Does Competition Solve the Hold-up Problem?*

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Abstract. In an environment in which both workers and firms undertake match specific investments, the presence of market competition for matches may solve the hold-up problems generated by the absence of complete contingent contracts. In particular, this paper shows that when matching is assortative and workers' investments precede market competition for matches investments are constrained efficient. Inefficiencies can arise in this framework as multiple equilibria of the competition game. Only one of these equilibria is efficient in the sense that the worker with the higher innate ability matches with the better firm. A different type of inefficiencies arise when firms undertake their match specific investment before market competition. These inefficiency leads to firms under-investments. However, we show that in this case the aggregate inefficiency is *small* in a well defined sense independent of the market size.

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1. Introduction

A central concern for economists is the extent to which market systems are efficient. In the idealized Arrow-Debreu model of general competitive equilibrium, efficiency follows under mild conditions, notably the absence of externalities. But in recent years, economists have become interested in studying market situations less idealized than in the Arrow-Debreu set-up and in examining the pervasive inefficiencies that may exist. The subject of the present paper, the "hold-up problem", is one example of a situation that is thought to give rise to significant inefficiencies.

The hold-up problem applies when an agent making an investment is unable to receive all the benefits that accrue from the investment. The existence of the problem is generally traced to incomplete contracts: with complete contracts, the inefficiency induced by the failure to capture benefits will not be permitted to persist. In the standard set-up of the problem, investments are chosen before agents interact and contracts can be determined only when agents meet. Prior investments will be a sunk cost and negotiation over the division of surplus resulting from an agreement is likely to lead to a sharing of the surplus enhancement made possible by one agent's investment (Williamson 1985, Grout 1984, Grossman and Hart 1986, Hart and Moore 1988).

What happens if agent interaction is through the marketplace? In an Arrow-Debreu competitive model, complete markets, with price-taking in each market, are assumed; if an agent chooses investment *ex-ante*, every different level of investment may be thought of as providing the agent with a different good to bring to the market. If the agent wishes to choose a particular level of investment over some other, and the "buyer" he trades with also prefers to trade with the agent in question, rather than with an "identical" agent with another investment level, then total surplus to be divided must be maximized by the investment level chosen: investment will be efficiently chosen and there is no hold-up problem. In this situation, the existence of complete markets implies that agents know the price that they will receive or pay whatever the investment level chosen: complete markets imply complete contracts.

An unrealistic failure of the Arrow-Debreu set-up is that markets are assumed to exist for every conceivable level of investment, irrespective of whether or not trade occurs in such a market. But without trade, it is far-fetched to assume that agents will believe that they can trade in inactive markets and that a competitive price will be posted in such market.

The purpose of this paper is to investigate the efficiency of investments when the trading pattern and terms of trade are determined explicitly by the interaction of buyers and sellers. To ensure that there are no inefficiencies resulting from market power, a model of Bertrand competition is analyzed where some agents invest prior to trade; however, this does not rule out the dependence of the pattern of outcomes on the initial investment of any agent and the analysis concentrates on the case of a finite number of traders to ensure this possibility. Contracts are the result of competition in the marketplace and we are interested in the degree to which the hold-up problem is mitigated by contracts that result from Bertrand competition. In this regard, it should be said that we shall not permit Bertrand competition in contingent contracts; in our analysis, contracts take the form of an agreement to trade at a particular price. We are thus investigating the efficiency of contracts implied by a simple trading structure rather than attempting explicitly to devise contracts that help address the hold-up problem (Aghion, Dewatripont, and Rey 1994, Nöldeke and Schmidt 1995, Maskin and Tirole 1999, Segal and Whinston 1998, e.g.).

To further tie our hands, we will restrict attention to markets where the Bertrand competitive outcome is robust to the way that markets are made to clear. Specifically, we assume that buyers and sellers can be ordered by their ability to generate surplus with a complementarity between buyers and sellers. This set-up gives rise to assortative matching in the quality of buyers and sellers. With investment choices, the quality of sellers and/or buyers is assumed to depend on such investments.

With Bertrand competition, there is an asymmetry between buyers and sellers in a market. As a convention, we assume that buyers bid for the right to trade with sellers by naming a price that they wish to receive.

We first consider a world in which only buyers' quality depends on their ex-ante investments, sellers' qualities are exogenously given. In this case we demonstrate that buyers' investment choices are constrained efficient. In particular, for given equilibrium match, a buyer bids just enough to win the right to trade with a seller and, if he were to have previously enhanced his quality and the value of the trade by extra investment, he would have been able to win the right with the same bid, as viewed by the seller, and so receive all the marginal benefits of the extra investment.

The constraints imposed on efficiency are given by the matches that are observed in equilibrium. Indeed, when the returns of investments in terms of buyers' quality are not too high it is possible that a buyer might undertake a high investment with the sole purpose of changing the seller with whom he will be matched. This may lead to inefficient equilibrium matches. Notice that, in such an environment, hold-up problems are solved and the only inefficiencies left are due to buyers' preemption strategies when choosing their investments. These inefficiencies may actually disappear if the returns from investments differ enough across buyers.[1]

We then consider a world in which the sellers' quality depends on their ex-ante investments. In this case we indeed show that sellers' investments are inefficient. However, the extent of the inefficiency is strictly limited. In particular, we show that the overall inefficiency in a market is less than that which could result from an under-investment by one seller in the market with all other sellers making efficient investments. This result holds irrespective of the number of buyers or sellers in the market. Moreover, surprisingly in this case all coordination problems are solved and the equilibrium matches are the efficient ones. In other words, the ordering of the sellers' qualities generated by ex-ante investments coincides with the ordering of sellers' innate qualities.

The structure of the paper is as follows. After a discussion of related literature in the next section, Section 3 lays down the basic model and the extensive form of the Bertrand competition game between workers and firms. Section 4 investigates the efficiency properties of the model where workers undertake ex-ante investments before competition occurs. We show that workers' investments are efficient given equilibrium

[1]For an analysis of how market competition may fail to solve coordination problems see Hart (1979) and Cooper and John (1988).

matches although, depending on parameters, coordination problems may arise that lead to inefficient matches. We then consider in Section 5.2 the model in which the firms undertake ex-ante investments. It is demonstrated that the inefficiency of equilibrium investments is small and can be bounded by an amount independent of the size of the market. Moreover, all coordination problems are solved and equilibrium matches are efficient. Section 6 provides concluding remarks.

2. Related Literature

The literature on the hold-up problem has mainly analyzed the bilateral relationship of two parties that may undertake match specific investments in isolation (Williamson 1985, Grout 1984, Grossman and Hart 1986, Hart and Moore 1988). In other words, these papers identify the inefficiencies that the absence of complete contingent contracts may induce in the absence of any competition for the parties to the match. This literature identifies the institutional (Grossman and Hart 1986, Hart and Moore 1990, Aghion and Tirole 1997) or contractual (Aghion, Dewatripont, and Rey 1994, Nöldeke and Schmidt 1995, Maskin and Tirole 1999, Segal and Whinston 1998) devices that might reduce and possibly eliminate these inefficiencies. We differ from this literature in that we do not alter either the institutional or contractual setting in which the hold-problem arises but rather analyze how competition among different sides of the market may eliminate the inefficiencies associated with such a problem.

The literature on bilateral matching, on the other hand, concentrates on the inefficiencies that arise because of frictions present in the matching process. These inefficiencies may lead to market power (Diamond 1971, Diamond 1982), unemployment (Mortensen and Pissarides 1994) and a class structure (Burdett and Coles 1997, Eeckhout 1999). A recent development of this literature shows how efficiency can be restored in a matching environment thanks to free entry into the market (Roberts 1996, Moen 1997) or Bertrand competition (Felli and Harris 1996). We differ from this literature in that we abstract from any friction in the matching process and focus on the presence of match specific investments before or after the matching process.

A small recent literature considers investments in a matching environment. Some of the papers focus on general investment that may be transferred across matches and identify the structure of contracts that may lead to efficiency (MacLeod and Malcomson 1993, Holmström 1999) or the inefficiencies due to the presence of an exogenous probability that the match will dissolve (Acemoglu 1997). A number of papers consider, instead, specific investments in a matching environment as we do (Acemoglu and Shimer 1998, Cole, Mailath, and Postlewaite 1998, de Meza and Lockwood 1998).

In particular, Acemoglu and Shimer (1998) consider a matching model with frictions. Firms post wage offers before choosing their investments. They obtain efficiency out of the free entry of firms and the fact that wages are announced before investment occurs. We differ in that we do not allow free entry of firms in the economy. As a matter of fact, the finite and discrete number of firms and workers in the market is critical in identifying the specific nature of the investment undertaken by both sides of the market. The mechanism leading to efficiency is therefore quite different in nature: we focus on the ability of Bertrand competition mechanism to achieve efficiency or near-inefficiency.

Cole, Mailath, and Postlewaite (1998) is the paper closest to ours. As in our setting they focus on *ex-ante* match specific investment and analyze efficiency when matches and the allocation of the shares of surplus are in the core of the assignment game. They demonstrate the existence of an equilibrium allocation that induces efficient investments as well as allocations that yield inefficiencies. This is done under a critical assumption. When the numbers of workers (sellers) and buyers (firms) are discrete they are able to pin down an allocation of the matches' surplus yielding efficient investments via a condition defined as 'double-overlapping'. This condition requires the presence of at least two workers (or two firms) with identical innate characteristics; it implies the existence of an immediate competitor for the worker or the firm in each match. In this case, the share of surplus a worker gets is exactly the worker's outside option and efficiency is promoted. In the absence of double-overlapping, investments may not be efficient because indeterminacy arises creating room for under-investment. Such a condition is not needed in our environment since, by specifying the extensive form of market competition as Bertrand competition, we obtain a binding outside option for any value of the workers' and firms' innate characteristics. Notice that double-overlapping is essentially an assumption on the specificity of the investments that both workers and firms choose. If double overlapping holds it means that investment is specific to a small group of workers or firms but among these workers and firms it is general. We do not need this assumption for efficiency.

Finally de Meza and Lockwood (1998) analyze a matching environment in which both sides of the market can undertake match specific investments but focus on a setup that delivers inefficient investments. As a result the presence of asset ownership and asset trading may enhance welfare as in Grossman and Hart (1986). They focus on whether one would observe asset trading before or after investment and match formation. In our setting, given that we obtain efficiency or near-efficiency, we do not need to explore the efficiency enhancing role of asset ownership. However, we do explore the different efficiency properties of an environment in which firms undertake investments both before and/or after matches are formed. The difference is that we take this timing to be exogenously rather than endogenously determined. As we argue in the Conclusions, Section 6 below, the only large inefficiency that could arise in a general model in which both workers and firms can undertake *ex-ante* and *ex-post* investments is generated by the *ex-post* investments of the agents on the side of the market whose remunerations are established in the employment contract.

3. The Framework

We consider a simple matching model: S workers match with T firms, we assume that the number of workers is higher than the number of firms $S > T$. Each firm is assumed to match only with one worker. Workers and firms are labelled, respectively, $s = 1, \ldots, S$ and $t = 1, \ldots, T$. Both workers and firms can make match specific investments, denoted respectively x_s and y_t, incurring costs $C(x_s)$ respectively $C(y_t)$.[2] The cost function $C(\cdot)$ is strictly convex and $C(0) = 0$. The surplus of each match is

[2] For simplicity we take both cost functions to be identical, none of our results depending on this assumption. If the cost functions were type specific we would require the marginal costs to increase with the identity of the worker or the firm.

then a function of the quality of the worker σ and the firm τ involved in the match: $v(\sigma, \tau)$. Each worker's quality is itself a function of the worker innate ability, indexed by the worker's identity s, and the worker specific investment x_s: $\sigma(s, x_s)$. In the same way, we assume that each firm's quality is a function of the firm's innate ability, indexed by the firm's identity t, and the firm's specific investment y_t: $\tau(t, y_t)$.

