Life-Cycle Models and Cross-Country Analysis of Saving*

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Abstract

This paper develops a rational expectations life-cycle model designed as a framework for the cross-country analysis of (private) saving decisions. It is shown that a broad range of life-cycle models that have been used in the literature to study aggregate time series on consumption and saving fail to deliver plausible predictions for the purpose of analyzing saving decisions across countries as they imply that the level of saving has a constant mean and that the long-run saving rate may tend to zero. Introducing a utility specification that ties the long-run evolution of consumers' aspired consumption paths to that of aggregate labor income, an analytically tractable life-cycle model is proposed that has plausible long-run properties, including the implication that the net asset-labor income ratio, the saving rate, and the consumption-labor income ratio have meaningful long-run distributions. The moments of the long-run saving rate are shown to depend in a precise way on various characteristics of consumers' preferences, the real rate of interest, the growth rate and volatility of labor income, the government consumption-labor income ratio, and the government debt-labor income ratio. Employing a data set on saving rates and asset holdings across OECD economies and using techniques for the estimation of dynamic heterogeneous panels, the paper will also adduce empirical evidence assessing the model's ability to explain differences in the saving patterns across these economies.

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1 Introduction

There are few economic issues that have more far reaching consequences than how households provide for their futures through savings, savings that in turn help finance the investment projects that firms intend to carry out, and that are critical for a country's welfare in the long run. In this paper, we develop a theoretical framework for the cross-country analysis of private saving decisions and employ this framework for an empirical analysis of private saving in the OECD economies.

Our theoretical framework is a life-cycle model of consumption and saving. Since pioneered by Modigliani and Brumberg (1954) and Ando and Modigliani (1963), the life-cycle approach has been used widely to understand consumption and saving decisions both at the household and at the aggregate level. In the last decade, empirical work based on life-cycle models has emphasized two issues: One distinguished strand of literature has tested the implications of these models for aggregate time series on consumption and saving within a single country, particularly the sensitivity of consumption to anticipated and unanticipated changes in labor income.[1] A second distinguished strand of literature has attempted to reconcile life-cycle theory with household data, including lifetime consumption profiles, focusing in particular on the role of precautionary savings and of liquidity constraints for households' consumption decisions.[2] More recently, a growing literature has been concerned with identifying the key determinants/correlates of private saving decisions across countries.[3] While many studies in this literature motivate their regressions on the basis of life-cycle theory, typically no tight link between life-cycle theory and econometric analysis has been provided.

In this paper, we develop a rational expectations life-cycle model designed as a framework for the cross-country analysis of (private) saving decisions. Such a model needs to have plausible implications for the cross-country distribution of private saving decisions in the short and long run. We show that a broad range of life-cycle models that have been used in the literature to study aggregate time series on consumption and saving fail to deliver plausible predictions for the cross-country analysis of saving decisions as they imply that the level of saving has a constant mean and that the long-run saving rate may tend to zero. We then propose an analytically tractable life-cycle model that has plausible long-run properties, including the implication that the net asset-labor income ratio, the saving rate, and the consumption-labor income ratio have meaningful long-run distributions. While our model is designed to adhere as closely as possible to the life-cycle models that have been used in the literature to study aggregate consumption and saving data, three features

[1] See, for example, Deaton (1992) and Muellbauer and Lattimore (1995) for reviews of this literature.

[2] See, for example, Attanasio (2000), for a discussion of some of this literature.

[3] Contributions to this literature include Edwards (1996), Ogaki, Ostry, and Reinhart (1996), Hartman and Engel (1998), Masson, Bayoumi, and Samiei (1998), and Haque, Pesaran, and Sharma (2000). See also the reviews in Deaton (1999) and Agenor (2000).

of our model distinguish it from those previously considered in the literature: First, we argue that the consumption paths that consumers aspire to achieve should reflect (anticipated) increases in their standard of living. We thus propose a utility specification that ties the long-run evolution of consumers' aspired consumption paths to that of aggregate labor income, while also allowing for shifts in taste around this path. Second, we show that for the life-cycle model to have plausible long-run implications for the level and rate of saving, its forcing variables, particularly labor income, need to be modelled as geometric rather than arithmetic processes.[4] Third, to capture important aspects of the cross-country distribution of private saving decisions identified in previous studies, we furthermore model the role of government consumption expenditures for private consumers, incorporating the government solvency constraint into consumers' decision rationale. We show, in particular, that our model implies that the moments of the long-run saving rate (as well as the net asset-labor income ratio and the consumption-labor income ratio) depend in a precise way on various characteristics of consumers' preferences, the real rate of interest, the growth rate and volatility of labor income, the government consumption-labor income ratio, and the government debt-labor income ratio. We furthermore discuss how prudence affects these long-run relations. Employing a data set on private saving rates and asset holdings across OECD economies and using techniques for the estimation of dynamic heterogeneous panels, empirical evidence will be adduced in future versions of this paper to assess the model's ability to explain differences in the private saving patterns across these economies.

The remainder of this paper is organized as follows: Section 2 reviews the long-run properties of various life-cycle models studied in the literature, both under quadratic and power (CRRA) utility, and allowing (in the context of the former utility specification) for factors such as habit formation and risk sensitivity. Section 3 proposes a life-cycle model whose long-run implications include that the net asset-labor income ratio and the saving rate have meaningful long-run distributions. The empirical validity of various implications of the life-cycle model introduced in Section 3 for the cross-country variation in private saving rates in the OECD is analyzed in Section 4. Section 5 summarizes and concludes with some suggestions for future research.

[4] Geometric specifications of the forcing variables can, of course, in any case capture the exponential trend in the level of many aggregate time series, as opposed to arithmetic specifications, and are thus *per se* preferable from an empirical perspective. Some of the recent literature on reconciling life-cycle theory with household data has also been working with geometric specifications of the labor income process. See, for example, Carroll (1997).

2 Long-Run Implications of Life-Cycle Models Studied in the Literature

2.1 Models With Quadratic Utility

We begin with the Hall (1978) life-cycle model. While it is overly simplistic (and we will consider much richer models in what follows), it serves as a useful first benchmark for the long-run implications of a life-cycle model.[5] Consider a representative, infinitely-lived consumer having access to perfect credit markets. Suppose that the real rate of interest on domestic assets, r_t, is given by

$$r_t = r > 0, \tag{2.1}$$

for all t, and that labor income received at the beginning of period t, y_t, is generated by the arithmetic autoregressive process[6]

$$(1-\phi_y L)\, y_t = \mu_y + (1-\phi_y)\,\gamma_y t + \varepsilon_{yt}, \qquad |\phi_y| \leq 1, \quad \gamma_y \geq 0, \quad \varepsilon_{yt} \sim iid\; N\left(0, \sigma_y^2\right), \tag{2.3}$$

where

$$\mu_y = (1-\phi_y)\,\varrho_y + \phi_y \gamma_y. \tag{2.4}$$

Also suppose that the consumer's preferences are given by

$$\sum_{j=0}^{\infty} \beta^j \left(-\frac{1}{2}\right) (c_{t+j} - b)^2\,, \tag{2.5}$$

where c_t denotes consumption expenditure at the beginning of period t, $\beta = (1+\rho)^{-1}$ $(\rho \leq r)$ is a constant discount factor, and b measures (the exogenous component to) the consumer's aspiration, in the Hall (1978) model assumed to be constant. The consumer maximizes (2.5) by choosing $\{c_{t+j}\}_{j=0}^{\infty}$ subject to the period-by-period budget constraints,

$$a_{t+j} = (1+r)\, a_{t+j-1} + y_{t+j} - c_{t+j}, \qquad j = 0, 1, \ldots\,, \tag{2.6}$$

[5]Our focus in this paper is on life-cycle models abstracting from capital investment decisions. This is consistent with the earlier work in this area (for example, Modigliani, 1970, 1991) attempting to relate observed cross-country private saving patterns to life-cycle analysis. See also Deaton (1999).

[6]Note that the specification (2.3) ensures that if there is a unit root in labor income, $\phi_y = 1$, then there is, as in the case where $|\phi_y| < 1$, only a linear time trend (with coefficient equal to γ_y) in the unconditional representation of the labor income process:

$$y_t = \phi^t y_0 + \sum_{j=0}^{t-1} \phi^j \mu + (1-\phi)\,\gamma \sum_{j=0}^{t-1} \phi^j\,(t-j) + \sum_{j=0}^{t-1} \phi^j \varepsilon_{y,t-j}. \tag{2.2}$$

where a_t denotes the wealth level at the beginning of period t, the transversality condition,

$$\lim_{j \to \infty} (1+r)^{-j} E\left(a_{t+j} | \Omega_t\right) = 0, \tag{2.7}$$

where Ω_t denotes the information set available to the consumer at the beginning of period t, and $E(\cdot|\Omega_t)$ is the mathematical conditional expectations operator, and given an initial wealth level, a_{t-1}.

The optimal decision rule for the life-cycle model (2.5), (2.6), (2.7), and (2.1) under the labor income specification (2.3) can be readily shown to be given by[7]

$$c_t = \lambda_2 a_{t-1} + \lambda_3 y_t + \lambda_4 + \lambda_5 t, \tag{2.8}$$

where

$$\lambda_2 = 1 + r - \delta, \tag{2.9}$$

$$\lambda_3 = \psi_y, \tag{2.10}$$

$$\lambda_4 = \left(\frac{\delta - 1}{r}\right) b + \left(\frac{1}{r}\right) \psi_y \mu_y + \left[\frac{(1-\phi_y)(1+r)}{r^2}\right] \psi_y \gamma_y, \tag{2.11}$$

$$\lambda_5 = \left(\frac{1-\phi_y}{r}\right) \psi_y \gamma_y, \tag{2.12}$$

with

$$\psi_y = \frac{1+r-\delta}{1+r-\phi_y}, \tag{2.13}$$

and

$$\delta = \frac{1}{\beta(1+r)} \in (0, 1]. \tag{2.14}$$

To understand the implications of the decision rule (2.8) for the long-run behavior of consumption and saving, we solve (2.8) for a_{t-1}, substitute the resultant expression as well as its counterpart for a_t back into the period t budget constraint, and use the specification of the labor income process (2.3) and the definitions of λ_4 and λ_5 given in (2.11) and (2.12), respectively, to obtain[8]

$$(1 - \delta L) c_t = \psi_y \varepsilon_{yt} + (1-\delta) b. \tag{2.15}$$

[7] We assume that a_0 is sufficiently large for c_1 to be positive.

[8] The representation (2.15) of the decision rule also reveals that unless $\delta = 1$, marginal utility will generally not be positive for all t, as consumption may (and generally will for many points on the sample path) exceed the bliss level b.

There are a number of noteworthy implications of the decision rule written in the form of (2.15) for the long-run behavior of consumption and saving: Clearly, the consumption process has no deterministic trend or drift, regardless of whether the labor income process has a constant mean, a deterministic trend, or a drift. Furthermore, the consumption process displays a unit root if and only if $\delta = 1$, that is, if the consumer's discount rate is equal to the market real rate of interest. Consequently, from the period-by-period budget constraints, if there is a deterministic trend or a drift in labor income, then there is a deterministic trend or drift of the opposite sign in the wealth level. As shown in Appendix B, disposable income, defined as

$$y_t^d = y_t + ra_{t-1}, \tag{2.16}$$

under $\delta < 1$ follows a covariance stationary process with mean $b - \gamma_y/r$, and under $\delta = 1$ is an $I(1)$ process with no drift. As also shown in Appendix B, the level of saving, defined as

$$s_t = a_t - a_{t-1} = y_t^d - c_t, \tag{2.17}$$

in either case follows a covariance stationary process with mean $-\gamma_y/r$, and the saving rate, sr_t, defined as

$$sr_t = \frac{s_t}{y_t^d}, \tag{2.18}$$

tends to zero as the mean of disposable income becomes large. These implications are crossly at odds with the empirical regularities for the OECD countries (and beyond): Consumption per capita data, whether considered in levels or logarithms display an upward trend/drift, as do the data on (private) saving per capita. The (private) saving rate, while varying significantly across countries, in no country shows a tendency to converge to zero. In the Hall (1978) life-cycle model, regardless of whether $\delta < 1$ (so that the consumer is patient) or $\delta = 1$ (so that the consumer is "time indifferent"), there is no motivation for the consumer to choose an upward sloping consumption profile. Rather, the long-run mean of consumption is chosen to be the bliss (if $\delta < 1$) or the initial level of consumption (if $\delta = 1$). The consumer anticipates any upward trend/drift in labor income, and can borrow against it. Subject to the transversality condition, there is no upper bound on the level of indebtedness, and the consumer is never liable for primary budget deficits. One may thus conjecture that more plausible long-run implications would result if the Hall model is modified such that consumers have a motivation to maintain an upward sloping consumption profile and/or if they are eventually liable for primary budget deficits. We therefore consider four modifications of the Hall model, some of which have been suggested in the literature to have such effects: habit formation, prudence (precautionary saving), trended/drifting aspiration, and finitely-lived overlapping generations.

Habit Formation

Suppose that the consumer's preferences are given by

$$\sum_{j=0}^{\infty} \beta^j \left(-\frac{1}{2}\right) \left(x_{t+j} - \eta x_{t+j-1}\right)^2, \tag{2.19}$$

where

$$x_{t+j} = c_{t+j} - b, \tag{2.20}$$

and $\eta \in [0, 1)$ measures the degree of preference for habit formation. Also assume now that the initial level of consumption, c_{t-1}, is given. Notice that under this preference specification involving habit formation, consumption levels must be continually increasing in order to offset the negative effect that past consumption has on current-period utility, and the consumption path might be expected to be upward trending even in the long run.

The optimal decision rule for the life-cycle model (2.19), (2.6), (2.7), and (2.1) under the labor income specification (2.3) can be shown to be given by[9]

$$c_t = \lambda_1 c_{t-1} + \lambda_2 a_{t-1} + \lambda_3 y_t + \lambda_4 + \lambda_5 t, \tag{2.21}$$

where

$$\lambda_1 = \eta \left(\frac{\delta}{1+r}\right), \tag{2.22}$$

$$\lambda_2 = \left(1 + r - \phi_y\right) \psi_y, \tag{2.23}$$

$$\lambda_4 = \left(\frac{\delta - 1}{r}\right) (1 - \eta)\, b + \left(\frac{1}{r}\right) \psi_y \mu_y + \left[\frac{\left(1 - \phi_y\right)(1 + r)}{r^2}\right] \psi_y \gamma_y, \tag{2.24}$$

and λ_3 and λ_5 are defined by (2.10) and (2.12), respectively, but with ψ_y now given by

$$\psi_y = \frac{(1 + r - \delta)(1 + r - \eta)}{\left(1 + r - \phi_y\right)(1 + r)} \tag{2.25}$$

rather than (2.13). Again solving the decision rule (2.21) for a_{t-1}, substituting the resultant expression as well as its counterpart for a_t back into the period t budget constraint, and using the specification of the labor income process (2.3) and the definitions of λ_4 and λ_5 given in (2.24) and (2.12), respectively, one obtains

$$(1 - \eta L)(1 - \delta L)\, c_t = \psi_y \varepsilon_{yt} + (1 - \delta)(1 - \eta)\, b. \tag{2.26}$$

[9] See Appendix A for a proof.

From (2.26), it is readily seen that the key long-run implications for consumption, wealth, and saving are unaffected by the presence of habit formation: The consumption process still has a constant mean, regardless of whether the labor income process has a constant mean, a deterministic trend, or a drift. The consumption process has a unit root if and only if $\delta = 1$, that is, if the discount rate is equal to the market real rate of interest. If there is a deterministic trend or a drift in labor income, then there is a deterministic trend or drift of the opposite sign in the wealth level. Disposable income under $\delta < 1$ follows a covariance stationary process with mean $b - \gamma_y / r$, and under $\delta = 1$ is an $I(1)$ process with no drift. The level of saving in either case follows a covariance stationary process with mean $-\gamma_y / r$. The saving rate therefore tends to zero as the mean of disposable income becomes large. Habit formation *per se* thus does not help to overcome the problematic long-run implications of the Hall model.

Prudence (Risk Sensitivity)

To introduce a precautionary saving motive into the above life-cycle model under habit formation, we extend Willassen's (1992) analysis of consumption under risk sensitivity to allow for habit formation.[10] Suppose that the consumer's preferences are given by

$$\left(\frac{1}{\theta}\right)\left\{1 - \exp\left[\theta \sum_{j=0}^{\infty} \beta^j \left(\frac{1}{2}\right)\left(x_{t+j} - \eta x_{t+j-1}\right)^2\right]\right\}, \tag{2.27}$$

where $\theta \geq 0$ measures the degree of risk sensitivity, with x_{t+j} still being given by (2.20). Under the preference specification (2.27), as under CARA and CRRA utility, the consumer will want to provide for future labor income contingencies through precautionary saving, inducing (at least temporarily) an upward sloping consumption profile.

To determine whether the consumption path is upward trending even in the long run, let us analyze the optimal decision rule for the life-cycle model (2.27), (2.6), (2.7), and (2.1) under the labor income specification (2.3). It can be shown to be given by[11]

$$c_t = \lambda_1 c_{t-1} + \lambda_2 a_{t-1} + \lambda_3 y_t + \lambda_4 + \lambda_5 t, \tag{2.28}$$

where

$$\lambda_1 = \eta\left(\frac{\delta - \overline{\theta}}{1 + r - \overline{\theta}}\right), \tag{2.29}$$

$$\lambda_4 = \left[1 - \frac{(1+r)(1+r-\delta)}{r\left(1 + r - \overline{\theta}\right)}\right](1-\eta)\, b + \left(\frac{1}{r}\right)\psi_y \mu_y + \left[\frac{(1-\phi_y)(1+r)}{r^2}\right]\psi_y \gamma_y, \tag{2.30}$$

[10] See also Weil (1993) and Hansen, Tallarini, and Sargent (1999) for life-cycle models under risk sensitivity.

[11] See Appendix A for a proof.

and λ_2, λ_3, and λ_5 given by (2.23), (2.10), and (2.12), respectively, with ψ_y now given by

$$\psi_y = \frac{(1+r-\delta)(1+r-\eta)}{(1+r-\phi_y)\left(1+r-\overline{\theta}\right)}, \tag{2.31}$$

and

$$\overline{\theta} = \theta\sigma_y^2 \left[\frac{1+r-\eta}{(1+r-\phi_y)^2}\right]. \tag{2.32}$$

Note that the propensities to consume out of past consumption and labor income are positive if $\overline{\theta} < \delta < 1+r$, or, equivalently, if $\theta < (1+r-\phi_y)^2 \delta / \left[(1+r-\eta)\sigma_y^2\right] < 1+r$. Precautionary saving is reflected in the fact that the marginal propensities to consume out of past consumption, labor income, and wealth are affected by the prudence motive. To see this, it is useful to rewrite the decision rule (2.28) as

$$c_t = \eta c_{t-1} + (1-\eta)\, b - \kappa\varphi_t, \tag{2.33}$$

where

$$\kappa = \frac{(1+r-\delta)(1+r-\eta)}{r\left(1+r-\overline{\theta}\right)}, \tag{2.34}$$

and

$$\begin{aligned}\varphi_t \quad = \quad & \left(\frac{\eta r}{1+r-\eta}\right) c_{t-1} + \left[1-\left(\frac{\eta r}{1+r-\eta}\right)\right] b - r a_{t-1} - \left(\frac{r}{1+r-\phi_y}\right) y_t \\ & - \left(\frac{1}{1+r-\phi_y}\right)\mu_y - \left[\frac{(1-\phi_y)(1+r)}{r(1+r-\phi_y)}\right]\gamma_y - \left(\frac{1-\phi_y}{1+r-\phi_y}\right)\gamma_y t. \end{aligned} \tag{2.35}$$

If the parameter restriction $\kappa\varphi_t \in (0, \eta c_{t-1} + (1-\eta)\, b)$ is satisfied, precautionary saving is assured to be positive, and the larger the degree of risk sensitivity, θ, or the volatility of labor income, σ_y^2, the larger the amount of precautionary saving. (Note that as θ and/or σ_y^2 increase, then so do $\overline{\theta}$ and κ; also note that φ_t depends neither on θ nor on σ_y^2.)

Again solving the decision rule (2.28) for a_{t-1}, substituting the resultant expression as well as its counterpart for a_t back into the period t budget constraint, and using the labor income specification (2.3) and the definitions of λ_4 and λ_5 given in (2.30) and (2.12), respectively, one obtains

$$(1-\eta L)(1-\xi L)\, c_t = \psi_y \varepsilon_{yt} + (1-\xi)(1-\eta)\, b. \tag{2.36}$$

where

$$\xi = \frac{(1+r)\left(\delta-\overline{\theta}\right)}{1+r-\overline{\theta}}. \tag{2.37}$$

In the presence of risk sensitivity, the consumption process and the disposable labor income process cannot have unit roots anymore, regardless of the properties of labor income. It is readily verified from (2.37) that for $\xi = 1$ we need

$$\theta = \left(\frac{1}{\sigma_y^2}\right)\left[\frac{(1+r-\phi_y)^2}{r(1+r-\eta)}\right](\rho - r). \tag{2.38}$$

However, since by assumption $|\phi_y| \leq 1$, $0 < \rho \leq r$, and $\theta \geq 0$, (2.38) can hold if and only if $\rho = r$ and thus $\theta = 0$. The other long-run implications for consumption, wealth, and saving, however, are unaffected by the presence of risk sensitivity: The consumption process still has a constant mean, regardless of whether the labor income process has a constant mean, deterministic trend, or drift. If there is a deterministic trend or a drift in labor income, then there is a deterministic trend or drift of the opposite sign in the wealth level, and the level of saving follows a covariance stationary process with mean $-\gamma_y/r$. The saving rate therefore tends to zero as the mean of disposable income becomes large. Habit formation and risk sensitivity *per se* thus do not help to overcome the problematic long-run implications of the Hall model.

