs $x_{i_m}^0$ and updates beliefs in a manner to be described presently. We will let x_i^t represent the average beliefs for agents of type i at time t, i.e.,

$$x_i^t = \int_0^1 x_{i_m}^t \, dm;$$

and x the vector $(x_1, x_2, \ldots, x_N)'$. We will focus on the average beliefs of each type, and in fact often refer to the "beliefs of type i" as though there were only a single agent of type i with beliefs x_i^t. In fact, the results of this Section can be stated in a setting in which there is only a single agent of each type. We introduce a continuum because we shall develop a competitive asset market model in Section 3.

We define a listening structure by a function S which maps from types in $\mathcal{N}$ into subsets of $\mathcal{N}$. The subset $S(i)$ of $\mathcal{N}$ is interpreted as the set of types which type i listens to. We will generally assume that $i \in S(i)$, i.e., that types listen to their own type, among others. We will presume that each period, an agent of type i listens to the beliefs of one agent, randomly drawn from the set of agents whose types are in $S(i)$.[11] The probability that an agent of

[10] When time is clear, we will often drop the superscript.

[11] In an earlier draft of this paper, we instead modeled agent i as listening to one agent of each type in $S(i)$ every period. In this alternative setting, the results in this section go through unchanged. In fact, convergence within groups follows in a much simpler manner – by the law of large numbers – since all agents of a given

type i listens to an agent of type j in any period is given by T_{ij}, where $T_{ij} > 0$ if $j \in S(i)$ and $T_{ij} = 0$ otherwise. Naturally,

$$\sum_{j \in S(i)} T_{ij} = 1.$$

In most examples, we will assume uniform listening, so that $T_{ij} = 1/\#S(i)$ for $j \in S(i)$. Our results do not depend on this, and we can allow for non-uniform randomization to capture that notion that some types may be more likely to encounter each other than other types. The matrix T summarizes the communication structure of the model.[12]

Example 1 Let $N = 3$, and $S(1) = \{1,2,3\}$; $S(2) = \{2,3\}$; $S(3) = \{1,3\}$. Then with uniform listening,

$$T = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \quad (4)$$

Note that S can also be depicted as a directed graph over N. The directed graph implied by this listening structure is depicted in figure 1, with an arrow from i to j indicating that $i \in S(j)$ and hence i "influences" j.

After listening to the agent he has been randomly matched with, each agent updates his beliefs accordingly. We do not address incentives for accurate reporting here – rather we presume throughout that reports are conveyed accurately, and can be verified. Note, however, that since agents are infinitesimal in this model, strictly speaking there is no incentive for agents to misreport their beliefs.[13]

In line with our notion of persuasion discussed in the introduction, an agent i_m updates as if the agents of $S(i)$ comprise all agents in the world, and does not take into account the presence of agents outside of $S(i)$. Note that this does not imply that these other agents will not ultimately affect the beliefs of an agent of type i; indeed, if agents of type i listen to agents of type j but not those of type k, but agents of type j do listen to those of type k,

type now deterministically sample from the same types each period. We instead adopt the assumption in the text because it captures the natural notion that individuals listen to the same number of other individuals irrespective of type (instead of those types whose set $S(i)$ is larger being able to undertake more sampling), and this in turn greatly simplifies the analysis in our asset pricing model of Section 3.

[12] For the case of a single, atomistic agent of each type, the listening process must follow our alternative specification (as discussed in footnote 11 above), whereby agent i listens to all agents in $S(i)$ each period. The weights T_{ij} can then be interpreted as reflecting differences in the weight the agent puts on the opinions of others in $S(i)$. The results developed here in the continuum case extend to the atomistic case.

[13] We will find in our asset market model that agents benefit from being influential. Therefore, even if reports are not verifiable and agents not infinitesimal, agents would have some incentive to report their beliefs truthfully, insofar as accurate reports are more likely to lead to beneficial influence. Banerjee and Somanathan (1997) consider a process which could be interpreted as persuasion, where the primary focus is on agents' incentives to reveal their information when it is costly to do so.

the beliefs of the type k agents will indeed affect type i, through their effect on j.

Specifically, in this Section we assume that in period t, an agent updates his beliefs by giving weight λ_t to the beliefs of the agent he listens to, and weight $1 - \lambda_t$ to his own beliefs. For $\lambda = 1/2$ this corresponds to the updating rule in equation 3. We allow for $\lambda \neq 1/2$ in order to explore the sensitivity of our results to the agents' beliefs about their own precision relative to others.[14]

In Section 3 we will show that this rule results in correct Bayesian inferences under a particular (potentially misspecified) model of the world. For now, we will simply resort to the two intuitive rationalizations of the introduction: either agents don't fully understand the extent to which the views they listen to are not representative, or agents (sometimes inaccurately) assess some individuals as possessing worthless information not worth listening to. We will also show that if agents do in fact listen to representative samples, then ultimate beliefs will correctly aggregate the available information.

Defining $T(\lambda)$ to be the matrix $T(\lambda) \equiv ((1-\lambda)I + \lambda T)$, our updating rule implies that average beliefs for types obey the following process:

$$x^t = T(\lambda_t)x^{t-1}; \quad t = 1, 2, \ldots; \tag{5}$$

and together with agents' initial beliefs, this implies that

$$x^t = \left[\prod_{s=1}^{t} T(\lambda_s)\right] x^0; \quad t = 1, 2, \ldots. \tag{6}$$

Equation (6) describes the evolution for the average belief x_i of each type i. At any point in time, however, agents of a given type will have heterogeneous beliefs. These differences exist due to the heterogeneity of their initial beliefs, together with randomness of the listening process as agents interact each period.

In general, the dynamics of beliefs may be quite complicated. However, as we now show, there will in general exist aggregate statistics of the beliefs which remain stationary over time. In particular, suppose $w' = (w_1, \ldots, w_n)$ is such that

$$w'T = w'. \tag{7}$$

That is, suppose w' is a row eigenvector of T with eigenvalue 1. Then it follows that for all

[14] For example, if agents believe that only they are updating over time, then $\lambda_t = \frac{1}{t+1}$ would be an appropriate rule.

λ, $w'T(\lambda) = w'$. Hence, for all t,

$$w'x^t = w' \left[\prod_{s=1}^{t} T(\lambda_s) \right] x^0 = w'x^0 . \tag{8}$$

Note also that since the rows of T sum to 1, T can be interpreted as a stochastic matrix defining a finite Markov chain. Any stationary distribution of this Markov chain will satisfy (7). Thus, we can restrict w to be non-negative and sum to 1. Since a finite Markov chain has a stationary distribution, we have:[15]

Theorem 1 *For any listening structure, there exist weights w corresponding to a stationary distribution of T such that the weighted average of population beliefs is stationary over time. That is, $w'x^t = w'x^0$ for all t.*

This result implies immediately that if individual beliefs converge, they must converge to $w'x^0$. In the next Section we explore the conditions that are sufficient for such convergence.

2.3 Belief Dynamics with Strongly Connected Types

We now turn to questions regarding the convergence of beliefs in the population. As we remarked earlier, there are two distinct sources of heterogeneity of beliefs. There is the heterogeneity of initial beliefs, together with the heterogeneity introduced over time through the random listening process. Intuitively, though, the communication process should bring agents' beliefs closer together, and the random listening process should diminish in importance over time. We will show that this intuition is correct provided that agents are not isolated but are linked to each other through the listening structure, and that they put sufficient weight λ_t on the beliefs of others.

The following definitions will be useful:

Definition 1 *We say type i is linked to type j, if there exists a sequence of types $k_1, k_2, \ldots, k_r \in N$ such that, $k_1 \in S(i)$; $k_2 \in S(k_1)$; $\ldots$; $k_r \in S(k_{r-1})$; $j \in S(k_r)$.*

Definition 2 *We call a subset of types $I \subset N$ strongly connected if $\forall i, j \in I$, i is linked to j.*

[15] While we use the theory of Markov chains here and elsewhere in our analysis, we remark that the dynamic process of beliefs in our model does not follow a finite Markov chain defined by the matrix T. We are post-multiplying $T(\lambda)$ by the actual beliefs, rather than pre-multiplying it by a distribution. Indeed, the process followed by beliefs cannot be described as a finite Markov chain. To describe it as a Markov process would require an infinite-dimensional state variable representing the distribution of population beliefs.

Recall that T is a stochastic matrix defining a finite Markov chain. It is immediate that this chain is irreducible since N is strongly connected, and aperiodic since the diagonal of T is strictly positive. It is well known that a finite, irreducible, aperiodic Markov chain has a unique stationary distribution $w' = (w_1; w_2; \ldots; w_n)$, with $w_i > 0$ for all i (see, for example, Theorem 8.8 in Billingsley (1986)).

The final condition we need for convergence is that the cumulative weight that agents put on the opinions of others is large enough, while at the same time they do not completely ignore their own prior beliefs. This is captured by the following:

Assumption 1 $\sum_{t=1}^{\infty} \lambda_t = \infty$; and there exists $\bar{\lambda} < 1$ such that for all t, $0 \le \lambda_t \le \bar{\lambda} < 1$.

We then have the following convergence result:

Theorem 2 Suppose N is strongly connected, and Assumption 1 holds. Then for any initial beliefs x^0, the beliefs of all agents converge to $w'x^0$. That is, for all i and m,

$$\lim_{t \to \infty} x^t_{i_m} = \lim_{t \to \infty} x^t_i = w'x^0;$$

where w is the unique stationary distribution of T.[16] [17]

Thus, provided the weight λ_t that agents put on others beliefs does not fall too quickly, in the limit all agents will converge to the belief equal to a weighted average of the initial population beliefs, with the weights given by the stationary distribution w.[18] The individual weights w_i can be interpreted as the influence that each type i exerts on the outcome through its members' initial beliefs. For the listening structure given above in example 1 and figure 1, these stationary weights are given by $w = (\frac{3}{9}; \frac{2}{9}; \frac{4}{9})$. It is not surprising that the initial average belief of type 3 in this example has the most influence on the final outcome, insofar as all types listen to type 3, while only two of the three types listen to each of types 1 and 2. What, however, is striking is that type 1 has more weight on the final outcome than type 2. This is because while type 1 is listened to by the influential type 3 (and type 1 itself), type 2 is only listened to by the less influential type 1 (and by type 2 itself). This example

[16] The statement regarding the beliefs of individual i_m apply almost everywhere.

[17] All proofs omitted from the text are in the Appendix.

[18] Intuitively, placing more weight on one's priors (decreasing λ) slows down the process of convergence, but doesn't affect the ultimate point to which the process converges, provided that T eventually gets infinite weight. This result implies that even when the weight $\lambda(t)$ put on the new period observation is proportional to $\frac{1}{t}$, beliefs still converge to $w'x^0$. In many settings, such as the model of Section 3, $\lambda = 1/2$ and the condition is obviously satisfied.

demonstrates that influence in our setting derives not only from being listened to by many types, but also from being listened to by influential types, where influence is determined endogenously. (We will later formalize this intuition in Theorem 3.) To further the notion that both the number and the influence of those who listens to a type determine this type's own influence, consider the following, more complex example.

Example 2 Let $N = 4$, and let $S(1) = \{1,2,3\}$; $S(2) = \{2,3,4\}$; $S(3) = \{3,4\}$; $S(4) = \{1,4\}$ with uniform listening. The directed graph implied by this listening structure is depicted in figure 2. Note that types 3 and 4 are both listened to by two other types, so we would expect them to be more influential. However, type 4 is listened to by 3 directly, so we would expect 4 to be more influential than 3. Finally, since 4 listens to 1, 1 should be more influential than 2. In fact, the influence weights for this example are given by $w = (\frac{6}{23}, \frac{3}{23}, \frac{6}{23}, \frac{8}{23})$.