We assume *positive assortative matching*. In other words, the higher is the quality of the worker and the firm the higher is the surplus generated by the match:[3] $v_1(\sigma, \tau) > 0$, $v_2(\sigma, \tau) > 0$. Further, the marginal surplus generated by a higher quality of the worker or of the firm in the match increases if the quality of the partner: $v_{12}(\sigma, \tau) > 0$. We further assume that the quality of the worker depends negatively on the worker's innate ability s, $\sigma_1(s, x_s) < 0$ (so that worker $s = 1$ is the highest ability worker) and positively on the worker's specific investment x_s. Similarly, the quality of a firm depends negatively on the firm's innate ability t, $\tau_1(t, y_t) < 0$, (firm $t = 1$ is the highest ability firm) and positively on the firm's investment y_t: $\tau_2(t, y_t) > 0$. We also assume that the quality of both the workers and the firms satisfy a *single crossing condition* requiring that the marginal productivity of both workers and firms investments decreases in their innate ability: $\sigma_{12}(s, x_s) < 0$ and $\tau_{12}(t, y_t) < 0$.

The combination of the assumption of positive assortative matching and the single crossing condition gives a particular meaning to the term specific investments we used for x_s and y_t. Indeed, in our setting the investments x_s and y_t have a use and value in matches other than (s, t); however, these values decrease with the identity of the partner implying that at least one component of this value is specific to the match in question, since we consider a discrete number of firms and workers.

We also assume that the surplus of each match is concave in the workers and firms quality — $v_{11} < 0$, $v_{22} < 0$ — and that the quality of both firms and workers exhibit decreasing marginal returns in their investments: $\sigma_{22} < 0$ and $\tau_{22} < 0$. [4]

[3]For convenience we denote with $v_l(\cdot, \cdot)$ the partial derivative of the surplus function $v(\cdot, \cdot)$ with respect to the l-th argument and with $v_{lk}(\cdot, \cdot)$ the cross-partial derivative with respect to the l-th and k-th argument or the second-partial derivatives if $l = k$. We use the same notation for the functions $\sigma(\cdot, \cdot)$ and $\tau(\cdot, \cdot)$ defined below.

[4]As established in Milgrom and Roberts (1990), Milgrom and Roberts (1994) and Edlin and

In Section 5.2 below we need stronger assumptions on the responsiveness of firms' investments to both the workers' quality and firms' identity and investments. These assumptions, labelled "responsive complementarity", can be described as follows. Denote, for a given level of worker's quality, each match surplus, net of the firm's investment cost as follows.[5]

$$w(\sigma, t, y_t) = v(\sigma, \tau(t, y_t)) - C(y_t). \tag{1}$$

We assume that:

$$\frac{\partial}{\partial s}\left(-\frac{w_{13}}{w_{33}}\right) > 0, \qquad \frac{\partial}{\partial y}\left(-\frac{w_{13}}{w_{33}}\right) < 0 \tag{2}$$

To be able to interpret these conditions, we first need to define the socially optimal investment choice when firm t matches with worker of given quality σ. This investment level, denoted $y(\sigma, t)$, is the solution to the following problem:

$$y(\sigma, t) = \underset{y}{\operatorname{argmax}}\, w(\sigma, t, y) \tag{3}$$

and is implicitly defined by the following first order condition:

$$w_3(\sigma, t, y(\sigma, t)) = 0 \tag{4}$$

Differentiating $y(\sigma, t)$ with respect to t gives

$$\frac{\partial y}{\partial t} = -\frac{w_{13}}{w_{33}} \tag{5}$$

so that (2) says that increases in s and decreases in y make investment more re-

Shannon (1998) our results can be derived with much weaker assumptions on the smoothness and concavity of the surplus function $v(\cdot,\cdot)$ and the two quality functions $\sigma(\cdot,\cdot)$ and $\tau(\cdot,\cdot)$ in the two investments x_s and y_t.

[5]The assumption of concavity of the surplus function $v(\cdot,\cdot)$ in the qualities σ and τ and of the function $\tau(t,\cdot)$ in the investment y_t imply the appropriate concavity properties for the net surplus function $w(\cdot, t, \cdot)$, as defined in (1), in σ and y_t.

sponsive to the type of the investor.[6] Responsiveness complementarity, and the other conditions that we have imposed, is satisfied by a standard iso-elastic specification of the model.

We analyze two different specification of our general framework.

We start with the analysis of a model in which only the workers choose ex-ante match specific investments x_s that determine the quality of each worker $\sigma(s, x_s)$ while firms are of exogenously given qualities: $\tau = \tau(t)$. Then workers Bertrand compete for the firms so as to determine the equilibrium matches and, at the same time, the share of the match surplus accruing to each party to the match. This model is analyzed in Section 5 below.

We then proceed to analyze (Section 5.2 below) the situation in which only firms choose ex-ante match specific investments y_t that determine each firm t's quality $\tau(t, y_t)$ while workers are of exogenously given quality $\sigma = \sigma(s)$. Then competition occurs in which, as before, workers Bertrand compete for firms and equilibrium matches and shares of surplus are determined.

Notice that in both cases, given the absence of uncertainty, both workers and firms can perfectly foresee the match they will end up with in equilibrium.

We assume the following extensive forms of the Bertrand competition game in which the T firms and the S workers engage. Workers Bertrand compete for firms. All workers simultaneously and independently make wage offers to every one of the T firms. Notice that we allow workers to make offers to more than one, possibly all firms. Each firm observes the offers she receives and decides which offer to accept. We assume that this decision is taken sequentially in order of the identity (innate ability) of firms. In other words the firm labelled 1 decides first which offer to accept. This commits the worker selected to work for firm 1 and automatically withdraws all offers this worker made to other firms. All other firms and workers observe this

[6] As y is a function of s and t, it is not freely variable. However, if investment is subject to a supplementary cost of p per unit then first-order conditions give $w_3 = p$ and an interpretation of (2) is that it is a responsiveness condition in compensated terms where p changes to induces the appropriate change in y.

decision and then firm 2 decides which offer to accept. This process is repeated until firm T decides which offer to accept. Notice that since $S > T$ even firm T, the last firm to decide, can potentially choose among multiple offers.

We choose to allow firms to select their preferred bid in order of innate ability so as to maximize the competition among workers for firms. In other words, the extensive form described above maximizes the depth of the active market for the match with every firm. As shown below, a worker of a given quality has a positive willingness to pay only for firms of a better quality than the firm he will be matched with in equilibrium. Therefore, if the market clears first for the low value matches the number of workers with positive willingness to pay for high quality firms is reduced. Some of the workers with a positive willingness to pay have exited the market since they have been matched already. Indeed, if the order according to which firms choose their most preferred bid is the inverse of the order of quality only two workers will compete for every firm: the worker with whom the firm ends up matched and the lowest quality worker that will not be matched with any firm in equilibrium. We conjecture that the equilibrium we characterize in Sections 4 and 5.2 below is the same as the equilibrium of the following alternative extensive form of the Bertrand competition game that endogenously determines which matches clear first. All workers submit simultaneously and independently offers to all firms. Firms simultaneously and independently decide which offer to accept. If a worker's offer is accepted by one firm only the worker is committed to work for that firm. If instead the same worker offer is accepted by more than one firm then the bidding process is repeated among the firms and workers who are not committed to a match yet. This process continues until all firms are matched.

We look for the *trembling-hand-perfect* equilibrium of our model.

4. Workers' Investments

In this section we analyze the model under the assumption that the quality of firms is exogenously give $\tau_t = \tau(t)$ while the quality of workers depends on both the workers' identity (innate ability) and their match specific investments $\sigma = \sigma(s, x_s)$. We are able to show that an efficient equilibrium always exist. However, although investment choices are efficient given the equilibrium matches, there may exist additional equilibria of the workers' investment game characterized by inefficient equilibrium matches.

4.1. Equilibrium Characterization of the Bertrand Competition Game

We proceed to solve our model backward and we start from the Bertrand competition game, as described in Section 3 above. Assume that workers' investments are chosen so as to obtain the following order of workers' qualities: $\sigma_{(1)} > \ldots > \sigma_{(N)}$. Recall also the firms' quality follows the order: $\tau_1 > \ldots > \tau_T$. The Bertrand competition game then determines the equilibrium matches and the shares of the gross surplus $v(\sigma, \tau)$ that accrue to the parties to the match.

We first identify an efficiency property of any equilibrium of the Bertrand game. Equilibrium matches are efficient in the sense that in every equilibrium of the Bertrand game the worker characterized by the k-th highest quality matches with the firm characterized by the k-th highest. We label this property — stated and proved in the following lemma — *efficiency in matching.*

Lemma 1. *Given an ordered vector* $(\sigma_{(1)}, \ldots, \sigma_{(S)})$ *every equilibrium of the Bertrand game is characterized by the equilibrium matches:* $(\sigma_{(k)}, \tau_k)$*, for every* $k = 1, \ldots, T$*.*

Proof: Assume by way of contradiction that this is not the case and there exist a pair of equilibrium matches $(\sigma_{(i)}, \tau'')$ and $(\sigma_{(j)}, \tau')$ such that $i < j$, or $\sigma_{(i)} > \sigma_{(j)}$, and $\tau' < \tau''$. Denote $B(\tau')$, respectively $B(\tau'')$, the bids that in equilibrium the firm of quality τ', respectively of quality τ'', accepts. For $(\sigma_{(i)}, \tau'')$ to be an equilibrium match we need that it is not convenient for the worker of quality $\sigma_{(i)}$ to match with the firm of quality τ', instead of τ''.

Notice that if the worker of quality $\sigma_{(i)}$ deviates and does not submit the bid $B(\tau'')$ to the firm of quality τ'' a different worker will be matched with firm τ''. Therefore, following this deviation, when competing for firm τ' the bid that worker $\sigma_{(i)}$ needs to submit to be matched with firm τ' is $\hat{B}(\tau') \leq B(\tau')$. The reason is that the set of bids submitted to firm τ' does not include the bid of the worker that matches with firm τ'' following the deviation of the worker of quality $\sigma_{(i)}$. Hence the maximum of these bids, $\hat{B}(\tau')$, is in general not higher than the equilibrium bid of the worker of quality $\sigma_{(i)}$: $B(\tau')$.

Therefore for $(\sigma_{(i)}, \tau'')$ to be an equilibrium match we need that

$$v(\sigma_{(i)}, \tau'') - B(\tau'') \geq v(\sigma_{(i)}, \tau') - \hat{B}(\tau').$$

or given that, as argued above, $\hat{B}(\tau') \leq B(\tau')$:

$$v(\sigma_{(i)}, \tau'') - B(\tau'') \geq v(\sigma_{(i)}, \tau') - B(\tau') \tag{6}$$

Moreover, for $(\sigma_{(j)}, \tau')$ to be an equilibrium match we need that it is not profitable for the worker of quality $\sigma_{(j)}$ to outbid, at an earlier stage of the Bertrand competition game, the worker of quality $\sigma_{(i)}$ (that submits bid $B(\tau'')$) so as to match with firm τ'':

$$v(\sigma_{(j)}, \tau') - B(\tau') \geq v(\sigma_{(j)}, \tau'') - B(\tau'') \tag{7}$$

The inequalities (6) and (7) imply:

$$v(\sigma_{(i)}, \tau'') + v(\sigma_{(j)}, \tau') \geq v(\sigma_{(i)}, \tau') + v(\sigma_{(j)}, \tau''). \tag{8}$$

Condition (8) contradicts the assortative matching assumption $v_{12}(\sigma, \tau) > 0$. ■

As argued in Section 4.2 below, Lemma 1 does not imply that the order of workers' qualities, which are endogenously determined, coincides with the order of the workers' innate abilities.