Trended/Drifting Aspiration

Suppose now that the exogenous component of the consumer's aspiration contains a deterministic trend or a drift reflecting (anticipated) increases in the standard of living as well as a stochastic component reflecting shifts in tastes:

$$x_{t+j} = c_{t+j} - b_{t+j}, \tag{2.39}$$

where b_t is generated by the arithmetic autoregressive process

$$(1-\phi_b L)\, b_t = \mu_b + (1-\phi_b)\,\gamma_b t + \varepsilon_{bt}, \qquad |\phi_b| < 1, \quad \gamma_b \geq 0, \quad \varepsilon_{bt} \sim iid\ N\left(0, \sigma_b^2\right), \tag{2.40}$$

with

$$\mu_b = (1-\phi_b)\,\varrho_b + \phi_b \gamma_b. \tag{2.41}$$

Since the consumer is now modelled as attempting to keep up with a trended/drifting (exogenous) aspiration process, one might expect that this model renders an upward sloping consumption profile. The optimal decision rule for the life-cycle model (2.27), (2.6), (2.7), and (2.1) under the specification of the exogenous component of the consumer's aspiration (2.40) and the labor income specification (2.3) can be shown to be given by[12]

$$c_t = \lambda_1 c_{t-1} + \lambda_2 a_{t-1} + \lambda_3 y_t + \lambda_6 b_t + \lambda_7 b_{t-1} + (\lambda_4 + \lambda_8) + (\lambda_5 + \lambda_9)\, t, \tag{2.42}$$

[12] See Appendix A for a proof.

where λ_1, λ_2, and λ_3 are given by (2.29), (2.23), and (2.10), respectively (with ψ_y still given by (2.31)), λ_4 is given by

$$\lambda_4 = \left(\frac{1}{r}\right) \psi_y \mu_y + \left[\frac{(1-\phi_y)(1+r)}{r^2}\right] \psi_y \gamma_y, \tag{2.43}$$

λ_5 is given by (2.12), and λ_6, λ_7, λ_8, and λ_9 are defined as follows:

$$\lambda_6 = 1 - \psi_b, \tag{2.44}$$

$$\lambda_7 = -\eta \left(\frac{\delta - \overline{\theta}}{1 + r - \overline{\theta}}\right), \tag{2.45}$$

$$\lambda_8 = -\left(\frac{1}{r}\right) \psi_b \mu_b - \left[\frac{(1-\phi_b)(1+r)}{r^2}\right] \psi_b \gamma_b, \tag{2.46}$$

and

$$\lambda_9 = -\left(\frac{1-\phi_b}{r}\right) \psi_b \gamma_b, \tag{2.47}$$

with

$$\psi_b = \frac{(1+r-\delta)(1+r-\eta)}{(1+r-\phi_b)\left(1+r-\overline{\theta}\right)}, \tag{2.48}$$

and where $\overline{\theta}$ is now defined as

$$\overline{\theta} = \theta \sigma_y^2 \left[\frac{1+r-\eta}{(1+r-\phi_y)^2}\right] + \theta \sigma_b^2 \left[\frac{(1+r-\eta)^2}{(1+r-\phi_b)^2 (1+r)}\right]. \tag{2.49}$$

Note that the propensities to consume out of past consumption, labor income, and the current exogenous component to aspiration are positive, and the propensity to consume out of the past exogenous component to aspiration is negative, if $\overline{\theta} < \delta < 1 + r$ and $\psi_b < 1$.

Solving the decision rule (2.42) for a_{t-1}, substituting the resultant expression as well as its counterpart for a_t back into the period t budget constraint, using the labor income specification (2.3) and noting that since $|\phi_b| < 1$, we can write b_t as

$$b_t = \varrho_b + \gamma_b t + (1 - \phi_b L)^{-1} \varepsilon_{bt}, \tag{2.50}$$

one obtains

$$(1-\phi_b L)(1-\eta L)(1-\xi L) c_t = \psi_y (1-\phi_b L) \varepsilon_{yt} \tag{2.51}$$

$$+ (1-\phi_b) \left\{ (1-\phi_b) \psi_b - r (1-\psi_b) \left[1 - \frac{\eta \delta}{(1+r)(1-\psi_b)}\right] \right\} \varrho_b$$

$$+\left\{r\phi_b(1-\psi_b)\left[1-\frac{\eta\delta}{(1+r)(1-\psi_b)}\right]+(1+r)(1-\phi_b)(1-\psi_b)-\eta(1+2r)\left(\frac{\delta-\overline{\theta}}{1+r-\overline{\theta}}\right)\right\}\gamma_b$$
$$+(1-\phi_b)\left\{(1-\phi_b)\psi_b-r(1-\psi_b)\left[1-\frac{\eta\delta}{(1+r)(1-\psi_b)}\right]\right\}\gamma_b t$$
$$+(1-\psi_b)\left[1-(1+r)L\right]\left[1-\frac{\eta\delta}{(1+r)(1-\psi_b)}\right]\varepsilon_{bt}.$$

The representation (2.51) of the decision rule reveals that the consumption process now contains a deterministic trend or drift. If $\theta > 0$ (so that there is risk sensitivity), consumption and disposable income follow trend stationary processes, regardless of whether $|\phi_y| < 1$ or $\phi_y = 1$, and regardless of whether $\rho < r$ or $\rho = r$. (This is also true if $\theta = 0$ and $\rho < r$). If $\theta = 0$ and $\rho = r$, then consumption and disposable income follow $I(1)$ processes with drift. Writing consumption and disposable income as functions of a constant, a trend term, and current and lagged values of ε_{yt} and ε_{bt} only, the coefficient on t for both variables reduces to γ_b. As shown in Appendix B, the level of saving in any case follows a covariance stationary process, now with mean $(\gamma_b - \gamma_y)/r$. Under $\gamma_b > 0$, the saving rate therefore converges to zero as $t \to \infty$. Thus, a trended/drifting (exogenous) aspiration process overcomes the problematic long-run implications of the Hall model only partially. While rendering a trend/drift in the consumption process, it does not *per se* deliver plausible long-run predictions for the level and the rate of saving.

Finitely-Lived Overlapping Generations

Following Yaari (1965), Blanchard (1985), and Gali (1990), let us relax the assumption of an infinitely-lived representative consumer and consider an economy of finitely-lived overlapping generations. Suppose that the probability of survival from period t to period $t+1$ is fixed at $1-p$, $p \in (0,1)$. There is no maximum lifetime. The size of each cohort at birth is normalized to p. Denoting by n_{qt} the size in period t of the cohort born in period q ($t \geq q$, 'cohort q'), we thus have

$$n_{qt} = p(1-p)^{t-q}, \tag{2.52}$$

and the total population at time t is given by

$$n_t = \sum_{q=-\infty}^{t} n_{qt} = 1. \tag{2.53}$$

(While there is survival uncertainty at the individual level, there is no uncertainty about the size of each cohort and the size of the total population if each cohort is sufficiently large for a law of large numbers to apply.) There is an annuity firm that pays to (receives from) each consumer holding positive (negative) wealth in period t an annuity, in return for inheriting the consumers' wealth (paying off his/her debt) when he/she dies. Thus, this annuity firm holds all of the consumers' gross assets and finances all of their gross borrowing. A zero profit condition for the annuity firm implies that the real return on wealth is equal to

$$z=\frac{1+r}{1-p}-1. \tag{2.54}$$

To introduce life-cycle savings, it is assumed that each consumer's labor supply is declining geometrically over his/her lifetime, at rate α:

$$l_{qt}=\left[\frac{1-(1-\alpha)(1-p)}{p}\right](1-\alpha)^{t-q}, \qquad q\leq t, \tag{2.55}$$

where l_{qt} denotes the labor supply in period t of a consumer born in period q, and $\alpha\in[0,1)$. Note that average per capita labor supply in period t then is given by

$$l_t=\sum_{q=-\infty}^{t} n_{qt}l_{qt}=1. \tag{2.56}$$

Labor income received at the beginning of period t by a consumer belonging to cohort q is given by

$$y_{qt}=l_{qt}y_t=\left[\frac{1-(1-\alpha)(1-p)}{p}\right](1-\alpha)^{t-q}y_t, \tag{2.57}$$

with average per capita labor income y_t,

$$y_t=\sum_{q=-\infty}^{t} n_{qt}y_{qt}, \tag{2.58}$$

generated by (2.3). At the beginning of period t each consumer belonging to cohort q solves[13]

$$\max_{\{c_{q,t+j}\}_{j=0}^{\infty}} E\left[\sum_{j=0}^{\infty}\left(\frac{1-p}{1+\rho}\right)^{j}\left(\frac{1}{2}\right)(c_{q,t+j}-b_q)^2\,|\Omega_{qt}\right], \tag{2.59}$$

subject to the period-by-period budget constraints,

$$a_{q,t+j}=(1+z)\,a_{q,t+j-1}+y_{q,t+j}-c_{q,t+j}, \qquad j=0,1,\dots, \tag{2.60}$$

the transversality condition,

$$\lim_{j\to\infty}(1+z)^{-j}E\left(a_{q,t+j}|\Omega_{qt}\right)=0, \tag{2.61}$$

and given an initial wealth level $a_{q,t-1}$. Note that $E\left(u_{q,t+j}|\Omega_{qt}\right)$ denotes the expected value of $u_{q,t+j}$ conditional on the consumer being alive at the beginning of period $t+j$ (Ω_{qt} denotes the

[13]For simplicity of exposition, we again abstract from habit formation and risk sensitivity, and assume a constant exogenous component to each consumer's aspiration.

information set available at the beginning of period t to a consumer belonging to cohort q). It is also assumed that $a_{q,q-1} = 0$, that is, consumers in each cohort are born holding no assets.

While it is clear from our analysis above that the decision rules of each individual consumer have the same long-run properties as those in the Hall model, it is natural to expect that average per capita consumption will be trended/drifting: Each cohort has a higher expected present discounted value of lifetime resources than the previously born cohort (due to the trend/drift in average per capita labor income), and thus average per capita consumption will be growing over time. The optimal period t decision rule of each consumer belonging to cohort q for the life-cycle model (2.59), (2.60), (2.61), and (2.1) under the labor income specification (2.57) and (2.3) is given by

$$c_{qt} = (1+z-\delta)\, a_{q,t-1} + \left(\frac{1+z-\delta}{1+z}\right) \sum_{j=0}^{\infty} \left(\frac{1}{1+z}\right)^{j} E\left(y_{q,t+j}|\Omega_{qt}\right) + \left(\frac{\delta-1}{z}\right) b_q. \tag{2.62}$$

Defining average per capita wealth at the beginning of period t as $a_t = \sum_{q=-\infty}^{t} n_{qt} a_{qt}$, from (2.60) its evolution can be shown to be given by

$$a_t = (1+r)\, a_{t-1} + y_t - c_t, \tag{2.63}$$

where c_t denotes average per capita consumption, $c_t = \sum_{q=-\infty}^{t} n_{qt} c_{qt}$. Note that in the aggregate the gross return on wealth is equal to $1+r$. Average per capita consumption from (2.62) may be verified to be given by

$$c_t = \lambda_2 a_{t-1} + \lambda_3 y_t + \lambda_4 + \lambda_5 t, \tag{2.64}$$

where

$$\lambda_2 = (1-p)\,(1+z-\delta)\,, \tag{2.65}$$

λ_3 is given by (2.10),

$$\lambda_4 = \left(\frac{\delta-1}{z}\right) b + \left(\frac{1-\alpha}{z+\alpha}\right) \psi_y \mu_y + \left[\frac{(1-\alpha)\,(1-\phi_y)\,(1+z)}{(z+\alpha)^2}\right] \psi_y \gamma_y, \tag{2.66}$$

and

$$\lambda_5 = \left[\frac{(1-\alpha)\,(1-\phi_y)}{z+\alpha}\right] \psi_y \gamma_y, \tag{2.67}$$

with

$$\psi_y = \frac{1+z-\delta}{1+z-(1-\alpha)\,\phi_y}, \tag{2.68}$$

and

$$b = \sum_{q=-\infty}^{t} n_{qt} b_q. \tag{2.69}$$

In Appendix B it is shown that if one writes the consumption decision rule in moving average form, that is, as a function of a constant, a trend term, and current and lagged values of ε_{yt} only, one obtains

$$c_t = \begin{cases} \mu_c + \gamma_c t + \gamma_c \sum_{j=1}^{t} \varepsilon_{yj} + \Phi_c(L)\,\varepsilon_{yt}, & \text{if } \phi_y = 1, \\ \mu_c + \gamma_c t + \Phi_c(L)\,\varepsilon_{yt}, & \text{if } |\phi_y| < 1, \end{cases} \tag{2.70}$$

where $\Phi_c(L)$ is an infinite-order lag polynomial, and

$$\gamma_c = \varpi_c \gamma_y, \tag{2.71}$$

with

$$\varpi_c = \frac{[(1-\alpha)\,p + \alpha]\,[r\,(2+r) + p - (1-p)\,\rho]}{[p + r + (1-p)\,\alpha]\,[p + r - (1-p)\,\rho]}. \tag{2.72}$$

It follows that consumption is trended if $p > 0$, and displays a unit root if and only if there is a unit root in labor income. This holds regardless of whether $\delta < 1$ or $\delta = 1$. If there is a unit root in labor income, then the trivariate VAR in $\begin{pmatrix} c_t, & a_{t-1}, & y_t \end{pmatrix}'$ has two cointegrating vectors, given by

$$\begin{pmatrix} 0 \\ -1/\varpi_a \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1/\varpi_c \\ 0 \\ 1 \end{pmatrix}, \tag{2.73}$$

with

$$\varpi_a = \frac{(1-p)\,[\alpha + \rho - (1-\alpha)\,r]}{[p + r + (1-p)\,\alpha]\,[p + r - (1-p)\,\rho]}. \tag{2.74}$$

Thus, in the aggregate, consumption and labor income as well as wealth and labor income are cointegrated.[14] The cointegrating vectors (2.73) are also cotrending vectors. Regardless of whether there is a unit root in labor income, the level of saving follows a covariance stationary process with mean $\varpi_a \gamma_a$. As disposable income has a deterministic trend (if $\delta < 1$) or a drift (if $\delta = 1$) (the trend/drift coefficient is given by $(\varpi_a r + 1)\,\gamma_y$), the saving rate converges to zero as $t \to \infty$. Thus, as for the case of a trended/drifting (exogenous) aspiration process, while the Yaari (1965)/Blanchard (1985)/ Gali (1990) overlapping generations model does render a trend/drift in the average per capita consumption process, and in addition in the case of a drift also renders consumption and labor income cointegrated, it fails to deliver plausible long-run predictions for the level and rate of saving.

[14] This result has also been obtained by Gali (1990) using different reasoning than the VAR framework considered in Appendix B.

2.2 Models With Power Utility

We consider next what role the curvature of the utility function plays for the long-run properties of the consumption and saving decisions, and consider the case of power utility. To keep the analysis analytically tractable, we continue to assume that $\rho \leq r$, and, partially in light of the results in Section 2.1, abstract from prudence.[15]

Suppose then that the representative consumer's preferences are given by

$$\sum_{j=0}^{\infty} \beta^j \left(\frac{c_{t+j}^{1-\sigma} - 1}{1-\sigma} \right), \tag{2.75}$$

where $\sigma > 0$ represents the coefficient of relative risk aversion (or the reciprocal of the intertemporal elasticity of substitution). The Euler equation for the life-cycle model (2.75), (2.6), and (2.1) is given by

$$c_t^{-\sigma} = \delta^{-1} E\left(c_{t+1}^{-\sigma} | \Omega_t\right). \tag{2.76}$$

From Jensen's inequality, it is observed that

$$E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} |\Omega_t\right] = \left[\frac{E\left(c_{t+1}|\Omega_t\right)}{c_t}\right]^{-\sigma} + d_t = \delta, \tag{2.77}$$

with $d_t > 0$. In what follows, we presume that certainty equivalence holds in the consumer's optimization problem, so that $d_t = 0$ for all t.[16]

The optimal decision rule for the life-cycle model (2.75), (2.6), (2.7), and (2.1) under the labor income specification (2.3) is given by

$$c_t = \lambda_2 a_{t-1} + \lambda_3 y_t + \lambda_4 + \lambda_5 t, \tag{2.78}$$

where

$$\lambda_2 = 1 + r - \delta^{-1/\sigma}, \tag{2.79}$$

$$\lambda_3 = \psi_y, \tag{2.80}$$

$$\lambda_4 = \left(\frac{1}{r}\right) \psi_y \mu_y + \left[\frac{(1-\phi_y)(1+r)}{r^2}\right] \psi_y \gamma_y, \tag{2.81}$$

[15] We do not consider buffer-stock life-cycle models/life-cycle models under liquidity constraints where $\rho > r$ by assumption. See, for example, Deaton (1991), Gourinchas and Parker (1996), and Carroll (1997) for a discussion of these models which cannot be solved analytically. On the latter point, see also Binder, Pesaran, and Samiei (2000).

[16] If d_t was (approximately) equal to a constant, say d, then the Euler equation would become

$$E\left(c_{t+1}|\Omega_t\right) = (\delta - d)^{-1/\sigma} c_t.$$

Note that the larger the risk adjustment d, the larger the expected growth rate of consumption.

$$\lambda_5 = \psi_y \left(\frac{1-\phi_y}{r^2} \right) \gamma_y, \tag{2.82}$$

with

$$\psi_y = \frac{1+r-\delta^{-1/\sigma}}{1+r-\phi_y}. \tag{2.83}$$

For the marginal propensities to consume out of wealth and labor income to be positive, it is necessary that $\delta^{-1/\sigma} < 1+r$. Solving the decision rule (2.78) for a_{t-1} and substituting the resultant expression as well as its counterpart for a_t back into the period t budget constraint, and using the definitions of λ_2 to λ_5 given in (2.79)-(2.83), respectively, one obtains

$$\left(1-\delta^{-1/\sigma} L\right) c_t = \psi_y \varepsilon_{yt}. \tag{2.84}$$

Under $\rho < r$ there is an explosive root in the autoregressive component of c_t, rendering the consumption path unstable. In contrast to life-cycle models with quadratic utility, life-cycle models under power utility have been widely studied under a geometric (rather than arithmetic) specification of the labor income process, and we will see next that doing so allows to derive more plausible long-run implications for the consumption process.

Suppose that labor income follows the geometric random walk process

$$\log y_t = \gamma_y - \frac{1}{2}\sigma_y^2 + \log y_{t-1} + \varepsilon_{yt}, \qquad \varepsilon_{yt} \sim iid\ N\left(0, \sigma_y^2\right). \tag{2.85}$$

Note that the specification (2.85) implies that the average logarithmic growth rate of labor income is given by

$$E\left(\triangle \log y_t\right) = \gamma_y - \frac{1}{2}\sigma_y^2, \tag{2.86}$$

and that the average proportional growth rate of labor income is given by

$$E\left(\frac{\triangle y_t}{y_{t-1}}\right) = \exp\left(\gamma_y\right) - 1 \equiv \mu_y. \tag{2.87}$$

The optimal decision rule for the life-cycle model (2.75), (2.6), (2.7), and (2.1) under the labor income specification (2.85) is given by

$$c_t = \lambda_2 a_{t-1} + \lambda_3 y_t, \tag{2.88}$$

where

$$\lambda_2 = 1+r-\delta^{-1/\sigma}, \tag{2.89}$$

and

$$\lambda_3 = \frac{1+r-\delta^{-1/\sigma}}{r-\mu_y}. \tag{2.90}$$

Note that for c_t positive, under $\mu_y < r$ it is necessary that $\delta^{-1/\sigma} < 1 + r$. Solving the decision rule (2.88) for a_{t-1} and substituting the resultant expression as well as its counterpart for a_t back into the period t budget constraint, and using the definitions of λ_2 and λ_3 given in (2.89) and (2.90), respectively, one obtains

$$\left(1 - \delta^{-1/\sigma} L\right) c_t = \left(\frac{1 + r - \delta^{-1/\sigma}}{r - \mu_y}\right) \left[1 - (1 + \mu_y) L\right] y_t. \tag{2.91}$$

Under $\rho < r$ there is again an explosive root in the autoregressive component of c_t. Note that in addition to the (potentially) explosive root in the autoregressive component of c_t, there is now also an explosive root in the lag polynomial in y_t whenever $\mu_y > 0$, that is, whenever there is an upward drift in y_t. For the consumption path to be stable, under $\mu_y > 0$ it must be the case that

$$1 + \mu_y = \delta^{-1/\sigma}, \tag{2.92}$$

or

$$r \approx \rho + \sigma g_y. \tag{2.93}$$

Given the drift in the labor income process, μ_y, the consumer's rate of time preference, ρ, and his risk aversion, σ, the real rate of return has to adjust, rendering the consumption path stable. Along this path consumption and labor income are cointegrated.

Note that the saving rate, sr_t, is given by

$$sr_t = \frac{y_t^d - c_t}{y_t^d} = 1 - \frac{c_t / y_t}{1 + r\widetilde{a}_{t-1}}, \tag{2.94}$$

where $\widetilde{a}_t = a_t / y_{t+1}$. From the representation (2.88) of the decision rule, imposing the stability restriction (2.92) it is readily verified that

$$\frac{c_t}{y_t} = (r - \mu_y)\, \widetilde{a}_{t-1} + 1. \tag{2.95}$$

Using (2.95) and the period t budget constraint, it follows that

$$\widetilde{a}_t = \exp\left(\frac{\sigma_y^2}{2} - \varepsilon_{y,t+1}\right) \widetilde{a}_{t-1}, \tag{2.96}$$

or

$$\widetilde{a}_t = \exp\left(\frac{t\sigma_y^2}{2} - \sum_{j=2}^{t+1} \varepsilon_{yj}\right) \widetilde{a}_0. \tag{2.97}$$

Substituting (2.95) into (2.94), one obtains

$$sr_t = \frac{\mu_y \widetilde{a}_{t-1}}{1 + r\widetilde{a}_{t-1}}. \tag{2.98}$$

The difference equations (2.98) and (2.96) jointly determine the saving path. The limiting behavior of the saving rate and of the asset-disposable income ratio,

$$\widetilde{a}_t^d = \frac{a_{t-1}}{y_t^d} = \frac{\widetilde{a}_{t-1}}{1 + r\widetilde{a}_{t-1}}, \tag{2.99}$$

is given in the following proposition.

Proposition 2.1 *Suppose $\widetilde{a}_0 > 0$. Then the saving rate sr_t converges in probability to μ_y/r, and the asset-disposable income ratio $\widetilde{a}_t^d$ converges in probability to $1/r$. The rate of convergence for both is $\sigma_y\sqrt{t-1}$.*

The proof of this and all other subsequent propositions is given in Appendix C. While the property of the life-cycle model under power utility and geometric random walk labor income that the long-run distribution of the saving rate is degenerate is disconcerting, this implication would be readily avoided in a richer model allowing, say, for utility effects of government consumption expenditure. Such additional features are, however, difficult to treat analytically in a model with power utility, and it is for this reason that in Section 3 we reconsider the quadratic utility specifications of Section 2.1 and ask whether they also render plausible long-run implications for consumption and saving once the critical role of the specification of the forcing variables as generated by geometric rather than arithmetic processes suggested by the analysis of this section is recognized. Before doing so, we consider the effects of allowing for finitely lived overlapping generations under the power utility specification.

Finitely-Lived Overlapping Generations

To relax the assumption of an infinitely-lived representative consumer, we embed the life-cycle model under power utility within the overlapping generations economy considered above.

In period t each consumer belonging to cohort q solves

$$\max_{\{c_{q,t+j}\}_{j=0}^{\infty}} E\left[\sum_{j=0}^{\infty}\left(\frac{1-p}{1+\rho}\right)^j\left(\frac{c_{q,t+j}^{1-\sigma}-1}{1-\sigma}\right)|\Omega_{qt}\right], \tag{2.100}$$

subject to the period-by-period budget constraints (2.60) and the transversality condition (2.61), and given an initial wealth level $a_{q,t-1}$. It is again assumed that $a_{q,q-1} = 0$.

The optimal decision rule of each consumer born in period q for the life-cycle model (2.100), (2.60), (2.61), and (2.1) under the specification (2.85) for average per capita labor income is given by

$$c_{qt} = \left(1 + z - \delta^{-1/\sigma}\right) a_{q,t-1} + \left(\frac{1+z-\delta^{-1/\sigma}}{1+z}\right)\sum_{j=0}^{\infty}\left(\frac{1}{1+z}\right)^j E\left(y_{q,t+j}|\Omega_{qt}\right). \tag{2.101}$$

Average per capita consumption from (2.101) can be shown to be given by

$$c_t = \lambda_2 a_{t-1} + \lambda_3 y_t, \tag{2.102}$$

where

$$\lambda_2 = (1-p)\left(1+z-\delta^{-1/\sigma}\right), \tag{2.103}$$

and

$$\lambda_3 = \frac{1+z-\delta^{-1/\sigma}}{1+z-(1-\alpha)(1+\mu_y)}. \tag{2.104}$$

The counterpart of the stability condition (2.92) is now

$$(1-\alpha)(1+\mu_y) = \delta^{-1/\sigma}, \tag{2.105}$$

the evolution of the wealth-labor income ratio a_t is given by

$$\widetilde{a}_t = (1-p)(1-\alpha)\exp\left(\frac{\sigma_y^2}{2} - \varepsilon_{y,t+1}\right)\widetilde{a}_{t-1}, \tag{2.106}$$

and the saving rate obeys

$$sr_t = \frac{(1-p)(1-\alpha)(1+\mu_y)-1}{1+r\widetilde{a}_{t-1}}\widetilde{a}_{t-1}. \tag{2.107}$$

The limiting behavior of the saving rate and the asset-disposable income ratio is given in the following proposition.