Given Theorem 2, it is straightforward to show that if agents do listen to representative samples and $S(i) = N$ for all i, then the influence weight of every type is equal. Thus, beliefs converge to a simple average of the information available in the population. In fact, this result can be generalized as shown below to include any "symmetric" listening structure. Thus, influence can be seen as emerging from asymmetries in the listening structure.

Corollary 1 Suppose that each type of agent listens to the same number of types, and also that each type of agent is listened to by the same number of types. Then under uniform listening,

$$w = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right). \tag{9}$$

Thus, if the listening structure is symmetric and if each type starts with information of "equal quality," the heuristic updating rule specified here leads agents to correctly aggregate information in the limit.[19]

Theorem 2 gives us one manner to characterize the weights that determine the beliefs that all agents will converge to when N is strongly connected — i.e., the stationary distribution of T. We now show that in the special case of uniform listening, these weights can be characterized in two additional manners, using trees derived from the directed graph representation of the listening structure.

[19] The notion of equal quality depends on the setting. It would be satisfied, for example, if as in Section 2.1 agents begin by observing data from equivalent experiments, and share frequency information regarding outcomes. In the normal/linear framework it would be satisfied if initial beliefs equal $\mu + \epsilon_i + \eta_{im}$, where ϵ are i.i.d., η are i.i.d., and all variables are joint normal.

In particular, a tree is a directed graph connecting all members of N, where every type in N save for one has a unique predecessor, there are no cycles, and one type (the root) has no predecessors. We will call a tree admissible to the listening structure S, if for every type i save for the type at the root of the tree, $S(i)$ contains the predecessor to i in the tree. In other words, trees are admissible if all branches directed from the root conform to the listening structure. We define g_i as the number of admissible trees for which i is the root. Intuitively, g_i gives the number of "routes" by which i influences others given listening structure S. In accord with the literature, we will refer to g_i as the number of i-trees defined by S, and to $g^0 = (g_1; g_2; \ldots; g_N)$ as the tree-count defined by S. Figure 3 gives the i-trees associated with the simple listening structure of example 1. There is one 1-tree, one 2-tree, and two 3-trees.

We can now state the following theorem which gives two additional interesting characterizations of the influence weights w.

Theorem 3 Let N be strongly connected, under the listening structure S, with associated tree-count g. Let $\delta_{ij} = 1$ if $j \in S(i)$ and 0 otherwise. Then under uniform listening, the influence weights w, which are given by the stationary distribution for T, can be characterized as follows: There exists $k > 0$ such that for all i,

$$kw_i = \sum_j g_j \delta_{ji} = g_i \sum_j \delta_{ij}: \qquad (10)$$

Theorem 3 gives two rather striking additional characterizations of w, which follow from Theorem 2 and the characterization of a stationary distribution of a Markov chain given by Freidlin and Wentzell (1984). The first states that w_i is proportional to the sum of the number of j-trees for all j who listen to i. This makes precise the earlier notion that the influence of a type can be characterized by the sum of a measure of influence of all types who listen to this type. The theorem also states that this is equivalent to the number of types who i listens to multiplied by g_i. Note that we have placed no restrictions on the relationship between δ_{ij} and δ_{ji}; that is, there is no connection between who i listens to and who listens to i. Nonetheless, these two expressions are equivalent, and both are proportional to w due to the manner in which trees are counted.

Example 3 Consider once again the listening structure S given in example 1. As noted above, admissible trees are shown in figure 3 and $g = (1; 1; 2)$. Theorem 3 gives us two manners to calculate weights w. Under the first method, the sum of the tree count over those who listen to each type is: 4 for type 3 (type 3 is listened to by all three types), 3 for type 1 (type 1 is listened to by types 1 and 3) and 2 for type 2 (type 2 is listened to by types

1 and 2). Normalizing, we obtain $w = (\frac{3}{9}, \frac{2}{9}, \frac{4}{9})$. Employing instead the second method, we multiply g_i by the number of types i listens to, yielding $(3, 2, 4)$, and normalizing once again gives w.

2.4 A General Characterization

So far we have only considered belief convergence when N is strongly connected. We now consider more general listening structures, though we maintain Assumption 1 throughout this section.

First we will define a useful partition of types based on the relationship of strong connectedness. In particular, consider the relationship of strong connectedness, defined by iCj if i is linked to j and j is linked to i. The relationship C is reflexive, symmetric and transitive, and therefore defines an equivalence class over types which is uniquely represented by a partition which we denote $\mathcal{I}$. By this construction, it is immediate both that for any $I \in \mathcal{I}$, I is strongly connected, and furthermore, that any strongly connected collection of types must belong to some element $I \in \mathcal{I}$.

Consider an element I of the partition $\mathcal{I}$. Let $x^0(I)$ be the vector x^0 restricted to the set of agents belonging to I, and $w(I)$ be the stationary distribution corresponding to partition element I when S is restricted to I. If no element of I listens to any type $j \notin I$, then the evolution of beliefs within I can be considered in isolation, and the results of the preceding Section apply. This motivates the following definition and Theorem.

Definition 3 A subset of types I is isolated if for all $i \in I$, $S(i) \subset I$.

Theorem 4 Suppose $I \in \mathcal{I}$ is isolated. Then the beliefs of all agents in I converge to $w(I)' x^0(I)$.

Thus, the isolated members of $\mathcal{I}$ can be treated independently. On the other hand, if $I \in \mathcal{I}$ is not isolated, the beliefs of its members will be influenced by the beliefs of types not in I. To identify the set of types that might influence I, we define the following.

Definition 4 For any $I, J \in \mathcal{I}$, we say that I is linked to J if there exists $i \in I$ that is linked to $j \in J$.

For any general listening structure with strong connections defined by the partition $\mathcal{I}$, define $\mathcal{J}$ to be the collection of isolated members of $\mathcal{I}$. We then have the following characterization for $\mathcal{J}$ and for the convergence of beliefs of all types.

Theorem 5 *The collection $\mathcal{J}$ of isolated members of $\mathcal{I}$ is nonempty, and each non-isolated member of $\mathcal{I}$ is linked to at least one isolated member. Furthermore,*

1. *For all $J \in \mathcal{J}$, the beliefs of all agents in J converge to $w(J)'x^0(J)$.*

2. *If $I \in \mathcal{I} \setminus \mathcal{J}$ is linked to a unique $J \in \mathcal{J}$, then the beliefs of all agents in I converge to $w(J)'x^0(J)$.*

3. *For all $I \in \mathcal{I} \setminus \mathcal{J}$, the beliefs of each type in I converge to a point in the interior of the convex hull (under the subspace topology) of the beliefs $w(J)'x^0(J)$, for all J such that $J \in \mathcal{J}$ and I linked to J.*[20]

If i and i' are two types in a non-isolated set $I \in \mathcal{I}$, it is possible in general for them to converge to different final beliefs, as shown in the following example.

Example 4 Let $N = 5$, and $S(1) = (1;2;3)$; $S(2) = (1;2;4)$; $S(3) = (3)$; $S(4) = S(5) = (4;5)$. Then $\mathcal{I} = \{\{1;2\};\{3\};\{4;5\}\}$. The isolated members of $\mathcal{I}$ are $\mathcal{J} = \{\{3\};\{4;5\}\}$. With uniform listening, Theorem 5 indicates that the beliefs of type 3 converge to x_3^0, and the beliefs of types 4 and 5 converge to $(x_4^0 + x_5^0)/2$. Furthermore, for the non-isolated set $\{1;2\}$, beliefs do not converge to the same point. Instead, the average beliefs of type 1 converge to $(2/3)x_3^0 + (1/6)x_4^0 + (1/6)x_5^0$, and the average beliefs of type 2 converge to $(1/3)x_3^0 + (1/3)x_4^0 + (1/3)x_5^0$.

We remark that in case 3 of Theorem 5, individual beliefs do not necessarily converge, though the (average) beliefs of each type do. To see why, consider the case in the previous example. Each period, some agents of type 1 will listen to agents of type 3, others will listen to agents of type 4, etc. Thus, there will exist a steady-state stationary distribution of the beliefs of the individual agents within types 1 and 2. This can be solved for numerically (in fact it is rather complex), and is illustrated in Figure 4.[21]

Before ending Section 2, we should note that while we have provided convergence results, we have not characterized the rate of convergence. For some listening structures, however, the rate of convergence may be more relevant than the convergent beliefs. Consider for example

[20] That is, if the convex hull is p-dimensional, beliefs converge to the interior of the convex hull in this p dimensional space. If I is only linked to a single member $J \in \mathcal{J}$, then beliefs converge to the point where the beliefs of types in J converge, which is the "interior" of this point in the subspace topology defined by this one point.

[21] Note that this non-convergence of individual beliefs is an artifact of the way we have specified the listening process. If agents instead listened to one agent of each type in their listening each period as discussed in footnote 11 above, all agents' beliefs would converge to the average.

a listening structure where all types save for one (the "hermit") are strongly connected and share many links, and the hermit listens only to himself, and is listened to by one other type. Our convergence results yield the rather counterintuitive outcome that the hermit will, in the limit, completely determine everyone's beliefs. However, if the other types communicate enough, they could all effectively converge to the same beliefs long before the hermit has any significant influence. If the outcome is realized between the time it takes for other types to converge and the time it takes for the hermit to influence beliefs, beliefs will be as if the hermit is not connected.

3 Persuasion and Asset Markets

In this Section we incorporate our model of persuasion into a model of asset markets. There are two reasons we believe this to be a natural and interesting application. First, most models of asset markets generate trading volume either through allocational motives, such as risk-sharing and liquidity, or through informational motives, such as private information about asset payoffs. In actuality, however, much trading volume seems to be generated because agents believe that they can process the same, public information better than others. Indeed, most speculative traders do not appear to have any source of private information regarding asset payoffs. Rather, they seem to make trading decisions based on media accounts, tips from friends, advice from analysts and market "experts," various data crunching or charting, etc. These traders thus seem to believe that their method of processing public information is better than that of the average trader. That is, they read the "right" media accounts, or get tips from the "right" set of people, or employ the "right" data crunching techniques, etc. Such notions readily correspond with our model of persuasion. Indeed, in our model, agents believe that only a subset of the population possesses useful information worth listening to, and that the beliefs of the remainder of the population consist of useless noise. Therefore, agents believe that they are better than others in identifying the useful information in the population.

Second, insofar as our model of persuasion endogenously characterizes influence as a function of the listening structure, it seems like an obvious setting to examine common questions relating to the role of influence in trading. We analyze to what extent influential traders move prices and can benefit from doing so. This allows us to examine the trade-off between influence and accuracy. We examine both how influential traders perform relative to accurate traders, i.e. traders who can accurately predict the asset payoff, and whether it is more beneficial for other traders to listen to those who are influential or to those who are accurate. We also consider to what extent influence can be self-perpetuating. Intuitively, when agents have differing equilibrium beliefs about expected payoffs,[22] agents' trades will on average be increasing in the difference between their beliefs and "average" population beliefs, represented in prices. If agents also communicate over time, however, then their beliefs will converge over time. Based on the model in Section 2, beliefs will converge towards beliefs that put greater weight on those agents with higher influence, and so prices will on average move towards the beliefs of influential agents. Thus, agents can profit by listening to the influential agents before their views are fully incorporated into prices. But if this leads agents to adjust their listening sets to include these influential agents, this will serve to increase further the influence of those agents. This positive feedback leads to "stable" listening structures in which all agents listen to the same market "guru."