Having established this property we can now move to the characterization of the unique equilibrium of the Bertrand competition game.

Proposition 1. *For any given ordered vector* $(\sigma_{(1)}, \ldots, \sigma_{(N)})$ *the unique equilibrium of the Bertrand competition subgame is such that the worker of quality* $\sigma_{(t)}$ *matches with the firm of quality* τ_t.

The share of the match surplus that each worker and each firm receive are:

$$\pi^W_{\sigma(t)} = \sum_{h=t}^{T} \left[v(\sigma_{(h)}, \tau_h) - v(\sigma_{(h+1)}, \tau_h) \right] \tag{9}$$

$$\pi^F_t = v(\sigma_{(t+1)}, \tau_t) - \sum_{h=t+1}^{T} \left[v(\sigma_{(h)}, \tau_h) - v(\sigma_{(h+1)}, \tau_h) \right] \tag{10}$$

Proof: We characterize the equilibrium proceeding by induction. Denote by t the class of subgames that starts with firm t having to choose among the submitted bids. These subgames differ depending on the bids previously accepted by firms $1, \ldots, t-1$. We first solve for the equilibrium of the T-th (the last) subgame in which all firms but firm T have selected a worker's bid.

Without loss in generality, we take $S = T + 1$. This subgame is then a simple decision problem for firm T that has to choose between the bids submitted by the two remaining workers. Denote $\alpha_{(T)}$ and $\alpha_{(T+1)}$ the qualities of these two workers such that $\alpha_{(T)} > \alpha_{(T+1)}$ and $B_{\alpha_{(T)}}$, respectively $B_{\alpha_{(T+1)}}$, their bids. Firm T clearly chooses the highest of these two bids.

Worker of quality $\alpha_{(T+1)}$ generates surplus $v(\alpha_{(T+1)}, \tau_T)$ if selected by firm T while worker of quality $\alpha_{(T)}$ generates surplus $v(\alpha_{(T)}, \tau_T)$ if selected. This implies that $v(\alpha_{(T+1)}, \tau_T)$ is worker $\alpha_{(T+1)}$'s maximum willingness to bid while $v(\alpha_{(T)}, \tau_T)$ is worker $\alpha_{(T)}$'s maximum willingness to bid. Notice that from $\alpha_{(T)} > \alpha_{(T+1)}$ and $v_1 > 0$ we have:

$$v(\alpha_{(T)}, \tau_T) > v(\alpha_{(T+1)}, \tau_T).$$

Worker $\alpha_{(T)}$ therefore submits a bid equal to the minimum necessary to outbid worker $\alpha_{(T+1)}$. In other words the equilibrium bid of worker $\alpha_{(T)}$ coincides with the equilibrium bid of worker $\alpha_{(T+1)}$: $B_{\alpha_{(T)}} = B_{\alpha_{(T+1)}}$. Worker $\alpha_{(T+1)}$, on his part, has an incentive to deviate and outbid worker $\alpha_{(T)}$ for any bid $B_{\alpha_{(T)}} < v(\alpha_{(T+1)}, \tau_T)$. Therefore the unique equilibrium is such that both workers' equilibrium bids are:

$$B_{\alpha_{(T)}} = B_{\alpha_{(T+1)}} = v(\alpha_{(T+1)}, \tau_T)$$

while, consistently with Lemma 1 above, the equilibrium match is the one between the worker of quality $\alpha_{(T)}$ and the firm of quality τ_T.[7] Notice that on the equilibrium path $\alpha_{(T)} = \sigma_{(T)}$ and $\alpha_{(T+1)} = \sigma_{(T+1)}$.

We now move to the t-th subgame, $(t < T)$. In this case firm t has to choose among the potential bids of the remaining $(T - t + 2)$ workers of qualities $\alpha_{(t)}, \ldots, \alpha_{(T+1)}$, where $\alpha_{(t)} > \ldots > \alpha_{(T+1)}$. Our induction hypothesis is that the continuation equilibria of the following $j = t + 1, \ldots, T$ subgames are such that the worker of quality $\alpha_{(j)}$ matches with firm j and the equilibrium payoffs are:

$$\hat{\pi}^W_{\alpha_{(j)}} = \sum_{h=j}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right] \qquad (11)$$

$$\hat{\pi}^F_j = v(\alpha_{(j+1)}, \tau_j) - \sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right]. \qquad (12)$$

Firm t clearly chooses the highest bid she receives.

The maximum willingness to bid of the worker of quality $\alpha_{(t)}$ is then exactly the surplus generated by the match of the worker of quality $\alpha_{(t)}$ and firm t minus the payoff that the worker would get according to the induction hypothesis (11) by moving to the next subgame:

$$v(\alpha_{(t)}, \tau_t) - \bar{\pi}^W_{\alpha_{(t)}} \qquad (13)$$

[7] This is just one of a whole continuum of subgame perfect equilibria of this simple Bertrand game *but* the unique trembling-hand-perfect equilibrium. Trembling-hand-perfection is here used in a completely standard way to insure that worker $\alpha_{(T+1)}$ does not choose an equilibrium bid (not selected by firm T) in excess of his maximum willingness to pay.

where

$$\bar{\pi}^W_{\alpha_{(t)}} = \left[v(\alpha_{(t)}, \tau_{t+1}) - v(\alpha_{(t+2)}, \tau_{t+1})\right] + \sum_{h=t+2}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h)\right]$$

The maximum willingness to bid of the worker of quality $\alpha_{(j)}$, $j = t+1, \ldots, N$, for firm t is instead

$$v(\alpha_{(j)}, \tau_t) - \hat{\pi}^W_{\alpha_{(j)}}. \tag{14}$$

where $\hat{\pi}^W_{\alpha_{(j)}}$ is defined in (11) above. Indeed, worker $\alpha_{(j)}$ is willing to pay the surplus he will be able to generate if matched with firm t in excess of the payoff $\hat{\pi}^W_{\alpha_{(j)}}$ he can guarantee himself, from our induction hypothesis, by not competing for firm t and moving to subgame j the only one in which his bid will be selected.

Comparing the willingness to pay of the workers of qualities $\alpha_{(t)}, \ldots, \alpha_{(N)}$ as defined in (13) and (14) we obtain:

$$v(\alpha_{(t)}, \tau_t) - \bar{\pi}^W_{\alpha_{(t)}} > v(\alpha_{(t+1)}, \tau_t) - \hat{\pi}^W_{\alpha_{(t+1)}}$$

or

$$v(\alpha_{(t)}, \tau_t) - v(\alpha_{(t)}, \tau_{t+1}) > v(\alpha_{(t+1)}, \tau_t) - v(\alpha_{(t+1)}, \tau_{t+1})$$

which is satisfied by the assortative matching assumption $v_{12}(\sigma, \tau) > 0$. We also obtain:

$$v(\alpha_{(j)}, \tau_t) - \bar{\pi}^W_{\alpha_{(j)}} > v(\alpha_{(j+1)}, \tau_t) - \hat{\pi}^W_{\alpha_{(j+1)}}$$

or

$$v(\alpha_{(j)}, \tau_t) - v(\alpha_{(j)}, \tau_j) > v(\alpha_{(j+1)}, \tau_t) - v(\alpha_{(j+1)}, \tau_j)$$

which is also satisfied by the assortative matching assumption $v_{12}(\sigma, \tau) > 0$.

Therefore the worker of quality $\alpha_{(t)}$ is the one with the highest willingness to bid for firm t followed by the worker of quality $\alpha_{(t+1)}$ and so on in decreasing order of quality. Using an argument symmetric to the one presented in the analysis of the T-th subgame, we then conclude that the equilibrium bids of worker $\alpha_{(t)}$ and $\alpha_{(t+1)}$

are

$$B_{\alpha_{(t)}} = B_{\alpha_{(t+1)}} = v(\alpha_{(t+1)}, \tau_t) - \hat{\pi}^W_{\alpha_{(t+1)}},$$

while, consistently with Lemma 1 above, the equilibrium match is the one between the worker of quality $\alpha_{(t)}$ and firm t. Notice that on the equilibrium path $\alpha_{(t)} = \sigma_{(t)}$ and $\alpha_{(j)} = \sigma_{(j)}$. Therefore (11) and (12) coincide with (9) and (10). ■

Notice that Lemma 1 and Proposition 1 provide us with a number of properties of the competition among workers for the matches.

In particular, as mentioned above, Lemma 1 shows that for given investment choices the equilibrium allocation of workers to firms of the Bertrand game is *efficient*. In other words, for given workers' investments a central planner would choose exactly the same matches as the ones observed in equilibrium. The efficiency of the allocation that matches a worker quality $\sigma_{(t)}$ with a t firm follows from our assumption of positive assortative matching.

Further from Proposition 1 above, the worker's equilibrium payoff $\pi^W_{\sigma_{(t)}}$ is the sum of the social surplus produced by the equilibrium match $v(\sigma_{(t)}, \tau_t)$ and an expression $\mathcal{W}_{\sigma_{(t)}}$ that does not depend on the quality $\sigma_{(t)}$ of the worker involved in the match. In particular this implies that $\mathcal{W}_{\sigma_{(t)}}$ does not depend on the match specific investment $x_{\sigma_{(t)}}$ of the worker of quality $\sigma_{(t)}$:

$$\pi^W_{\sigma_{(t)}} = v(\sigma_{(t)}, \tau_t) + \mathcal{W}_{\sigma_{(t)}}. \tag{15}$$

The firm's equilibrium payoff π^F_t is also the sum of two terms: the surplus generated by the match of firm t with worker $\sigma_{(t+1)}$ — an inefficient match if it were to occur — and an expression $\mathcal{P}_t$ that does not depend on firm t's quality τ_t:

$$\pi^F_t = v(\sigma_{(t+1)}, \tau_t) + \mathcal{P}_t.$$

These properties will play a crucial role when we analyze the efficiency of the workers' investment choices.

4.2. The Workers' Equilibrium Investments

We present now the characterization of the equilibrium of the workers' investment game. We first show that an equilibrium of this simultaneous move investment game always exist and that this equilibrium is efficient: the order of the induced qualities $\sigma_{(i)}$, $i = 1, \ldots, S$, coincides with the order of the workers' identities s, $s = 1, \ldots, S$. We then show that an inefficiency may arise, depending on the distribution of firms' qualities and workers' innate abilities. This inefficiency takes the form of additional inefficient equilibria, such that the order of the workers' identities differs from the order of induced qualities.

Notice first that each worker's investment choice is efficient given the equilibrium match the worker is involved in. Indeed, the Bertrand competition game will make each worker residual claimant of the surplus produced in his equilibrium match. Therefore, the worker is able to appropriate the marginal returns from his investment and hence his investment choice is efficient given the equilibrium match.