Proposition 2.2 *Suppose $\widetilde{a}_0 > 0$. If p and α are sufficiently small such that*

$$\log\left[(1-p)(1-\alpha)\exp\left(\frac{\sigma_y^2}{2}\right)\right] > 0, \tag{2.108}$$

then the saving rate sr_t converges in probability to $[(1-p)(1-\alpha)(1+\mu_y)-1]/r$, and the asset-disposable income ratio $\widetilde{a}_t^d$ converges in probability to $1/r$. The rate of convergence for both is $\sigma_y\sqrt{t-1}$.

Note that under (2.108) it will always be the case that $(1-p)(1-\alpha)(1+\mu_y) > 1$, and so the long-run saving rate is assured to be nonnegative. Even if (2.108) holds, the interpretation of the long-run saving rate suffers from an unfortunate implication: An increase in α leads to a decrease in the long-run saving rate, which is counterintuitive (as an increase in α implies that every cohort needs to provide for a larger amount of life-cycle savings). If (2.108) is not satisfied, the asset-labor income ratio and the saving rate both converge to zero.

3 Long-Run Implications of a Life-Cycle Model With Quadratic Utility, Habit Formation, and Government Consumption (Under Geometric Processes for the Forcing Variables)

In this section, we return to a quadratic specification of the current-period utility function, showing that as for the case of power utility, plausible long-run implications for consumption and saving are obtained if the forcing variables (including the exogenous component to the consumer's aspiration) are modelled as generated by geometric rather than arithmetic processes. In contrast to the case of power utility, however, the quadratic utility model remains analytically tractable under additional features such as habit formation and private valuation of government consumption expenditure, and thus provides an ideal theoretical framework for the cross-country analysis of (private) saving decisions.

As in Section 2, we suppose that the real rate of return on domestic assets, both those issued by the private sector and those issued by the government, r_t, is given by

$$r_t = r > 0, \tag{3.1}$$

for all t. We furthermore suppose that labor income received at the beginning of period t, y_t, is generated by the geometric random walk process[17]

$$\log y_t = \gamma - \frac{1}{2}\sigma_y^2 + \log y_{t-1} + \varepsilon_{yt}, \qquad \gamma \geq 0, \quad \varepsilon_{yt} \sim iid\ N\left(0, \sigma_y^2\right), \tag{3.2}$$

and define the average proportional growth rate of labor income, μ, by

$$\mu = E\left(\frac{\triangle y_t}{y_{t-1}}\right) = \exp\left(\gamma\right) - 1. \tag{3.3}$$

The consumer's preferences are given by[18]

$$\sum_{j=0}^{\infty} \beta^j \left(-\frac{1}{2}\right) \left(x_{t+j} - \eta x_{t+j-1}\right)^2, \qquad \theta \in [0,1], \tag{3.4}$$

where

$$x_t = c_t + \theta g_t - b_t, \tag{3.5}$$

[17] This is the same process as (2.85) in Section 2.2, albeit for compactness we use slightly different notation.

[18] Under (3.4), the consumer does not consider the social value of government consumption expenditure, and our set-up thus makes no statement about the combined private and social value of government consumption expenditure. We do not need to specify the magnitude of the social value of government consumption expenditure as our concern here is with private consumption and saving decisions only. It should also be noted that we do not model government investment expenditure. Our model set-up is consistent with the real rate of return being influenced by such investment expenditure.

with c_t denoting private consumption expenditure at the beginning of period t, g_t government consumption expenditure at the beginning of period t, and b_t measuring the exogenous component of the consumer's aspiration, generated by

$$\log\left(\frac{b_t}{y_t}\right) = \alpha_b - \frac{1}{2}\sigma_b^2 + \varepsilon_{bt}, \qquad \varepsilon_{bt} \sim iid\ N\left(0, \sigma_b^2\right); \tag{3.6}$$

as before, $\beta = (1+\rho)^{-1}$ $(\rho \leq r)$ is a constant discount factor, and $\eta \in [0,1)$ measures the degree of preference for habit formation. The specification of b_t reflects an aspired standard of living that is increasing over time.[19] Under α_b sufficiently large the consumer's aspired standard of living (with probability one) will always exceed his labor income, and the taste shocks $\{\varepsilon_{bt}\}$ lead to temporary shifts in the gap between the aspired standard of living and labor income. Note that the specification (3.6) implies that the mean exogenous aspiration rate is given by

$$E\left(\frac{b}{y}\right) = \exp(\alpha_b). \tag{3.7}$$

The consumer maximizes (3.4) by choosing $\{c_{t+j}\}_{j=0}^{\infty}$ subject to the period-by-period private budget constraints,

$$a_{t+j} = (1+r)\, a_{t+j-1} + y_{t+j} - \tau_{t+j} - c_{t+j}, \qquad j = 0, 1, \ldots, \tag{3.8}$$

where a_t denotes the private wealth level at the beginning of period t, and τ_t the amount of labor income taxed at the beginning of period t, and subject to the transversality condition (2.7),

$$\lim_{j\to\infty} (1+r)^{-j} E\left(a_{t+j}|\Omega_t\right) = 0.$$

The consumer also recognizes that the government operates under the solvency constraint

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \tau_{t+j} = (1+r)\, d_{t-1} + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j g_{t+j}, \qquad j = 0, 1, \ldots, \tag{3.9}$$

where d_t denotes the government debt level (bonds issued to the consumer) at the beginning of period t. The consumer's maximization is also subject to given initial wealth/debt levels, a_{t-1} and d_{t-1}, initial levels of private and government consumption, c_{t-1} and g_{t-1}, respectively, and given processes generating g_t and τ_t.

Suppose that the government sets its consumption expenditure rate as

$$\log\left(\frac{g_t}{y_t}\right) = \alpha_g - \frac{1}{2}\sigma_g^2 + \varepsilon_{gt}, \qquad \varepsilon_{gt} \sim iid\ N\left(0, \sigma_g^2\right). \tag{3.10}$$

Note that the specification (3.10) implies that the mean government expenditure rate is given by

$$E\left(\frac{g}{y}\right) = \exp(\alpha_g). \tag{3.11}$$

[19] The implications of assuming that the exogenous component of the consumer's aspiration was constant rather than trended are discussed below.

Since the government operates under the solvency constraint (3.9), the tax rate τ_t/y_t must be responsive to the current (and/or lagged) value(s) of the government debt-labor income ratio, d_{t-1}/y_t. We therefore specify the following tax rule:

$$\frac{g_t - \tau_t}{y_t} = \kappa_1 - \kappa_2 \left(\frac{d_{t-1}}{y_t} \right) + \varepsilon_{\tau t}, \qquad \kappa_1, \kappa_2 \geq 0, \quad \varepsilon_{\tau t} \sim iid\ N\left(0, \sigma_\tau^2\right). \tag{3.12}$$

Note that under (3.12) the government primary budget deficit, $g_t - \tau_t$, is "error correcting" in the government debt-labor income ratio.

The long-run behavior of the government debt-labor income ratio under the tax rule (3.12) is given in the following proposition.

Proposition 3.1 *If*

$$\frac{(1 + r - \kappa_2) \exp\left(\frac{1}{2}\sigma_y^2\right)}{1 + \mu} < 1, \tag{3.13}$$

then the government debt-labor income ratio, d_t/y_{t+1}, is an ergodic process in that it globally converges to a steady-state probability distribution function that does not depend on the initial state, d_0/y_1. Furthermore, if the stronger condition

$$\frac{(1 + r - \kappa_2) \exp\left(\sigma_y^2\right)}{1 + \mu} < 1 \tag{3.14}$$

holds, then the mean of the steady-state distribution of the government debt-labor income ratio, $\lim_{t\to\infty} E\left(d_t/y_{t+1}\right)$, exists and is given by

$$E\left(\frac{d}{y}\right) = \left[\frac{\exp\left(\sigma_y^2\right)}{1 + \mu - (1 + r - \kappa_2)\exp\left(\sigma_y^2\right)}\right]\kappa_1. \tag{3.15}$$

Finally, if the yet stronger condition

$$\frac{(1 + r - \kappa_2)^2 \exp\left(3\sigma_y^2\right)}{(1 + \mu)^2} < 1 \tag{3.16}$$

holds, then the variance of the steady-state distribution of the government debt-labor income ratio, $\lim_{t\to\infty} V\left(d_t/y_{t+1}\right)$, exists and is given by

$$\begin{aligned} V\left(\frac{d}{y}\right) &= \left[\frac{\exp\left(3\sigma_y^2\right)}{(1+\mu)^2 - \exp\left(3\sigma_y^2\right)(1 + r - \kappa_2)^2}\right]\left\{\sigma_\tau^2 + \left[\frac{1 + \mu + (1 + r - \kappa_2)\exp\left(\sigma_y^2\right)}{1 + \mu - (1 + r - \kappa_2)\exp\left(\sigma_y^2\right)}\right]\kappa_1^2\right\} \\ &\quad - \left[\frac{\exp\left(\sigma_y^2\right)}{1 + \mu - (1 + r - \kappa_2)\exp\left(\sigma_y^2\right)}\right]^2 \kappa_1^2. \end{aligned} \tag{3.17}$$

See Appendix C for a proof of Proposition 3.1. Clearly, (3.16) implies (3.14), which in turn implies (3.13). As is discussed in the proof given in Appendix C, our assumption that ε_{yt} and

$\varepsilon_{\tau t}$ are normally distributed is much stronger than is needed for (a qualitatively similar result as) Proposition 3.1 to be valid. Ergodicity of the government debt-labor income ratio could be proved even if the moments of ε_{yt} and $\varepsilon_{\tau t}$ did not exist, as long as they are finite with probability one.[20] This also distinguishes our proof of ergodicity from proofs available in the stochastic processes literature, for example, Rao, Rao, and Walker (1983), Pham (1986), and Meyn and Tweedie (1993), that establish geometric ergodicity for random coefficient autoregressive models (of which the process generating the government debt-labor income ratio is a special case) under existence of moment conditions. The existence of the mean and the variance of the steady-state distribution of the government debt-labor income ratio requires the existence of the first two moments of $\varepsilon_{\tau t}$ and the existence of $E\left[\exp\left(q\,\varepsilon_{yt}\right)\right]$, for $q=-1,-2$, but again not the normality assumption.

In Appendix A it is shown that upon ruling out explosive private consumption decisions the optimal decision rule for the life-cycle model for the life-cycle model (3.4), (3.8), (2.7), (3.9), and (3.1) is given by

$$
\begin{aligned}
c_t \;=\;& \lambda_1 c_{t-1}+\lambda_2\left(a_{t-1}-d_{t-1}\right)+\left(\frac{\lambda_2}{1+r}\right)\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E\left(y_{t+j}|\Omega_t\right)\right] \\
&+\left(\frac{\delta}{1+r}\right)\left\{\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E\left[\left(\frac{\eta+\delta}{\delta}\right)b_{t+j}-\eta b_{t+j-1}-\delta^{-1}b_{t+j+1}|\Omega_t\right]\right\} \\
&-\theta g_t-(1-\theta)\left(\frac{\lambda_2}{1+r}\right)\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E\left(g_{t+j}|\Omega_t\right)\right]+\lambda_1\theta g_{t-1},
\end{aligned}
\tag{3.18}
$$

where

$$\lambda_1=\frac{\eta\delta}{1+r}, \tag{3.19}$$

and

$$\lambda_2=\frac{\left(1+r-\eta\right)\left(1+r-\delta\right)}{1+r}. \tag{3.20}$$

Under the labor income specification (3.2), the specification of the exogenous component of the consumer's aspiration (3.6), and the government consumption expenditure specification (3.10) the decision rule (3.18) can be rewritten as

$$c_t=\lambda_1 c_{t-1}+\lambda_2\left(a_{t-1}-d_{t-1}\right)+\lambda_3 y_t+\lambda_4 b_t+\lambda_5 b_{t-1}+\lambda_6 g_t+\lambda_7 g_{t-1}, \tag{3.21}$$

where

$$\lambda_3=\left(\frac{\lambda_2}{r-\mu}\right)\left[1-\frac{(1+\mu)\exp\left(\alpha_b\right)}{1+r}-\frac{(1-\theta)(1+\mu)\exp\left(\alpha_g\right)}{1+r}\right], \tag{3.22}$$

[20] Note that the variance of ε_{yt}, σ_y^2, appears in (3.13) only because we have defined the drift of $\{\log y_t\}$ in (3.2) to be equal to $\gamma-\frac{1}{2}\sigma_y^2$.

$$\lambda_4 = \frac{(1+r)(\eta+\delta) - \eta\delta}{(1+r)^2}, \tag{3.23}$$

$$\lambda_5 = -\lambda_1, \tag{3.24}$$

$$\lambda_6 = -\theta - (1-\theta)\left(\frac{\lambda_2}{1+r}\right), \tag{3.25}$$

and

$$\lambda_7 = \theta\lambda_1. \tag{3.26}$$

Note that under the period-by-period private budget constraint (3.8), private saving in period t is defined as

$$s_t = ra_{t-1} + y_t - \tau_t - c_t. \tag{3.27}$$

We define the private saving rate in period t as

$$\frac{s_t}{y_t}. \tag{3.28}$$

Defining the saving rate as a fraction relative to labor income rather than to disposable income greatly simplifies the algebraic analysis of the long-run properties of the saving rate. The long-run properties of the consumer's saving and consumption decisions are given in the following proposition.

Proposition 3.2 *If*

$$\frac{\eta\exp\left(\frac{1}{2}\sigma_y^2\right)}{1+\mu} < 1, \tag{3.29}$$

$$\frac{\delta\exp\left(\frac{1}{2}\sigma_y^2\right)}{1+\mu} < 1, \tag{3.30}$$

and (3.13) holds, then the net asset-labor income ratio, $(a_t - d_t)/y_{t+1}$*, the private saving rate,* s_t/y_t*, and the consumption-labor income ratio,* c_t/y_t*, are ergodic processes in that they globally converge to steady-state probability distribution functions that do not depend on the initial states,* $(a_{-1} - d_{-1})/y_0$*,* $(a_0 - d_0)/y_1$*,* s_0/y_0*, and* c_0/y_0*, respectively. Furthermore, under the stronger conditions*

$$\frac{\eta\exp\left(\sigma_y^2\right)}{1+\mu} < 1, \tag{3.31}$$

$$\frac{\delta \exp\left(\sigma_y^2\right)}{1+\mu} < 1, \tag{3.32}$$

as well as (3.13), the mean of the steady-state distribution of the net asset-labor income ratio, $\lim_{t\to\infty} E\left[\left(a_t - d_t\right)/y_{t+1}\right]$*, exists and is given by*

$$E\left(\frac{a-d}{y}\right) = \pi\left[-1 + E\left(\frac{b}{y}\right) + (1-\theta)\, E\left(\frac{g}{y}\right)\right], \tag{3.33}$$

where

$$\pi = \left[\left(1 - \frac{\eta \exp\left(\sigma_y^2\right)}{1+\mu}\right)\left(1 - \frac{\delta \exp\left(\sigma_y^2\right)}{1+\mu}\right)\right]^{-1}\left[\frac{\exp\left(\sigma_y^2\right)}{1+\mu}\right]\upsilon \geq 0, \tag{3.34}$$

and

$$\upsilon = \frac{\eta\delta + (1+r)(1+\mu-\eta-\delta)}{(1+r)(r-\mu)} + \frac{\eta\delta\exp\left(\sigma_y^2\right)}{(1+r)(1+\mu)}. \tag{3.35}$$

Also, the mean of the steady-state distribution of the private saving rate, $\lim_{t\to\infty} E\left(s_t/y_t\right)$*, then exists and is given by*[21]

$$E\left(\frac{s}{y}\right) = \pi\mu\left[-1 + E\left(\frac{b}{y}\right) + (1-\theta)\, E\left(\frac{g}{y}\right)\right] + \left(\frac{1+\mu}{\exp\left(\sigma_y^2\right)} - 1\right) E\left(\frac{d}{y}\right), \tag{3.36}$$

and the mean of the steady-state distribution of the consumption-labor income ratio, $\lim_{t\to\infty} E\left(c_t/y_t\right)$*, then exists and is given by*

$$E\left(\frac{c}{y}\right) = 1 + \pi(r-\mu)\left[-1 + E\left(\frac{b}{y}\right)\right] + \left[\pi(r-\mu)(1-\theta) - 1\right] E\left(\frac{g}{y}\right). \tag{3.37}$$

See Appendix C for a proof of Proposition 3.2. Clearly, (3.31) and (3.32) imply (3.29) and (3.30).

Remark 3.1 *In the special case where* $\eta = 0$*,* π *becomes*

$$\pi = \left(\frac{1+\mu-\delta}{1+\mu-\delta\exp\left(\sigma_y^2\right)}\right)\left(\frac{\exp\left(\sigma_y^2\right)}{r-\mu}\right). \tag{3.38}$$

Remark 3.2 *Under the yet stronger conditions*

$$\frac{\eta \exp\left(3\sigma_y^2\right)}{1+\mu} < 1, \tag{3.39}$$

$$\frac{\delta \exp\left(3\sigma_y^2\right)}{1+\mu} < 1, \tag{3.40}$$

[21] Note that (3.36) suggests that in a cross-country regression of private saving on its determinants one should include government debt, not the government primary budget deficit/surplus as an explanatory variable. This stands in contrast to the variable choice in Masson, Bayoumi, and Samiei (1998), for example.

as well as (3.13), the variance of the steady-state distribution of the net asset-labor income ratio, $\lim_{t\to\infty} V\left[(a_t - d_t)/y_{t+1}\right]$*, exists and is given by*

$$V\left(\frac{a-d}{y}\right) = \left[\frac{\exp\left(\sigma_y^2/2\right)}{(1+\mu)\left(\overline{\delta}-\overline{\eta}\right)}\right]^2 \left[\chi_1 E\left(\varsigma_0^2\right) + \chi_2 E\left(\varsigma_1^2\right) + \chi_3 E\left(\varsigma_2^2\right)\right], \tag{3.41}$$

where

$$\overline{\eta} = \frac{\eta \exp\left(\sigma_y^2\right)}{1+\mu}, \tag{3.42}$$

$$\overline{\delta} = \frac{\delta \exp\left(\sigma_y^2\right)}{1+\mu}, \tag{3.43}$$

$$\chi_1 = \frac{\exp\left(2\sigma_y^2\right)\left(\overline{\delta}-\overline{\eta}\right)^2\left[1+\overline{\eta}\overline{\delta}\exp\left(\sigma_y^2\right)\right]}{\left[1-\overline{\delta}^2\exp\left(2\sigma_y^2\right)\right]\left[1-\overline{\eta}^2\exp\left(2\sigma_y^2\right)\right]\left[1-\overline{\eta}\overline{\delta}\exp\left(2\sigma_y^2\right)\right]}, \tag{3.44}$$

$$\chi_2 = \frac{2\exp\left(5\sigma_y^2/2\right)\left(\overline{\delta}-\overline{\eta}\right)^2\left(\overline{\eta}+\overline{\delta}\right)}{\left[1-\overline{\delta}^2\exp\left(2\sigma_y^2\right)\right]\left[1-\overline{\eta}^2\exp\left(2\sigma_y^2\right)\right]\left[1-\overline{\eta}\overline{\delta}\exp\left(2\sigma_y^2\right)\right]}, \tag{3.45}$$

$$\begin{aligned}\chi_3 \quad = \quad & \left\{\frac{2\exp\left(3\sigma_y^2\right)\left(\overline{\delta}-\overline{\eta}\right)^2\left[\overline{\delta}^2+\overline{\eta}\overline{\delta}+\overline{\eta}^2-\overline{\eta}^2\overline{\delta}^2\exp\left(2\sigma_y^2\right)-\overline{\eta}\overline{\delta}\left(\overline{\eta}+\overline{\delta}\right)\exp\left(\sigma_y^2/2\right)\right]}{\left[1-\overline{\eta}^2\exp\left(2\sigma_y^2\right)\right]\left[1-\overline{\delta}^2\exp\left(2\sigma_y^2\right)\right]}\right\} \\ & \left\{\frac{1}{\left[1-\overline{\eta}\exp\left(\sigma_y^2/2\right)\right]\left[1-\overline{\delta}\exp\left(\sigma_y^2/2\right)\right]\left[1-\overline{\eta}\overline{\delta}\exp\left(2\sigma_y^2\right)\right]}\right\},\end{aligned} \tag{3.46}$$

$$E\left(\varsigma_0^2\right) = \lim_{t\to\infty} E\left(\varsigma_t^2\right), \tag{3.47}$$

$$E\left(\varsigma_1^2\right) = \lim_{t\to\infty} E\left(\varsigma_t\varsigma_{t-1}\right), \tag{3.48}$$

and

$$E\left(\varsigma_2^2\right) = \lim_{t\to\infty} E\left(\varsigma_t\varsigma_{t-2}\right), \tag{3.49}$$

with

$$\varsigma_t = 1-\lambda_3-\lambda_1\left(\frac{y_{t-1}}{y_t}\right)-\lambda_4\left(\frac{b_t}{y_t}\right)-\lambda_5\left(\frac{b_{t-1}}{y_t}\right)-(1+\lambda_6)\left(\frac{g_t}{y_t}\right)+(\lambda_1-\lambda_7)\left(\frac{g_{t-1}}{y_t}\right). \tag{3.50}$$

See Appendix C for a proof of Remark 3.2. Clearly, (3.42) and (3.43) imply (3.31) and (3.32). As is discussed in the proof given in Appendix C, our assumption that ε_{yt}, ε_{bt}, ε_{gt}, and $\varepsilon_{\tau t}$ are normally distributed is much stronger than is needed for (qualitatively similar results as) Proposition 3.1 and Remark 3.2 to be valid. See also the comments made in this respect following Proposition 3.1.