In Section 3.1 we present an asset market model where agents both listen to each other and trade. In Section 3.2 we present behavioral rules, in line with our notion of persuasion, that the agents use to update beliefs and trade. In Section 3.3 we demonstrate that these behavioral rules can be rationalized as rational expectations equilibrium (REE) behavior for fully rational individuals operating under a "slight" misspecification of the world. Such a setting adds insight into our process of persuasion. In Section 3.4 we determine agents' payoffs, and show that it is beneficial to listen to both influential and accurate agents. Finally, in Section 3.5 we make the listening structure endogenous. We examine the trade-off between influence and accuracy, and obtain conditions under which influence is self-perpetuating.

3.1 The Asset Market Model

Agents can invest in a riskless and a risky asset. The rate of return on the riskless asset is zero. The risky asset is in zero net supply, and pays off

$$V + \varepsilon = \sum_{h=1}^{H} v_h \theta_h + \varepsilon: \qquad (11)$$

The first component, V, is a weighted sum of information coming from H "sources" such as company reports, macroeconomic announcements, etc. The information coming from source h is θ_h and is weighted by v_h. The θ_h's are independent and normal with mean zero and variance σ_θ^2. Therefore, V is normal with mean 0 and variance $\left(\sum_{h=1}^{H} v_h^2\right) \sigma_\theta^2$. We normalize both $\sum_{h=1}^{H} v_h^2$ and σ_θ^2 to 1.[23] The second component of the asset's payoff, ε, is independent

[22] Agents' beliefs will generally differ in equilibrium as long as agents do not believe prices are fully revealing.

[23] This is without loss of generality. We can first normalize σ_θ^2 to 1 by changing $\sum_{h=1}^{H} v_h^2$. We can then normalize $\sum_{h=1}^{H} v_h^2$ to 1 by rescaling the shares of the asset.

of the ε_h's and normal with mean 0 and variance σ_ζ^2.

Initially, agents observe information from each of the H sources. Agent m of type i observes a signal

$$s_{i_m,h} = \varepsilon_h + \eta_{i_m,h} \tag{12}$$

from source h. The noise $\eta_{i_m,h}$ is independent across sources and agents (and independent of all other variables in the model), and is normal with mean 0 and variance σ_η^2.[24]

To explore the effects we are interested in requires that we allow for multiple opportunities for both trade and communication. To keep the model as simple as possible, we allow for two rounds of trade. Prior to each opportunity for trade, communication occurs. Thus, the model can be described with four periods: at period 0, agents observe the signals. Between periods 0 and 1 there is one round of communication. At period 1 agents trade. Between periods 1 and 2 there are $T - 1$ rounds of communication, where $2 \leq T \leq \infty$. At period 2 agents trade again. Finally, at period 3 the asset pays off and agents consume.[25]

3.2 Behavioral Rules

In this Section we describe behavioral rules, in line with our notion of persuasion, that the agents use to update beliefs and trade. In Section 3.3 we will describe a misspecified model of the world under which the behavioral rules can be rationalized. At period 0 agents use their signals to form an initial estimate of the asset's payoff. A sufficient statistic of agent i_m's signals $s_{i_m,h}$, is the weighted sum of these signals with weights v_h. We assume that the agent uses some weights $v_{i,h}$ instead of the true weights v_h, and forms the sufficient statistic

$$s_{i_m} \equiv \sum_{h=1}^{H} v_{i,h}\, s_{i_m,h}. \tag{13}$$

The weights $v_{i,h}$ depend on i, and are thus type-specific. Since different types use different weights, they interpret the same information differently, and trade on the basis of their

[24] We introduce the private, agent-specific signals in order for agents to have different information and therefore a motive to communicate. It is important to note that the private signals are consistent with our statement that, in this model, agents trade not because of private information but because they interpret public information differently. Indeed, all agents have signals of the same quality, and understand this to be the case. Therefore, the signals do not introduce in and of themselves any motive to trade. Rather, different types differ on how they interpret these signals (see equation (13) below), and which other types' interpretations they listen to.

[25] Ideally, one could consider many rounds of trade, with communication between each round. Fortunately, the main intuitions emerge from this simpler setting. As we will see, what is critical is the total amount of communication that occurs prior to the last round of trade. It is for this reason that we will consider the effect of varying T.

interpretations. We introduce type-specific weights in order to allow for systematic differences in beliefs across types and, in particular, for differences in accuracy. The more accurate types use weights that are closer to the true weights. We assume that $\sum_{h=1}^{H} v_{i,h}^2$ is independent of i and equal to 1, as for the true weights. We also assume that $\sum_{h=1}^{H} v_{i,h} v_{j,h}$ is independent of i and j, and denote it by ρ. Finally, we use ρ_i to denote $\sum_{h=1}^{H} v_h v_{i,h}$. The parameter ρ_i belongs to the set $[-1, 1]$, and is close to 1 if type i's weights are close to the true weights. Therefore, ρ_i is a measure of type i's accuracy.[26]

Communication and updating is as in Section 2. At each communication round, agent i_m listens to one agent, randomly drawn from the set of agents whose types are in $S(i)$. We assume that agents communicate sufficient statistics of their information. We denote by $s_{i_m}^t$ the statistic of agent i_m at communication round t, and by s_i^t the average of $s_{i_m}^t$ over agents of type i. We refer to s_i^t as the statistic of type i at round t. The dynamics of s_i^t are given by the recursive formula

$$s_i^0 = \sum_{h=1}^{H} v_{i,h}\eta_h \equiv V_i \quad \text{and} \quad s_i^t = \frac{1}{2}s_i^{t-1} + \frac{1}{2}\left(\frac{1}{\#S(i)}\sum_{S(i)} s_j^{t-1}\right).$$

The intuition for this is the following. Initially, agent i_m forms his statistic s_{i_m}. Averaging over all agents of type i eliminates the idiosyncratic noise, $\varepsilon_{i_m,h}$. At each subsequent round, agent i_m learns the current statistic of an agent chosen at random from his listening set. Since both agents' statistics are based on an equivalent amount of information (they have each heard from the same number of other agents), they put equal weight on the two statistics. Finally, the idiosyncratic risk of the listening process is again eliminated by averaging over all agents of type i, so that we only need to consider the average statistic in $S(i)$.

Using vector notation and introducing the matrix $T(\lambda)$ defined in Section 2, we can write

[26] The assumptions that $\sum_{h=1}^{H} v_{i,h}^2$ is independent of i and $\sum_{h=1}^{H} v_{i,h} v_{j,h}$ is independent of i and j are for tractability. These assumptions imply some symmetry across types. Our goal is to have types as symmetric as possible, and differing only in their influence (measured by their weights in group beliefs) and accuracy (measured by ρ_i). Symmetry across types simplifies the computation of agents' payoffs in Theorem 7. We believe that our results would not be qualitatively altered if we allowed for less symmetry across types.

A technical question is whether there exist weights v_h and $v_{i,h}$ such that $\sum_{h=1}^{H} v_{i,h}^2 = 1$, $\sum_{h=1}^{H} v_{i,h} v_{j,h} = \rho$, and $\sum_{h=1}^{H} v_h v_{i,h} = \rho_i$. We can show that such weights exist if there are at least as many sources of information as types, and

$$\sum_{i=1}^{N}\left(\rho_i - \frac{\sum_{j=1}^{N}\rho_j}{N}\right)^2 \le (1-\rho)\left(1 - \frac{(\sum_{j=1}^{N}\rho_j)^2}{N(1+\rho(N-1))}\right).$$

The first condition ensures that there exist weights $v_{i,h}$ such that $\sum_{h=1}^{H} v_{i,h}^2 = 1$ and $\sum_{h=1}^{H} v_{i,h} v_{j,h} = \rho$. The second condition ensures that there also exist weights v_h such that $\sum_{h=1}^{H} v_h v_{i,h} = \rho_i$. The second condition bounds the dispersion in types' accuracies as a function of ρ. If, for instance $\rho = 1$, all types must use the same weights. Therefore, they must be equally accurate.

the dynamics of s_i^t as

$$s^0 = V \quad \text{and} \quad s^t = T\left(\frac{1}{2}\right) s^{t-1} = T\left(\frac{1}{2}\right)^t s^0 .$$

The dynamics of $\left(s_i^t\right)$ are thus exactly as in Section 2. Denote by w_{ij}^t the (i,j)-th element of the matrix $T\left(\frac{1}{2}\right)^t$; the influence of type j on type i as of round t. Thus,

$$s_i^t = \sum_j w_{ij}^t V_j; \tag{14}$$

and recall that the results of Section 2 imply that the weights w_{ij}^t sum to 1 and that, when types are strongly connected, $w_{ij}^t \to w_j$ as $t \to \infty$.

We assume that agents submit simple linear demand functions in each period. The demand of agent i_m is

$$\frac{1}{B_1}\left(s_{i_m}^1 - \frac{1}{A_1} p_1\right) \tag{15}$$

and

$$\frac{1}{B_2}\left(s_{i_m}^T + \frac{C_2}{A_2} p_1 - \frac{1}{A_2} p_2\right); \tag{16}$$

for some constants A_1, B_1, A_2, B_2, and C_2. The demand at period 1 depends on the agent's period 1 statistic $s_{i;m}^1$, and on the price. The demand at period 2 depends on the agent's period 2 statistic $s_{i;m}^T$, on the period 2 price, and on the period 1 price. Demand can depend on the period 1 price, since the price can affect, together with the period 2 statistic, the period 2 beliefs.[27] The market prices at periods 1 and 2 are obtained by aggregating demands. In particular, if we let

$$s^t = \frac{1}{N} \sum_{i=1}^{N} s_i^t$$

represent the average population statistic at round t, then market-clearing prices are given by

$$p_1 = A_1 s^1 \quad \text{and} \quad p_2 = A_2 s^T + C_2 p_1 . \tag{17}$$

The period 1 price is linear in the average statistic at round 1. The period 2 price is linear in the average statistic at round T, and in the period 1 price. Plugging prices into demands, we can find agents' positions in the risky asset. The positions of agent i_m at periods 1 and 2

[27] We assume linear demands because they are tractable and can be supported as part of an REE. However, the key aspect of demands is that they are increasing in the agent's beliefs and decreasing in the price, and it is this qualitative feature that drives the results.

are

$$y_1(s^1_{i_m}) \equiv \frac{1}{B_1}\left(s^1_{i_m} - s^1\right) \quad \text{and} \quad y_2(s^T_{i_m}) \equiv \frac{1}{B_2}\left(s^T_{i_m} - s^T\right); \tag{18}$$

respectively. The positions are linear in the difference between agent i_m's statistic and the average statistic. Thus, individuals trade based on the difference between their beliefs and average population beliefs.

We assume that $C_2 < 1$ and $A_2 + C_2 A_1 > A_1$. Using equation (17) that gives the prices at periods 1 and 2, we can give a natural interpretation to these inequalities. The first inequality implies that an increase in the round 1 average statistic, holding the round T average statistic constant, has a greater impact on the period 1 than on the period 2 price. The second inequality implies that an increase in both the round 1 and round T average statistics, has a greater impact on the period 2 than on the period 1 price. In Section 3.3 we show that both inequalities are satisfied when demands are optimal under a misspecified model of the world.

3.3 A Rationalization of the Behavioral Rules

In this Section we present a misspecified model of the world under which fully rational agents' actions are given by the above behavioral rules. We are agnostic as to whether agents' behavior as described in our model is likely to arise due to inaccurate updating (for example, due to the psychological impact of persuasive activity), or due to accurate updating under such a misspecification of the world. Rather, we present this setting here to demonstrate that the behavior we examine could be attributed to a relatively slight misspecification agents have of the world.