Assume that the equilibrium match is the one between the s worker and the t firm, from equation (15) worker s's optimal investment choice $x_s(t)$ is the solution to the following problem:

$$x_s(t) = \underset{x}{\operatorname{argmax}}\ \pi^W_{\sigma(s,x)} - C(x) = v(\sigma(s,x), \tau_t) - \mathcal{W}_{\sigma(s,x)} - C(x). \tag{16}$$

This investment choice is defined by the following necessary and sufficient first order conditions of problem (16):

$$v_1(\sigma(s, x_s(t)), \tau_t)\ \sigma_2(s, x_s(t)) = C'(x_s(t)) \tag{17}$$

where $C'(\cdot)$ is the first derivative of the cost function $C(\cdot)$.

Notice that (17) follows from the fact that $\mathcal{W}_{\sigma(s,x)}$ does not depend on worker s's quality $\sigma(s, x)$, and hence on worker s's match specific investment x_s. The following two lemmas derive the properties of worker s's investment choice $x_s(t)$ and his quality $\sigma(s, x_s(t))$.

Lemma 2. *For any given equilibrium match* $(\sigma(s, x_s(t)), \tau_t)$ *worker* s*'s investment choice* $x_s(t)$*, as defined in (17), is constrained efficient.*

Proof: Notice first that if a central planner is constrained to choose the match between worker s and firm t worker s's constrained efficient investment is the solution to the following problem:

$$x^*(s,t) = \operatorname*{argmax}_x v(\sigma(s,x), \tau(t)) - C(x). \tag{18}$$

This investment $x^*(s,t)$ is defined by the following necessary and sufficient first order conditions of problem (18):

$$v_1(\sigma(s,x^*), \tau(t))\ \sigma_2(s,x^*) = C'(x^*). \tag{19}$$

The result then follows from the observation that the definition of the constrained efficient investment x^*, equation (19), coincides with the definition of worker s's optimal investment $x_s(t)$, equation (17) above. ■

Lemma 3. *For any given equilibrium match* $(\sigma(s, x_s(t)), \tau_t)$ *worker* s*'s optimally chosen quality* $\sigma(s, x_s(t))$ *decreases both in the worker's identity* s *and in the firm identity* t*:*

$$\frac{d\,\sigma(s,x_s(t))}{d\,s} < 0, \qquad \frac{d\,\sigma(s,x_s(t))}{d\,t} < 0.$$

Proof: The result follows from condition (17) that implies:

$$\frac{d\,\sigma(s,x_s(t))}{d\,s} = \frac{\sigma_1\ v_1\ \sigma_{22} - \sigma_1\ C'' - v_1\ v_2\ \sigma_{12}}{v_{11}\ (\sigma_2)^2 + v_1\ \sigma_{22} - C''} < 0,$$

and

$$\frac{d\,\sigma(s,x_s(t))}{d\,t} = \frac{v_{12}\ (\sigma_2)^2}{v_{11}\ (\sigma_2)^2 + v_1\ \sigma_{22} - C''} < 0,$$

where the functions σ_h and σ_{hk}, $h, k \in \{1, 2\}$, are computed at $(s, x_s(t))$; the functions v_h and v_{hk}, $h, k \in \{1, 2\}$, are computed at $(\sigma(s, x_s(t)), \tau_t)$ and C'' denotes the second derivative of the cost function computed at $x_s(t)$. ■

We can now define an equilibrium of the workers' investment game. Let $(s_1, \dots, s_S)$ denote a permutation of the vector of workers' identities $(1, \dots, S)$. An equilibrium of the workers' investment game is then a vector of investment choices $x_{s_i}(i)$, as defined in (17) above, such that the resulting workers' qualities have the same order as the identity of the associated firms:

$$\sigma_{(i)} = \sigma(s_i, x_{s_i}(i)) < \sigma_{(i-1)} = \sigma(s_{i-1}, x_{s_{i-1}}(i-1)) \qquad \forall i = 2, \dots, S, \tag{20}$$

where $\sigma_{(i)}$ denotes the i-th element of the equilibrium ordered vector of qualities $(\sigma_{(1)}, \dots, \sigma_{(S)})$.

Notice that this equilibrium definition allows for the order of workers' identities to differ from the order of their qualities and therefore from the order of the identities of the firms each worker is matched with.

We can now proceed to show the existence of the efficient equilibrium of the worker investment game. This is the equilibrium characterized by the coincidence of the order of workers' identities and the order of their qualities. From Lemma 1 the efficient equilibrium matches are $(\sigma(i, x_i(i)), \tau_i)$, $i = 1, \dots, N$.

Proposition 2. *The equilibrium of the workers' investment game characterized by* $s_i = i$, $i = 1, \dots, S$ *always exists and is efficient.*

Proof: We prove this result in three steps. We first show that the workers' equilibrium qualities $\sigma(i, x_i(i))$ associated with the equilibrium $s_i = i$ satisfy condition (20). We then show that the net payoff to worker i associated with any given quality σ of this worker is continuous in σ. This result is not obvious since, from Lemma 1 — given the investment choices of other workers — worker i can change his equilibrium match by changing his quality σ. Finally, we show that this net payoff has a unique global maximum and this maximum is such that the corresponding quality σ is in the interval in which worker i is matched with firm i. These steps clearly imply that each worker i has no incentive to deviate and choose an investment different from the one that maximizes his net payoff and yields an equilibrium match with firm i.

Let $\pi_i^W(\sigma) - C(x(i,\sigma))$ be the net payoff to worker i where $x(i,\sigma)$ denotes worker i's investment level associated with quality σ:

$$\sigma(i, x(i,\sigma)) \equiv \sigma. \tag{21}$$

Step 1. Worker i's equilibrium quality $\sigma(i, x_i(i))$ is such that:

$$\sigma(i, x_i(i)) = \sigma_{(i)} < \sigma(i-1, x_{i-1}(i-1)) = \sigma_{(i-1)}, \qquad \forall i = 2, \ldots, S.$$

The proof follows directly from Lemma 3 above.

Step 2. The net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is continuous in σ.

Let $(\sigma_{(1)}, \ldots, \sigma_{(i-1)}, \sigma_{(i+1)}, \ldots, \sigma_{(S)})$ be the given ordered vector of the qualities of the workers, other than i. Notice that if $\sigma \in (\sigma_{(i-1)}, \sigma_{(i+1)})$ by Lemma 1 worker i is matched with firm i. Then by Proposition 1 and the definition of $v(\cdot,\cdot)$, $C(\cdot)$, $\sigma(\cdot,\cdot)$ and (21) the payoff function $\pi_i^W(\sigma) - C(x(i,\sigma))$ is continuous in σ.

Consider now the limit from the right, for $\sigma \to \sigma_{(i-1)}^-$, of the net payoff to worker i when it is matched with firm i, $\sigma \in (\sigma_{(i+1)}, \sigma_{(i-1)})$. From (9) this limit is

$$\begin{aligned} \pi_i^W(\sigma_{(i-1)}^-) \quad - \quad C(x(i,\sigma_{(i-1)}^-)) \; = \; v(\sigma_{(i-1)}, \tau_i) - v(\sigma_{(i+1)}, \tau_i) \; + \\ + \quad \sum_{h=i+1}^{T} \left[v(\sigma_{(h)}, \tau_h) - v(\sigma_{(h+1)}, \tau_h) \right] - C(x(i, \sigma_{(i-1)})). \end{aligned} \tag{22}$$

Conversely, if $\sigma \in (\sigma_{(i-1)}, \sigma_{(i-2)})$ then by Lemma 1 worker i is matched with firm $(i-1)$. Then from (9) the limit from the left, for $\sigma \to \sigma_{(i-1)}^+$, of the net payoff to worker i when matched with firm $i-1$ is

$$\begin{aligned}\pi_i^W(\sigma^+_{(i-1)}) \;-\; C(x(i,\sigma^+_{(i-1)})) \;&=\; v(\sigma_{(i-1)},\tau_{i-1}) - v(\sigma_{(i-1)},\tau_{i-1}) \;+ \\ &+\; v(\sigma_{(i-1)},\tau_i) - v(\sigma_{(i+1)},\tau_i) \;+ \\ &+\; \sum_{h=i+1}^{T}\left[v(\sigma_{(h)},\tau_h) - v(\sigma_{(h+1)},\tau_h)\right] - C(x(i,\sigma_{(i-1)})).\end{aligned} \tag{23}$$

In this case while the worker of quality σ is matched with firm $i-1$ the worker of quality σ_{i-1} is matched with firm i.

Equation (22) coincides with equation (23) since the first two terms of the left-hand-side of equation (23) are identical. A similar argument shows continuity of the net payoff function at $\sigma = \sigma_{(h)}$, $h = 1,\ldots,i-2, i+1,\ldots,N$.

Step 3. The net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ has a unique global maximum in the interval $(\sigma_{(i+1)},\sigma_{(i-1)})$.

Notice first that in the interval $(\sigma_{(i+1)},\sigma_{(i-1)})$, by Lemma 1 and Proposition 1, the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ takes the following expression.

$$\begin{aligned}\pi_i^W(\sigma) - C(x(i,\sigma)) \;&=\; v(\sigma,\tau_i) - v(\sigma_{(i+1)},\tau_i) \;+ \\ &+\; \sum_{h=i+1}^{T}\left[v(\sigma_{(h)},\tau_h) - v(\sigma_{(h+1)},\tau_h)\right] - C(x(i,\sigma)).\end{aligned} \tag{24}$$

This expression, and therefore the net payoff $\pi_i^W(\sigma) - C(x_\sigma)$ in the interval $(\sigma_{(i+1)},\sigma_{(i-1)})$, is strictly concave in σ (by strict concavity of $v(\cdot,\tau_i)$, $\sigma(i,\cdot)$ and strict convexity of $C(\cdot)$) and reaches a maximum at $\sigma_{(i)} = \sigma(i,x_i(i))$ as defined in (17) above.

Notice, further, that in the right adjoining interval $(\sigma_{(i-1)},\sigma_{(i-2)})$, by Lemma 1 and Proposition 1, the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ takes the following expression

— different from (24).

$$\begin{aligned}\pi_i^W(\sigma) \; - \; C(x(i,\sigma)) \; &= \; v(\sigma,\tau_{i-1}) - v(\sigma_{(i-1)},\tau_{i-1}) \; + \\ &+ \; v(\sigma_{(i-1)},\tau_i) - v(\sigma_{(i+1)},\tau_i) \; + \\ &+ \; \sum_{h=i+1}^{T} \big[v(\sigma_{(h)},\tau_h) - v(\sigma_{(h+1)},\tau_h)\big] - C(x(i,\sigma)).\end{aligned} \tag{25}$$

This new expression of the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is also strictly concave (by strict concavity of $v(\cdot,\tau_{i-1})$, $\sigma(i,\cdot)$ and strict convexity of $C(\cdot)$) and reaches a maximum at $\sigma(i,x_i(i-1))$. From Lemma 3 above we know that

$$\sigma(i,x_i(i-1)) < \sigma_{(i-1)} = \sigma(i-1,x_{i-1}(i-1)).$$

This implies that in the interval $(\sigma_{(i-1)},\sigma_{(i-2)})$ the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is strictly decreasing in σ.

A symmetric argument shows that the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is strictly decreasing in σ in any interval $(\sigma_{(h)},\sigma_{(h-1)})$ for every $h = 2,\ldots,i-2$.