Proposition 3.2 shows that an infinitely lived representative consumer based life-cycle economy can render plausible long-run implications for private consumption and saving decisions. Consumption and labor income as well as saving and labor income are "cointegrated" in the sense that their ratios tend to steady-state probability distribution functions, the moments of which depend in a precise way on the exogenous aspiration rate, and the government expenditure-labor income and government debt-labor income ratios. These results stand in contrast to the perception in the literature that a standard life-cycle economy, at least in the absence of (occasionally binding) liquidity constraints or a buffer-stock saving motive, cannot render meaningful long-run relations between consumption and labor income (see, for example, Gali, 1990, and Deaton, 1992, for a summary of this view). What is critical for the existence of meaningful long-run relations between consumption and labor income and saving and labor income are both a trended/drifting exogenous aspiration component (which seems plausible, reflecting aspired increases in the standard of living in line with the trend/drift in labor income), and a geometric specification of the forcing variables, in particular labor income and the exogenous component to aspiration. While a geometric specification in any case would seem a better representation of the observed data on average per capita labor income (in the OECD economies and beyond), it appears that the critical role of a geometric specification of the forcing variables for the long-run properties in life-cycle economies has not yet been recognized in the literature.[22]

We had already seen in Section 2 that introducing a trended/drifting exogenous aspiration component into the Hall model by itself does not yield plausible implications for the long-run private saving decisions. Let us conclude our discussion of Proposition 3.2 by considering the role of the specification (3.6) of the exogenous component of the consumer's aspiration. If it was assumed that $b_t = b$, then $E\left(b/y\right) = 0$, and it is readily seen from Proposition 3.2 that the mean of the steady-state distribution of the net asset-labor income ratio would be given by

$$E\left(\frac{a-d}{y}\right) = \pi\left[-1 + (1-\theta)\,E\left(\frac{g}{y}\right)\right], \tag{3.51}$$

with π still defined by (3.34). Furthermore, the mean of the steady-state distribution of the private saving rate would be given by

$$E\left(\frac{sr}{y}\right) = \pi\mu\left[-1 + (1-\theta)\,E\left(\frac{g}{y}\right)\right] + \left(\frac{1+\mu}{\exp\left(\sigma_y^2\right)} - 1\right)E\left(\frac{d}{y}\right), \tag{3.52}$$

[22]See, for example, Hansen and Sargent (1999), who provide an extensive discussion of the solution and estimation of linear-quadratic economies, including life-cycle economies, in which the forcing variables are modelled as arithmetic processes.

and the mean of the steady-state distribution of the consumption-labor income ratio would be given by

$$E\left(\frac{c}{y}\right)=1-\pi\left(r-\mu\right)+\left[\pi\left(r-\mu\right)\left(1-\theta\right)-1\right]E\left(\frac{g}{y}\right). \tag{3.53}$$

Clearly, then, under $E\left(g/y\right)=E\left(d/y\right)=0$, the mean of the steady-state distribution of the (net) asset-labor income ratio and the mean of the steady-state distribution of the private saving rate are both negative. The consumer continuously borrows and accumulates debt (without violating the transversality condition). In the presence of government consumption expenditure and government debt, the means of the steady-state distributions of the net asset-labor income ratio and the private saving rate may be positive. However, even in the latter case, under plausible parameterizations it is very likely that the mean of the steady-state distribution of the consumption-labor income ratio is negative. We therefore model b_t as upward trending; again, doing so anyway seems realistic in an economy where there is (real) labor income growth. The specification (3.6) ensures that the growth rate of labor income plays an immediate role in determining the growth rate of consumption.

Finitely-Lived Overlapping Generations

Next, we reconsider the Yaari (1965), Blanchard (1985), and Gali (1990) economy of finitely-lived overlapping generations. In doing so, our particular interest is on the robustness of the Ricardian equivalence feature of (3.21) to allowing for finitely-lived overlapping generations. The set-up of the overlapping generations economy is similar to the overlapping generations settings we had discussed in Section 2, with average per capita labor income y_t, generated by the geometric random walk process (3.2). The period t preferences of each consumer belonging to cohort q are given by

$$\sum_{j=0}^{\infty}\left(\frac{1-p}{1+\rho}\right)^{j}\left(-\frac{1}{2}\right)\left(x_{q,t+j}-\eta x_{q,t+j-1}\right)^{2}, \tag{3.54}$$

where

$$x_{qt}=c_{qt}+\theta g_{qt}-b_{qt}, \tag{3.55}$$

with g_{qt} government consumption expenditure at the beginning of period t accruing to a consumer belonging to cohort q, and b_{qt} measuring the exogenous component of this consumer's aspiration at the beginning of period t, generated by

$$\log\left(\frac{b_{qt}}{y_t}\right)=\log\left(\frac{b_t}{y_t}\right)=\alpha_B-\frac{1}{2}\sigma_B^2+\varepsilon_{Bt},\qquad \varepsilon_{Bt}\sim iid\ N\left(0,\sigma_B^2\right),\qquad q\leq t. \tag{3.56}$$

The specification of b_{qt} again reflects an aspired standard of living that is increasing over time.[23]

Each consumer belonging to cohort q in period t maximizes (3.54) by choosing $\{c_{q,t+j}\}_{j=0}^{\infty}$ subject to the period-by-period private budget constraints

$$a_{q,t+j} = (1+z)\, a_{q,t+j-1} + y_{q,t+j} - \tau_{q,t+j} - c_{q,t+j}, \qquad j = 0, 1, \dots, \tag{3.57}$$

where a_{qt} denotes the private wealth level at the beginning of period t of a consumer belonging to cohort q, and τ_{qt} the amount of labor income taxes levied upon this consumer at the beginning of period t. The maximization is also subject to the transversality condition

$$\lim_{j\to\infty} (1+z)^{-j}\, E\left(a_{q,t+j}|\Omega_{qt}\right) = 0. \tag{3.58}$$

It is again assumed that $a_{q,q-1} = 0$, that is, consumers in each cohort are born holding no assets.

Consumers also recognize that the government operates under the solvency constraint (3.9),

$$\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^{j} \tau_{t+j} = (1+r)\, d_{t-1} + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^{j} g_{t+j}, \qquad j = 0, 1, \dots,$$

where d_t denotes the average per capita government debt level (bonds issued to the consumers) at the beginning of period t, τ_t denotes the average per capita government tax revenue,

$$\tau_t = \sum_{q=-\infty}^{t} n_{qt}\tau_{qt}, \tag{3.59}$$

and g_t denotes the average per capita government consumption expenditure,

$$g_t = \sum_{q=-\infty}^{t} n_{qt}g_{qt}. \tag{3.60}$$

The consumers' maximization is also subject to a given initial private wealth level, $a_{q,t-1}$ and a given initial government debt level, d_{t-1}, initial levels of private and government consumption, $c_{q,t-1}$ and $g_{q,t-1}$, respectively, and given processes generating g_{qt} and τ_{qt}.

Suppose as before that the government sets its consumption expenditure rate as

$$\log\left(\frac{g_t}{y_t}\right) = \alpha_g - \frac{1}{2}\sigma_g^2 + \varepsilon_{gt}, \qquad \varepsilon_{gt} \sim iid\, N\left(0, \sigma_g^2\right), \tag{3.61}$$

and that it obeys the following tax rule:

$$\frac{g_t - \tau_t}{y_t} = \kappa_1 - \kappa_2\left(\frac{d_{t-1}}{y_t}\right) + \varepsilon_{\tau t}, \qquad \kappa_1 \geq 0, \quad \kappa_2 \in (0,1], \quad \varepsilon_{\tau t} \sim iid\, N\left(0, \sigma_\tau^2\right). \tag{3.62}$$

[23] In (3.56), the exogenous aspiration component of a consumer belonging to cohort q is specified as a ratio relative to average per capita current-period labor income, rather than the consumer's own current-period labor income. Thus, an individual consumer's aspired standard of living does not fluctuate with variations in his/her labor supply, and all members of all cohorts have the same aspired standard of living in period t. A consumer's lifetime aspired standard of living, however, depends, through the productivity growth in y_t, on when the consumer was born.

To complete the model, we also need to specify relations between the average per capita government consumption expenditure (its average per capita taxation) and government consumption expenditure accruing to (taxes levied upon) a consumer belonging to cohort q, $q = -\infty, \ldots, t-1, t$. We adopt the following specification:

$$g_{qt} = g_t, \qquad q \leq t, \tag{3.63}$$

so that government consumption expenditure in any given period in per capita terms accrues equally to all cohorts, and

$$\tau_{qt} = l_{qt}\tau_t, \qquad q \leq t, \tag{3.64}$$

so that the consumers' tax dues vary proportionately with their labor income levels. (Note that under (3.63) and (3.64) government fiscal policy in any given period generally has redistributive effects across cohorts.)

As is shown in Appendix A, the optimal period t decision rule of each consumer belonging to cohort q for this life-cycle model is given by

$$\begin{aligned} c_{qt} &= \psi_1 c_{q,t-1} + \psi_2 a_{q,t-1} + \psi_3 \sum_{j=0}^{\infty} \left(\frac{1}{1+z}\right)^j E\left(y_{q,t+j} - \tau_{q,t+j} + \theta g_{q,t+j} | \Omega_{qt}\right) \\ &\quad + \psi_4 \sum_{j=0}^{\infty} \left(\frac{1}{1+z}\right)^j E\left(x_{q,t+j} | \Omega_{qt}\right) + \psi_5 g_{qt} + \psi_6 g_{q,t-1}, \end{aligned} \tag{3.65}$$

where

$$x_{qt} = \left(1 + \eta\delta^{-1}\right) b_{qt} - \eta b_{q,t-1} - \delta^{-1} b_{q,t+1}, \tag{3.66}$$

$$\delta = \frac{1}{\beta\left(1+r\right)}, \tag{3.67}$$

$$\psi_1 = \frac{\eta\delta}{1+z}, \tag{3.68}$$

$$\psi_2 = \frac{\left(1+z-\eta\right)\left(1+z-\delta\right)}{1+z}, \tag{3.69}$$

$$\psi_3 = \frac{\psi_2}{1+z}, \tag{3.70}$$

$$\psi_4 = \frac{\delta}{1+z}, \tag{3.71}$$

$$\psi_5 = -\theta, \tag{3.72}$$

and

$$\psi_6 = \theta\psi_1. \tag{3.73}$$

Average per capita consumption from (3.65) may be verified to be given by (under $\Omega_{qt} = \Omega_t$)

$$\begin{aligned} c_t &= \psi_1 (1-p) c_{t-1} + \psi_2 (1-p) a_{t-1} + \psi_3 \sum_{j=0}^{\infty} \left(\frac{1-\alpha}{1+z}\right)^j E\left(y_{t+j} - \tau_{t+j} | \Omega_t\right) \\ &+ \psi_3 \theta \sum_{j=0}^{\infty} \left(\frac{1}{1+z}\right)^j E\left(g_{t+j} | \Omega_t\right) + \psi_4 \sum_{j=0}^{\infty} \left(\frac{1}{1+z}\right)^j E\left(x_{t+j} | \Omega_t\right) + \psi_5 g_t + \psi_6 (1-p) g_{t-1}, \end{aligned} \tag{3.74}$$

where

$$x_t = \left(1 + \eta\delta^{-1}\right) b_t - \eta b_{t-1} - \delta^{-1} b_{t+1}. \tag{3.75}$$

In contrast to the case of an economy made up of a representative, infinitely-lived consumer, the expected present discounted value of taxes cannot be *directly* eliminated from (3.74) using the government solvency constraint, (3.9), as the government and the consumer discount future taxes at different rates, implying that τ_{t+j} in the average per capita decision rule (3.74) is discounted at rate $(1-\alpha)/(1+z)$, and in the government solvency constraint (3.9) is discounted at rate $1/(1+r)$. In Appendix A, it is shown how the expected present discounted value of taxes (discounted at rate $(1-\alpha)/(1+z)$) may be eliminated from (3.74) as a function of forcing and predetermined variables only, using the government period-by-period budget constraint underlying (3.9) and the government tax rule (3.12). Under the average per capita labor income specification (3.2), the specification of the exogenous component of the average per capita aspiration (3.6), and the government consumption expenditure specification (3.10), the average per capita decision rule (3.74) can then be rewritten as

$$c_t = \lambda_1 c_{t-1} + \lambda_2 a_{t-1} + \lambda_3 d_{t-1} + \lambda_4 y_t + \lambda_5 y_t \varepsilon_{\tau t} + \lambda_6 b_t + \lambda_7 b_{t-1} + \lambda_8 g_t + \lambda_9 g_{t-1}, \tag{3.76}$$

where

$$\lambda_1 = \frac{\eta\delta(1-p)}{1+z}, \tag{3.77}$$

$$\lambda_2 = \frac{(1+z-\eta)(1+z-\delta)(1-p)}{1+z}, \tag{3.78}$$

$$\lambda_3 = -\left(\frac{\lambda_2}{1-p}\right)\left[\frac{\kappa_2}{1+z-(1-\alpha)(1+r-\kappa_2)}\right], \tag{3.79}$$

$$\lambda_4 = \left[\frac{\lambda_2}{(1-p)(1+z)}\right]\left[\frac{1+z}{1+z-(1+\mu)(1-\alpha)}\right]\left[1+\kappa_1\frac{1+z-(1+r)(1-\alpha)}{1+z-(1-\alpha)(1+r-\kappa_2)}\right]$$
$$+\left[\frac{\lambda_2}{(1-p)(1+z)}\right]\left[\theta\frac{(1+\mu)\exp(\alpha_g)}{z-\mu}-\frac{(1+\mu)(1-\alpha)\exp(\alpha_g)}{1+z-(1+\mu)(1-\alpha)}\right]$$
$$-\left[\frac{\lambda_2(1+\mu)\exp(\alpha_b)}{(1-p)(1+z)(z-\mu)}\right], \tag{3.80}$$

$$\lambda_5 = \left[\frac{\lambda_2}{(1-p)(1+z)}\right]\left[\frac{1+z-(1+r)(1-\alpha)}{1+z-(1-\alpha)(1+r-\kappa_2)}\right], \tag{3.81}$$

$$\lambda_6 = \frac{(1+z)(\eta+\delta)-\eta\delta}{(1+z)^2}, \tag{3.82}$$

$$\lambda_7 = -\frac{\lambda_1}{1-p}, \tag{3.83}$$

$$\lambda_8 = -\theta-(1-\theta)\left[\frac{\lambda_2}{(1+z)(1-p)}\right], \tag{3.84}$$

and

$$\lambda_9 = \theta\lambda_1. \tag{3.85}$$

Note that under $p = 0$ (infinite lifetime) and $\alpha = 0$ (no life-cycle savings), the decision rule reduces to the one for the representative, infinitely-lived consumer economy, (3.21).[24] Ergodicity of the average per capita net asset-labor income ratio, $(a_t - d_t)/y_{t+1}$, the average per capita private saving rate, s_t/y_t, and the average per capita consumption-labor income ratio, c_t/y_t, can be established in conceptually the same way as for the representative, infinitely-lived consumer economy. In contrast to (3.21), Ricardian equivalence in general does not hold, however. This is easily seen from the coefficients on a_{t-1} and d_{t-1}, which in general do not sum to zero. As $\lambda_2 > \lambda_3$, a fraction of government bonds is net wealth to the consumers. The size of this fraction is inversely related to κ_2, which is plausible, since a lower value of κ_2 *ceteris paribus* corresponds to a higher mean government debt-labor income ratio.

Prudence

Finally in this section, we consider how consumers' prudence might be incorporated into a life-cycle model with habit formation and government consumption under geometric processes for the forcing variables. We will now also relax the assumption of a constant real rate of return on

[24]Note that under $p = 0$ there are also no overlapping generations, as all cohorts of new-born consumers have size zero, and there is only one "initial cohort". Furthermore, under $p = 0$ clearly only $\alpha = 0$ is meaningful to consider.

domestic assets, and instead suppose that the real return follows a stochastic process. The Euler equation of a life-cycle model with habit formation and government consumption as specified above, but a general utility function $u(\cdot)$ allowing for prudence (exhibiting strictly convex marginal utility) and a stochastic process for the real return is given by

$$u^{'}\left[c_t + \theta g_t - \eta\left(c_{t-1} + \theta g_{t-1}\right)\right] = \beta E\left\{\left(1 + r_{t+1}\right) u^{'}\left[c_{t+1} + \theta g_{t+1} - \eta\left(c_t + \theta g_t\right)\right] |\Omega_t\right\}, \tag{3.86}$$

which may be rewritten as

$$\begin{aligned} u^{'}\left[c_t + \theta g_t - \eta\left(c_{t-1} + \theta g_{t-1}\right)\right] &= \beta\left[1 + E\left(r_{t+1}|\Omega_t\right)\right]\left\{u^{'}\left[E\left(c_{t+1}|\Omega_t\right) + \theta E\left(g_{t+1}|\Omega_t\right)\right.\right. \\ &\quad \left.\left. -\eta\left(c_t + \theta g_t\right)\right] + ps_t\right\} + cov_t. \end{aligned} \tag{3.87}$$

where

$$cov_t = \beta Cov\left\{r_{t+1}, u^{'}\left[c_{t+1} + \theta g_{t+1} - \eta\left(c_t + \theta g_t\right)\right] |\Omega_t\right\}. \tag{3.88}$$

Using Jensen's inequality, under strictly convex marginal utility it can be shown that $ps_t > 0$. If r_t and y_t are independent or positively correlated, cov_t under some additional conditions can be shown to be strictly negative, whereas cov_t may be strictly positive if r_t and y_t are positively correlated. As is discussed, for example, in Binder, Pesaran, and Samiei (2000), life-cycle models with Euler equation given by (3.86) and strictly convex marginal utility can in general not be solved analytically. An analytically tractable and empirically powerful approach to the analysis of such life-cycle models is to approximate ps_t and cov_t as smooth functions of potentially relevant determinants: ps_t reflects the consumers' desire to provide for future contingencies such as cuts in labor income, reductions in government consumption expenditure, or unusually low asset returns through precautionary saving. Determinants of the likelihood of such contingencies arising in time series could include the probability of a recession occurring or the probability of a change in government occurring, and in a cross section could include the volatility of transitory output fluctuations or the amount of unemployment insurance benefits granted. cov_t, if strictly negative, reflects the consumers' desire to limit the exposure to losses of funds saved for consumption smoothing and precautionary purposes due to unusually low asset returns (and thus acts counter the precautionary saving effect). A relevant determinant in time series would seem the probability of a (significant) drop in asset returns and in cross section would seem the volatility of (transitory) rate of return fluctuations. Once functional forms have been specified, the Euler equation (3.87) can be log-linearized and then analytically analyzed regarding its long-run properties. While subject to the Lucas critique, such decision rules capturing the effects of a broad range of uncertainties would seem well suited for cross-country analysis of private saving rates.

4 Empirical Evidence for OECD Economies

[To Be Added.]

5 Conclusion

[To Be Added.]

Appendix A: Derivation of Decision Rules

Quadratic Utility, Habit Formation, Risk Sensitivity, Arithmetic Processes for the Exogenous Component to the Consumer's Aspiration and for Labor Income, and Infinite Lifetime

It will be useful to split $\{y_t\}$ and $\{b_t\}$ into their deterministic and stochastic components. Accordingly, let

$$y_t = z_{yt} + \widetilde{y}_t, \qquad b_t = z_{bt} + \widetilde{b}_t, \tag{A.1}$$

where

$$z_{yt} = \left(\sum_{j=0}^{t-1} \phi_y^j \mu_y\right) + (1-\phi_y)\gamma_y \left[\sum_{j=0}^{t-1} \phi_y^{t-j-1}(j+1)\right] + \phi_y^t y_0, \tag{A.2}$$

$$z_{bt} = \left(\sum_{j=0}^{t-1} \phi_b^j \mu_b\right) + (1-\phi_b)\gamma_b \left[\sum_{j=0}^{t-1} \phi_b^{t-j-1}(j+1)\right] + \phi_b^t b_0, \tag{A.3}$$

$$\widetilde{y}_t = \sum_{j=1}^{t} \phi_y^{t-j} \varepsilon_{yj}, \tag{A.4}$$

and

$$\widetilde{b}_t = \sum_{j=1}^{t} \phi_b^{t-j} \varepsilon_{bj}. \tag{A.5}$$

Suppose first that $\eta = 0$. For this case, it follows from Whittle's (1990) risk-sensitive certainty equivalence principle that maximization of the exponential-of-quadratic objective function (2.27) (under $\eta = 0$) subject to the linear period-by-period budget constraints (2.6) and subject to the linear autoregressive processes with Gaussian innovations (2.3) and (2.40) is equivalent to extremizing the Lagrangian

$$\mathcal{L}_t = \lim_{T\to\infty} \left\{ \begin{array}{c} \sum_{h=0}^{T-t} \beta^h \left(\frac{1}{2}\right) \left(c_{t+h} - z_{b,t+h} - \widetilde{b}_{t+h}\right)^2 \\ -\sum_{h=0}^{T-t} \beta^h \lambda_{t+h} \left[a_{t+h} - (1+r)\, a_{t+h-1} - z_{y,t+h} - \widetilde{y}_{t+h} + c_{t+h}\right] \\ -\sum_{h=1}^{T-t} \beta^{h-1} \left(2\theta\sigma_y^2\right)^{-1} \left(\widetilde{y}_{t+h} - \phi_y \widetilde{y}_{t+h-1}\right)^2 \\ -\sum_{h=1}^{T-t} \beta^{h-1} \left(2\theta\sigma_b^2\right)^{-1} \left(\widetilde{b}_{t+h} - \phi_b \widetilde{b}_{t+h-1}\right)^2 \end{array} \right\} \tag{A.6}$$

with respect to all decisions not yet made at the beginning of period t (that is, c_t, c_{t+1}, $\ldots$; a_t, a_{t+1}, $\ldots$) and all variables not observed at the beginning of period t (that is, λ_t, λ_{t+1}, $\ldots$; $\widetilde{y}_{t+1}$, $\widetilde{y}_{t+2}$, $\ldots$; $\widetilde{b}_{t+1}$, $\widetilde{b}_{t+2}$, $\ldots$).[25] Let $q_{t+h|t}$ denote the planned value at the beginning of period t of the variable q for period $t+h$ (if q is an endogenous variable), or, if q is an exogenous variable, the (risk-adjusted) anticipated value at the beginning of period t for period $t+h$. Then the optimality conditions are given by

$$c_{t+h|t} - z_{b,t+h} - \widetilde{b}_{t+h|t} = \lambda_{t+h|t}, \qquad h = 0, 1, \ldots, T-t, \tag{A.7}$$

[25] See Whittle (1990), Chapters 6 and 7, for further details.

$$\lambda_{t+h|t} = \beta (1+r) \lambda_{t+h+1|t}, \qquad h = 0, 1, \ldots, T-t-1, \tag{A.8}$$

$$a_{t+h|t} - (1+r) a_{t+h-1|t} - z_{y,t+h} - \widetilde{y}_{t+h|t} + c_{t+h|t} = 0, \qquad h = 0, 1, \ldots, T-t, \tag{A.9}$$

$$\beta \lambda_{t+h|t} - \left(\frac{1}{\theta \sigma_y^2}\right) \left(\widetilde{y}_{t+h|t} - \phi_y \widetilde{y}_{t+h-1|t}\right) + \beta \left(\frac{\phi_y}{\theta \sigma_y^2}\right) \left(\widetilde{y}_{t+h+1|t} - \phi_y \widetilde{y}_{t+h|t}\right) = 0, \tag{A.10}$$

$h = 1, 2, \ldots, T-t-1$,

$$\beta \lambda_{T|t} - \left(\frac{1}{\theta \sigma_y^2}\right) \left(\widetilde{y}_{T|t} - \phi_y \widetilde{y}_{T-1|t}\right) = 0, \tag{A.11}$$

$$-\beta \lambda_{t+h|t} - \left(\frac{1}{\theta \sigma_b^2}\right) \left(\widetilde{b}_{t+h|t} - \phi_b \widetilde{b}_{t+h-1|t}\right) + \beta \left(\frac{\phi_b}{\theta \sigma_b^2}\right) \left(\widetilde{b}_{t+h+1|t} - \phi_b \widetilde{b}_{t+h|t}\right) = 0, \tag{A.12}$$