Suppose that agents have exponential utility functions over final wealth, and the following model of the world. First, an agent assumes that the asset's payoff is given by

$$V_i + \varepsilon = \sum_{h=1}^{H} v_{i,h}\theta_h + \varepsilon: \tag{19}$$

Note that this is identical to the true model of equation (11), save for the fact that the agent employs the type-specific weights $v_{i,h}$ instead of the true weights v_h. Second, the agent assumes that the world is made of two sets of types, “smart” and “dumb.” The smart types are the types that he listens to, i.e. the types in $S(i)$. The agent believes that these smart types act exactly like he does. In particular, he believes that they assume that the asset's payoff is given by equation (19), and that their listening set is $S(i)$. Dumb types, on the other hand, are taken to behave like "noise traders." They are believed to supply inelastically u_1 and u_2 units of the asset at periods 1 and 2. The variables u_1 and u_2 are independent of each other and of the other variables of the model. They are normal with mean 0 and standard deviation $\# S(i)\sigma_{u_1}$ and $\# S(i)\sigma_{u_2}$, respectively.[28] [29]

The first error in agents' model obviously rationalizes the use of the type-specific weights instead of the true weights, for forming the sufficient statistic at period 0. Intuitively, agents may inaccurately assess the relative weight that should be placed on chartist information versus industry information versus accounting information. The second error rationalizes the communication and the agents' manner of updating their information. Recall that central to our notion of persuasion is that agents only listen to some sources of information, and that they do not accurately adjust for the biases in their sources. Under this assumption, such behavior follows rationally. Agents only choose to listen to some types since they take others to be noise. Furthermore, agents weigh their information equally with those that they listen to, since they believe these agents behave exactly like themselves.[30]

The second error in agents' model also rationalizes the demands. Agents form demands in accord with a standard noisy REE computed according to their model of the world. In this equilibrium, smart agents submit demands that have the form of equations (15) and (16), and the dumb agents behave like noise traders. Agents submit price-elastic demands, and are thus willing to trade, since they assume that prices are partly driven by the noise traders. In theorem 6 we show that agents' demands are consistent with REE.

Theorem 6 There exist coefficients A_1, B_1, A_2, B_2, and C_2, such that equations (15) and

[28] A criticism of the assumption that dumb types behave like noise traders, is that the agent's model of the world becomes very different from the true model. Indeed, the agent is not aware that dumb types receive signals as per equation (12), and form beliefs by communicating with other types. An alternative assumption that addresses this criticism, is the following. The agent believes that some sources of information are pure noise, and gives them 0 weight. Moreover, he assumes that dumb types give positive weight only to these sources of information, and communicate only with dumb types. (For instance, fundamentalists might believe that chartists have access to the same sources of information that they have, but simply choose to base their beliefs on the "misleading" chartist sources.) The two assumptions are not completely equivalent, since the behavior of dumb types, and thus their effect on price, could be predicted by observing the sources of information to which they give positive weight. Rather than pursue this complication, we adopt the assumption that dumb types behave like noise traders for simplicity.

[29] We assume that the standard deviation of the noise is proportional to $\# S(i)$ for simplicity. It implies that agents share the same "signal-to-noise" ratio for the economy, and hence the coefficients A_1, A_2, B_1, B_2, and C_2, of the REE are the same across types. Intuitively, if $\# S(i)$ is large, type i believes that there are many smart types. He must also believe that the noise is large, in order to compute the same effect of the noise on the price as a type j with a small $\# S(j)$.

[30] It is worth reemphasizing that agents' listening sets may be determined by accessibility and transaction costs as well as an assessment about quality. In a setting where some agents are inaccessible, an agent should endeavor, over time, to place more weight on those who listen to sources that are not readily accessible, and less on those who listen to familiar sources. The failure to correctly adjust weights in this manner when some agents are inaccessible is another way to interpret our persuasion bias.

(16) are equilibrium demands in a REE computed according to the agent's model. Moreover, $C_2 < 1$ and $A_2 + C_2 A_1 > A_1$.

3.4 Agents' Payoffs

For simplicity we assume that agents start with zero positions in both the risky and the riskless asset. The consumption of agent i_m is

$$y_1(s^1_{i_m})(p_2 - p_1) + y_2(s^T_{i_m})(V + \zeta - p_2):$$

The first term is the capital gain between periods 1 and 2, while the second term is the capital gain between periods 2 and 3. Consumption is the sum of capital gains.

In theorem 7 we determine the expected consumption of agent i_m. We take expected consumption as a measure of the agent's payoff, both in this Section and in Section 3.5.[31] To state the theorem we introduce some notation. We denote by

$$w^t_j = \frac{1}{N}\sum_{k=1}^{N} w^t_{kj};$$

the "average" influence of type j over the population at communication round t. We denote by $Y_{i;1}$ the vector whose j'th component is

$$\frac{1}{B_1}\left(w^1_{ij} - w^1_j\right);$$

which measures type j's impact on type i's position at period 1. That is, i's position is determined by the amount i is influenced by j relative to the average amount the population is influenced by j. Similarly, we similarly denote by $Y_{i;2}$ the vector whose j'th component is

$$\frac{1}{B_2}\left(w^T_{ij} - w^T_j\right);$$

which measures type j's impact on type i's position at period 2. We denote by P_1 the vector whose j'th component is $A_1 w^1_j$: The j'th component of P_1 measures type j's impact on the period 1 price. We denote by P_2 the vector whose j'th component is

$$A_2 w^T_j + C_2 A_1 w^1_j;$$

[31] This is consistent with our emphasis on the behavioral rules as primitive. Under the rational interpretation of the behavioral rules, however, one could also consider expected utility as a criterion. While we use expected consumption for tractability, we can show that for σ^2_ζ and T are large, the expected consumption and expected utility criteria converge.

which measures type j's impact on the period 2 price. Finally, we denote by R the vector whose j'th component is ρ_j, i.e. the vector of types' accuracies. We then have the following result.

Theorem 7 *The expected payoff of agent i_m is*

$$(1-\rho)Y'_{i,1}(P_2-P_1)+Y'_{i,2}(R-(1-\rho)P_2). \qquad (20)$$

The agent's payoff decomposes neatly into two terms, that capture the benefits of listening to influential types and the benefits of listening to accurate types respectively. The first term corresponds to the capital gain between periods 1 and 2. This capital gain is large when the agent listens more than the average agent to influential types. Indeed, suppose that agent i_m listens to an influential type j. When j is optimistic, agent i_m will buy at period 1. Since j is influential, he will "pull" the price towards his beliefs. Therefore, the period 2 price will be higher than the period 1 price, and agent i_m will realize a capital gain. Formally, the j'th component of the vector $Y_{i,1}$ is positive (agent i_m buys when j is optimistic) since j has more influence on type i than on the average type, and the j'th component of the vector P_2-P_1 is positive (the period 2 price is higher than the period 1 price) since j has more influence on the period 2 beliefs than on the period 1 beliefs. The second term in equation (20) corresponds to the capital gain between periods 2 and 3. This capital gain is large when the agent listens to accurate types. Indeed, suppose that type j is accurate. When j's beliefs are optimistic, if agent i_m listens to agent j, agent i_m will buy at period 2. Since j is accurate, the true V will be high, and agent i_m will realize a capital gain. Formally, the j'th component of the vector $Y_{i,2}$ is positive and the j'th component of the vector R is large.

An interesting special case of theorem 7 is when there is an infinite number of communication rounds ($T=\infty$) and all types converge to the same belief. Since all types converge to the same belief, they all have a zero position in the risky asset at period 2, i.e. the vector $Y_{i,2}$ is equal to 0. The payoff of agent i_m consists only of the capital gain between periods 1 and 2, i.e. of the benefit of listening to influential types. Intuitively, there is no benefit of listening to accurate types for the following reason. The information of the accurate types is useful for establishing a position at period 2. However, at period 2 the accurate types have been "influenced" by the influential types. Since the beliefs of all agents converge, listening to an accurate type is no different than listening to any other type at that stage. In fact, in period 2 all agents hold identical portfolios (which are equal to zero in the case of zero net supply securities), independent of their initial accuracy, listening sets, or initial beliefs.

There is a benefit of listening to accurate types when there is a finite number of communication rounds. With a finite number of rounds, the accurate types are not "fully" influenced by the influential types. Beliefs will not have yet converged, and consequently, those with accurate models and those who listen to them will have, in expectation, more accurate valuations. Consequently, when the truth is revealed, they will stand to gain from their period 2 position. Whether this benefit of listening to accurate types outweighs the benefit of listening to influential types depends on the relative size of the two terms in equation (20).

Note that there could also be a benefit of listening to accurate types even with an infinite number of updating rounds provided that all types are not strongly connected. In this case beliefs of all types do not converge to the same point, and consequently, those whose beliefs have converged to a more accurate assessment can benefit in second period trading with those who have not.

3.5 Endogenous Listening Structure

In this Section we make the listening structure endogenous. We allow agents to change their listening set, and define "stable" listening structures where agents do not want to change that set. We next examine whether a particular class of listening structures, called "guru listening structures" (GLS), are stable, and then study the full set of stable listening structures. Finally, we derive the implications of our results for the persistence of influence and for the trade-off between influence and accuracy.

Definition 5 A stable listening structure S is such that the expected payoff of an agent does not increase when the agent changes his listening set. The new set has to include the agent's type.

We motivate this definition by the following evolutionary story. Suppose that the asset market model is repeated many times. An agent may question his assumption about who are the smart types, and conduct an experiment. He may adopt a different listening set and compute his average consumption after a large number of repetitions. For a large number of repetitions, average consumption is close to the agent's true payoff. If the listening structure is not stable, some agents will adopt a different listening set.

In theorem 8 we examine whether GLS are stable. In these listening structures, there is a "guru type", say type i. Agents of the guru type listen only to their type, i.e. $S(i) = \{i\}$. Agents of other types listen to their type and the guru type, i.e. $S(j) = \{i; j\}$ for $j \neq i$.

Theorem 8 *The GLS with guru type i is stable if and only if*

$$\frac{1-\rho_i}{B_1}\left[\left(1-\frac{3^T}{4^T}\right)A_2 - \frac{1}{4}\left(A_1 - C_2A_1\right)\right]$$

$$+\frac{1}{B_2}\frac{3^T-2^T}{4^{T-1}}\left[\rho_i + \min_{j\neq i}\rho_j - 2\max_{j\neq i}\rho_j - (1-\rho_i)\left[\left(1-\frac{3^T}{4^T}\right)A_2 + \frac{1}{4}C_2A_1\right]\right] \geq 0. \qquad (21)$$

Furthermore, there exists a smallest $T_i^* < \infty$ with the property that (21) holds for all $T > T_i^*$. Also, if (21) holds for i, then (21) holds for all j such that $\rho_j \geq \rho_i$.

For $T = \infty$, any GLS is stable, independently of the accuracy of the guru. The intuition is that when $T = \infty$ there is no benefit of listening to accurate types. There is only a benefit of listening to influential types, and the guru is the only influential type.