Notice, further, that in the left adjoining interval $(\sigma_{(i+2)},\sigma_{(i+1)})$, by Lemma 1 and Proposition 1, the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ takes the following expression — different from (24) and (25).

$$\begin{aligned}\pi_i^W(\sigma) \; - \; C(x(i,\sigma)) \; &= \; v(\sigma,\tau_{i+1}) - v(\sigma_{(i+2)},\tau_{i+1}) \; + \\ &+ \; \sum_{h=i+2}^{T} \big[v(\sigma_{(h)},\tau_h) - v(\sigma_{(h+1)},\tau_h)\big] - C(x(i,\sigma)).\end{aligned} \tag{26}$$

This new expression of the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is also strictly concave in σ (by strict concavity of $v(\cdot,\tau_{i+1})$, $\sigma(i,\cdot)$ and strict convexity of $C(\cdot)$) and reaches a maximum at $\sigma(i,x_i(i+1))$ that from Lemma 3 is such that

$$\sigma_{(i+1)} = \sigma(i+1,x_{i+1}(i+1)) < \sigma(i,x_i(i+1)).$$

This implies that in the interval $(\sigma_{(i+2)},\sigma_{(i+1)})$ the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is strictly increasing in σ.

A symmetric argument shows that the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is strictly increasing in σ in any interval $(\sigma_{(k+1)}, \sigma_{(k)})$ for every $k = i+2, \ldots, T-1$. ■

The intuition behind the proof of this result is simple to describe. The payoff to worker i, $\pi_i^W(\sigma) - C(x(i,\sigma))$, changes expression as worker i increases his investment so as to improve his quality and get matched with a higher quality firm. This payoff however is continuous at any point, such as $\sigma_{(i-1)}$, in which in the continuation Bertrand game the worker gets matched with a different firm, but has a kink at such points.[8]

However, if the equilibrium considered is the efficient one — $s_i = i$ for every $i = 1, \ldots, S$ — the payoff to worker i is monotonic decreasing in any interval to the right of the $(\sigma_{(i+1)}, \sigma_{(i-1)})$ and increasing in any interval to the left. Therefore, this payoff has a unique global maximum. Hence worker i has no incentive to deviate and change his investment choice.

If instead we consider an inefficient equilibrium — an equilibrium where $s_1, \ldots, s_S$ differs from $1, \ldots, S$ — then the payoff to worker i is still continuous at any point, such as $\sigma(s_i, x_{s_i}(i))$, in which in the continuation Bertrand game the worker gets matched with a different firm. However, this payoff is not any more monotonic decreasing in any interval to the right of the $(\sigma(s_{i+1}, x_{s_{i+1}}(i+1)), \sigma(s_{i-1}, x_{s_{i-1}}(i-1)))$ and increasing in any interval to the left. In particular, this payoff is increasing at least in the right neighborhood of the switching points $\sigma(s_h, x_{s_h}(h))$ for $h = 1, \ldots, i-1$ and decreasing in the left neighborhood of the switching points $\sigma(s_k, x_{s_k}(k))$ for $k = i+1, \ldots, N$.

This implies that depending on the values of parameters these inefficient equilibria may or may not exist. We show below that for given firms' qualities it is possible to construct inefficient equilibria if two workers' qualities are close enough. Alternatively,

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[8]Indeed, from (22) and (23) we get that $\frac{\partial\left[\pi_i^W(\sigma_{(i-1)}^-) - C\left(x(i,\sigma_{(i-1)}^-)\right)\right]}{\partial\sigma} = v_1(\sigma_{(i-1)}, \tau_i) - \frac{C'\left(x(i,\sigma_{(i-1)})\right)}{\sigma_2\left(i, x(i,\sigma_{(i-1)})\right)}$ and $\frac{\partial\left[\pi_i^W(\sigma_{(i-1)}^+) - C\left(x(i,\sigma_{(i-1)}^+)\right)\right]}{\partial\sigma} = v_1(\sigma_{(i-1)}, \tau_{i-1}) - \frac{C'\left(x(i,\sigma_{(i-1)})\right)}{\sigma_2\left(i, x(i,\sigma_{(i-1)})\right)}$. Therefore, from $v_{12} > 0$, we conclude that $\frac{\partial\left[\pi_i^W(\sigma_{(i-1)}^+) C\left(x(i,\sigma_{(i-1)}^+)\right)\right]}{\partial\sigma} > \frac{\partial\left[\pi_i^W(\sigma_{(i-1)}^-) - C\left(x(i,\sigma_{(i-1)}^-)\right)\right]}{\partial\sigma}$.

for given workers' qualities inefficient equilibria do not exist if the firms qualities are close enough.

Proposition 3. *Given any ordered vector of firms' qualities $(\tau_1, \ldots, \tau_T)$, it is possible to construct an inefficient equilibrium of the workers' investment game such that there exists at least an i such that $s_i < s_{i-1}$.*

Moreover, given any vector of workers' quality functions $(\sigma(s_1, \cdot), \ldots, \sigma(s_S, \cdot))$, it is possible to construct an ordered vector of firms' qualities $(\tau_1, \ldots, \tau_T)$ such that there does not exist any inefficient equilibrium of the workers' investment game.

Proof: First, for a given ordered vector of firms' qualities $(\tau_1, \ldots, \tau_T)$ we construct an inefficient equilibrium of the workers' investment game such that there exist one worker, labelled s_j, $j \in \{2, \ldots, S\}$, such that $s_j < s_{j-1}$.

To show that a vector $(s_1, \ldots, s_j, \ldots, s_S)$ is an equilibrium of the workers' investment game we need to verify that condition (20) holds for every $i = 2, \ldots, S$ and no worker s_i has an incentive to deviate and choose an investment x different from $x_{s_i}(i)$, as defined in (16).

Notice first that for every worker, other than s_j and s_{j-1} Proposition 2 above applies and hence it is an equilibrium for each worker to choose investment level $x_{s_i}(i)$, as defined in (16), such that (20) is satisfied.

We can therefore restrict attention on worker s_j and s_{j-1}. In particular we need to consider a worker s_{j-1} of a quality arbitrarily close to the one of worker s_j. This is achieved by considering a sequence of quality functions $\sigma^n(s_{j-1}, \cdot)$ that converges uniformly to $\sigma(s_j, \cdot)$.[9] Then from definition (16), the continuity and strict concavity of $v(\cdot, \tau)$ and $\sigma(s, \cdot)$, the continuity and strict convexity of $C(\cdot)$ and the continuity of

[9] The sequence $\sigma^n(s_{j-1}, \cdot)$ converges uniformly to $\sigma(s_j, \cdot)$ if and only if

$$\lim_{n\to\infty} \sup_x |\sigma^n(s_{j-1}, x) - \sigma(s_j, x)| = 0.$$

$v_1(\cdot, \tau)$, $\sigma_2(s, \cdot)$ and $C'(\cdot)$ for any given $\varepsilon > 0$ there exists an index n_ε such that from every $n > n_\varepsilon$:

$$\left|\sigma^n(s_{j-1}, x_{s_{j-1}}(j-1)) - \sigma(s_j, x_{s_j}(j-1))\right| < \varepsilon. \tag{27}$$

From Lemma 3 and the assumptions $s_j > s_{j-1}$ we also know that for every $n > n_\varepsilon$:

$$\sigma^n(s_{j-1}, x_{s_{j-1}}(i-1)) < \sigma(s_j, x_{s_j}(j-1)). \tag{28}$$

While from the assumption $\tau_j < \tau_{j-1}$ we have that:

$$\sigma(s_j, x_{s_j}(j)) < \sigma(s_j, x_{s_j}(j-1)). \tag{29}$$

Inequalities (27), (28) and (29) imply that for any worker s_{j-1} characterized by the quality function $\sigma^n(s_{j-1}, \cdot)$ where $n > n_\varepsilon$, the equilibrium condition (20) is satisfied:

$$\sigma(s_j, x_{s_j}(j)) < \sigma^n(s_{j-1}, x_{s_{j-1}}(j-1)). \tag{30}$$

To conclude that $(s_1, \ldots, s_j, \ldots, s_S)$ is an equilibrium of the workers' investment game we still need to show that neither worker s_j nor worker s_{j-1} want to deviate a choose an investment different from $x_{s_j}(j)$ and $x_{s_{j-1}}(j-1)$, where the quality function associated with worker s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$ for $n > n_\varepsilon$.

Consider the net payoff to worker s_j: $\pi^W_{s_j}(\sigma) - C(x(s_j, \sigma))$. An argument symmetric to the one used in Step 2 of Proposition 2 shows that this payoff function is continuous in σ. Moreover, from the definition of $\sigma_{(j)}$, Lemma 3, (28) and (30) we obtain that

$$\sigma_{(j)} < \sigma^n_{(j-1)} \ < \sigma(s_j, x_{s_j}(j-1)) \ < \sigma_{(j-2)}.$$

Then using an argument symmetric to the one used in Step 3 of the proof of Proposition 2 we conclude that this net payoff function has two local maxima at $\sigma_{(j)}$ and $\sigma(s_j, x_{s_j}(j-1))$ and a kink at $\sigma^n_{(j-1)}$. We then need to show that there exist at least an element of the sequence $\sigma^n_{(j-1)}$ such that the net payoff $\pi^W_{s_j}(\sigma) - C(x(s_j, \sigma))$ reaches a global maximum at $\sigma_{(j)}$. Therefore when the quality function of worker s_{j-1} is

$\sigma^n(s_{j-1},\cdot)$ worker s_j has no incentive to deviate and choose a different investment.

From (9) the net payoff $\pi^W_{s_j}(\sigma) - C(x(s_j,\sigma))$ computed at $\sigma_{(j)}$ is greater than the same net payoff computed at $\sigma(s_j, x_{s_j}(j-1))$ if and only if

$$\begin{aligned} v(\sigma_{(j)},\tau_j) \quad - \quad & C\left(x(s_j,\sigma_{(j)})\right) \geq \\ \geq \quad & v(\sigma(s_j,x_{s_j}(j-1)),\tau_{j-1}) \quad - \quad v(\sigma^n_{(j-1)},\tau_{j-1}) + \\ + \quad & v(\sigma^n_{(j-1)},\tau_j) - C\left(x\left(s_j,\sigma(s_j,x_{s_j}(j-1))\right)\right) \end{aligned} \tag{31}$$

Inequality (27) above and the continuity of $v(\cdot,\tau_{j-1})$, $\sigma(s_j,\cdot)$ and $C(\cdot)$ imply that for any given $\varepsilon > 0$ there exist a ξ_ε and a n_{ξ_ε} such that for every $n > n_{\xi_\varepsilon}$

$$\left|v(\sigma(s_j,x_{s_j}(j-1)),\tau_{j-1}) - v(\sigma^n_{(j-1)},\tau_{j-1})\right| < \xi_\varepsilon$$

and

$$\left|C\left(x\left(s_j,\sigma(s_j,x_{s_j}(j-1))\right)\right) - C\left(x(s_j,\sigma^n_{(j-1)})\right)\right| < \xi_\varepsilon$$

These two inequalities imply that a necessary condition for (31) to be satisfied is

$$v(\sigma_{(j)},\tau_j) - C\left(x(s_j,\sigma_{(j)})\right) \geq v(\sigma^n_{(j-1)},\tau_j) - C\left(x(s_j,\sigma^n_{(j-1)})\right) + 2\xi_\varepsilon. \tag{32}$$

We can now conclude that there exist an $\varepsilon > 0$ such that for every $n > n_{\xi_\varepsilon}$ condition (32) is satisfied with strict inequality. This is because (by strict concavity of $v(\cdot,\tau_j)$, $\sigma(s_j,\cdot)$ and strict convexity of $C(\cdot)$) the function $v(\sigma,\tau_j) - C\left(x(s_j,\sigma)\right)$ is strictly concave and has a unique interior maximum at $\sigma_{(j)}$.