$h = 1, 2, \ldots, T-t-1$, and

$$\beta \lambda_{T|t} - \left(\frac{1}{\theta \sigma_b^2}\right) \left(\widetilde{b}_{T|t} - \phi_b \widetilde{b}_{T-1|t}\right) = 0, \tag{A.13}$$

where we are interested in the case where $T \to \infty$. Note that the optimality condition (A.7) equates the shadow price $\lambda_{t+h|t}$ to the marginal utility obtained from the period $t+h$ consumption of the additional funds obtained *ceteris paribus* through an infinitesimal relaxation of the period $t+h$ budget constraint (conditional on information available at the beginning of period t). The optimality condition (A.8) equates the marginal utility of period $t+h$ consumption to the marginal utility of period $t+h+1$ consumption discounted back to period $t+h$ (again conditional on information available at the beginning of period t). It may be verified that these two optimality conditions in the case where $\theta = 0$ but $\eta \geq 0$ can be reduced to

$$c_{t+h|t} - \eta c_{t+h-1|t} - z_{b,t+h} - \widetilde{b}_{t+h|t} + \eta z_{b,t+h-1} + \eta \widetilde{b}_{t+h-1|t} = \lambda_{t+h|t}, \qquad h = 0, 1, \ldots, T-t, \tag{A.14}$$

and (A.8), respectively.[26] Observing the risk-sensitive certainty equivalence principle, the optimality conditions for extremizing the Lagrangian

$$\mathcal{L}_t = \lim_{T \to \infty} \left\{ \begin{array}{c} \sum_{h=0}^{T-t} \beta^h \left(\frac{1}{2}\right) \left(c_{t+h} - \eta c_{t+h-1} - z_{b,t+h} - \widetilde{b}_{t+h} + \eta z_{b,t+h-1} + \eta \widetilde{b}_{t+h-1}\right)^2 \\ - \sum_{h=0}^{T-t} \beta^h \lambda_{t+h} \left[a_{t+h} - (1+r) a_{t+h-1} - z_{y,t+h} - \widetilde{y}_{t+h} + c_{t+h}\right] \\ - \sum_{h=1}^{T-t} \beta^{h-1} \left(2\theta \sigma_y^2\right)^{-1} \left(\widetilde{y}_{t+h} - \phi_y \widetilde{y}_{t+h-1}\right)^2 \\ - \sum_{h=1}^{T-t} \beta^{h-1} \left(2\theta \sigma_b^2\right)^{-1} \left(\widetilde{b}_{t+h} - \phi_b \widetilde{b}_{t+h-1}\right)^2 \end{array} \right\} \tag{A.15}$$

with respect to c_t, c_{t+1}, $\ldots$; a_t, a_{t+1}, $\ldots$; λ_t, λ_{t+1}, $\ldots$; $\widetilde{y}_{t+1}$, $\widetilde{y}_{t+2}$, $\ldots$, and $\widetilde{b}_{t+1}$, $\widetilde{b}_{t+2}$, $\ldots$, are then given by (A.14), (A.8)-(A.11), the counterpart of the difference equation (A.12) under $\eta > 0$,

$$-\beta \left(\frac{1+r-\eta}{1+r}\right) \lambda_{t+h|t} - \left(\frac{1}{\theta \sigma_b^2}\right) \left(\widetilde{b}_{t+h|t} - \phi_b \widetilde{b}_{t+h-1|t}\right) + \beta \left(\frac{\phi_b}{\theta \sigma_b^2}\right) \left(\widetilde{b}_{t+h+1|t} - \phi_b \widetilde{b}_{t+h|t}\right) = 0, \tag{A.16}$$

[26]See the derivation of the decision rule under quadratic utility, habit formation, government consumption, geometric processes for the exogenous component to the consumer's aspiration and for labor income, and finite lifetime, particularly (A.35)-(A.45), for a more detailed argument. As the derivation would be conceptually similar to (A.35)-(A.45), we omit it here to keep the presentation as compact as possible.

and (A.13). Conjecture that

$$\lambda_{t+h|t} = \delta^{t+h} k_t, \qquad h = 0, 1, \ldots, T - t, \tag{A.17}$$

with k_t undetermined. To determine k_t, we shall solve the difference equations (A.10) subject to the boundary conditions $\widetilde{y}_{t|t} = \widetilde{y}_t$ and (A.11), and (A.16) subject to the boundary conditions $\widetilde{b}_{t|t} = \widetilde{b}_t$ and (A.13), and then combine the solutions of these difference equation with (A.14), (A.8), and (A.9). Note from (A.14) that once k_t is determined, the optimal decision rule is given by:[27]

$$c_t = \eta c_{t-1} + b_t - \eta b_{t-1} + \delta^t k_t. \tag{A.18}$$

It may be verified that the solution of the difference equation (A.10) subject to the boundary conditions $\widetilde{y}_{t|t} = \widetilde{y}_t$ and (A.11) is given by:

$$\begin{aligned} \widetilde{y}_{t+h|t} &= \phi_y^h \widetilde{y}_t + \left[\frac{\beta\theta\sigma_y^2}{1 - (1+r)^{-1}\phi_y} \right] \\ & \left\{ \delta^{t+h} \left[\frac{1 - (\phi_y/\delta)^h}{1 - \phi_y/\delta} \right] - \left[\frac{1 - \left(\beta\phi_y^2\right)^h}{1 - \beta\phi_y^2} \right] \left[(1+r)^{-1}\phi_y \right]^{T-t+1} (\beta\phi_y)^{-(t+h)} \right\} k_t. \end{aligned} \tag{A.19}$$

Note that as $T \to \infty$, (A.20) becomes

$$\widetilde{y}_{t+h|t} = \phi_y^h \widetilde{y}_t + \left[\frac{\beta\theta\sigma_y^2}{1 - (1+r)^{-1}\phi_y} \right] \left\{ \delta^{t+h} \left[\frac{1 - (\phi_y/\delta)^h}{1 - \phi_y/\delta} \right] \right\} k_t. \tag{A.20}$$

Similarly, as $T \to \infty$, the solution of the difference equation (A.16) subject to the boundary conditions $\widetilde{b}_{t|t} = \widetilde{b}_t$ and (A.13) is given by

$$\widetilde{b}_{t+h|t} = \phi_b^h \widetilde{b}_t - \left[\frac{\beta\theta\sigma_b^2 (1 + r - \eta)}{(1+r) - \phi_b} \right] \left\{ \delta^{t+h} \left[\frac{1 - (\phi_b/\delta)^h}{1 - \phi_b/\delta} \right] \right\} k_t. \tag{A.21}$$

It is worth noting that the anticipated future values of y and b are risk adjusted, and that the anticipated values at t only equal the conditional expectations at t if $\theta = 0$. These risk adjustments drive the difference between the decision rule under (2.27) and the decision rule in the case where the objective function is additively separable with quadratic felicity functions. Returning to the solution under $\theta > 0$, define $\widetilde{c}_{t+h|t} = c_{t+h|t} - \eta c_{t+h-1|t}$. Then it is readily verified that

$$\sum_{h=0}^{\infty} (1+r)^{-h} \widetilde{c}_{t+h|t} = \left(\frac{1 + r - \eta}{1 + r} \right) \left[\sum_{h=0}^{\infty} (1+r)^{-h} c_{t+h|t} \right] - \eta c_{t-1}. \tag{A.22}$$

Substituting (A.22) into the lifetime budget constraint

$$(1+r) a_{t-1} + \sum_{h=0}^{\infty} (1+r)^{-h} \left(z_{t+h} + \widetilde{y}_{t+h|t} - c_{t+h|t} \right) = 0, \tag{A.23}$$

the latter becomes

$$\begin{aligned} & (1+r) a_{t-1} + \left[\sum_{h=0}^{\infty} (1+r)^{-h} \left(z_{t+h} + \widetilde{y}_{t+h|t} \right) \right] \\ = & \left(\frac{1+r}{1+r-\eta} \right) \left\{ \left[\sum_{h=0}^{\infty} (1+r)^{-h} \widetilde{c}_{t+h|t} \right] + \eta c_{t-1} \right\}. \end{aligned} \tag{A.24}$$

[27]Note that the optimal plans $c_{t+s|t}$, $s = 1, 2, \ldots$, are then also readily determined from (A.14).

Substituting (A.20) (to eliminate $\widetilde{y}_{t+h|t}$), (A.14) (to eliminate $\widetilde{c}_{t+h|t}$), (A.21) (to eliminate $\widetilde{b}_{t+h|t}$), and (A.17) (to eliminate $\lambda_{t+h|t}$) into (A.24), and noting that $z_{y,t+h} + \phi_y^h \widetilde{y}_t = E\left(y_{t+h}|\Omega_t\right)$, with

$$E\left(y_{t+h}|\Omega_t\right) = \phi_y^h y_t + \left(\sum_{j=0}^{h-1} \phi_y^j \mu_y\right) + \left(1-\phi_y\right)\gamma_y \left[\sum_{j=0}^{h-1} \phi_y^{h-1-j}\left(t+j+1\right)\right], \tag{A.25}$$

for $h \geq 1$, and that therefore

$$\sum_{h=0}^{\infty} \frac{E\left(y_{t+h}|\Omega_t\right)}{\left(1+r\right)^h} = \left(\frac{1+r}{1+r-\phi_y}\right)\left(y_t + \vartheta_{0y} + \vartheta_{1y} t\right), \tag{A.26}$$

where

$$\vartheta_{0y} = \left(\frac{1}{r}\right)\mu_y + \left[\frac{\left(1-\phi_y\right)\left(1+r\right)\gamma_y}{r^2}\right], \tag{A.27}$$

and

$$\vartheta_{1y} = \frac{\left(1-\phi_y\right)\gamma_y}{r}, \tag{A.28}$$

and that $z_{b,t+h} + \phi_b^h \widetilde{b}_t = E\left(b_{t+h}|\Omega_t\right)$, with

$$E\left(b_{t+h}|\Omega_t\right) = \phi_b^h b_t + \left(\sum_{j=0}^{h-1} \phi_b^j \mu_b\right) + \left(1-\phi_b\right)\gamma_b \left[\sum_{j=0}^{h-1} \phi_b^{h-1-j}\left(t+j+1\right)\right], \tag{A.29}$$

for $h \geq 1$, and that therefore

$$\sum_{h=0}^{\infty} \frac{E\left(b_{t+h}|\Omega_t\right)}{\left(1+r\right)^h} = \left(\frac{1+r}{1+r-\phi_b}\right)\left(b_t + \vartheta_{0b} + \vartheta_{1b} t\right), \tag{A.30}$$

where

$$\vartheta_{0b} = \left(\frac{1}{r}\right)\mu_b + \left[\frac{\left(1-\phi_b\right)\left(1+r\right)\gamma_b}{r^2}\right], \tag{A.31}$$

and

$$\vartheta_{1b} = \frac{\left(1-\phi_b\right)\gamma_b}{r}, \tag{A.32}$$

one obtains after some algebra that

$$k_t = \left[\frac{\left(1+r-\delta\right)\left(1+r-\eta\right)}{r\left(1+r-\overline{\theta}\right)}\right] g_t, \tag{A.33}$$

where

$$\begin{aligned} g_t \quad = \quad & r a_{t-1} + \left(\frac{r}{1+r-\phi_y}\right) y_t + \left(\frac{1}{1+r-\phi_y}\right)\mu_y + \left[\frac{\left(1-\phi_y\right)\left(1+r\right)}{r\left(1+r-\phi_y\right)}\right]\gamma_y \\ & + \left(\frac{1-\phi_y}{1+r-\phi_y}\right)\gamma_y t - \left(\frac{\eta r}{1+r-\eta}\right) c_{t-1} - \left(\frac{r}{1+r-\phi_b}\right) b_t \\ & - \left(\frac{\eta r}{1+r-\eta}\right) b_{t-1} - \left(\frac{1}{1+r-\phi_b}\right)\mu_b - \left[\frac{\left(1-\phi_b\right)\left(1+r\right)}{r\left(1+r-\phi_b\right)}\right]\gamma_b \\ & - \left(\frac{1-\phi_b}{1+r-\phi_b}\right)\gamma_b t. \end{aligned} \tag{A.34}$$

Substituting (A.33) and (A.34) back into (A.18), and collecting the terms in c_{t-1} and b_t, one obtains (2.42). Note that (2.8) is a special case of (2.42) with $\eta = 0$, $\theta = 0$, $\phi_b = 1$, $\mu_b = \sigma_b^2 = 0$, and $b_0 = b$, that (2.21) is a special case of (2.42) with $\theta = 0$, $\phi_b = 1$, $\mu_b = \sigma_b^2 = 0$, and $b_0 = b$, and that (2.28) is a special case of (2.42) with $\phi_b = 1$, $\mu_b = \sigma_b^2 = 0$, and $b_0 = b$. ■

Quadratic Utility, Habit Formation, Government Consumption, Geometric Processes for the Exogenous Component to the Consumer's Aspiration and for Labor Income, and Finite Lifetime

The period t Euler equation for each consumer belonging to cohort q can be written as

$$z_{qt} - \beta\left[1 + r + \eta\left(1-p\right)\right] E\left(z_{q,t+1}|\Omega_{qt}\right) + \beta^2\eta\left(1-p\right)\left(1+r\right) E\left(z_{q,t+2}|\Omega_{qt}\right) = 0, \tag{A.35}$$

where

$$z_{qt} = \widetilde{c}_{qt} - \eta\widetilde{c}_{q,t-1} - \widetilde{b}_{qt}, \tag{A.36}$$

$$\widetilde{c}_{qt} = c_{qt} + \theta g_{qt}, \tag{A.37}$$

and

$$\widetilde{b}_{qt} = b_{qt} - \eta b_{q,t-1}. \tag{A.38}$$

Note that (A.35) is a fourth-order difference equation under rational expectations in c_{qt}. Ruling out explosive solutions, (A.35) can be reduced to a second-order rational expectations equation in c_{qt}.[28] To see this, define the expectations revision process

$$\varepsilon_{qt}^{j} = E\left(z_{q,t+h}|\Omega_{qt}\right) - E\left(z_{q,t+h}|\Omega_{q,t-1}\right), \tag{A.39}$$

and use it in (A.35) to obtain

$$\beta^2\eta\left(1-p\right)\left(1+r\right) z_{q,t+2} - \beta\left[1 + r + \eta\left(1-p\right)\right] z_{q,t+1} + z_{qt} = \xi_{qt}, \tag{A.40}$$

where ξ_{qt} is a martingale difference process defined by

$$\xi_{qt} = \beta^2\eta\left(1-p\right)\left(1+r\right)\varepsilon_{q,t+2}^{0} + \beta^2\eta\left(1-p\right)\left(1+r\right)\varepsilon_{q,t+1}^{1} - \beta\left[1 + r + \eta\left(1-p\right)\right]\varepsilon_{q,t+1}^{0}. \tag{A.41}$$

The roots of the characteristic equation associated with (A.40) are given by $1/\left[\beta\eta\left(1-p\right)\right]$, and $1/\left[\beta\left(1+r\right)\right]$. By assumption the former root falls outside the unit circle, and can be readily shown to result in explosive individual-specific consumption decisions. Writing the left-hand side of (A.40) as

$$\left\{\beta^2\eta\left(1-p\right)\left(1+r\right)F - \beta\left[1 + r + \eta\left(1-p\right)\right] + F^{-1}\right\} z_{q,t+1}, \tag{A.42}$$

[28] See also Binder and Pesaran (2000), who discuss such a reduction in order in the context of an infinite-horizon life-cycle model under social interactions, but with no government consumption and a constant exogenous component to the consumer's aspiration.

where F denotes the forward operator, and factorizing the resultant operator so that[29]

$$\left[\beta\left(1+r\right)-F^{-1}\right]\left[1-\eta\left(1-p\right)\beta F\right]z_{q,t+1}=0, \tag{A.43}$$

it follows that for the individual-specific optimal consumption decisions to be non-explosive, we must have:

$$\left[\frac{1}{1-\eta\left(1-p\right)\beta F}\right]\xi_{qt}=\beta\left(1+r\right)\varepsilon^{0}_{q,t+1}. \tag{A.44}$$

Under (A.44), we therefore need to solve

$$z_{qt}=\beta\left(1+r\right)E\left(z_{q,t+1}|\Omega_{qt}\right), \tag{A.45}$$

subject to the private budget constraint (3.57) and the government solvency constraint (3.9). Using (A.36) to substitute for z_{qt} in terms of $\widetilde{c}_{qt}$ and $\widetilde{b}_{qt}$, we have

$$\widetilde{c}_{qt}=\left[\frac{\eta}{1+\beta\left(1+r\right)\eta}\right]\widetilde{c}_{q,t-1}+\left[\frac{\beta\left(1+r\right)}{1+\beta\left(1+r\right)\eta}\right]E\left(\widetilde{c}_{q,t+1}|\Omega_{qt}\right)+\frac{E\left(x_{qt}|\Omega_{qt}\right)}{1+\beta\left(1+r\right)\eta}, \tag{A.46}$$

where

$$x_{qt}=\widetilde{b}_{qt}-\beta\left(1+r\right)\widetilde{b}_{q,t+1}. \tag{A.47}$$

Consider now the quasi-difference transformation[30]

$$m_{qt}=\widetilde{c}_{qt}-u_{1}\widetilde{c}_{q,t-1}, \tag{A.48}$$

where u_1 is any root of the quadratic equation (associated with (A.46))

$$\left[\frac{\beta\left(1+r\right)}{1+\beta\left(1+r\right)\eta}\right]u_{1}^{2}-u_{1}+\left[\frac{\eta}{1+\beta\left(1+r\right)\eta}\right]=0 \tag{A.49}$$

that falls inside the unit circle. Applying this transformation to (A.46), one obtains

$$E\left(m_{q,t+1}|\Omega_{qt}\right)=u_{2}\left[m_{qt}-\frac{x_{qt}}{1+\beta\left(1+r\right)\left(\eta-u_{1}\right)}\right], \tag{A.50}$$

with u_2 defined as

$$u_{2}=\frac{1+\beta\left(1+r\right)\left(\eta-u_{1}\right)}{\beta\left(1+r\right)}. \tag{A.51}$$

Leading (A.50) j periods forward, $j=1,2,\ldots$, taking conditional expectations with respect to Ω_{qt}, and substituting recursively to obtain $E(m_{q,t+j}|\Omega_{qt})$ as a function of m_{qt}, x_{qt}, $E\left(x_{q,t+1}|\Omega_{qt}\right)$, $\ldots$, $E\left(x_{q,t+j-1}|\Omega_{qt}\right)$, one obtains

$$E(m_{q,t+j}|\Omega_{qt})=u_{2}^{j}m_{qt}-\left[\frac{1}{1+\beta\left(1+r\right)\left(\eta-u_{1}\right)}\right]\sum_{k=0}^{j-1}u_{2}^{j-k}E\left(x_{q,t+k}|\Omega_{qt}\right). \tag{A.52}$$

[29] Related factorizations have been used, for example, by Muellbauer (1988) and Deaton (1992).

[30] See, for example, Binder and Pesaran (1997) for a detailed discussion of the use of this quasi-difference transformation for the solution of linear rational expectations models.

From the expected lifetime budget constraint of a consumer belonging to cohort q,

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+z}\right)^{j} E\left(\widetilde{c}_{q,t+j}|\Omega_{qt}\right) = E\left(lr_{qt}|\Omega_{qt}\right), \tag{A.53}$$

where

$$lr_{qt} = (1+z)\,a_{q,t-1} + \sum_{j=0}^{\infty}\left(\frac{1}{1+z}\right)^{j}\left(y_{q,t+j} - \tau_{q,t+j} + \theta g_{q,t+j}\right), \tag{A.54}$$

it follows that

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+z}\right)^{j} E\left(m_{q,t+j}|\Omega_{qt}\right) = \left(\frac{1+z-u_1}{1+z}\right) E\left(lr_{qt}|\Omega_{qt}\right) - u_1\widetilde{c}_{q,t-1}. \tag{A.55}$$

Substituting (A.52) into (A.55), one obtains

$$\begin{aligned}\widetilde{c}_{qt} \;=\;& \left(\frac{u_1u_2}{1+z}\right)\widetilde{c}_{q,t-1} + \left[\frac{\left(1+z-u_1\right)\left(1+z-u_2\right)}{(1+z)^2}\right] E\left(lr_{qt}|\Omega_{qt}\right) \\ &+ \left(\frac{u_2}{1+z}\right)\sum_{j=0}^{\infty}\left(\frac{1}{1+z}\right)^{j} E\left(x_{q,t+j}|\Omega_{qt}\right). \end{aligned} \tag{A.56}$$

It is now easily established that (A.56) is invariant to the choice of u_1 in (A.49); namely, the same expression results on the right-hand side of (A.56) irrespective of whether $u_1 = \eta$ (implying $u_2 = \delta$) or $u_1 = \delta$ (implying $u_2 = \eta$) is used for u_1 in (A.56). Thus (A.56) can be rewritten as

$$\widetilde{c}_{qt} = \left(\frac{\eta\delta}{1+z}\right)\widetilde{c}_{q,t-1} + \left[\frac{\left(1+z-\eta\right)\left(1+z-\delta\right)}{(1+z)^2}\right] E\left(lr_{qt}|\Omega_{qt}\right) + \left(\frac{\delta}{1+z}\right)\sum_{j=0}^{\infty}\left(\frac{1}{1+z}\right)^{j} E\left(x_{q,t+j}|\Omega_{qt}\right), \tag{A.57}$$

which may be further rewritten as (3.65). Aggregating (3.65) across all cohorts, one obtains (3.74). The expected present discounted value of taxes,

$$\sum_{j=0}^{\infty}\left(\frac{1-\alpha}{1+z}\right)^{j} E\left(\tau_{t+j}|\Omega_{qt}\right),$$

may be eliminated from (3.74) as a function of forcing and predetermined variables only using the government period-by-period budget constraint,

$$d_t = (1+r)\,d_{t-1} + g_t - \tau_t, \tag{A.58}$$

and the government tax rule (3.12). Solving (A.58) forward for d_{t+j} and then substituting into (3.12) evaluated at $t+j$, after some algebra one obtains the following recursive relation for τ_{t+j}:

$$\tau_{t+j} = g_{t+j} - \kappa_1 y_{t+j} - \kappa_2\sum_{k=1}^{j}(1+r)^{k-1}\left(\tau_{t+j-k} - g_{t+j-k}\right) + \kappa_2\,(1+r)^{j}\,d_{t-1} - \varepsilon_{\tau,t+j}y_{t+j}, \tag{A.59}$$

which may be further rewritten as

$$\sum_{j=0}^{\infty}\left(\frac{1-\alpha}{1+z}\right)^{j}E\left(\tau_{t+j}|\Omega_{qt}\right)=\sum_{j=0}^{\infty}\left(\frac{1-\alpha}{1+z}\right)^{j}E\left(g_{t+j}|\Omega_{qt}\right)+\left[\frac{\left(1+z\right)\kappa_{2}}{1+z-\left(1-\alpha\right)\left(1+r-\kappa_{2}\right)}\right]d_{t-1}$$

$$-\left[\frac{1+z-\left(1+r\right)\left(1-\alpha\right)}{1+z-\left(1-\alpha\right)\left(1+r-\kappa_{2}\right)}\right]\sum_{j=0}^{\infty}\left(\frac{1-\alpha}{1+z}\right)^{j}E\left[\left(\kappa_{1}+\varepsilon_{\tau,t+j}\right)y_{t+j}|\Omega_{qt}\right]. \tag{A.60}$$

Substituting (A.60) back into (3.74), and using the average per capita labor income specification (3.2), the specification of the exogenous component of the average per capita aspiration (3.6), and the government consumption expenditure specification (3.10), the decision rule (3.76) for average per capita consumption is now readily obtained. The decision rule (3.21) for the infinitely lived representative consumer economy is a special case of (3.76) for $p=0$ and $\alpha=0$. ■