For finite T, equation (21) may or may not be satisfied, depending on types' accuracies. It is harder to satisfy when ρ_i is small and $\max_{j\neq i}\rho_j$ large, that is, when the guru type is not very accurate and the best non-guru type is very accurate. Intuitively, agents may drop the guru type and listen to the best non-guru type. Equation (21) is also harder to satisfy when $\min_{j\neq i}\rho_j$ is small, i.e. when the worst non-guru type is very inaccurate. To understand this result, assume that an agent of the worst non-guru type decides to listen to the best non-guru type, in addition to his type and the guru type. Since the cardinality of the agent's listening set increases, the weight on his type will decrease. If the agent's type is very inaccurate, the agent's payoff will increase.[32]

We next study the full set of stable listening structures for large T. In theorem 9 we provide necessary conditions for a listening structure to be stable. These conditions are generic in types' accuracies, i.e. they hold for all ρ_i except a set of measure 0.

Theorem 9 *Generically in types' accuracies:*

(i) For large but finite T, only GLS listening structures can be stable.

(ii) For $T = \infty$, any GLS is stable, but other stable structures are possible. Any stable listening structures has the following properties: (a) there is a set I such that $S(i) \subseteq I$ for $i \in I$ and $S(i) = I \cup \{i\}$ for $i \notin I$, (b) I is strongly connected, and (c) $(P_2 - P_1)_i$ is independent of i for $i \in I$.

[32] Intuitively, if one's own type is not very accurate, it pays to associate with other more accurate types as well as the guru type in part to reduce the amount of time spent talking to one's own type.

For large but finite T, all stable listening structures are GLS. To understand the intuition for this result, we proceed in two steps. First, we show that we can rank types according to the benefit of listening to them, which is a combination of their influence and accuracy. Generically in types' accuracies, the ranking is strict. Thus we can without loss of generality label the types $1, 2, \ldots, N$ in order of decreasing benefit. An agent of type n listens to his type, since he is constrained to, and to type 1. He may also listen to "intermediate" types, between 1 and n, since this reduces the weight of his type. Therefore, for each n

$$S(n) = \{1, 2, \ldots, n'\} \cup \{n\} \quad \text{for some} \quad n' < n. \tag{22}$$

To complete the proof it remains to show that $n' = 1$ for all n. Suppose not; i.e. suppose that some type other than type 2 also listens to type 2. We show that this implies that the benefit of listening to type 2 must be less than the benefit of listening to type N, contradicting our initial ordering. The reason for this is that for large T, the benefit of a type derives mostly from the type's influence rather than its accuracy. This influence of a type can be measured by the impact of the type's beliefs on the difference between the period 2 price and the period 1 price. Since type 2 is listened to by more types than type N, 2's beliefs have a larger impact on the period 1 price. Moreover, the beliefs of both types have a very small impact on the period 2 price, since beliefs converge to the beliefs of type 1. Thus, the influence of type 2 would be smaller than the influence of type N, a contradiction. Hence a stable structure must be a GLS for large finite T.

For $T = \infty$, the stable configurations can be characterized by a "guru set" I such that agents listen only to their own type and types in this set. Second, the guru set is strongly connected, and thus all types converge to the same belief. Third, all types in the guru set have the same influence. Notice that GLS satisfy the necessary conditions, with I the singleton that contains the guru. However, and in contrast to the case where T is finite but large, larger guru sets are possible. Such listening structures involve ties, i.e. the benefit of listening to all types in I is the same. Ties are generic in types' accuracies, since accuracy does not matter when $T = \infty$ and all types converge to the same belief.

Theorems 8 and 9 imply that for large T, influence is self-perpetuating. Indeed, theorem 8 implies that for large T, GLS are stable, regardless of the guru's accuracy. Moreover, theorem 9 implies that for large but finite T, GLS are the only stable listening structures. For $T = \infty$ other listening structures can also be stable, but the necessary conditions are independent of types' accuracies.[33]

[33] For general values of T, we are unable to determine the full set of stable listening structures. However, using theorem 8, we can obtain some results on the trade-off between influence and accuracy. First, as T

4 Conclusion

In this paper we have analyzed a model of social updating and communication meant to capture a wide range of persuasive processes. In particular, our agents, cognizant that they possess limited information, communicate with one another and update their beliefs accordingly. Agents use plausible rules to update their beliefs, but do not give enough weight to the views of agents they do not come into contact with.

Such an updating process admits several different natural interpretations. Agents may end up overweighting those they are in contact with in the manner prescribed in our model simply because they do not consider the possibility that the views of those they listen to may not be representative of all available information. Even if agents understand such views are not necessarily representative, agents might not be able to undertake the complicated inference and updating problem of trying to infer from their reports whom the agents that they listen to are themselves listening to. Instead, upon hearing a similar opinion from two sources, an agent simply counts this as two pieces of useful information without trying to determine whether one of the sources was influenced (either directly or indirectly) by the other. Finally, we can also give our process the interpretation (as we have in our finance application) that agents believe that only some fraction of other agents have useful information, and that not all agents are correct about who has useful models of who does not.

Our view is that such biases are widespread, and reflected in many processes and the design of many institutions, ranging from debating procedures to court trials to marketing. In perhaps every activity where agents are engaged in persuasive activity, they seem to care greatly about the amount of contact they have with the targets of their persuasion. It is very hard to rationalize this without some notion that repeated interaction, on average, affects beliefs in predictable manners. Capturing this notion in as simple and natural a manner as possible is the essence of our model.

An interesting question is whether such a persuasion bias in updating is due to a psychological tendency for overweighting one's own experiences/peers/contacts/..., or whether it can be rationalized from a more foundational behavioral setting, perhaps as a boundedly rational response to information in an incomplete setting.[34] For many day-to-day topics, an agent's peer group may indeed contain of all individuals with useful information, or might indeed be representative of the universe of information. Consequently, applying such an up-

becomes smaller, the number of stable GLS decreases. Second, as the number of stable GLS decreases, those with less accurate guru types become unstable first.

[34] These two views, of course, need not be mutually exclusive.

dating rule might often be quite rational. One could further imagine that it is likely to be very taxing for an agent to try to discern in what settings their sources of information are representative of all useful information and in what settings outside expertise beyond their circles would be valuable, especially if the question at hand concerns a new topic that the agent has little idea how to evaluate. Loosely speaking, an agent facing processing costs might be best served by following a simple updating rule, and perhaps, over time changing who she listens to based on the outcome. We are currently considering such a foundational basis for our model.

While remaining somewhat agnostic about the source of our \persuasion bias"," we analyze its consequences, both in a general setting and in an asset market setting. In general, we analyze convergence of beliefs. Under the condition of strong connectedness, beliefs of all agents converge to a weighted average of initial beliefs, where the weights associated with each type of agent have a number of rather striking features. These weights, which are readily interpreted as in°uence, follow both from characterizing the stationary distribution of a Markov matrix that represents the listening structure, and from two di®erent manners of \counting trees" which derive from the graphical nature of the listening structure. Intuitively, agents are in°uential if other in°uential agents listen to them, where in°uence is de¯ned endogenously through the listening structure. In the asset market setting, we explore the trade-o® between in°uence and accuracy. Our analysis indicates that listening structures that do not necessarily aggregate information e±ciently can be \stable". For example, provided that su±cient updating occurs before more information on fundamentals is learned, it can be in everyone's interests to listen to one market \guru" due to the fact that others are listening to him, despite his being very inaccurate.

A few words on the testing of our model are probably in order. We readily acknowledge that a satisfactory empirical test of the implications of our model is likely to be di±cult. Our model yields a number of precise implications for beliefs and in°uence. However, these predictions follow from the primitive of the listening structure. In most empirical settings, information on the listening structure seems likely to be hard to come by, making satisfactory empirical testing di±cult.

In contrast, however, we would argue that our model is very well suited for experimental testing. Precise listening structures should be easy to create in an experimental setting. Our model yields precise and unique implications for the propagation of beliefs in general, and market trading in particular, as a function of this listening structure. Furthermore, our model yields unique and precise predictions for how simple changes in the listening structure { such as removing or adding a listening link { will impact the beliefs of an entire population. It is worth noting that the quantitative nature of our model's predictions go well beyond the predictions of many behavioral models. Consequently, we believe that testing some of these implications experimentally would be quite interesting and would provide for a strong test of our framework. But perhaps this is because we have listened to one another too much, and have therefore been unduly persuaded about the merits of our ideas.

Appendix – Proofs

Proof to Theorem 2: We first show that (average) type beliefs converge to $w'x'$. It is well-known that the matrix T^t converges to a limit T^∞ when t goes to ∞, and that each row of T^∞ is equal to w' (see, for example, Theorem 8.8 in Billingsley (1986)). To show that type beliefs converge to $w'x'$, we will show that the matrix $\prod_{s=1}^{t} T(\lambda_s)$ converges to T^∞. Define the random variable Λ_t to be equal to 1 with probability λ_t and 0 otherwise. Assume also that Λ_t are independent over time. Define the (random) matrix Z_t by

$$Z_t = \prod_{s=1}^{t} [(1 - \Lambda_s) I + \Lambda_s T] = T^{\sum_{s=1}^{t} \Lambda_s}.$$

Then

$$E(Z_t) = \prod_{s=1}^{t} T(\lambda_s).$$

By the Borel-Cantelli lemma, if $\sum_{t=1}^{\infty} \lambda_t = \infty$, then

$$Pr(\Lambda_t = 1 \text{ infinitely often}) = Pr\left(\sum_{t=1}^{\infty} \Lambda_t = \infty\right) = 1.$$

Since the matrix T^t is bounded uniformly in t, the dominated convergence theorem implies that

$$\lim_{t\to\infty} E(Z_t) = \lim_{t\to\infty} E\left(T^{\sum_{s=1}^{t} \Lambda_s}\right) = T^\infty.$$

We next show that the beliefs of all agents converge to $w'x'$. We fix a time t_1 and define two processes for each agent i_m, both starting at t_1. The first process is

$$\varepsilon^t_{i_m} = (1 - \lambda_t)\varepsilon^{t-1}_{i_m} + \lambda_t \varepsilon^{t-1}_{j_{m(t)}}; \quad \varepsilon^{t_1}_{i_m} = x^{t_1}_{i_m} - x^{t_1}_i; \tag{23}$$

and the second is

$$y^t_{i_m} = (1 - \lambda_t) y^{t-1}_{i_m} + \lambda_t y^{t-1}_{j_{m(t)}}; \quad y^{t_1}_{i_m} = x^{t_1}_i; \tag{24}$$

where $j_{m(t)}$ denotes the agent that i_m listens to at time t. It is easy to check that

$$x^t_{i_m} = \varepsilon^t_{i_m} + y^t_{i_m};$$

and thus the two processes are a decomposition of the agent's beliefs. The first process describes the evolution of beliefs if at time t_1 agents within a type have different beliefs, but (average) type beliefs are identical and equal to 0. The second process describes the evolution of beliefs if at time t_1 agents within a type have the same beliefs, which are equal to the true type beliefs $x^{t_1}_i$. (Subsequent to t_1, agents within a type will have different beliefs because they will randomly listen to different types.) The first process represents the "initial" heterogeneity, i.e. the within type heterogeneity induced from the heterogeneity at t_1. The second process represents the "random listening" heterogeneity, i.e. the within type heterogeneity induced from random listening subsequent to t_1.