Consider now the net payoff to worker s_{j-1}: $\pi^W_{s_{j-1}}(\sigma) - C(x(s_{j-1},\sigma))$. An argument symmetric to the one used above allow us to prove that this payoff function is continuous in σ. Further, from the definition of $\sigma_{(j)}$, Lemma 3, and (30) we have that

$$\sigma_{(j+1)} < \sigma^n(s_{j-1}, x_{s_{j-1}}(j)) \; < \sigma_{(j)} < \sigma^n_{(j-1)}.$$

Therefore we conclude that this net payoff function has two local maxima at $\sigma^n_{(j-1)}$ and $\sigma^n(s_{j-1}, x_{s_{j-1}}(j))$ and a kink at $\sigma_{(j)}$. We still need to prove that there exist at least an element of the sequence $\sigma^n_{(j-1)}$ such that the net payoff $\pi^W_{s_{j-1}}(\sigma)-C(x(s_{j-1},\sigma))$ reaches a global maximum at $\sigma^n_{(j-1)}$ which implies that when the quality function of worker s_{j-1} is $\sigma^n(s_{j-1},\cdot)$ this worker has no incentive to deviate and choose a different investment.

From (9) the net payoff $\pi^W_{s_{j-1}}(\sigma)-C(x(s_{j-1},\sigma))$ computed at $\sigma^n_{(j-1)}$ is greater than the same net payoff computed at $\sigma^n(s_{j-1},x_{s_{j-1}}(j))$ if and only if

$$\begin{aligned} v(\sigma^n_{(j-1)},\tau_{j-1}) \quad &- \quad v(\sigma_{(j)},\tau_{j-1})+v(\sigma_{(j)},\tau_j)-C\left(x(s_{j-1},\sigma^n_{(j-1)})\right)\geq \\ &\geq \quad v(\sigma^n(s_{j-1},x_{s_{j-1}}(j)),\tau_j)-C\Big(x\left(s_{j-1},\sigma^n(s_{j-1},x_{s_{j-1}}(j))\right)\Big) \end{aligned} \tag{33}$$

Definition (16), the continuity and strict concavity of $v(\cdot,\tau_j)$ and $\sigma(s_{j-1},\cdot)$, the continuity and strict convexity of $C(\cdot)$ and the continuity of $v_1(\cdot,\tau_j)$, $\sigma_2(s_j,\cdot)$ and $C'(\cdot)$ imply that for given $\varepsilon'>0$ there exists a $n_{\varepsilon'}$, a $\xi_{\varepsilon'}$ and a $n_{\xi_{\varepsilon'}}$ such that from every $n>n_{\varepsilon'}$:

$$\left|\sigma^n(s_{j-1},x_{s_{j-1}}(j))-\sigma_{(j)}\right|<\varepsilon';$$

while for every $n>n_{\xi_{\varepsilon'}}$

$$\left|v(\sigma_{(j)},\tau_j)-v(\sigma^n(s_{j-1},x_{s_{j-1}}(j)),\tau_j)\right|<\xi_{\varepsilon'}$$

and

$$\left|C\left(x(s_{j-1},\sigma_{(j)})\right)-C\Big(x\left(s_{j-1},\sigma^n(s_{j-1},x_{s_{j-1}}(j))\right)\Big)\right|<\xi_{\varepsilon}.$$

The last two inequalities imply that a necessary condition for (33) to be satisfied is

$$v(\sigma^n_{(j-1)},\tau_{j-1})-C\left(x(s_{j-1},\sigma^n_{(j-1)})\right)\geq v(\sigma_{(j)},\tau_{j-1})-C\left(x(s_{j-1},\sigma_{(j)})\right)+2\xi_{\varepsilon'}. \tag{34}$$

We can now conclude that there exists a $\varepsilon'>0$ such that for every $n>n_{\xi_{\varepsilon'}}$ condition (34) is satisfied with strict inequality. This is because (by strict concavity of $v(\cdot,\tau_{j-1})$, $\sigma^n(s_{j-1},\cdot)$ and strict convexity of $C(\cdot)$) the function $v(\sigma,\tau_{j-1})-C\left(x(s_{j-1},\sigma)\right)$ is strictly concave and has a unique interior maximum at $\sigma^n_{(j-1)}$.

This concludes the construction of the inefficient equilibrium of the workers' investment game.

We need now to show that for any given vector of workers' quality functions $(\sigma(s_1,\cdot),\ldots,\sigma(s_S,\cdot))$ it is possible to construct an ordered vector of firms qualities $(\tau_1,\ldots,\tau_T)$ such that no inefficient equilibrium exist.

Assume, by way of contradiction, that an inefficient equilibrium exists for any ordered vector of firms' qualities $(\tau_1,\ldots,\tau_T)$. Consider first the case in which this inefficient equilibrium is such that there exist only one worker s_j such that $s_j < s_{j-1}$. Let τ^n_{j-1} be a sequence of quality levels of firm $(j-1)$ such that $\tau^n_{j-1} > \tau_j$ and τ^n_{j-1} converges to τ_j.

From Lemma 3 and the assumption $s_j > s_{j-1}$ we have that

$$\sigma(s_j, x_{s_j}(j)) > \sigma(s_{j-1}, x_{s_{j-1}}(j)) \tag{35}$$

where $x_{s_j}(j)$ and $x_{s_{j-1}}(j)$ are defined in (16). Further, denote $x^n_{s_{j-1}}(j-1)$ the optimal investment defined, as in (17), by the following set of first order conditions:

$$v_1(\sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)), \tau^n_{j-1})\ \sigma_2(s_{j-1}, x^n_{s_{j-1}}(j-1)), \tau^n_{j-1}) = C'(x^n_{s_{j-1}}(j-1)).$$

Then from Lemma 3 we have that

$$\sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)) > \sigma(s_{j-1}, x_{s_{j-1}}(j)). \tag{36}$$

Further, continuity of the functions $v(\sigma,\cdot)$, $v_1(\sigma,\cdot)$, $\sigma(s,\cdot)$, $\sigma_2(s,\cdot)$, $C(\cdot)$ and $C'(\cdot)$ imply that for given $\hat{\varepsilon} > 0$ there exist an $n_{\hat{\varepsilon}}$ such that for every $n > n_{\hat{\varepsilon}}$

$$\left|\sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)) - \sigma(s_{j-1}, x_{s_{j-1}}(j))\right| < \hat{\varepsilon}. \tag{37}$$

Then from (35), (36) and (37) there exists an $\hat{\varepsilon} > 0$ and hence an $n_{\hat{\varepsilon}}$ such that for every $n > n_{\hat{\varepsilon}}$

$$\sigma(s_j, x_{s_j}(j)) > \sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)). \tag{38}$$

Inequality (38) clearly contradicts the necessary condition (20) for the existence of the inefficient equilibrium.

A similar construction leads to a contradiction in the case the inefficient equilibrium is characterized by more than one worker s_j such that $s_j < s_{j-1}$. ■

The intuition behind this result is simple. The continuity of each worker's payoff implies that when two workers have similar innate abilities exactly as it is not optimal for each worker to deviate when he is matched efficiently it is also not optimal for him to deviate when he is inefficiently assigned to a match. Indeed, the difference in workers' qualities is almost entirely determined by the difference in firms' qualities rather than by the difference in workers' innate ability.

Conversely, if firms' qualities are similar then the difference in workers qualities is almost entirely explained by the difference in workers' innate abilities implying that it is not possible to construct an inefficient equilibrium of the workers' investment game. The reason being that the improvement in the worker's incentives to invest due to a match with a better firm are more than compensated by the decrease in the worker's incentives induced by a lower innate ability of the worker. Hence it is not optimal for two workers of decreasing innate abilities to generate increasing qualities so as to be matched with increasing quality firms.

We then conclude that when workers are undertaking ex-ante match specific investments and then Bertrand compete for a match with a firm investments are constrained efficient. However, if workers are similar in innate ability inefficiencies may arise that take the form of additional equilibria characterized by inefficient matches. Hence the higher is the degree of specificity due to the workers' characteristics with respect to the specificity due to the firms' characteristics the less likely is this inefficiency.

5. Firm's investments

We move now to the model in which the qualities of workers are exogenously given $\sigma_s = \sigma(s)$ while the qualities of firms are a function of firms' ex-ante match specific investments y and the firm's identity t: $\tau(t, y)$. In this model we show that firms' investments are not any more constrained efficient. Firms under-invest since their marginal incentives to undertake investments are determined by their outside option that depends on the surplus of the match between the firm and the immediate competitor to the worker the firm is matched with in equilibrium (a strictly lower surplus than the equilibrium one). However, we are able to provide an upper-bound on the aggregate inefficiency of the under-investment. Indeed, aggregate inefficiency is strictly lower than the inefficiency that would be generated in the same environment if the best firm matches with the best worker in isolation. In other words, competition strictly decreases the inefficiency due to the hold-up problem.

Furthermore, as a counterpart to this 'near efficiency' result we show that equilibrium matches are always efficient: the order of firms innate abilities coincides with the order of their derived qualities. In other words, all coordination problems are solved and no inefficient equilibrium arise.

5.1. Equilibrium Characterization of the Bertrand Competition Game

We solve our model backward and start from the Bertrand competition game as described in Section 3 above. Assume that firms investments are given so as to obtain an ordered vector of firms' qualities $(\tau_{(1)}, \ldots, \tau_{(T)})$ where $\tau_{(h)} < \tau_{(h-1)}$ for every $h = 2, \ldots, T$. The ordered vector of workers' qualities is instead $(\sigma_1, \ldots, \sigma_S)$, where $\sigma_k < \sigma_{k-1}$ for every $k = 2, \ldots, S$.

The analysis of the Bertrand competition subgame differs from the one presented in the section above. Indeed, the order in which firms choose among bids in this subgame is determined by the firms' identities rather than by their qualities. This implies that unless firms' qualities (which are endogenously determined) have the same order of firms' identities it is possible that firms do not choose among bids in the decreasing order of their marginal contribution to a match (at least off the equilibrium path). Therefore competition among firms for each match might not be maximized as it was necessarily the case on and off the equilibrium path in the model in which the workers make ex-ante investments.

We start by showing that as in Section 4.1 above equilibrium matches are efficient for given firms' investments. In other words we are able to prove a result equivalent to Lemma 1 above.

Lemma 4. *Given an ordered vector* $(\tau_{(1)}, \ldots, \tau_{(S)})$ *every equilibrium of the Bertrand competition subgame is characterized by the equilibrium matches:* $(\sigma_k, \tau_{(k)})$*, for every* $k = 1, \ldots, T$.

Proof: Assume by way of contradiction that this is not the case and there exist a pair of equilibrium matches $(\sigma', \tau_{(i)})$ and $(\sigma'', \tau_{(j)})$ such that $i < j$, or $\tau_{(i)} > \tau_{(j)}$, and $\sigma' < \sigma''$. Denote $B(\tau_{(i)})$, respectively $B(\tau_{(j)})$, the bids that in equilibrium the firm of quality $\tau_{(i)}$, respectively of quality $\tau_{(j)}$, accepts.