Appendix B: VAR Representations of Decision Rules and Their Long-Run Properties

Quadratic Utility, Habit Formation, Risk Sensitivity, Constant Exogenous Component to the Consumer's Aspiration, Arithmetic Process for Labor Income, and Infinitely Lived Representative Consumer

As in Section 2, we begin by considering the scenario where $b_t = b$, that is, a constant exogenous component to the aspiration process. Let $\mathbf{x}_t = \begin{pmatrix} c_t, & a_{t-1}, & y_t \end{pmatrix}'$, and write (2.28), (2.6) (for $j = -1$), and (2.3) as a VAR(1) in $\mathbf{x}_t$:

$$\mathbf{D}_0 \mathbf{x}_t = \mathbf{D}_1 \mathbf{x}_{t-1} + \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{v}_t, \tag{A.61}$$

where

$$\mathbf{D}_0 = \begin{pmatrix} 1 & -\lambda_2 & -\lambda_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.62}$$

$$\mathbf{D}_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ -1 & 1+r & 1 \\ 0 & 0 & \phi_y \end{pmatrix}, \tag{A.63}$$

$$\mathbf{a}_0 = \begin{pmatrix} \lambda_4 \\ 0 \\ \mu_y \end{pmatrix}, \quad \mathbf{a}_1 = \begin{pmatrix} \lambda_5 \\ 0 \\ (1-\phi_y)\gamma_y \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_t = \begin{pmatrix} 0 \\ 0 \\ \varepsilon_{yt} \end{pmatrix}. \tag{A.64}$$

The VAR (A.61) can be rewritten as

$$(\mathbf{D}_0 - \mathbf{D}_1 L)(\mathbf{x}_t - \boldsymbol{\mu} - \boldsymbol{\gamma} t) = \mathbf{v}_t, \tag{A.65}$$

or

$$(\mathbf{I}_3 - \Phi L)(\mathbf{x}_t - \boldsymbol{\mu} - \boldsymbol{\gamma} t) = \mathbf{u}_t, \tag{A.66}$$

where $\Phi = \mathbf{D}_0^{-1}\mathbf{D}_1$,

$$\mathbf{u}_t = \mathbf{D}_0^{-1}\mathbf{v}_t = \begin{pmatrix} \psi_y \varepsilon_{yt} \\ 0 \\ \varepsilon_{yt} \end{pmatrix}, \tag{A.67}$$

and $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are defined through $(\mathbf{D}_0 - \mathbf{D}_1)\boldsymbol{\gamma} = \mathbf{a}_1$ and $(\mathbf{D}_0 - \mathbf{D}_1)\boldsymbol{\mu} = \mathbf{a}_0 - \mathbf{D}_1\boldsymbol{\gamma}$.[31]

[31]In what follows we distinguish four cases: $\theta > 0$, $\rho < r$, $|\phi_y| < 1$ (Case 1), $\theta > 0$, $\rho < r$, $\phi_y = 1$ (Case 2), $\theta > 0$, $\rho = r$, $|\phi_y| < 1$ (Case 3), and $\theta > 0$, $\rho = r$, $\phi_y = 1$ (Case 4). When $\theta > 0$, none of the long-run properties of interest depend on whether $\rho < r$ or $\rho = r$, and the case where $\rho = r$ can always be obtained simply by imposing $\rho = r$ on all expressions in Case 1 and in Case 2. Also, when $\rho < r$, none of the long-run properties of interest depend on whether $\theta > 0$ or $\theta = 0$, and the case where $\theta = 0$ can always be obtained simply by imposing $\theta = 0$ on all expressions in Case 1 and in Case 2. Finally, none of the long-run properties of interest depend on whether $\eta > 0$ or $\eta = 0$, and the case where $\eta = 0$ can always be obtained simply by imposing $\eta = 0$ on all expressions in Cases 1 to 4.

Case 1: $\theta > 0$, $\rho < r$, $|\phi_y| < 1$

The eigenvalues of Φ are given by η, $(1+r)\left(\delta - \overline{\theta}\right) / \left(1 + r - \overline{\theta}\right)$, and ϕ_y. This establishes that the transversality condition (2.7) is satisfied for the decision rule (2.28).[32] The infinite moving average representation of (A.66) is given by

$$\mathbf{x}_t = \boldsymbol{\mu} + \boldsymbol{\gamma} t + \mathbf{C}^* (L) \mathbf{u}_t, \tag{A.68}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} b \\ (rb - \gamma_y - r\varrho_y)/r^2 \\ \varrho_y \end{pmatrix}, \tag{A.69}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 \\ -\gamma_y/r \\ \gamma_y \end{pmatrix}, \tag{A.70}$$

and $\mathbf{C}^*(L) = \sum_{h=0}^{\infty} \mathbf{C}_h^* L^h$, with $\mathbf{C}_0^* = \mathbf{I}_3$, and $\mathbf{C}_h^* = \Phi^h$ for $h \geq 1$. Thus c_t is covariance stationary, the negative deterministic trend in ra_{t-1} offsetting the positive deterministic trend in y_t (a_{t-1} and y_t are cotrending with coefficient vector $\begin{pmatrix} r, & 1 \end{pmatrix}'$, rendering disposable income covariance stationary). Defining the vector $\boldsymbol{\kappa}$,

$$\boldsymbol{\kappa} = \begin{pmatrix} -1, & r, & 1 \end{pmatrix}', \tag{A.71}$$

note that saving is given by

$$S_t = \boldsymbol{\kappa}' \mathbf{x}_t = -\gamma_y/r + \boldsymbol{\kappa}' \mathbf{C}^*(L) \mathbf{u}_t. \tag{A.72}$$

Case 2: $\theta > 0$, $\rho < r$, $\phi_y = 1$

The eigenvalues of Φ are given by η, $(1+r)\left(\delta - \overline{\theta}\right) / \left(1 + r - \overline{\theta}\right)$, and 1. This again establishes that the transversality condition (2.7) is satisfied for the decision rule (2.28). The infinite moving average representation of (A.66) is given by

$$\mathbf{x}_t = \boldsymbol{\mu} + \boldsymbol{\gamma} t + \mathbf{C} \sum_{h=1}^{t} \mathbf{u}_h + \mathbf{C}^*(L) \mathbf{u}_t, \tag{A.73}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} b \\ (rb - \gamma_y)/r^2 \\ 0 \end{pmatrix}, \tag{A.74}$$

[32] Under $0 < \overline{\theta} < \delta < 1 + r$ it is readily verified that $(1+r)\left(\delta - \overline{\theta}\right) / \left(1 + r - \overline{\theta}\right)$ falls inside the unit circle.

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 \\ -\gamma_y/r \\ \gamma_y \end{pmatrix}, \tag{A.75}$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/r \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.76}$$

and $\mathbf{C}^*(L) = \sum_{h=0}^{\infty} \mathbf{C}_h^* L^h$, with $\mathbf{C}_0^* = \mathbf{I}_3 - \mathbf{C}$, and $\mathbf{C}_h^* = \Phi^h - \mathbf{C}$ for $h \geq 1$. Thus c_t is again covariance stationary, and a_{t-1} and y_t are cointegrated and cotrending (with coefficient vector $\begin{pmatrix} r, & 1 \end{pmatrix}'$), rendering disposable income covariance stationary.

To formally derive the cointegrating vector and corresponding adjustment matrix, consider the bivariate VAR in $\mathbf{z}_t = \begin{pmatrix} a_{t-1}, & y_t \end{pmatrix}'$

$$\mathbf{z}_t = \mathbf{M}_1 \mathbf{z}_{t-1} + \mathbf{M}_2 \mathbf{z}_{t-2} + \mathbf{b}_0 + \mathbf{w}_t, \tag{A.77}$$

where

$$\mathbf{M}_1 = \begin{pmatrix} 1 + r + \lambda_1 - \lambda_2 & 1 - \lambda_3 \\ 0 & 1 \end{pmatrix}, \tag{A.78}$$

$$\mathbf{M}_2 = \begin{pmatrix} -(1+r)\lambda_1 & -\lambda_1 \\ 0 & 0 \end{pmatrix}, \tag{A.79}$$

$$\mathbf{b}_0 = \begin{pmatrix} -\lambda_4 \\ \gamma_y \end{pmatrix}, \qquad \text{and} \qquad \mathbf{w}_t = \begin{pmatrix} 0 \\ \varepsilon_{yt} \end{pmatrix}. \tag{A.80}$$

The matrix $(\mathbf{I}_2 - \mathbf{M}_1 - \mathbf{M}_2)$ is of rank one, and can be decomposed as $\mathbf{I}_2 - \mathbf{M}_1 - \mathbf{M}_2 = \boldsymbol{\alpha}\boldsymbol{\beta}'$, with (imposing the exact-identifying restriction that $\beta_1 = 1$)

$$\boldsymbol{\alpha} = \begin{pmatrix} (1-\eta)\left[1 - \delta + r\left(1 + \overline{\theta} - \delta\right)\right] / \left(1 + r - \overline{\theta}\right) \\ 0 \end{pmatrix}, \tag{A.81}$$

and

$$\boldsymbol{\beta} = \begin{pmatrix} 1 \\ 1/r \end{pmatrix}. \tag{A.82}$$

As for saving, note that

$$\boldsymbol{\kappa}'\mathbf{x}_t = S_t = -\gamma_y/r + \boldsymbol{\kappa}'\mathbf{C}^*(L)\mathbf{u}_t. \tag{A.83}$$

Case 3: $\theta = 0$, $\rho = r$, $|\phi_y| < 1$

The eigenvalues of Φ are given by η, 1, and ϕ_y. This establishes that the transversality condition (2.7) is satisfied for the decision rule (2.28). The infinite moving average representation of (A.66) is given by

$$\mathbf{x}_t = \boldsymbol{\mu} + \boldsymbol{\gamma} t + \mathbf{C} \sum_{h=1}^{t} \mathbf{u}_h + \mathbf{C}^* (L) \mathbf{u}_t, \tag{A.84}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ -\left(r\varrho_y + \gamma_y\right)/r^2 \\ \varrho_y \end{pmatrix}, \tag{A.85}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 \\ -\gamma_y/r \\ \gamma_y \end{pmatrix}, \tag{A.86}$$

$$\mathbf{C} = \begin{pmatrix} -r/(1-\eta) & r + r^2/(1-\eta) & r(1+r-\eta)/\left[(1-\eta)(1+r-\phi_y)\right] \\ -1/(1-\eta) & 1 + r/(1-\eta) & (1+r-\eta)/\left[(1-\eta)(1+r-\phi_y)\right] \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.87}$$

and $\mathbf{C}^*(L) = \sum_{h=0}^{\infty} \mathbf{C}_h^* L^h$, with $\mathbf{C}_0^* = \mathbf{I}_3 - \mathbf{C}$, and $\mathbf{C}_h^* = \Phi^h - \mathbf{C}$ for $h \geq 1$. Thus c_t and a_{t-1} are both $I(1)$, c_t has no drift, a_{t-1} has a negative drift, and c_t and a_{t-1} are cointegrated, though not cotrended.

To formally derive the cointegrating vector and corresponding adjustment matrix, consider the bivariate VAR in $\mathbf{z}_t = \begin{pmatrix} c_t, & a_{t-1} \end{pmatrix}'$

$$\mathbf{M}_0 \mathbf{z}_t = \mathbf{M}_1 \mathbf{z}_{t-1} + \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{w}_t, \tag{A.88}$$

where

$$\mathbf{M}_0 = \begin{pmatrix} 1 & -\lambda_2 \\ 0 & 1 \end{pmatrix}, \tag{A.89}$$

$$\mathbf{M}_1 = \begin{pmatrix} \lambda_1 & 0 \\ -1 & 1+r \end{pmatrix}, \tag{A.90}$$

$$\mathbf{b}_0 = \begin{pmatrix} \lambda_4 \\ 0 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} \lambda_5 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{w}_t = \begin{pmatrix} \lambda_3 y_t \\ y_{t-1} \end{pmatrix}. \tag{A.91}$$

The matrix $(\mathbf{M}_0 - \mathbf{M}_1)$ is of rank one, and can be decomposed as $\mathbf{M}_0 - \mathbf{M}_1 = \boldsymbol{\alpha}\boldsymbol{\beta}'$, with (imposing the exact-identifying restriction that $\beta_1 = 1$)

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 - \eta/(1+r) \\ 0 \end{pmatrix}, \tag{A.92}$$

and

$$\boldsymbol{\beta} = \begin{pmatrix} 1 \\ -r \end{pmatrix}. \tag{A.93}$$

As for saving, note that

$$\boldsymbol{\kappa}^{'}\mathbf{x}_t = S_t = -\gamma_y/r + \boldsymbol{\kappa}^{'}\mathbf{C}^*(L)\,\mathbf{u}_t. \tag{A.94}$$

Case 4: $\theta = 0$, $\rho = r$, $\phi_y = 1$

The eigenvalues of Φ are given by η, 1, and 1. This again establishes that the transversality condition (2.7) is satisfied for the decision rule (2.28). The infinite moving average representation of (A.66) is given by

$$\mathbf{x}_t = \boldsymbol{\mu} + \boldsymbol{\gamma}t + \mathbf{C}\sum_{h=1}^{t}\mathbf{u}_h + \mathbf{C}^*(L)\,\mathbf{u}_t, \tag{A.95}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ -(\gamma_y + r\varrho_y)/r^2 \\ \varrho_y \end{pmatrix}, \tag{A.96}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 \\ -\gamma_y/r \\ \gamma_y \end{pmatrix}, \tag{A.97}$$

$$\mathbf{C} = \begin{pmatrix} -r/(1-\eta) & r + r^2/(1-\eta) & 1 + r/(1-\eta) \\ -1/(1-\eta) & 1 + r/(1-\eta) & 1/(1-\eta) \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.98}$$

and $\mathbf{C}^*(L) = \sum_{h=0}^{\infty}\mathbf{C}_h^* L^h$, with $\mathbf{C}_0^* = \mathbf{I}_3 - \mathbf{C}$, and $\mathbf{C}_h^* = \Phi^h - \mathbf{C}$ for $h \geq 1$. Thus c_t, a_{t-1}, and y_t are all $I(1)$, c_t has no drift, a_{t-1} has a negative drift, and y_t has a positive drift. There is now one cointegrating relationship, between c_t and y_t^d: The matrix $(\mathbf{D}_0 - \mathbf{D}_1)$ is of rank one, and can be decomposed as $\mathbf{D}_0 - \mathbf{D}_1 = \boldsymbol{\alpha}\boldsymbol{\beta}^{'}$, with (imposing the exact-identifying restriction that $\beta_1 = 1$)

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 - \eta/(1+r) \\ 1 \\ 0 \end{pmatrix}, \tag{A.99}$$

and

$$\boldsymbol{\beta} = \begin{pmatrix} 1 \\ -r \\ -1 \end{pmatrix}. \tag{A.100}$$

As for saving, note that

$$\boldsymbol{\kappa}'\mathbf{x}_t = S_t = -\gamma_y/r + \boldsymbol{\beta}'\mathbf{C}^*(L)\,\mathbf{u}_t. \tag{A.101}$$

Quadratic Utility, Habit Formation, Risk Sensitivity, Arithmetic Processes for the Exogenous Component to the Consumer's Aspiration and for Labor Income, and Infinitely Lived Representative Consumer

Consider next the scenario where b_t is generated by the stochastic process (2.40). Let

$$\mathbf{x}_t = \begin{pmatrix} c_t, & a_{t-1}, & y_t, & b_t \end{pmatrix}',$$

and write (2.42), (2.6) (for $j = -1$), (2.3), and (2.40) as a VAR(1) in $\mathbf{x}_t$:

$$\mathbf{D}_0\mathbf{x}_t = \mathbf{D}_1\mathbf{x}_{t-1} + \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{v}_t, \tag{A.102}$$

where

$$\mathbf{D}_0 = \begin{pmatrix} 1 & -\lambda_2 & -\lambda_3 & -\lambda_6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.103}$$

$$\mathbf{D}_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & \lambda_7 \\ -1 & 1+r & 1 & 0 \\ 0 & 0 & \phi_y & 0 \\ 0 & 0 & 0 & \phi_b \end{pmatrix}, \tag{A.104}$$

$$\mathbf{a}_0 = \begin{pmatrix} \lambda_4 + \lambda_8 \\ 0 \\ \mu_y \\ \mu_b \end{pmatrix}, \quad \mathbf{a}_1 = \begin{pmatrix} \lambda_5 + \lambda_9 \\ 0 \\ (1-\phi_y)\,\gamma_y \\ (1-\phi_b)\,\gamma_b \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_t = \begin{pmatrix} 0 \\ 0 \\ \varepsilon_{yt} \\ \varepsilon_{bt} \end{pmatrix}. \tag{A.105}$$

The VAR (A.102) can be rewritten as

$$(\mathbf{D}_0 - \mathbf{D}_1 L)\,(\mathbf{x}_t - \boldsymbol{\mu} - \boldsymbol{\gamma} t) = \mathbf{v}_t, \tag{A.106}$$

or

$$(\mathbf{I}_4 - \Phi L)\,(\mathbf{x}_t - \boldsymbol{\mu} - \boldsymbol{\gamma} t) = \mathbf{u}_t, \tag{A.107}$$

where $\Phi = \mathbf{D}_0^{-1}\mathbf{D}_1$,

$$\mathbf{u}_t = \mathbf{D}_0^{-1}\mathbf{v}_t = \begin{pmatrix} \iota_b\varepsilon_{bt} + \psi_y\varepsilon_{yt} \\ 0 \\ \varepsilon_{yt} \\ \varepsilon_{bt} \end{pmatrix}, \tag{A.108}$$

where

$$\iota_b = \frac{(1+r)\left(\delta+\eta-\overline{\theta}\right)-\delta\eta-\left(1+r-\overline{\theta}\right)\phi_b}{\left(1+r-\overline{\theta}\right)(1+r-\phi_b)}, \tag{A.109}$$

and $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are again defined through $(\mathbf{D}_0-\mathbf{D}_1)\boldsymbol{\gamma}=\mathbf{a}_1$ and $(\mathbf{D}_0-\mathbf{D}_1)\boldsymbol{\mu}=\mathbf{a}_0-\mathbf{D}_1\boldsymbol{\gamma}$.

Case 1: $\theta>0$, $\rho<r$, $|\phi_y|<1$

The eigenvalues of Φ are given by η, $(1+r)\left(\delta-\overline{\theta}\right)/\left(1+r-\overline{\theta}\right)$, ϕ_y, and ϕ_b. This establishes that the transversality condition (2.7) is satisfied for the decision rule (2.42). The infinite moving average representation of (A.107) is given by

$$\mathbf{x}_t=\boldsymbol{\mu}+\boldsymbol{\gamma}t+\mathbf{C}^*(L)\,\mathbf{u}_t, \tag{A.110}$$

where

$$\boldsymbol{\mu}=\begin{pmatrix}\varrho_b\\ \left[\gamma_b-\gamma_y+r\left(\varrho_b-\varrho_y\right)\right]/r^2\\ \varrho_y\\ \varrho_b\end{pmatrix}, \tag{A.111}$$

$$\boldsymbol{\gamma}=\begin{pmatrix}\gamma_b\\ \left(\gamma_b-\gamma_y\right)/r\\ \gamma_y\\ \gamma_b\end{pmatrix}, \tag{A.112}$$

and $\mathbf{C}^*(L)=\sum_{h=0}^{\infty}\mathbf{C}_h^*L^h$, with $\mathbf{C}_0^*=\mathbf{I}_4$, and $\mathbf{C}_h^*=\Phi^h$ for $h\geq 1$. Thus c_t is trend stationary, as is a_{t-1}.

Defining the vector $\boldsymbol{\kappa}$,

$$\boldsymbol{\kappa}=\begin{pmatrix}-1, & r, & 1, & 0\end{pmatrix}^{'}, \tag{A.113}$$

note that saving is given by

$$S_t=\boldsymbol{\kappa}^{'}\mathbf{x}_t=\left(\gamma_b-\gamma_y\right)/r+\boldsymbol{\kappa}^{'}\mathbf{C}^*(L)\,\mathbf{u}_t. \tag{A.114}$$

Case 2: $\theta>0$, $\rho<r$, $\phi_y=1$

The eigenvalues of Φ are given by η, $(1+r)\left(\delta-\overline{\theta}\right)/\left(1+r-\overline{\theta}\right)$, 1, and ϕ_b. This again establishes that the transversality condition (2.7) is satisfied for the decision rule (2.42). The infinite moving average representation of (A.107) is given by

$$\mathbf{x}_t=\boldsymbol{\mu}+\boldsymbol{\gamma}t+\mathbf{C}\sum_{h=1}^{t}\mathbf{u}_h+\mathbf{C}^*(L)\,\mathbf{u}_t, \tag{A.115}$$

where

$$\boldsymbol{\mu}=\begin{pmatrix}\varrho_b\\ \left(\gamma_b-\gamma_y+r\varrho_b\right)/r^2\\ 0\\ \varrho_b\end{pmatrix}, \tag{A.116}$$

$$\boldsymbol{\gamma}=\begin{pmatrix}\gamma_b\\ (\gamma_b-\gamma_y)/r\\ \gamma_y\\ \gamma_b\end{pmatrix}, \tag{A.117}$$

$$\mathbf{C}=\begin{pmatrix}0&0&0&0\\ 0&0&-1/r&0\\ 0&0&1&0\\ 0&0&0&0\end{pmatrix}, \tag{A.118}$$

and $\mathbf{C}^*(L)=\sum_{h=0}^{\infty}\mathbf{C}_h^* L^h$, with $\mathbf{C}_0^*=\mathbf{I}_4-\mathbf{C}$, and $\mathbf{C}_h^*=\Phi^h-\mathbf{C}$ for $h\geq 1$. Thus c_t is again trend stationary, and a_{t-1} is $I(1)$ with drift.