For $z = \varepsilon; y; x$, we denote by $\sigma_{z_i}^t$ the standard deviation of z_{im}^t across agents of type i, i.e.

$$(\sigma_{z_i}^t)^2 \equiv \int_0^1 \left(z_{im}^t - \int_0^1 z_{im'}^t \, dm' \right)^2 dm.$$

We also denote by σ_z^t the maximum of $\sigma_{z_i}^t$ across types. We use σ_ε^t, σ_y^t, and σ_x^t as measures of the initial, random listening, and total within type heterogeneity at time t. Since the processes ε_{im}^t and y_{im}^t are independent across agents of type i, we have

$$(\sigma_{x_i}^t)^2 = (\sigma_{\varepsilon_i}^t)^2 + (\sigma_{y_i}^t)^2.$$

Therefore,

$$(\sigma_x^t)^2 \le (\sigma_\varepsilon^t)^2 + (\sigma_y^t)^2.$$

We first show that σ_ε^t goes to 0 when t goes to ∞. Equation (23) implies that

$$(\sigma_\varepsilon^t)^2 \le (1-\lambda_t)^2 (\sigma_\varepsilon^{t-1})^2 + (\lambda_t)^2 (\sigma_\varepsilon^{t-1})^2 = (1-2\lambda_t(1-\lambda_t))(\sigma_\varepsilon^{t-1})^2.$$

Therefore, σ_ε^t goes to 0, if

$$\prod_{s=t_1+1}^{t} (1-2\lambda_s(1-\lambda_s))$$

goes to 0, or its logarithm goes to $-\infty$. The logarithm is

$$\sum_{s=t_1+1}^{t} \log(1-2\lambda_s(1-\lambda_s)) \le -2\sum_{s=t_1+1}^{t} \lambda_s(1-\lambda_s) \le -2(1-\lambda^1)\sum_{s=t_1+1}^{t} \lambda_s;$$

and goes to $-\infty$.

We next show that by choosing t_1 large enough, we can make σ_y^t arbitrarily small. Define s_y^t by

$$s_y^t \equiv \sup_{im} |y_{im}^t - w'x'|.$$

Since the y_{im}^t are obtained by convex combinations of y_{im}^{t-1}, s_y^t cannot increase over time. Therefore,

$$s_y^t \le s_y^{t_1} = \sup_i |x_i^{t_1} - w'x'|.$$

Since all types converge to $w'x'$, s_y^t and thus σ_y^t can be made arbitrarily small for t_1 large enough. Therefore, σ_x^t goes to 0 when t goes to ∞. ∎

Proof to Corollary 1: The hypothesis implies $\#S(i) = n$ for all i and $\#S^{-1}(i) = m$ for all i. Then by an "adding up" constraint, $m = n$. Let $w = (1/N, \ldots, 1/N)$. Then for all j,

$$\sum_i w_i T_{ij} = \frac{1}{N}\sum_i T_{ij} = \frac{1}{N}\sum_{\{i: j \in S(i)\}} \#S(i)^{-1} = \frac{1}{N}\frac{m}{n} = \frac{1}{N}:$$

thereby showing that w is the stationary distribution of T. ∎

Proof to Theorem 3: First, suppose that for all i $\sum_j g_j \delta_{ji} = g_i \sum_j \delta_{ij}$. Then to show that

w is a stationary distribution, we need to show that for all j, $\sum_i w_i T_{ij} = w_j$. Using the second definition of w_i,

$$\sum_i w_i T_{ij} = \frac{1}{C}\sum_i \frac{g_i(\sum_j \delta_{ij})\delta_{ij}}{\#S(i)} = \frac{1}{C}\sum_i g_i \delta_{ij} = w_j;$$

where we use the first definition of w_j and the fact that $\#S(i) = \sum_j \delta_{ij}$.

Thus it remains to show that $\sum_j g_j \delta_{ji} = g_i \sum_j \delta_{ij}$.

For a given listening structure, consider the set Y_i of all admissible trees with any root k such that $\delta_{ki} = 1$. We can apply the following procedure to this tree:

B_i: If $k = i$, leave the tree unchanged. Otherwise, delete the link from i to its predecessor, and add a link from the root to i. This creates an i-tree.

Next, starting with any i-tree, consider the following operation defined for any j:

A_j: If $j = i$, leave the tree unchanged. Otherwise, there exists a k such that i immediately succeeds k and j is in the subtree with root k (it is of course possible that $j = k$). Delete the link from k to i, and add a link from i to j. Note that this yields a new tree with root k.

It is easy to verify the following: (1) if $A_j(x) = y$, then $B_i(y) = x$, and (2) if $B_i(y) = x$, then $A_j(x) = y$ for some j such that $\delta_{ij} = 1$.

By construction, $\#Y_i = \sum_j g_j \delta_{ji}$. Alternatively, from the above we can construct Y_i by starting from each i-tree and applying A_j for each j such that $\delta_{ij} = 1$. Properties (1) and (2) above guarantee that this construction is 1-1 and onto respectively. Hence, $\#Y_i = g_i \sum_j \delta_{ij}$, proving the result. ∎

Proof to Theorem 4: Follows immediately by noting that if I is isolated, then types in I only listen to one another, and therefore, theorem 2 applies directly to this subset of types. ∎

Proof to Theorem 5: First we show that at least one element of $\mathcal{I}$ is isolated. Suppose not. For any $I, J \in \mathcal{I}$, we say that I listens to J if there exists $i \in I$, $j \in J$ such that $j \in S(i)$. Consider the graph among members of $\mathcal{I}$ defined by this listening structure. Since by assumption, no element is isolated, it follows that each element of $\mathcal{I}$ listens to at least one other element. However, since $\mathcal{I}$ contains a finite number of elements, this implies at least one cycle in the listening structure consisting of more than one element. But then all types belonging to elements in this cycle are strongly connected, and therefore $\mathcal{I}$ is not the partition defined by strong connectedness. Hence, the subset $\mathcal{J}$ of isolated members of $\mathcal{I}$ is nonempty.

Point 1 of the theorem follows immediately from theorem 4. To show point 2, we consider a type $i \in \mathcal{I}$, and note that from point 3, i's beliefs converge to $w(J)'x'(J)$. To show that all agents in i converge to $w(J)'x'(J)$, we proceed as in theorem 2. To show point 3, we denote by $x(i)$ the point to which beliefs of type $i \in \mathcal{I}$ converge, $L(i)$ the set of all types to which type i is linked, $I(i) \subset N$ the set of all types belonging to strongly connected isolated sets to which type i is linked, and $CH(S) \subset \mathbb{R}^p$ the convex hull of any set of points $S \in \mathbb{R}^p$. Abusing notation, we denote by $x(J)$ the image under x of a set of types J. First note that all types converge to a point in the interior of the convex hull of the convergent points of those to whom they listen, since the vector of convergent beliefs x is given by $Tx = x$.

We now claim that for any i, $CH(x(I(i))) = CH(x(L(i)))$. $CH(x(I(i))) \subseteq CH(x(L(i)))$ is immediate, since by definition, $I(i) \subseteq L(i)$. Now suppose $CH(x(I(i))) \not\supseteq CH(x(L(i)))$. Then it must follow that there exists some type $j \in L(i)$ that converges to a point x that is an extreme point of $CH(x(L(i)))$ and is not in $CH(x(I(i)))$. The type $j \in L(i)$ which converges to this point j cannot belong to an isolated set, since he is not in $I(i)$. Therefore, he must listen to other types. But since all types converge to a point in the interior of the convex hull of the convergent points of those to whom they listen, these types that j listens to must all converge to x as well, or one of them must converge to a point outside $CH(x(L(i)))$. Both of these possibilities yield a contradiction. In particular, for the latter case, i is linked to any type that j listens to, and therefore, any such type must converge to a point inside $CH(x(L(i)))$. For the former, any type that j listens to that converges to x as well must in turn be linked to another new type that also converges to this point, since none of these types can belong to an isolated set of types, by the reasoning above. And these types must in turn be linked to other such types. But there are a finite number of types, implying a contradiction. Hence $CH(x(I(i))) = CH(x(L(i)))$, and since i converges to a point in the interior of $CH(x(L(i)))$, we are done. ‖

Proof to Theorem 6: The equilibrium computed according to agent i_m's model is as follows. The period 1 price is

$$p_1 = A_1 \left(V_i - B_1 \frac{u_1}{\#S(i)} \right); \tag{25}$$

and the period 2 price is

$$p_2 = A_2 \left(V_i - B_2 \frac{u_2}{\#S(i)} \right) + C_2 p_1. \tag{26}$$

The demands of agent i_m are given by equations (15) and (16). The demands of any other smart agent are given by the same equations, but for that agent's statistics rather than agent i_m's. To show that prices and demands constitute a rational expectations equilibrium, we need to show that demands are optimal given prices and that prices clear the market. We first show market-clearing. We next show that demands are optimal given prices, provided that the constants A_1, B_1, A_2, B_2, and C_2 satisfy a system of non-linear equations. We finally show that this system has a solution.

Market-Clearing

According to the model of agent i_m, the statistic of a smart agent at round t is equal to V_i plus idiosyncratic noise, which is normal with variance $\sigma_\epsilon^2/2^t$. Indeed, the initial statistic is equal to V_i plus idiosyncratic noise, which is normal with variance σ_ϵ^2. Moreover, the statistic at round t is obtained by averaging two round $t-1$ statistics.

To show market-clearing at period 1, we note that the aggregate demand of the smart agents is

$$\frac{\#S(i)}{B_1} \left(V_i - \frac{1}{A_1} p_1 \right).$$

This is because the average statistic of the smart agents is V_i, and the mass of smart agents is $\#S(i)$. The demand of the dumb agents is $-u_1$. The market thus clears if the price p_1 is given by equation (25). Market-clearing at period 2 follows similarly.

Demands are Optimal Given Prices

We denote by y_1 and y_2 the first and second period positions of agent i_m in the risky asset. Agent i_m's consumption is

$$y_1(p_2 - p_1) + y_2(V_i + \varepsilon - p_2).$$

At period 2, agent i_m chooses his demand y_2 to maximize

$$-E_2 \exp(-a y_2(V_i + \varepsilon - p_2)) = -\exp(-a(y_2(E_2(V_i) - p_2) - \frac{1}{2} a y_2^2 (Var_2(V_i) + \sigma_\varepsilon^2))).$$

The optimal demand is

$$y_2 = \frac{E_2(V_i) - p_2}{a(Var_2(V_i) + \sigma_\varepsilon^2)}. \tag{27}$$

The subscript 2 (1) in an expectation or variance means that these are conditional on information available at period 2 (1). This information consists of the round 0 statistic of agent i_m, the round 0 statistics of the agents that agent i_m has listened to, both directly and indirectly, and the period 1 and 2 prices. We denote by $S^t(i_m)$ the set that includes agent i_m and the agents he has listened to, both directly and indirectly, by round t. The cardinality of $S^t(i_m)$ is 2^t. We denote by τ_η the precision of the round 0 statistic of a smart agent, i.e. $\tau_\eta = 1/\sigma_\eta^2$. Finally, we denote by τ_{p_1} and τ_{p_2} the precisions of $B_1 u_1/\#S(i)$ and $B_2 u_2/\#S(i)$, respectively, i.e. $\tau_{p_1} = 1/(B_1\sigma_{u_1})^2$ and $\tau_{p_2} = 1/(B_2\sigma_{u_2})^2$. We have

$$1/Var_2(V_i) \equiv \tau_2 = 1 + 2^T\tau_\eta + \tau_{p_1} + \tau_{p_2}.$$

and

$$E_2(V_i) = \frac{2^T\tau_\eta}{\tau_2}\frac{\sum_{S^T(i_m)} s_{j,0}}{2^T} + \frac{\tau_{p_1}}{\tau_2}\frac{1}{A_1}p_1 + \frac{\tau_{p_2}}{\tau_2}\frac{1}{A_2}(p_2 - C_2 p_1). \tag{28}$$