Consider first the match $(\sigma'', \tau_{(i)})$. For this match to occur in equilibrium we need that it is not convenient for the worker of quality σ'' to match with the firm of quality $\tau_{(j)}$ rather than $\tau_{(i)}$. If worker σ'' deviates and does not submit a bid that will be selected by firm $\tau_{(i)}$ then two situations may occur depending on whether the firm of quality $\tau_{(i)}$ chooses her bid before or after the firm of quality $\tau_{(j)}$. In particular if $\tau_{(i)}$ chooses her bid before $\tau_{(j)}$ then following the deviation of the worker of quality σ'' a different worker will be matched with firm $\tau_{(i)}$. As in the proof of Lemma 1 above this may reduce the bids submitted to the firm of quality $\tau_{(j)}$ of exactly one bid reducing the maximum of these bids $\hat{B}(\tau_{(j)})$ that the worker of quality σ'' need to submit so as to be matched, following his deviation, with the firm of quality $\tau_{(j)}$. This implies the following necessary condition for $(\sigma'', \tau_{(i)})$ to be an equilibrium match.

$$v(\sigma'', \tau_{(i)}) - B(\tau_{(i)}) \geq v(\sigma'', \tau_{(j)}) - B(\tau_{(j)}) \tag{39}$$

Alternatively if $\tau_{(i)}$ chooses her bid after $\tau_{(j)}$ then for $(\sigma'', \tau_{(i)})$ to be an equilibrium match we need that worker σ'' does not find convenient to deviate and outbid the worker of quality σ' by submitting bid $B(\tau_{(j)}$. This equilibrium condition therefore coincides with (39) above.

Consider now the equilibrium match $(\sigma', \tau_{(j)})$. For this match to occur in equilibrium we need that the worker of quality σ' does not want to deviate and be matched with the firm of quality $\tau_{(i)}$ rather than $\tau_{(j)}$. As discussed above, depending on whether $\tau_{(j)}$ chooses her bid before or after $\tau_{(i)}$, the following is a necessary or a necessary and sufficient condition for $(\sigma', \tau_{(j)})$ to be an equilibrium match.

$$v(\sigma', \tau_{(j)}) - B(\tau_{(j)}) \geq v(\sigma', \tau_{(i)}) - B(\tau_{(i)}). \tag{40}$$

The inequalities (39) and (40) imply:

$$v(\sigma'', \tau_{(i)}) + v(\sigma', \tau_{(j)}) \geq v(\sigma', \tau_{(i)}) + v(\sigma'', \tau_{(j)}). \tag{41}$$

Condition (41) contradicts the assortative matching assumption $v_{12}(\sigma, \tau) > 0$. ■

Having established this property we can now move to the characterization of the unique equilibrium of the Bertrand competition game.

Consider first the match between the firm of identity (innate ability) t with the worker of identity s or quality σ_s. We define the level of investment $y_t(s)$ as follows

$$y_t(s) = \underset{y}{\text{argmax}}\ v(\sigma_s, \tau(t, y)) - C(y). \tag{42}$$

In other words, $y_t(s)$ is defined by the following necessary and sufficient condition:

$$v_2(\sigma_s, \tau(t, y_t(s)))\ \tau_2(t, y_t(s)) = C'(y_t(s)). \tag{43}$$

Notice then that, in contrast with what we did in Section 4 above, we cannot here solve fort the Bertrand competition game without solving at the same time for the firms' investment game. Indeed, the workers' bids depend on the firms' investment choice. In other words the Bertrand competition game differs depending on the relative size of each firm's investment choice.

Proposition 4. *The unique equilibrium of the Bertrand competition subgame in the case in which firms undertake ex-ante investments is such that firm t chooses* *investment* $y_t(t+1)$*, as defined in (43), generates quality* $\tau(t, y_t(t+1)) = \tau_t$*, and matches with worker* t *of quality* σ_t*,* $t = 1, \ldots, T$*.*

The share of the surplus that each worker and each firm receive are:

$$\Pi^W_{\sigma_t} = \sum_{h=t}^{T} [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] \tag{44}$$

$$\Pi^F_{\tau(t)} = v(\sigma_{t+1}, \tau_t) - \sum_{h=t+1}^{T} [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)]\,. \tag{45}$$

Proof: We prove this result in three steps. We first show that if firms choose investments $y_t(t+1)$, for $t = 1, \ldots, T$, (labelled *simple* investments, for convenience) then the order of firms' identities coincides with the inverse order of firms' qualities. We then proceed to show that in this case the equilibrium matches are such that worker t matches with firm t, for every $t = 1, \ldots, T$, and the shares of the surplus accruing to each worker and each firm are the ones defined in (44) and (45) above. We then conclude the proof by showing that the unique equilibrium of the firms' investment subgame is for firm t to choose the simple investment $y_t(t+1)$, $t = 1, \ldots, T$.

Step 1. If each firm t chooses the simple investment $y_t(t+1)$, as defined in (42), then

$$\tau_1 = \tau(1, y_1(2)) > \ldots > \tau_T = \tau(T, y_T(T+1)).$$

The proof follows from the fact that from (43) we obtain:

$$\frac{\partial \tau(t, y_t(s))}{\partial t} = \frac{v_2\ \tau_1\ \tau_{22} - \tau_1\ C'' - v_2\ \tau_2\ \tau_{12}}{v_{22}(\tau_2)^2 + v_2\ \tau_{22} - C''} < 0 \tag{46}$$

and

$$\frac{\partial \tau(t, y_t(s))}{\partial s} = \frac{v_{12}(\tau_2)^2}{v_{22}(\tau_2)^2 + v_2\ \tau_{22} - C''} < 0 \tag{47}$$

where (with an abuse of notation) we denote with τ_h and τ_{hk}, $h, k \in \{1, 2\}$ the first and second order derivatives of the quality functions $\tau(\cdot, \cdot)$ computed at $(t, y_t(s))$.

Moreover the first and second order derivative (v_h and v_{hk}, $h, k \in \{1, 2\}$) of the functions $v(\cdot, \cdot)$ are computed at $(\sigma_s, \tau(t, y_t(s)))$.

Step 2. If firms' investments are such that $\tau_1 > \ldots > \tau_T$ then firm t is matched with worker t, $t = 1, \ldots, T$ and the shares of surplus to each worker and each firm are the ones specified in (44) and (45).

Notice that since $\tau_1 > \ldots > \tau_T$ and $\sigma_1 > \ldots \sigma_S$ then from Lemma 4 firm t matches with worker t, $t = 1, \ldots, T$. Moreover Proposition 1 above applies and hence the shares of surplus to each worker and each firm in (9) and (10) coincides with the shares in (44) and (45).

Step 3. The unique equilibrium of the firms' investment subgame is such that firm t chooses the simple investment $y_t(t + 1)$ for every $t = 1, \ldots, T$.

We prove this result starting from firm T. In the T-th (the last) subgame of the Bertrand competition game all firms, but firm T, have selected a worker's bid. Denote τ_T the quality of this firm.

Assume for simplicity that $S = T + 1$. We use the same notation as in the proof of Proposition 1 above. In particular since we want to show that firm T chooses a simple investment independently from the investment choice of the other firms we denote $\alpha_{(T)}$ and $\alpha_{(T+1)}$ the qualities of the two workers that are still un-matched in the T-th subgame, such that $\alpha_{(T)} > \alpha_{(T+1)}$. Indeed, from Lemma 4 the identity of the two workers left will depend on the order of firms' qualities and therefore on the investment choices of the other $(T - 1)$ firms.

From Step 2 above we have that the worker of quality $\alpha_{(T)}$ matches with firm T. Firm T's payoff is $v(\alpha_{(T+1)}, \tau_T)$ while the payoff of the worker of quality $\alpha_{(T)}$ is $\left[v(\alpha_{(T)}, \tau_T) - v(\alpha_{(T+1)}, \tau_T)\right]$ and the payoff of the worker of quality $\alpha_{(T+1)}$ is zero.

Denote now $a_{(T)}$, respectively $a_{(T+1)}$, the identity of the workers of quality $\alpha_{(T)}$, respectively $\alpha_{(T+1)}$: $a_{(T)} < a_{(T+1)}$. Firm T's optimal investment y_T is then defined as follows

$$y_T = \underset{y}{\text{argmax}}\, v(\alpha(T+1), \tau(T, y)) - C(y).$$

This implies that the optimal investment of firm T is the simple investment $y_T = y_T(a_{(T+1)})$, as defined in (43), whatever is the pair of workers left in the T-th subgame. If all other firms undertake a simple investment then from Step 1: $a_{(T)} = T$ and $a_{(T+1)} = T+1$. Hence firm T's optimal investment is $y_T(T+1)$.

Denote now $t+1$, $(t < T)$, the last firm that undertakes a simple investment $y_{t+1}(t+2)$. We then show that also firm t will choose a simple investment $y_t(t+1)$. Consider the t-th subgame in which firm t has to choose among the potential bids of the remaining $(T-t+2)$ workers labelled $a_{(t)} < \ldots < a_{(T+1)}$, with associated qualities $\alpha_{(t)} > \ldots > \alpha_{(T+1)}$, respectively.[10] From the assumption that every firm $j = t+1, \ldots, T$ undertakes a simple investment $y_j(\alpha_{(j+1)})$ and Step 1 we obtain that $\tau_{t+1} > \ldots > \tau_T$. We first show that the quality associated with firm t is such that $\tau_t > \tau_{t+1}$.

Assume by way of contradiction that firm t chooses investment y^* that yields a quality τ^* such that $\tau_{j+1} \leq \tau^* \leq \tau_j$ for some $j \in \{t+1, \ldots, T-1\}$. Then from Lemma 4 and Step 2 we have that firm t matches with worker $a_{(j)}$ and firm t's payoff is:

$$\Pi^F_{\tau^*} = v(\alpha_{(j+1)}, \tau(t, y^*)) - \sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right] \tag{48}$$

where $\tau(t, y^*) = \tau^*$. From (48) we obtain that y^* is then the solution to the following problem:

$$y^* = \underset{y}{\text{argmax}}\, v(\alpha(j+1), \tau(t, y)) - C(y). \tag{49}$$

From the assumption that all firm $j \in \{t+1, \ldots, T\}$ undertakes a simple investment and definition (42) we also have that firm j's investment choice $y_j(a_{(j+1)})$ is defined

[10] Once again we want to show that firm t undertakes a simple investment independently of the investment choice of firms $1, \ldots, t-1$ that determines the exact identities of the un-matched workers in the t-th subgame of the Bertrand competition game.

as follows:

$$y_j(a_{(j+1)}) = \underset{y}{\operatorname{argmax}}\, v(\alpha(j+1), \tau(j, y)) - C(y). \tag{50}$$

Notice further that the payoff to firm t in (48) is continuous in τ^*. Indeed the limit for τ^* that converges from the right to τ_j is equal to

$$\Pi^F_{\tau_j} = v(\alpha_{(j+1)}, \tau_j) - \sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right]. \tag{51}$$

If instead $\tau_j < \tau^* \leq \tau_{j-1}$ then from Step 2 the payoff to the firm with quality τ^* is

$$\begin{aligned} \Pi^F_{\tau^*} &= v(\alpha_{(j)}, \tau^*) - v(\alpha_{(j)}, \tau_j) + \\ &+ v(\alpha_{(j+1)}, \tau_j) - \textstyle\sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right]. \end{aligned} \tag{52}$$

Therefore the limit for τ^* that converges to τ_j from the left is, from (52), equal to $\Pi^F_{\tau_j}$ in (51). This proves the continuity in τ^* of the payoff function in (48).

Continuity of the payoff function in (48) together with definitions (49), (50) and condition (46) imply that $y* > y_j(a_{(j+1)}$ or $\tau^* > \tau_j$ a contradiction to the hypothesis $\tau^* \leq \tau_j$.