To derive the cointegrating relations and corresponding adjustment matrix, consider the bivariate VAR in $\mathbf{z}_t=\begin{pmatrix}a_{t-1}, & y_t\end{pmatrix}'$

$$\mathbf{z}_t=\mathbf{M}_1\mathbf{z}_{t-1}+\mathbf{M}_2\mathbf{z}_{t-2}+\mathbf{b}_0+\mathbf{b}_1 t+\mathbf{w}_t, \tag{A.119}$$

where

$$\mathbf{M}_1=\begin{pmatrix}1+r+\lambda_1-\lambda_2 & 1-\lambda_3\\ 0 & 1\end{pmatrix}, \tag{A.120}$$

$$\mathbf{M}_2=\begin{pmatrix}-(1+r)\lambda_1 & -\lambda_1\\ 0 & 0\end{pmatrix}, \tag{A.121}$$

$$\mathbf{b}_0=\begin{pmatrix}-\lambda_4-\lambda_8+\lambda_9\\ \gamma_y\end{pmatrix}, \qquad \mathbf{b}_1=\begin{pmatrix}-\lambda_9\\ 0\end{pmatrix}, \qquad \text{and} \qquad \mathbf{w}_t=\begin{pmatrix}-\lambda_6 b_{t-1}-\lambda_7 b_{t-2}\\ \varepsilon_{yt}\end{pmatrix}. \tag{A.122}$$

The matrix $(\mathbf{I}_2-\mathbf{M}_1-\mathbf{M}_2)$ is of rank one, and can be decomposed as $\mathbf{I}_2-\mathbf{M}_1-\mathbf{M}_2=\boldsymbol{\alpha}\boldsymbol{\beta}'$, with (imposing the exact-identifying restriction that $\beta_1=1$)

$$\boldsymbol{\alpha}=\begin{pmatrix}(1-\eta)\left[1-\delta+r\left(1+\overline{\theta}-\delta\right)\right]/(1+r-\theta)\\ 0\end{pmatrix}, \tag{A.123}$$

and

$$\boldsymbol{\beta}=\begin{pmatrix}1\\ 1/r\end{pmatrix}. \tag{A.124}$$

As for saving, note that

$$\boldsymbol{\kappa}'\mathbf{x}_t=S_t=(\gamma_b-\gamma_y)/r+\boldsymbol{\kappa}'\mathbf{C}^*(L)\mathbf{u}_t. \tag{A.125}$$

Case 3: $\theta = 0$, $\rho = r$, $|\phi_y| < 1$

The eigenvalues of Φ are given by η, 1, ϕ_y, and ϕ_b. This establishes that the transversality condition (2.7) is satisfied for the decision rule (2.42). The infinite moving average representation of (A.107) is given by

$$\mathbf{x}_t = \boldsymbol{\mu} + \boldsymbol{\gamma} t + \mathbf{C} \sum_{h=1}^{t} \mathbf{u}_h + \mathbf{C}^* (L) \mathbf{u}_t, \tag{A.126}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \varrho_b \\ [\gamma_b - \gamma_y + r(\varrho_b - \varrho_y)]/r^2 \\ \varrho_y \\ \varrho_b \end{pmatrix}, \tag{A.127}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma_b \\ (\gamma_b - \gamma_y)/r \\ \gamma_y \\ \gamma_b \end{pmatrix}, \tag{A.128}$$

$$\mathbf{C} = \begin{pmatrix} -r/(1-\eta) & r + r^2/(1-\eta) & r(1+r-\eta)/\varpi_y & r(\eta - \phi_b)/\varpi_b \\ -1/(1-\eta) & 1 + r/(1-\eta) & (1+r-\eta)/\varpi_y & (\eta - \phi_b)/\varpi_b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.129}$$

with

$$\varpi_y = (1-\eta)(1+r-\phi_y), \tag{A.130}$$

and

$$\varpi_b = (1-\eta)(1+r-\phi_b), \tag{A.131}$$

and $\mathbf{C}^*(L) = \sum_{h=0}^{\infty} \mathbf{C}_h^* L^h$, with $\mathbf{C}_0^* = \mathbf{I}_4 - \mathbf{C}$, and $\mathbf{C}_h^* = \Phi^h - \mathbf{C}$ for $h \geq 1$. Thus c_t and a_{t-1} are both $I(1)$ with drift.

To derive the cointegrating vector and corresponding adjustment matrix, consider the bivariate VAR in $\mathbf{z}_t = \begin{pmatrix} c_t, & a_{t-1} \end{pmatrix}'$

$$\mathbf{M}_0 \mathbf{z}_t = \mathbf{M}_1 \mathbf{z}_{t-1} + \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{w}_t, \tag{A.132}$$

where

$$\mathbf{M}_0 = \begin{pmatrix} 1 & -\lambda_2 \\ 0 & 1 \end{pmatrix}, \tag{A.133}$$

$$\mathbf{M}_1 = \begin{pmatrix} \lambda_1 & 0 \\ -1 & 1+r \end{pmatrix}, \tag{A.134}$$

$$\mathbf{b}_0 = \begin{pmatrix} \lambda_4 + \lambda_8 \\ 0 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} \lambda_5 + \lambda_9 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{w}_t = \begin{pmatrix} \lambda_3 y_t + \lambda_6 b_t + \lambda_7 b_{t-1} \\ y_{t-1} \end{pmatrix}. \tag{A.135}$$

The matrix $(\mathbf{M}_0 - \mathbf{M}_1)$ is of rank one, and can be decomposed as $\mathbf{M}_0 - \mathbf{M}_1 = \boldsymbol{\alpha}\boldsymbol{\beta}'$, with (imposing the exact-identifying restriction that $\beta_1 = 1$)

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 - \eta/(1+r) \\ 0 \end{pmatrix}, \tag{A.136}$$

and

$$\boldsymbol{\beta} = \begin{pmatrix} 1 \\ -r \end{pmatrix}. \tag{A.137}$$

As for saving, note that

$$\boldsymbol{\kappa}'\mathbf{x}_t = S_t = (\gamma_b - \gamma_y)/r + \boldsymbol{\kappa}'\mathbf{C}^*(L)\,\mathbf{u}_t. \tag{A.138}$$

Case 4: $\theta = 0$, $\rho = r$, $\phi_y = 1$

The eigenvalues of Φ are given by η, 1, 1, and ϕ_b. This again establishes that the transversality condition (2.7) is satisfied for the decision rule (2.42). The infinite moving average representation of (A.107) is given by

$$\mathbf{x}_t = \boldsymbol{\mu} + \boldsymbol{\gamma}t + \mathbf{C}\sum_{h=1}^{t}\mathbf{u}_h + \mathbf{C}^*(L)\,\mathbf{u}_t, \tag{A.139}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \varrho_b \\ (\gamma_b - \gamma_y + r\varrho_b)/r^2 \\ 0 \\ \varrho_b \end{pmatrix}, \tag{A.140}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma_b \\ (\gamma_b - \gamma_y)/r \\ \gamma_y \\ \gamma_b \end{pmatrix}, \tag{A.141}$$

$$\mathbf{C} = \begin{pmatrix} -r/(1-\eta) & r + r^2/(1-\eta) & 1 + r/(1-\eta) & r(\eta - \phi_b)/\varpi_b \\ -1/(1-\eta) & 1 + r/(1-\eta) & 1/(1-\eta) & (\eta - \phi_b)/\varpi_b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.142}$$

and $\mathbf{C}^*(L) = \sum_{h=0}^{\infty}\mathbf{C}_h^* L^h$, with $\mathbf{C}_0^* = \mathbf{I}_4 - \mathbf{C}$, and $\mathbf{C}_h^* = \Phi^h - \mathbf{C}$ for $h \geq 1$. Thus c_t, a_{t-1}, and y_t are all $I(1)$ with drift.

To derive the cointegrating relations and corresponding adjustment matrix, consider the trivariate VAR in $\mathbf{z}_t = \begin{pmatrix} c_t, & a_{t-1}, & y_t \end{pmatrix}'$

$$\mathbf{M}_0\mathbf{z}_t = \mathbf{M}_1\mathbf{z}_{t-1} + \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{w}_t, \tag{A.143}$$

where

$$\mathbf{M}_0 = \begin{pmatrix} 1 & -\lambda_2 & -\lambda_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.144}$$

$$\mathbf{M}_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ -1 & 1+r & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.145}$$

$$\mathbf{b}_0 = \begin{pmatrix} \lambda_4 + \lambda_8 \\ \gamma_y \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} \lambda_9 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{w}_t = \begin{pmatrix} \lambda_6 b_t + \lambda_7 b_{t-1} \\ \varepsilon_{yt} \end{pmatrix}. \tag{A.146}$$

The matrix $(\mathbf{M}_0 - \mathbf{M}_1)$ is of rank one, and can be decomposed as $\mathbf{M}_0 - \mathbf{M}_1 = \boldsymbol{\alpha}\boldsymbol{\beta}'$, with (imposing the exact-identifying restriction that $\beta_1 = 1$)

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 - \eta/(1+r) \\ 1 \\ 0 \end{pmatrix}, \tag{A.147}$$

and

$$\boldsymbol{\beta} = \begin{pmatrix} 1 \\ -r \\ -1 \end{pmatrix}. \tag{A.148}$$

As for saving, note that

$$\boldsymbol{\kappa}'\mathbf{x}_t = S_t = (\gamma_b - \gamma_y)/r + \boldsymbol{\beta}'\mathbf{C}^*(L)\mathbf{u}_t. \tag{A.149}$$

Quadratic Utility, Constant Exogenous Component to the Consumer's Aspiration, and Finitely Lived Overlapping Generations

Consider next the case where there are overlapping generations of finitely lived consumers. Let $\mathbf{x}_t = \begin{pmatrix} c_t, & a_{t-1}, & y_t \end{pmatrix}'$, and write (2.64), (2.63) (lagged one period), and (2.3) as a VAR(1) in $\mathbf{x}_t$:

$$\mathbf{D}_0\mathbf{x}_t = \mathbf{D}_1\mathbf{x}_{t-1} + \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{v}_t, \tag{A.150}$$

where

$$\mathbf{D}_0 = \begin{pmatrix} 1 & -\lambda_2 & -\lambda_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.151}$$

$$\mathbf{D}_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ -1 & 1+r & 1 \\ 0 & 0 & \phi_y \end{pmatrix}, \tag{A.152}$$

$$\mathbf{a}_0 = \begin{pmatrix} \lambda_4 \\ 0 \\ \mu_y \end{pmatrix}, \quad \mathbf{a}_1 = \begin{pmatrix} \lambda_5 \\ 0 \\ (1-\phi_y)\,\gamma_y \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_t = \begin{pmatrix} 0 \\ 0 \\ \varepsilon_{yt} \end{pmatrix}. \tag{A.153}$$

The VAR (A.150) can be rewritten as

$$(\mathbf{D}_0 - \mathbf{D}_1 L)\,(\mathbf{x}_t - \boldsymbol{\mu} - \boldsymbol{\gamma} t) = \mathbf{v}_t, \tag{A.154}$$

or

$$(\mathbf{I}_3 - \Phi L)\,(\mathbf{x}_t - \boldsymbol{\mu} - \boldsymbol{\gamma} t) = \mathbf{u}_t, \tag{A.155}$$

where $\Phi = \mathbf{D}_0^{-1}\mathbf{D}_1$,

$$\mathbf{u}_t = \mathbf{D}_0^{-1}\mathbf{v}_t = \begin{pmatrix} \varpi_y \varepsilon_{yt} \\ 0 \\ \varepsilon_{yt} \end{pmatrix}, \tag{A.156}$$

$$\varpi_y = \frac{r\,(2+r) + p - (1-p)\,\rho}{(1+r)\,[(1+r) - (1-p)\,(1-\alpha)\,\phi_y]}, \tag{A.157}$$

and $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are defined through $(\mathbf{D}_0 - \mathbf{D}_1)\,\boldsymbol{\gamma} = \mathbf{a}_1$ and $(\mathbf{D}_0 - \mathbf{D}_1)\,\boldsymbol{\mu} = \mathbf{a}_0 - \mathbf{D}_1\boldsymbol{\gamma}$.[33]

Case 1: $\rho < r$, $|\phi_y| < 1$

The eigenvalues of Φ are given by 0, $(1-p)\,(1+\rho)\,/\,(1+r)$, and ϕ_y. The infinite moving average representation of (A.155) is given by

$$\mathbf{x}_t = \boldsymbol{\mu} + \boldsymbol{\gamma} t + \mathbf{C}^*(L)\,\mathbf{u}_t, \tag{A.158}$$

where

$$\boldsymbol{\gamma} = \begin{pmatrix} \varpi_c \gamma_y \\ \varpi_A \gamma_y \\ \gamma_y \end{pmatrix}, \tag{A.159}$$

[33] In what follows we distinguish two cases: $\rho < r$, $|\phi_y| < 1$ (Case 1), and $\rho < r$, $\phi_y = 1$ (Case 2). When $p > 0$, none of the long-run properties of interest depend on whether $\rho < r$ or $\rho = r$, and the case where $\rho = r$ can always be obtained simply by imposing $\rho = r$ on all expressions in Case 1 and Case 2.

$$\varpi_c = \frac{[(1-\alpha)p+\alpha][r(2+r)+p-(1-p)\rho]}{[p+r+(1-p)\alpha][p+r-(1-p)\rho]}, \tag{A.160}$$

$$\varpi_A = \frac{(1-p)[\alpha+\rho-(1-\alpha)r]}{[p+r+(1-p)\alpha][p+r-(1-p)\rho]}, \tag{A.161}$$

$\mathbf{C}^*(L) = \sum_{h=0}^{\infty} \mathbf{C}_h^* L^h$, with $\mathbf{C}_0^* = \mathbf{I}_3$, and $\mathbf{C}_h^* = \Phi^h$ for $h \geq 1$, and the explicit expression for $\boldsymbol{\mu}$ is available upon request. Thus c_t, A_{t-1}, and y_t are all stationary. Defining the vector $\boldsymbol{\kappa}$,

$$\boldsymbol{\kappa} = \begin{pmatrix} -1, & r, & 1 \end{pmatrix}', \tag{A.162}$$

note that saving is given by

$$S_t = \boldsymbol{\kappa}'\mathbf{x}_t = -\varpi_A\gamma_y + \boldsymbol{\kappa}'\mathbf{C}^*(L)\mathbf{u}_t. \tag{A.163}$$

Case 2: $\rho < r$, $\phi_y = 1$

The eigenvalues of Φ are given by 0, $1-p$, and 1. The infinite moving average representation of (A.155) is given by

$$\mathbf{x}_t = \boldsymbol{\mu} + \boldsymbol{\gamma}t + \mathbf{C}\sum_{h=1}^{t}\mathbf{u}_h + \mathbf{C}^*(L)\mathbf{u}_t, \tag{A.164}$$

where

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & \varpi_c \\ 0 & 0 & \varpi_A \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.165}$$

$\boldsymbol{\gamma}$ is given by (A.159), $\mathbf{C}^*(L) = \sum_{h=0}^{\infty} \mathbf{C}_h^* L^h$, with $\mathbf{C}_0^* = \mathbf{I}_3$, and $\mathbf{C}_h^* = \Phi^h$ for $h \geq 1$, and the explicit expression for $\boldsymbol{\mu}$ is available upon request. Thus c_t and A_{t-1} are both $I(1)$ with drift; c_t and y_t are cointegrated and cotrended, as are A_t and y_t (though disposable income is $I(1)$ with drift $(\varpi_A r + 1)\gamma_y$).

The matrix $(\mathbf{D}_0 - \mathbf{D}_1)$ is of rank two, and can be decomposed as $\mathbf{D}_0 - \mathbf{D}_1 = \boldsymbol{\alpha}\boldsymbol{\beta}'$, with (imposing the exact-identifying restrictions that $\beta_{11} = \beta_{22} = 0$ and $\beta_{31} = \beta_{33} = 1$)

$$\boldsymbol{\beta} = \begin{pmatrix} 0 & -1/\varpi_c \\ -1/\varpi_A & 0 \\ 1 & 1 \end{pmatrix}. \tag{A.166}$$

(The exact expression for $\boldsymbol{\alpha}$ is available upon request.) As for saving, note that

$$\boldsymbol{\kappa}'\mathbf{x}_t = S_t = -\varpi_A\gamma_y + \boldsymbol{\kappa}'\mathbf{C}^*(L)\mathbf{u}_t. \tag{A.167}$$

■

Appendix C: Proofs of Propositions

Proposition 2.1:

To show that $\widetilde{a}_t^d$ converges in probability to $1/r$, from (2.99) note that

$$\widetilde{a}_t^d - \frac{1}{r} = -\frac{1}{r\left(1 + r\widetilde{a}_{t-1}\right)}. \tag{A.168}$$

Therefore, defining $\zeta_{yt} = \sum_{j=2}^{t} \varepsilon_{yj}$,

$$\begin{aligned}
\Pr\left(\left|\widetilde{a}_t^d - \frac{1}{r}\right| > \eta\right) &= \Pr\left[\frac{1}{r\left(1 + r\widetilde{a}_{t-1}\right)} > \eta\right] \\
&= \Pr\left(\widetilde{a}_{t-1} < \frac{1 - r\eta}{r^2\eta}\right) \\
&= \Pr\left\{\exp\left[\frac{(t-1)\,\sigma_y^2}{2} - \zeta_{yt}\right] < \frac{1 - r\eta}{r^2\eta\widetilde{a}_0}\right\} \\
&= \Pr\left[-\zeta_{yt} < -\frac{(t-1)\,\sigma_y^2}{2} + \log\left(\frac{1 - r\eta}{r^2\eta\widetilde{a}_0}\right)\right] \\
&= \Phi\left[-\frac{\sigma_y\sqrt{t-1}}{2} + \log\left(\frac{1 - r\eta}{r^2\eta\widetilde{a}_0}\right) / \left(\sigma_y\sqrt{t-1}\right)\right] \qquad \text{(A.169)}
\end{aligned}$$

(where Φ denotes the Normal cumulative distribution function), and

$$\widetilde{a}_t^d \xrightarrow{p} \frac{1}{r}, \tag{A.170}$$

for any arbitrarily small $\eta \in (0, 1/r)$. Since the saving rate obeys

$$sr_t = \frac{\mu_y \widetilde{a}_{t-1}}{1 + r\widetilde{a}_{t-1}}, \tag{A.171}$$

it is readily seen that

$$sr_t - \frac{\mu_y}{r} = -\frac{\mu_y}{r\left(1 + r\widetilde{a}_{t-1}\right)}. \tag{A.172}$$

Using (2.99), (A.172) may be rewritten as

$$sr_t - \frac{\mu_y}{r} = -\mu_y\left(\frac{1}{r} - \widetilde{a}_{t-1}^d\right), \tag{A.173}$$

and

$$\text{plim}_{t\to\infty}\left(sr_t\right) - \frac{\mu_y}{r} = -\mu_y\left[\frac{1}{r} - \text{plim}_{t\to\infty}\left(\widetilde{a}_{t-1}^d\right)\right].$$

Observing that $\widetilde{a}_t^d \xrightarrow{p} 1/r$, it readily follows that $sr_t \xrightarrow{p} \mu_y/r$. Furthermore, sr_t converges at the same rate as $\widetilde{a}_t^d$, which from (A.169) is equal to $\sigma_y\sqrt{t-1}$. ■

Proposition 2.2:

Under (2.106) it is readily shown that

$$\begin{aligned}\Pr\left(\left|\widetilde{a}_t^d - \frac{1}{r}\right| > \eta\right) &= \Phi\left\{-\sigma_y\sqrt{t-1}\log\left[(1-p)(1-\alpha)\exp\left(\frac{\sigma_y^2}{2}\right)\right]/\sigma_y^2\right.\\ &\quad \left. + \log\left(\frac{1-r\eta}{r^2\eta\sigma\widetilde{a}_0}\right)/\left(\sigma_y\sqrt{t-1}\right)\right\}, \end{aligned} \tag{A.174}$$

and thus

$$\widetilde{a}_t^d \xrightarrow{p} \frac{1}{r}, \tag{A.175}$$

for any arbitrarily small $\eta \in (0, 1/r)$. Given that the asset-disposable income ratio converges in probability to $1/r$, the proof that

$$sr_t \xrightarrow{p} [(1-p)(1-\alpha)(1+\mu_y) - 1]/r$$

is conceptually the same as the second part of the proof of Proposition 2.1, and thus need not be repeated. ■

Proposition 3.1:[34]

Upon substituting the tax rule (3.12) into the period-by-period government budget constraint

$$d_t = (1+r)\, d_{t-1} + g_t - \tau_t, \tag{A.176}$$

dividing on both sides of the resultant equation by y_{t+1}, and rearranging terms, (A.176) becomes[35]

$$\begin{aligned} w_t &= \exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{y,t+1}\right)(1+r-\kappa_2)\, w_{t-1} + \exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{y,t+1}\right)\kappa_1 \\ &\quad + \exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{y,t+1}\right)\varepsilon_{\tau t}, \end{aligned} \tag{A.178}$$

where $w_t = d_t/y_{t+1}$. To derive the asymptotic properties of $\{w_t\}$, iterate on (A.178) forward from $t = 1$, with the initial condition w_0. Then w_h is given by

$$w_h = \lambda^h \exp\left(-\sum_{j=1}^{h}\varepsilon_{y,j+1}\right) w_0 + \exp\left(-\gamma + \frac{1}{2}\sigma_y^2\right)\sum_{j=0}^{h-1}\lambda^j \exp\left(-\sum_{k=0}^{j}\varepsilon_{y,h+1-k}\right)(\varepsilon_{\tau,h-j} + \kappa_1), \tag{A.179}$$

with

$$\lambda = \exp\left(-\gamma + \frac{1}{2}\sigma_y^2\right)(1+r-\kappa_2). \tag{A.180}$$

[34] We are grateful to Sören Johansen for suggesting to us the use of the law of the iterated logarithm in the proof of Propositions 3.1 and 3.2.

[35] Note that under the labor income specification (3.2) we have

$$\frac{y_t}{y_{t+1}} = \exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{y,t+1}\right). \tag{A.177}$$

We first establish that with probability one $\{w_t\}$ asymptotically does not depend on its initial condition, w_0. By the law of the iterated logarithm, we have that with probability one

$$\sup_{h\to\infty} \frac{\left|\sum_{j=1}^{h} \varepsilon_{y,j+1}\right|}{\sigma_y\sqrt{2h\log\left[\log\left(h\right)\right]}} \leq 1, \tag{A.181}$$

and thus, also with probability one,

$$\left|\sum_{j=1}^{h} \varepsilon_{y,j+1}\right| \leq m\sqrt{h\log\left[\log\left(h\right)\right]}, \tag{A.182}$$

for some finite $m > 0$. Considering the coefficient on w_0, $\lambda^h \exp\left(-\sum_{j=1}^{h} \varepsilon_{y,j+1}\right)$, this coefficient must then go to zero with probability one as $h \to \infty$:

$$\begin{aligned}\lim_{h\to\infty} \lambda^h \exp\left(-\sum_{j=1}^{h} \varepsilon_{y,j+1}\right) &\leq \lim_{h\to\infty} \lambda^h \exp\left(m\sqrt{h\log\left[\log\left(h\right)\right]}\right) \\ &= \lim_{h\to\infty} \exp\left[h\log\lambda\left(1+\frac{m}{\log\lambda}\sqrt{\frac{\log\left[\log\left(h\right)\right]}{h}}\right)\right].\end{aligned} \tag{A.183}$$

Noting that under (3.13) $\lambda \in (0, 1)$, it is readily seen that

$$h\log\lambda \to -\infty \quad \text{as} \quad h \to \infty, \tag{A.184}$$

and that

$$\sqrt{\frac{\log\left[\log\left(h\right)\right]}{h}} \to 0 \quad \text{as} \quad h \to \infty. \tag{A.185}$$

It follows that with probability one

$$\lim_{h\to\infty} \lambda^h \exp\left(-\sum_{j=1}^{h} \varepsilon_{y,j+1}\right) = 0. \tag{A.186}$$

Note that the condition that ε_{yt} is normally distributed is much stronger than is needed for (A.186) to hold. See, for example, Petrov (1995) for a discussion of the law of the iterated logarithm if not only the normality assumption is dropped but if it is also not supposed that any moments of ε_{yt} exist.