Using equation (28), and noting that $\sum_{S^T(i_m)} s_{j,0}/2^T = s^T_{i_m}$, it is easy to check that the demand (16) coincides with the optimal demand 27 if

$$B_2 = \frac{a\left(\frac{1}{\tau_2} + \sigma_\varepsilon^2\right)\tau_2}{2^T\tau_\eta}, \tag{29}$$

$$A_2 = \frac{2^T\tau_\eta + \tau_{p_2}}{\tau_2}, \tag{30}$$

and

$$C_2 = \frac{\tau_{p_1}}{\tau_2}\frac{1}{A_1}. \tag{31}$$

At period 1 agent i_m chooses his demand y_1 to maximize

$$-E_1 \exp(-a(y_1(p_2 - p_1) + y_2(E_2(V_i) - p_2) - \frac{1}{2} a y_2^2\left(\frac{1}{\tau_2} + \sigma_\varepsilon^2\right))),$$

where the subscript 1 means that the expectation is conditional on information available at round 1 and on the price p_1. To simplify this equation, we first substitute y_2 from equation

27. We get

$$-E_1\exp\left(-a\left(y_1(p_2 - p_1) + \frac{(E_2(V_i) - p_2)^2}{2a(\frac{1}{\tau_2}+\sigma_3^2)}\right)\right).$$

We next substitute p_2 and $E_2(V_i)$ from equations (25) and (28) respectively. Using equations (30) and (31) and setting

$$\beta = \left(\frac{2^T\tau_\epsilon}{\tau_2}\right)^2 \frac{1}{a\left(\frac{1}{\tau_2}+\sigma_3^2\right)};$$

we get

$$-E_1\exp\left(-a\left(y_1\left(A_2\left(V_i - B_2\frac{u_2}{\#S(i)}\right)+C_2p_1 - p_1\right)+\frac{1}{2}\beta\left(\frac{\sum_{S^T(i_m)}\mathbf{s}_{j_m\mathbf{o}}}{2^T} - V_i + B_2\frac{u_2}{\#S(i)}\right)^2\right)\right).$$

We finally separate the signals in $S^1(i_m)$ that the agent knows at period 1, and the signals in $S^T(i_m)\backslash S^1(i_m)$ that the agent will obtain between periods 1 and 2. Setting

$$\epsilon = \frac{\sum_{S^T(i_m)\backslash S^1(i_m)}\sum_{h=1}^{H}\mathbf{v}_{i,h}\epsilon_{j_m\mathbf{o},h}}{2^T};$$

we get

$$-E_1\exp\left(-a\left(y_1\left(A_2\left(V_i - B_2\frac{u_2}{\#S(i)}\right)+C_2p_1 - p_1\right)\right.\right.$$
$$\left.\left.+\frac{1}{2}\beta\left(\epsilon + \left(\frac{\sum_{S^1(i_m)}\mathbf{s}_{j_m\mathbf{o}}}{2} - V_i\right)\frac{1}{2^{T-1}} + B_2\frac{u_2}{\#S(i)}\right)^2\right)\right). \quad (32)$$

The expectation in equation (32) has to be computed w.r.t. V_i, u_2, and ϵ.

We will use the formula

$$E(\exp(-a(b_0 + b'x + \frac{1}{2}x'Cx))) = \frac{1}{\sqrt{|I + aC\Sigma^2|}}\exp(-a(b_0 - \frac{1}{2}ab'\Sigma^2(I + aC\Sigma^2)^{-1}b)); \quad (33)$$

where x is a $n\times 1$ normal vector with mean 0 and variance-covariance matrix Σ^2, I the $n\times n$ identity matrix, b_0 a number, b an $n\times 1$ vector, and C an $n\times n$ symmetric matrix. We set $n = 2$,

$$x_1 = (V_i - E_1(V_i)) - B_2\frac{u_2}{\#S(i)};$$

$$x_2 = \epsilon - (V_i - E_1(V_i))\frac{1}{2^{T-1}} + B_2\frac{u_2}{\#S(i)};$$

$$b_1 = y_1A_2;$$

$$b_2 = \beta\left(\frac{\sum_{S^1(i_m)}\mathbf{s}_{j_m\mathbf{o}}}{2} - E_1(V_i)\right)\frac{1}{2^{T-1}};$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix};$$

and

$$b_0 = y_1\left(A_2E_1(V_{-}) + C_2p_1 - p_1\right) + \frac{1}{2}\beta\left(\sum\frac{S^1(i_m)\,S'_{j_m}}{2} - E_1(V_{-})\right)^2\frac{1}{2^{2(T-1)}}.$$

Omitting the terms that do not depend on y_1 and thus do not affect the optimization w.r.t. y_1, we can write equation (32) as

$$-\exp\left(-a\left(y_1\left(A_2E_1(V_{-}) + C_2p_1 - p_1\right) + y_1A_2k_2\left(\sum\frac{S^1(i_m)\,S'_{j_m}}{2} - E_1(V_{-})\right) - \frac{1}{2}y_1^2A_2^2k_3\right)\right), \quad (34)$$

where the constants k_2 and k_3 are

$$k_2 = \frac{a\left(\frac{1}{2^{T-1}\tau_1} + \frac{1}{\tau_{p_2}}\right)\beta}{1 + a\beta\left(Var(\eta) + \frac{1}{2^{2(T-1)}\tau_1} + \frac{1}{\tau_{p_2}}\right)}\frac{1}{2^{T-1}},$$

and

$$k_3 = a\frac{\frac{1}{\tau_1} + \frac{1}{\tau_{p_2}} + a\beta\left[Var(\eta)\left(\frac{1}{\tau_1} + \frac{1}{\tau_{p_2}}\right) + \frac{1}{\tau_1\tau_{p_2}}\left(1 - \frac{1}{2^{T-1}}\right)^2\right]}{1 + a\beta\left(Var(\eta) + \frac{1}{2^{2(T-1)}\tau_1} + \frac{1}{\tau_{p_2}}\right)},$$

and

$$\tau_1 \equiv Var_1(V_{-}) = 1 + 2\tau_\eta + \tau_{p_1}.$$

The agent chooses his demand y_1 to maximize (34). The optimal demand is

$$y_1 = \frac{A_2E_1(V_{-}) + C_2p_1 - p_1 + A_2k_2\left(\sum\frac{S^1(i_m)\,S'_{j_m}}{2} - E_1(V_{-})\right)}{A_2^2k_3}. \quad (35)$$

To identify the demand (35) with the demand (15), we have to compute $E_1(V_1)$. We have

$$E_1(V_{-}) = \frac{2\tau_\eta}{\tau_1}\frac{\sum S^1(i_m)\,S'_{j_m}}{2} + \frac{\tau_{p_1}}{\tau_1}\frac{1}{A_1}p_1. \quad (36)$$

Using equations (30), (31), and (36), it is easy to check that the demand (15) coincides with the demand (35) if

$$B_1 = \frac{A_2k_3}{\frac{2\tau_\eta}{\tau_1} + k_2\left(1 - \frac{2\tau_\eta}{\tau_1}\right)}, \quad (37)$$

and

$$A_1 = \frac{2^T\tau_\eta + \tau_{p_2}}{\tau_2}\left(\frac{2\tau_\eta + \tau_{p_1}}{\tau_1} + k_2\frac{1}{\tau_1}\right) + \frac{\tau_{p_1}}{\tau_2}. \quad (38)$$

To show that $C_2 < 1$ and $A_1 < A_2 + C_2A_1$ we use equations (30), (31), (38), and the fact that $k_2 < 1$.

The System has a Solution

Equations (29), (30), (31), (37), and (38), constitute a system of 5 non-linear equations in the 5 unknowns A_1, B_1, A_2, B_2, and C_2. The unknowns A_1, A_2, and C_2 can be determined from B_1 and B_2 using equations (38), (30), and (31), respectively. To determine B_1 and B_2, we study the system of equation (29) and equation

$$B_1 = \frac{\left(\frac{2^T\tau_\epsilon + \tau_{p_2}}{\tau_2} k_3\right)}{\frac{2\tau_\epsilon}{\tau_1} + k_2\left(1 - \frac{2\tau_\epsilon}{\tau_1}\right)}; \tag{39}$$

which is obtained by substituting A_2 into equation (37). Using the definitions of τ_{p_1}, τ_{p_2}, and τ_2, we can write equation (29) as

$$B_2 - \frac{a\left(1 + \left(1 + 2^T\tau_\epsilon + \frac{1}{B_1^2\sigma_{u_1}^2} + \frac{1}{B_2^2\sigma_{u_2}^2}\right)\sigma_3^2\right)}{2^T\tau_\epsilon} = 0.$$

The LHS is increasing in B_2, goes to $-\infty$ when B_2 goes to 0, and goes to ∞ when B_2 goes to ∞. Therefore, equation (29) has a unique solution $B_2(B_1)$. $B_2(B_1)$ is decreasing in B_1, is of order $1/B_1^2$ when B_1 is small, and goes to a strictly positive limit when B_1 goes to ∞. Consider now equation (39) where we replace B_2 by $B_2(B_1)$. It is easy to check that the RHS is of order $1/B_1^2$ when B_1 is small, and goes to a strictly positive limit when B_1 goes to ∞. Therefore, equation (39) has a solution B_1. ∥

Proof to Theorem 7: Using equations (17) and (18), we can write agent i_m's consumption as

$$\frac{1}{B_1}(s^1_{i_m} - s^1)(A_2 s^T + (C_2A_1 - A_1)s^1) + \frac{1}{B_2}(s^T_{i_m} - s^T)(V + \epsilon - A_2 s^T - C_2A_1 s^1). \tag{40}$$

Taking expectations w.r.t. the idiosyncratic noise, we get

$$\frac{1}{B_1}(s^1_i - s^1)(A_2 s^T + (C_2A_1 - A_1)s^1) + \frac{1}{B_2}(s^T_i - s^T)(V + \epsilon - A_2 s^T - C_2A_1 s^1).$$

Using equation (14) and the definitions of $y_{i;1}$, $y_{i;2}$, p_1, and p_2, we get

$$\left(y'_{i;1}V\right)\left[(p_2 - p_1)'V\right] + \left(y'_{i;2}V\right)(V + \epsilon - p_2 V).$$

Taking expectations w.r.t. V and ϵ, and denoting by $\mathrm{cov}(V)$ the variance-covariance matrix of V, we get

$$y'_{i;1}\mathrm{cov}(V)(p_2 - p_1) + y'_{i;2}(R - \mathrm{cov}(V)p_2).$$

To obtain the theorem, we note that the elements of both $y_{i;1}$ and $y_{i;2}$ sum to 0, that all diagonal elements of $\mathrm{cov}(V)$ are equal to 1, and that all non-diagonal elements are equal to ρ. Therefore, $y'_{i;1}\mathrm{cov}(V) = (1-\rho)y'_1$ and $y'_{i;2}\mathrm{cov}(V) = (1-\rho)y'_2$. ∥

Proof to Theorem 8: We first assume a general listening structure, and compute agent i_m's expected payoff when he adopts a listening set $\tilde{S}(i)$ instead of $S(i)$. We next show that the GLS with guru type 1 is stable, if and only if condition (21) holds.