To establish that $\{w_t\}$ converges globally to a well-defined steady-state probability distribution function, it now remains to show that

$$\sum_{j=0}^{\infty} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{yk}\right) \tag{A.187}$$

and

$$\sum_{j=0}^{\infty} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{yk}\right) \varepsilon_{\tau j} \tag{A.188}$$

exist, noting that if the infinite sums in (A.187) and (A.188) exist, then so does

$$\lim_{h\to\infty} \exp\left(-\gamma + \frac{1}{2}\sigma_y^2\right) \sum_{j=0}^{h-1} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{y,h+1-k}\right) \left(\varepsilon_{\tau,h-j} + \kappa_1\right), \tag{A.189}$$

the second component of w_h in (A.179) as $h \to \infty$. Consider first the infinite sum in (A.187). From (A.182) it follows that with probability one for some finite $m > 0$

$$\sum_{j=0}^{\infty} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{yk}\right) \le \sum_{j=0}^{\infty} \lambda^j \exp\left(m\sqrt{(j+1)\log\left[\log(j+1)\right]}\right)$$

$$= \sum_{j=0}^{\infty} \exp\left(j \log \lambda \left\{1 + \left(\frac{j+1}{j}\right)\left(\frac{m}{\log \lambda}\right)\sqrt{\frac{\log\left[\log(j+1)\right]}{j+1}}\right\}\right). \tag{A.190}$$

Observing (A.185), it is clear that

$$1 + \left(\frac{j+1}{j}\right)\left(\frac{m}{\log \lambda}\right)\sqrt{\frac{\log\left[\log(j+1)\right]}{j+1}} \to 1 \quad \text{as} \quad j \to \infty, \tag{A.191}$$

and thus $\sum_{j=0}^{\infty} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{yk}\right)$ exists with probability one. Consider next the infinite sum in (A.188). From the triangle inequality

$$\begin{aligned}\left|\sum_{j=0}^{\infty} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{yk}\right)\varepsilon_{\tau j}\right| &\le \sum_{j=0}^{\infty} |\lambda|^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{yk}\right)|\varepsilon_{\tau j}| \\ &\le \left(\sup_j |\varepsilon_{\tau j}|\right)\left[\sum_{j=0}^{\infty} |\lambda|^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{yk}\right)\right].\end{aligned} \tag{A.192}$$

As $\varepsilon_{\tau j}$ is finite-valued with probability one, from (A.190) and (A.191) it is clear that

$$\sum_{j=0}^{\infty} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{yk}\right)\varepsilon_{\tau h}$$

exists with probability one. Again clearly neither the condition that ε_{yt} is normally distributed nor the condition that $\varepsilon_{\tau t}$ is normally distributed are needed for this argument to apply. As for ε_{yt}, it is not necessary that $\varepsilon_{\tau t}$ have moments, it only needs to be finite-valued with probability one.

We turn next to the (first two) moments of the steady state probability distribution function of the government debt-labor income ratio. From the above it follows that as t tends to infinity,

$$w_t = \lim_{h \to \infty} \exp\left(-\gamma + \frac{1}{2}\sigma_y^2\right) \sum_{j=0}^{h-1} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{y,h+1-k}\right)\left(\varepsilon_{\tau,h-j} + \kappa_1\right). \tag{A.193}$$

Thus, under (3.14)

$$\begin{aligned}\lim_{t\to\infty} E(w_t) &= \exp\left(-\gamma + \frac{1}{2}\sigma_y^2\right)\sum_{j=0}^{\infty} \lambda^j \exp\left[\left(\frac{j+1}{2}\right)\sigma_y^2\right]\kappa_1 \\ &= \left[\frac{\exp\left(\sigma_y^2\right)}{\exp(\gamma) - (1 + r - \kappa_2)\exp\left(\sigma_y^2\right)}\right]\kappa_1.\end{aligned} \tag{A.194}$$

Squaring both sides of (A.193) and then taking unconditional expectations, one obtains

$$\begin{aligned}\lim_{t\to\infty} E\left(w_t^2\right) &= \exp\left(-2\gamma + \sigma_y^2\right) E\left\{\left[\lim_{h\to\infty}\sum_{j=0}^{h-1} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{y,h+1-k}\right)\left(\varepsilon_{\tau,h-j} + \kappa_1\right)\right]\right. \\ &\left.\left[\lim_{h\to\infty}\sum_{j=0}^{h-1} \lambda^j \exp\left(-\sum_{k=0}^{j} \varepsilon_{y,h+1-k}\right)\left(\varepsilon_{\tau,h-j} + \kappa_1\right)\right]\right\}.\end{aligned} \tag{A.195}$$

Upon some algebra, (A.195) under (3.16) can be simplified to

$$\lim_{t\to\infty} E\left(w_t^2\right) = \left[\frac{\exp\left(3\sigma_y^2\right)}{\exp\left(2\gamma\right) - \exp\left(3\sigma_y^2\right)\left(1+r-\kappa_2\right)^2}\right]\left\{\sigma_\tau^2 + \left[\frac{\exp\left(\gamma\right) + \left(1+r-\kappa_2\right)\exp\left(\sigma_y^2\right)}{\exp\left(\gamma\right) - \left(1+r-\kappa_2\right)\exp\left(\sigma_y^2\right)}\right]\kappa_1^2\right\}. \tag{A.196}$$

Thus, under (3.16) the variance of the steady-state distribution of w_t is given by (3.17). Finally, note that the mean and variance of the steady-state distribution of w_t can in general be computed using the moment generating function of ε_{yt}, $E\left[\exp\left(q\,\varepsilon_{yt}\right)\right]$, for $q = -1, -2$, and the first two moments of $\varepsilon_{\tau t}$. Thus, the normality assumptions regarding ε_{yt} and $\varepsilon_{\tau t}$ are not necessary conditions for the mean and variance of the steady-state distribution of w_t to exist. ■

Proposition 3.2:

Subtracting the period-by-period budget constraint for the government, (A.176), from that of the consumer, (3.8), and substituting the decision rule (3.21) into the resulting expression one obtains the second-order difference equation

$$\begin{aligned}\widetilde{a}_t - \left(\eta+\delta\right)\widetilde{a}_{t-1} + \eta\delta\widetilde{a}_{t-2} \quad &= \quad \left(1-\lambda_3\right)y_t - \lambda_1 y_{t-1} - \lambda_4 b_t - \lambda_5 b_{t-1} \\ &\quad - \left(1+\lambda_6\right)g_t + \left(\lambda_1-\lambda_7\right)g_{t-1},\end{aligned} \tag{A.197}$$

where $\widetilde{a}_t = a_t - d_t$. Dividing on both sides of (A.197) by y_{t+1}, one obtains

$$\widetilde{w}_t - \left(\eta+\delta\right)\exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{y,t+1}\right)\widetilde{w}_{t-1} + \eta\delta\exp\left(-2\gamma + \sigma_y^2 - \varepsilon_{yt} - \varepsilon_{y,t+1}\right)\widetilde{w}_{t-2} = \vartheta_t, \tag{A.198}$$

where $\widetilde{w}_t = \left(a_t - d_t\right)/y_{t+1}$, and

$$\begin{aligned}\vartheta_t \quad = \quad &\exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{y,t+1}\right)\left[1 - \lambda_3 - \lambda_1\exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{yt}\right) - \lambda_4\exp\left(\alpha_b - \frac{1}{2}\sigma_b^2 + \varepsilon_{bt}\right)\right. \\ &-\lambda_5\exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{yt}\right)\exp\left(\alpha_b - \frac{1}{2}\sigma_b^2 + \varepsilon_{b,t-1}\right) - \left(1+\lambda_6\right)\exp\left(\alpha_g - \frac{1}{2}\sigma_g^2 + \varepsilon_{gt}\right) \\ &\left.+\left(\lambda_1-\lambda_7\right)\exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{yt}\right)\exp\left(\alpha_g - \frac{1}{2}\sigma_g^2 + \varepsilon_{g,t-1}\right)\right].\end{aligned} \tag{A.199}$$

Defining $\mathbf{z}_t = \begin{pmatrix} \widetilde{w}_t, & \widetilde{w}_{t-1} \end{pmatrix}'$, (A.198) may be rewritten in first-order form as

$$\mathbf{z}_t = \Phi_t \mathbf{z}_{t-1} + \Gamma_t, \tag{A.200}$$

where

$$\Phi_t = \begin{pmatrix} \left(\overline{\eta}+\overline{\delta}\right)\exp\left(-\varepsilon_{y,t+1}\right) & -\overline{\eta}\overline{\delta}\exp\left(-\varepsilon_{yt} - \varepsilon_{y,t+1}\right) \\ 1 & 0 \end{pmatrix}, \tag{A.201}$$

$$\overline{\eta} = \eta\exp\left(-\gamma + \frac{1}{2}\sigma_y^2\right), \tag{A.202}$$

$$\overline{\delta} = \delta\exp\left(-\gamma + \frac{1}{2}\sigma_y^2\right), \tag{A.203}$$

and

$$\Gamma_t = \begin{pmatrix} \vartheta_t \\ 0 \end{pmatrix}. \tag{A.204}$$

Defining

$$\mathbf{q}_t = \begin{pmatrix} \exp(\varepsilon_{y,t+1})\,\widetilde{w}_t \\ \widetilde{w}_{t-1} \end{pmatrix}, \tag{A.205}$$

$$\Theta = \begin{pmatrix} \overline{\eta} + \overline{\delta} & -\overline{\eta}\overline{\delta} \\ 1 & 0 \end{pmatrix}, \tag{A.206}$$

and

$$\widetilde{\Gamma}_t = \begin{pmatrix} \exp(\varepsilon_{y,t+1})\,\vartheta_t \\ 0 \end{pmatrix}, \tag{A.207}$$

it is easily verified that (A.200) may be rewritten as

$$\mathbf{q}_t = \exp(-\varepsilon_{yt})\,\Theta\mathbf{q}_{t-1} + \widetilde{\Gamma}_t. \tag{A.208}$$

To derive the asymptotic properties of $\{\mathbf{q}_t\}$ (and hence also $\{\widetilde{w}_t\}$), iterate on (A.208) forward from $t = 1$, with the initial condition $\mathbf{q}_0$. Then $\mathbf{q}_h$ is given by

$$\mathbf{q}_h = \Theta^h \exp\left(-\sum_{j=1}^{h} \varepsilon_{yj}\right)\mathbf{q}_0 + \sum_{j=0}^{h-1} \Theta^j \exp\left(-\sum_{k=1}^{j} \varepsilon_{y,h+1-k}\right)\widetilde{\Gamma}_{h-j}. \tag{A.209}$$

We first establish that with probability one $\{\mathbf{q}_t\}$ (and hence also $\{\widetilde{w}_t\}$) asymptotically does not depend on its initial condition, $\mathbf{q}_0$ ($\widetilde{w}_0$), by showing that the coefficient matrix on $\mathbf{q}_0$ in (A.209),

$$\Theta^h \exp\left(-\sum_{j=1}^{h} \varepsilon_{yj}\right), \tag{A.210}$$

with probability one converges to the zero matrix. Diagonalizing Θ as

$$\Theta = \mathbf{P}\begin{pmatrix} \overline{\eta} & 0 \\ 0 & \overline{\delta} \end{pmatrix}\mathbf{P}^{-1}, \tag{A.211}$$

with

$$\mathbf{P} = \begin{pmatrix} \overline{\delta} & \overline{\eta} \\ 1 & 1 \end{pmatrix}, \tag{A.212}$$

the coefficient matrix on $\mathbf{q}_0$ in (A.209) can be written as

$$\mathbf{P}\begin{pmatrix} \overline{\eta}^h \exp\left(-\sum_{j=1}^{h} \varepsilon_{yj}\right) & 0 \\ 0 & \overline{\delta}^h \exp\left(-\sum_{j=1}^{h} \varepsilon_{yj}\right) \end{pmatrix}\mathbf{P}^{-1}. \tag{A.213}$$

Define the coefficient λ such that

$$\lambda = \max\left\{\overline{\eta}, \overline{\delta}\right\}. \tag{A.214}$$

Note that under (3.29) and (3.30) $\lambda < 1$. Applying the law of the iterated logarithm to $\exp\left(-\sum_{j=1}^{h} \varepsilon_{yj}\right)$ (see (A.181) to (A.186)), it is clear that with probability one

$$\lim_{h\to\infty} \lambda^h \exp\left(-\sum_{j=1}^{h} \varepsilon_{yj}\right) = 0. \tag{A.215}$$

Consider now the second term on the right-hand side of (A.209),

$$\sum_{j=0}^{h-1} \mathbf{z}_j, \tag{A.216}$$

where

$$\mathbf{z}_j = \Theta^j \exp\left(-\sum_{k=1}^{j} \varepsilon_{y,h+1-k}\right) \widetilde{\Gamma}_{h-j}. \tag{A.217}$$

To establish that $\{\mathbf{q}_t\}$ (and hence also $\{\widetilde{w}_t\}$) converges globally to a well defined steady-state probability distribution function, it remains to show that

$$\lim_{h\to\infty} \sum_{j=0}^{h-1} \mathbf{z}_j \tag{A.218}$$

exists. Using (A.211) and (A.212), Θ^j can be written as

$$\mathbf{P}\begin{pmatrix} \overline{\eta}^j & 0 \\ 0 & \overline{\delta}^j \end{pmatrix}\mathbf{P}^{-1} = \left(\frac{1}{\overline{\delta} - \overline{\eta}}\right)\begin{pmatrix} \overline{\delta}^{j+1} - \overline{\eta}^{j+1} & \overline{\delta}\overline{\eta}^{j+1} - \overline{\eta}\overline{\delta}^{j+1} \\ \overline{\delta}^j - \overline{\eta}^j & \overline{\delta}\overline{\eta}^j - \overline{\eta}\overline{\delta}^j \end{pmatrix}. \tag{A.219}$$

From (A.199) and (A.207) it is easily seen that the first row of $\widetilde{\Gamma}_{h-j}$ is given by

$$\exp\left(-\gamma + \frac{1}{2}\sigma_y^2\right)\left(\varrho_1 + \varrho_{2,h-j} + \varrho_{3,h-j} + \varrho_{4,h-j} + \varrho_{5,h-j} + \varrho_{6,h-j}\right), \tag{A.220}$$

where

$$\varrho_1 = 1 - \lambda_3, \tag{A.221}$$

$$\varrho_{2,h-j} = -\lambda_1 \exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{y,h-j}\right), \tag{A.222}$$

$$\varrho_{3,h-j} = -\lambda_4 \exp\left(\alpha_b - \frac{1}{2}\sigma_b^2 + \varepsilon_{b,h-j}\right), \tag{A.223}$$

$$\varrho_{4,h-j} = -\lambda_5 \exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{y,h-j}\right)\exp\left(\alpha_b - \frac{1}{2}\sigma_b^2 + \varepsilon_{b,h-j-1}\right), \tag{A.224}$$

$$\varrho_{5,h-j} = -\left(1+\lambda_6\right)\exp\left(\alpha_g - \frac{1}{2}\sigma_g^2 + \varepsilon_{g,h-j}\right), \tag{A.225}$$

and

$$\varrho_{6,h-j} = \left(\lambda_1 - \lambda_7\right)\exp\left(-\gamma + \frac{1}{2}\sigma_y^2 - \varepsilon_{y,h-j}\right)\exp\left(\alpha_g - \frac{1}{2}\sigma_g^2 + \varepsilon_{g,h-j-1}\right). \tag{A.226}$$

Also note that the second row of $\widetilde{\Gamma}_{h-j}$ is equal to zero. Noting this and using (A.219) and (A.220), $\mathbf{z}_j$ is therefore equal to

$$\left(\frac{1}{\delta-\eta}\right)\begin{pmatrix} \overline{\delta}^{j+1} - \overline{\eta}^{j+1} \\ \overline{\delta}^{j} - \overline{\eta}^{j} \end{pmatrix}\exp\left(-\sum_{k=1}^{j}\varepsilon_{y,h+1-k}\right)\left(\varrho_1 + \varrho_{2,h-j} + \varrho_{3,h-j} + \varrho_{4,h-j} + \varrho_{5,h-j} + \varrho_{6,h-j}\right). \tag{A.227}$$

Using the definition of λ given in (A.214) and recalling that under (3.29) and (3.30) $\lambda < 1$, then $\lim_{h\to\infty}\sum_{j=0}^{h-1}\mathbf{z}_j$ exists if

$$\lim_{h\to\infty}\sum_{j=0}^{h-1}\lambda^{j+1}\exp\left(-\sum_{k=1}^{j}\varepsilon_{y,h+1-k}\right)\varrho_{l,h-j} \tag{A.228}$$

exists for $l = 1, 2, \ldots, 6$. Since $\sup_j\left|\varrho_{l,h-j}\right|$ is finite with probability one, from (A.192) it is clear that the limit in (A.228) exists for $l = 1, 2, \ldots, 6$. Yet again, for the reasons discussed in the proof of Proposition 3.1, normality of ε_{yt}, ε_{bt}, and ε_{gt} is not required for this argument. We have thus established the ergodicity of $\{\widetilde{w}_t\}$ under the conditions (3.29) and (3.30).

Substituting for $\tau_t - g_t$ in

$$y_t - \tau_t - c_t = a_t - d_t - (1+r)\left(a_{t-1} - d_{t-1}\right) - \left(\tau_t - g_t\right), \tag{A.229}$$

from the tax rule (3.12), and substituting the resultant expression into (3.27), it is readily seen that if the net asset-labor income ratio follows an ergodic process, then so does the private saving rate. Furthermore, solving (3.27) for c_t and substituting for τ_t in the resultant expression from the tax rule (3.12), it is also readily seen that if the net asset-labor income ratio follows an ergodic process, then so does the consumption-labor income ratio.

We finally turn to the first moment of the steady-state probability distribution function of the net asset-labor income ratio, the private saving rate, and the consumption-labor income ratio. From the above it follows that as t tends to infinity,

$$\mathbf{q}_t = \lim_{h\to\infty}\sum_{j=0}^{h-1}\Theta^j\exp\left(-\sum_{k=1}^{j}\varepsilon_{y,h+1-k}\right)\widetilde{\Gamma}_{h-j}, \tag{A.230}$$

and thus

$$\widetilde{w}_{t-1} = \lim_{h\to\infty}\sum_{j=0}^{h-1}\left(\frac{\overline{\delta}^j - \overline{\eta}^j}{\overline{\delta} - \overline{\eta}}\right)\exp\left(-\sum_{k=1}^{j}\varepsilon_{y,h+1-k}\right)\exp\left(\varepsilon_{y,h+1-j}\right)\vartheta_{h-j}. \tag{A.231}$$

where ϑ_t is given by (A.199). Thus, under (3.31), (3.32), and (3.13)

$$\begin{aligned}\lim_{t\to\infty}E\left(\widetilde{w}_t\right) &= \sum_{j=0}^{\infty}\left(\frac{\overline{\delta}^j - \overline{\eta}^j}{\overline{\delta} - \overline{\eta}}\right)\exp\left[\left(\frac{j-1}{2}\right)\sigma_y^2\right]E\left(\vartheta\right) \\ &= \frac{E\left(\vartheta\right)}{\left[1-\eta\exp\left(-\gamma+\sigma_y^2\right)\right]\left[1-\delta\exp\left(-\gamma+\sigma_y^2\right)\right]},\end{aligned} \tag{A.232}$$

with

$$E(\vartheta) = \exp\left(-\gamma+\sigma_y^2\right)\left\{1-\lambda_3-\lambda_1\exp\left(-\gamma+\sigma_y^2\right)-\left[\lambda_4+\lambda_5\exp\left(-\gamma+\sigma_y^2\right)\right]E\left(\frac{b}{y}\right)\right.$$
$$\left.-\left[1+\lambda_6-(\lambda_1-\lambda_7)\exp\left(-\gamma+\sigma_y^2\right)\right]E\left(\frac{g}{y}\right)\right\}. \quad \text{(A.233)}$$

Upon some algebra, (A.232) may be further simplified to (3.33). To establish that $\pi \geq 0$, note that

$$\upsilon \geq \frac{\eta\delta+(1+r)(1+\mu-\eta-\delta)}{(1+r)(r-\mu)}+\frac{\eta\delta}{(1+r)(1+\mu)}=\widetilde{\upsilon}. \quad \text{(A.234)}$$

It may be readily verified that

$$\widetilde{\upsilon}=\frac{(1+\mu)^2+(1+\mu)(\eta+\delta)-\eta\delta}{(1+\mu)(r-\mu)}=\frac{(1+\mu-\eta)(1+\mu-\delta)}{(1+\mu)(r-\mu)}. \quad \text{(A.235)}$$

Under $r-\mu>0$, $\mu\geq 0$, $\eta\in[0,1)$, and $\delta\in(0,1]$, it is clearly true that $\widetilde{\upsilon}\geq 0$. If $\widetilde{\upsilon}\geq 0$, it then readily follows from (3.31) and (3.32) that $\pi \geq 0$. To derive the mean of the steady state distribution of the private saving rate and the mean of the steady state distribution of the consumption-labor income ratio, note (A.229). Substituting for $\tau_t - g_t$ in (A.229) from the tax rule (3.12), and substituting the resultant expression into (3.27), the mean of the steady state distribution of the private saving rate (3.36) is readily derived. Solving (3.27) for c_t and substituting for τ_t in the resultant expression from the tax rule (3.12), the mean of the steady state distribution of the consumption-labor income ratio (3.37) is also readily obtained. Note that the means of the steady-state distributions of the net asset-labor income ratio, the private saving rate, and the consumption-labor income ratio can in general be computed using the moment generating functions of ε_{yt}, ε_{bt}, ε_{gt}, $E\left[\exp\left(q\,\varepsilon_{lt}\right)\right]$, for $l=y,b,g$, and $q=-1,1$. Thus, the normality assumptions regarding ε_{yt}, ε_{bt}, and ε_{gt} are not necessary conditions for the means of these steady-state distributions to exist. ■

Remark 3.2:

Squaring both sides of (A.193) and then taking unconditional expectations, one obtains

$$\mathbf{q}_t\mathbf{q}_t' = \left(\frac{1}{\overline{\delta}-\overline{\eta}}\right)^2\left[\lim_{h\to\infty}\sum_{j=0}^{h-1}\Theta^j\exp\left(-\sum_{k=1}^{j}\varepsilon_{y,h+1-k}\right)\widetilde{\Gamma}_{h-j}\right]$$
$$\left[\lim_{h\to\infty}\sum_{j=0}^{h-1}\Theta^j\exp\left(-\sum_{k=1}^{j}\varepsilon_{y,h+1-k}\right)\widetilde{\Gamma}_{h-j}\right]', \quad \text{(A.236)}$$

and thus

$$\widetilde{w}_{t-1}^2 = \left(\frac{1}{\overline{\delta}-\overline{\eta}}\right)^2\left[\lim_{h\to\infty}\sum_{j=0}^{h-1}\left(\overline{\delta}^j-\overline{\eta}^j\right)\exp\left(-\sum_{k=1}^{j}\varepsilon_{y,h+1-k}\right)\exp\left(\varepsilon_{y,h+1-j}\right)\vartheta_{h-j}\right]$$
$$\left[\lim_{h\to\infty}\sum_{j=0}^{h-1}\left(\overline{\delta}^j-\overline{\eta}^j\right)\exp\left(-\sum_{k=1}^{j}\varepsilon_{y,h+1-k}\right)\exp\left(\varepsilon_{y,h+1-j}\right)\vartheta_{h-j}\right]. \quad \text{(A.237)}$$

Taking unconditional expectations of (A.237), under (3.39) and (3.40) upon some algebra one obtains (3.41). Note that the variance of the steady-state distribution of $\widetilde{w}_t$ can in general be computed using the moment generating functions of ε_{yt}, ε_{bt}, ε_{gt}, $E\left[\exp\left(q\,\varepsilon_{lt}\right)\right]$, for $l = y, b, g$, and $q = -2, -1, 1, 2$. Thus, the normality assumptions regarding ε_{yt}, ε_{bt}, and ε_{gt} are not necessary conditions for the variance of the steady-state distribution of $\widetilde{w}_t$ to exist. ■

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