Expected Payoff

The agent's consumption is given by equation (40). However, the expectations of $s^1_{i_m}$

and $s^T_{i_m}$ w.r.t. the idiosyncratic noise are no longer equal to s^1_i and s^T_i. To compute the expectation of $s^1_{i_m}$, we note that $s^1_{i_m}$ is the average of $s^0_{i_m}$ and $s^0_{k_{m'}}$ for some agent $k_{m'}$ with $k \in \tilde{S}(i)$. Therefore, the expectation is

$$\frac{1}{2}s^0_i + \frac{1}{2}\frac{1}{\#\tilde{S}(i)}\sum_{k \in \tilde{S}(i)} s^0_k.$$

Notice that when $\tilde{S}(i) = S(i)$, this is equal to s^1_i. Similarly, the expectation of $s^T_{i_m}$ is

$$\frac{1}{2^T}s^0_i + \frac{1}{\#\tilde{S}(i)}\sum_{k \in \tilde{S}(i)}\left(\frac{1}{2^T}s^0_k + \frac{1}{2^{T-1}}s^1_k + \cdots + \frac{1}{2}s^{T-1}_k\right).$$

To compute the expectations w.r.t. V and η, we proceed as in theorem 7. We denote by $\tilde{y}_{i;1}$ the vector whose j'th component is

$$\frac{1}{B_1}\left(\frac{1}{2}w^0_{ij} + \frac{1}{2}\frac{1}{\#\tilde{S}(i)}\sum_{k \in \tilde{S}(i)} w^0_{kj} - w^1_j\right).$$

We similarly denote by $\tilde{y}_{i;2}$ the vector whose j'th component is

$$\frac{1}{B_2}\left(\frac{1}{2^T}w^0_{ij} + \frac{1}{\#\tilde{S}(i)}\sum_{k \in \tilde{S}(i)}\left(\frac{1}{2^T}w^0_{kj} + \frac{1}{2^{T-1}}w^1_{kj} + \cdots + \frac{1}{2}w^{T-1}_{kj}\right) - w^T_j\right).$$

Proceeding as in theorem 7, expected payoff is

$$(1-\rho)\tilde{y}^0_{i;1}(p_2 - p_1) + \tilde{y}^0_{i;2}(R - (1-\rho)p_2). \tag{41}$$

In equation (41), we will only consider the terms that depend on $\tilde{S}(i)$. To simplify these terms, we introduce some notation. We denote by $z_{k;1}$ the vector whose j'th component is $(1/B_1)w^0_{kj}$. We also denote by $z_{k;2}$ the vector whose j'th component is

$$\frac{1}{B_2}\left(\frac{1}{2^{T-1}}w^0_{kj} + \frac{1}{2^{T-2}}w^1_{kj} + \cdots + w^{T-1}_{kj}\right).$$

Finally, we define b_k by

$$b_k = (1-\rho)z^0_{k;1}(p_2 - p_1) + z^0_{k;2}(R - (1-\rho)p_2). \tag{42}$$

We can interpret b_k as the benefit of listening to type k. Going back to equation (41), we can write the terms that depend on $\tilde{S}(i)$ as

$$\frac{1}{\#\tilde{S}(i)}\sum_{k \in \tilde{S}(i)} b_k.$$

The benefit of adopting $\tilde{S}(i)$ as listening set, is thus the average benefit of listening to types in $\tilde{S}(i)$.

GLS is Stable

A necessary and sufficient condition for the GLS with guru type 1 to be stable is

$$b_1 + \min_{i \neq 1} b_i - 2 \max_{i \neq 1} b_i \geq 0. \tag{43}$$

Indeed, suppose that this condition holds. An agent of the guru type 1 does not want to adopt a listening set $\{1\} \cup I$ instead of $\{1\}$ if

$$b_1 \geq \frac{b_1 + \sum_{i \in I} b_i}{1 + \# I}.$$

Condition (43) implies that $b_1 \geq \max_{i \neq 1} b_i$, therefore this inequality holds. An agent of a non-guru type j does not want to adopt a listening set $\{1\} \cup I$ instead of $\{1, j\}$ if

$$\frac{b_1 + b_j}{2} \geq \frac{b_j + \sum_{i \in I} b_i}{1 + \# I}.$$

If $1 \in I$, we can write this condition as

$$b_1 + b_j \geq 2 \frac{\sum_{i \in I \setminus \{1\}} b_i}{\# I - 1};$$

and if $1 \notin I$, as

$$b_1 + b_j + \frac{b_1 - b_j}{\# I} \geq 2 \frac{\sum_{i \in I} b_i}{\# I}.$$

Condition (43) implies that either inequality holds. To show that condition (43) is necessary, assume that j is the non-guru type with the lowest b_j, and that $I = \{1, i\}$ where i is the non-guru type with the highest b_i.

We next compute the b_i's and show that equations (43) and (21) are equivalent. The influence weights as of round t are

$$w^t_{1,1} = 1; \quad w^t_{1,j} = 0; \quad w^t_{i,1} = 1 - \left(\frac{3}{4}\right)^t; \quad w^t_{i,i} = \left(\frac{3}{4}\right)^t; \quad \text{and} \quad w^t_{ij} = 0;$$

for $i, j \neq 1$ and $i \neq j$. Therefore,

$$b_1 = \frac{1 - \rho}{B_1}(p_2 - p_1)_1 + \frac{1}{B_2} \frac{2^T - 1}{2^{T-1}} (R - (1 - \rho) p_2)_1;$$

and

$$b_i = \frac{1 - \rho}{B_1}(p_2 - p_1)_i + \frac{1}{B_2} \frac{3^T - 2^T}{4^{T-1}} (R - (1 - \rho) p_2)_i + \frac{1}{B_2} \left(\frac{2^T - 1}{2^{T-1}} - \frac{3^T - 2^T}{4^{T-1}} \right) (R - (1 - \rho) p_2)_1.$$

Moreover,

$$(p_1)_1 = A_1\frac{(N-1)\pi + 1}{N}; \quad (p_2)_1 = A_2\frac{(N-1)\left(1-\left(\sigma\right)^T\right)+1}{N} + C_2A_1\frac{(N-1)\pi+1}{N};$$

and

$$(p_1)_i = A_1\frac{\sigma}{N}; \quad (p_2)_i = A_2\frac{\left(\sigma\right)^T}{N} + C_2A_1\frac{\sigma}{N};$$

for $i \neq 1$. Plugging p_1 and p_2 into b_1 and b_i, it is easy to check that equations (43) and (21) are equivalent. Therefore, the GLS with guru type 1 is stable if (21) holds.

For $T = \infty$, (21) becomes

$$A_2 - \frac{1}{4}(A_1 - C_2A_1) \geq 0;$$

and is satisfied with strict inequality since $A_2 > A_1 - C_2A_1 > 0$. Therefore, by continuity (21) is satisfied for large T. If $\rho_j \geq \rho_i$, then $\max_{k\neq j}\rho_k \leq \max_{k\neq i}\rho_k$, and

$$\rho_j + \min_{k\neq j}\rho_k \geq \rho_i + \min_{k\neq i}\rho_k.$$

(To show the last inequality we distinguish cases according to whether i has the minimum accuracy among all types.) Therefore, if (21) holds for i, it holds for j. ∥

Proof to Theorem 9: We consider a stable configuration and show that it must be as in the theorem. We first assume large but finite T, and then $T = \infty$.

Large But Finite T

We first show that, generically in types' accuracies, the benefits b_i, defined by equation (42), differ across types. To show that $b_i \neq b_j$ generically, it suffices to show that $z_{i,2} \neq z_{j,2}$. We will show that the i'th components of $z_{i,2}$ and $z_{j,2}$, denoted by $f_i(T)$ and $f_j(T)$, differ. Both $f_i(T)$ and $f_j(T)$ are linear combinations of exponentials in T. Since $f_i(\infty) = 1/B_2 \neq f_j(\infty) = 0$, the coefficients of the exponentials differ. Therefore, $f_i(T)$ and $f_j(T)$ can be equal only for T in a finite set, and differ for large T.

Since the benefits b_i differ across types, we can use them to rank the types. Without loss of generality, assume that type n is ranked n. To determine the listening set of type n, we note that the average benefit of listening to types in this set, has to exceed the average benefit for all other sets that include type n. Therefore, the listening set includes types 1, n, and may include types 2 to n' for $n' < n$.

We next show that $n' = 1$, i.e. that the listening structure is a GLS. We proceed by contradiction and assume that for some n, $n' > 1$, i.e. that type 2 is listened to by at least one more type except himself. Since all beliefs converge to type 1's beliefs, $w^t_{i,1}$ goes to 1, and w^t_{ij} for $j \neq 1$ goes to 0, as t goes to ∞. Therefore, b_2 goes to

$$(1-\rho)(p_2 - p_1)_2 + (R - (1-\rho)p_2)_1 = -(1-\rho)(A_1 - C_2A_1)w_2^{\infty} + (R - (1-\rho)p_2)_1;$$

and b_N goes to

$$(1-\rho)(p_2-p_1)_N + (R-(1-\rho)p_2)_1 = -(1-\rho)(A_1-C_2A_1)w^\infty_N + (R-(1-\rho)p_2)_1;$$

We will show that $w^\infty_2 > w^\infty_N$. Using $C_2 < 1$, we will conclude that for large T, $b_2 < b_N$, a contradiction. Since type N is only listened to by himself, we have

$$w^\infty_N = \frac{w^\infty_{N,N}}{N} = \frac{1}{N}\left(\frac{1}{2}+\frac{1}{2}\frac{1}{\#S(N)}\right).$$

Since type 2 is listened to by at least one more type except himself, we have

$$w^\infty_2 > \frac{w^\infty_{2,2}}{N} = \frac{1}{N}\left(\frac{1}{2}+\frac{1}{2}\frac{1}{\#S(2)}\right) \geq \frac{1}{N}\left(\frac{1}{2}+\frac{1}{2}\frac{1}{\#S(N)}\right) = w^\infty_N.$$

Infinite T

We proceed as for large but finite T, and rank types according to the benefits b_i. We denote by I the set of types with the highest benefit. (We consider sets because for $T=\infty$ there can be ties.) The listening set of a type in I is a subset of I. The listening set of a type not in I includes this type, all types in I, and may include some other types as well.

If types $i,j \in I$ converge to different beliefs, then $z_{i,2} \neq z_{j,2}$. Therefore, generically in types' accuracies, $b_i \neq b_j$, a contradiction. All types in I thus converge to the same belief, which is a weighted average of their initial beliefs. Moreover, all types not in I converge to that weighted average as well. We denote by w_i the weight of type $i \in I$, and by w the vector whose i'th component is w_i for $i \in I$ and 0 for $i \notin I$. For $T=\infty$, the vector $z_{i,2}$ equals $(2/B_2)w$. Therefore,

$$b_i = (1-\rho)(p_2-p_1)_i + \frac{2}{B_2}w'(R-(1-\rho)p_2). \qquad (44)$$

Using equation (44), and proceeding as in the case where T is finite but large, we can show that a type not in I listens only to himself and to the types in I. Equation (44) also implies that $(p_2-p_1)_i$ is independent of i for $i \in I$.

To show that I is strongly connected, we proceed by contradiction. Suppose that it is not. Theorem 5 implies that I has a strongly connected isolated subset J. (Theorem 5 applies with I instead of N since I is isolated.) Since all types in I converge to the same belief, this belief is a weighted average of the initial beliefs of the types in J. Since

$$\sum_{j\in J} w_j = 1 > \sum_{j\in J} w^\infty_j;$$

there exists a type $j \in J$ such that $w_j > w^\infty_j$. Using $A_2 > A_1 - C_2A_1$, we have

$$(p_2-p_1)_j = A_2 w_j - (A_1 - C_2A_1)w^\infty_j > 0.$$

For a type $i \in I \setminus J$, we have

$$(p_2 - p_1)_i = A_2 w_i - (A_1 - C_2 A_1) w_i^1 = -(A_1 - C_2 A_1) w_i^1 < 0.$$

This is a contradiction since $(p_2 - p_1)_i$ is independent of i for $i \in I$. $\|$

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