We now show that firm t will choose a simple investment $y_t(\alpha_{(t+1)})$. From the result we just obtained $\tau_t > \tau_{t+1} > \ldots > \tau_T$ and the assumption that $\alpha_{(t)} > \ldots > \alpha_{(S)}$ are the qualities of the unmatched workers in the t-th subgame of the Bertrand competition game we conclude, using Step 2 above, that the payoff to firm t is:

$$\Pi^F_{\tau_t} = v(\alpha_{(t+1)}, \tau_t) - \sum_{h=t+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right] \tag{53}$$

Firm t's investment choice is then the simple investment $y_t(a_{(t+1)}$ defined as follows:

$$y_t(a_{(t+1)} = \underset{y}{\operatorname{argmax}}\, v(\alpha(t+1), \tau(t, y)) - C(y). \tag{54}$$

To conclude that a simple investment $y_t(a_{(t+1)})$ is the unique equilibrium choice for firm t in the firms' investment game we still need to show that firm t has no incentive to deviate and choose an investment y^*, and hence a quality τ^*, that exceeds the quality τ_k of one of the $(t-1)$ firms that are already matched at the t-th subgame of the Bertrand competition game: $k < t$. The reason why this choice of investment might be optimal for firm t is that it changes the pool of workers $a_{(t)}, \ldots, a_{(S)}$ unmatched in subgame t. Of course this choice will change the simple nature of firm t's investment only if $\tau_k > \tau_{t+1}$. Indeed we already showed that if $\tau_k < \tau_{t+1}$ then $\tau_t > \tau_k$ and from (54) firm t's investment choice is $y_t(a_{(t+1)})$ a simple investment for any given set of unmatched workers.

Consider the following deviation by firm t: firm t chooses an investment $y^* > y_t(a_{(t+1)})$ that yields quality $\tau^* > \tau_k > \tau_{t+1}$. Recall that Lemma 4 implies that the ranking of each firm in the ordered vector of firms' qualities determines the worker each firm is allocated to. Hence, firm t's deviation changes the ranking and the allocation of all firms whose quality τ is smaller than τ^* and greater than τ_{t+1}. However, this deviation does not alter the ranking of the $T+1-t$ firms with identities $(t+1, \ldots, T)$ and qualities $(\tau_{t+1}, \ldots, \tau_T)$. Therefore, the only difference between the set of un-matched workers in the t-th subgame of the Bertrand competition game on the equilibrium path and the set of un-matched workers in the same subgame following firm t's deviation is the identity and quality of the worker that matches with firm t.[11] The remaining set of workers' identities and qualities $(\alpha_{(t+1)}, \ldots, \alpha_{(S)})$ is unchanged.

Hence, following firm t's deviation the un-matched workers' qualities are $\alpha^* > \alpha_{(t+1)} > \ldots > \alpha_{(T)}$, where α^* is the quality of the worker that according to Lemma 4 is matched with firm t when the quality of this firm is τ^*. Step 2 implies that firm t's payoff following this deviation is then:

$$\Pi^F_{\tau^*} = v(\alpha_{(t+1)}, \tau^*) - \sum_{h=t+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right] \quad (55)$$

[11] Recall that all firms with identities $(k, \ldots, t-1)$ have already been matched in the t-th subgame of the Bertrand competition game.

Continuity of the payoff function in (54) together with (55) imply that firm t's net payoff is maximized at $y_t(a_{(t+1)})$. Hence, firm t cannot gain from choosing an investment $y^* > y_t(a_{(t+1)}$. This proves that firm t will choose a simple investment $y_t(a_{(t+1)})$. This argument holds for every $t < T$ implying that all firm choose a simple investment. Therefore $a_{(t)} = t$ and firm t's equilibrium investment choice is $y_t = y_t(t+1)$. ■

Notice that the same efficiency properties we discuss in relation to Proposition 1 hold in this case as well.

As in the sequential investment case, the worker's equilibrium payoff $\Pi^W(t, t, y_t, x_t)$ is equal to the sum of the social surplus, $v(t, t, y_t, x_t)$ and an expression W_t that does not depend on worker t's match specific investment x_t:

$$\Pi^W(t, t, y_t, x_t) = v(t, t, y_t, x_t) + W_t. \tag{56}$$

Similarly, the firm's equilibrium payoff $\Pi^F(t, t, y_t, x_t)$ is the sum of the surplus generated by the (inefficient) match of firm t with worker $(t + 1)$ and an expression P_t that does not depend on firm t's match-specific investment y_t:

$$\Pi^F(t, t, y_t, x_t) = v(t, t + 1, y_t, x_{t+1}) + P_t. \tag{57}$$

5.2. The Inefficiencies of Non-Sequential Investment

In Section 5 above we have argued that the agents on the side of the market that is responsible for bidding for matches in the Bertrand competition game make constrained efficient *ex-ante* investments.[12] In our model, these are the workers. This section analyses the potential inefficiencies that arise if firms also make *ex-ante* investments that precede the Bertrand competition game.

For sake of simplicity denote $w(\sigma, t, y)$ the net surplus to firm t when it matches

[12]The constraint on the workers' investment choices is represented by the firms' investment choices that affect directly the marginal returns of the workers' choices.

with the worker of quality σ:

$$w(t, s, y) = v(t, s, y) - C(y)$$

Further recall that we assume that this net surplus function satisfies the "responsive complementarity" assumptions as stated in (2) above.

From Proposition 4 we know that the return to firm t is given, from (57), by:

$$\Pi^F(t, t, y_t) = w(t, t+1, y_t) + P_t \tag{58}$$

where P_t depends upon investments made by firms of a higher identity than t. If firm t must make an *ex-ante* investment then, recognising that competition will follow leading to the return given by (58), y_t will be chosen to maximize (58) and we have:[13]

$$y_t = \operatorname*{argmax}_y w(t, t+1, y) \tag{59}$$

On the other hand, efficiency calls for the maximization of total surplus. As the surplus from the match between firm t and worker t is $w(t, t, y_t)$, efficiency requires an investment of y_t^* satisfying

$$y_t^* = \operatorname*{argmax}_y w(t, t, y). \tag{60}$$

The inefficiency of *ex-ante* investment by all firms is therefore given by

$$L = \sum_1^T w(t, t, y_t^*) - \sum_1^T w(t, t, y_t) \tag{61}$$

How large is this loss L? First, notice that the difference between y_t^* and y_t is

[13]Notice that if the argument x is not suppressed in the function $w(\cdot, \cdot, \cdot, \cdot)$ then (59) below defines firm t's reaction function $y(t, t, x)$. Our complementarity and concavity assumption on the surplus function imply that $y(t, t, x)$ is strictly monotonic in x and the Nash equilibrium of this investment game is unique.

approximately proportional to the difference in characteristics between worker t and $t+1$ (given that y is differentiable in s). On the other hand, as y_t^* solves (60) , the difference between $w(t,t,y_t)$ and $w(t,t,y_t^*)$ will be approximately proportional to the *square* of the difference between y_t and y_t^* which will be small if worker t and worker $t+1$ have similar characteristics. To give an example of how this affects L, consider a situation where the characteristics of a worker are captured by a real number c with workers 0 through T having characteristics which are evenly spaced between $\overline{c}$ and $\underline{c}$. How is L affected by the size of the market T? The difference between y_t^* and y_t is approximately proportional to $(\overline{c}-\underline{c})/T$ and the difference between $w(t,t,y_t)$ and $w(t,t,y_t^*)$ will be approximately proportional to $\left[(\overline{c}-\underline{c})/T\right]^2$. Summing over t then gives a total loss L that is proportional to $(\overline{c}-\underline{c})^2/T$: in large markets the aggregate inefficiency created by *ex-ante* investment will be arbitrarily small.[14]

This is a result that changes the degree of specificity of the workers' investment choices. Increasing the number and hence the density of firms evenly spaced in the interval $[\overline{c},\underline{c}]$ is equivalent to introducing firms with closer and closer characteristics. This is equivalent to reducing the loss in productivity generated by the match of a worker that made a given investment with the firm that is immediately below in characteristics levels. Hence, there is a sense in which this result is not fully satisfactory since we know that if the worker's investment is general in nature the firms' investment choices are efficient.

Therefore, in the rest of this section, we identify an upper-bound on the aggregate inefficiency present in the economy that is independent of the number of firms and does not alter the specificity of the workers investment choices. Whatever the size of T, it is possible to get a precise upper-bound on the loss L. Indeed, the inefficiency created by the firms' *ex-ante* under-investment is less than that which could be created by the under-investment of only one firm (the best 1) in a match with a worker (the worst T).

[14] See Kaneko (1982).

Proposition 5. *Assume that there are at least two firms* ($T \geq 2$). *Let* M *be the efficiency loss resulting from firm* 1 *choosing an investment level given by* $\tilde{y} = \operatorname{argmax}_y w(1, T+1, y)$:

$$M = w(1, 1, y_1^*) - w(1, 1, \tilde{y}). \tag{62}$$

If types and investments are complementary (in a sense that second cross-partial derivatives are positive and (2) is satisfied) then

$$L < M. \tag{63}$$

Proof: If $y(t, s)$ is the efficient investment level when worker of type s is matched with a firm of type t then L and M can be written as

$$L = \sum_1^T w(t, t, y(t, t)) - \sum_1^T w(t, t, y(t, t+1)) \tag{64}$$

$$M = \sum_1^T w(1, 1, y(1, t)) - \sum_1^T w(1, 1, y(1, t+1)) \tag{65}$$

so that

$$\begin{aligned} M - L = \sum_1^T \Big\{ \Big[w(1, 1, y(1, t)) \ - \ w(t, t, y(t, t)) \Big] - \\ - \ \Big[w(1, 1, y(1, t+1)) - w(t, t, y(t, t+1)) \Big] \Big\} \end{aligned} \tag{66}$$

Define a function f as

$$f_t(\alpha, \beta) = w(t - \beta, t - \beta, y(t - \beta, t + \alpha)) \tag{67}$$

so that (66) becomes

$$M - L = \sum_{t=1}^T \{ [f_t(0, t-1) - f_t(0, 0)] - [f_t(1, t-1) - f_t(1, 0)] \} \tag{68}$$

From (68), it is clear that, as $T > 1$, each bracketed term in the summation will be positive with some strictly positive if

$$\frac{\partial^2 f_t}{\partial\alpha\partial\beta} < 0 \tag{69}$$

which, using (67), corresponds to

$$d = -y_2\ (w_{23} + w_{13} + w_{33}\ y_1) - w_3\ y_{12} < 0 \tag{70}$$

with each derivative on the right-hand-side of (70) being evaluated at $(t-\beta, t-\beta, y)$ where y is evaluated at $(t-\beta, t+\alpha)$.

To investigate the actual sign of d, we must investigate the function $y(t,s)$ which is defined by (4). Differentiating (4) and denoting the evaluation of each derivative w_{ij} at $(t-\beta, t+\alpha, y(t-\beta, t+\alpha))$ with $\hat{w}_{ij}$ gives

$$y_2 = -\left(\frac{\hat{w}_{23}}{\hat{w}_{33}}\right) \tag{71}$$

$$y_1 = -\left(\frac{\hat{w}_{13}}{\hat{w}_{33}}\right) \tag{72}$$

$$y_{12} = -\frac{1}{\hat{w}_{33}}\left[\hat{w}_{312} - \frac{\hat{w}_{323}\hat{w}_{13}}{\hat{w}_{33}} - \frac{\hat{w}_{133}\hat{w}_{23}}{\hat{w}_{33}} + \frac{\hat{w}_{333}\hat{w}_{23}\hat{w}_{13}}{\hat{w}_{33}^2}\right] \tag{73}$$

Using (71), (72) and (73) in (70) gives

$$\begin{aligned} d = \frac{\hat{w}_{23}}{\hat{w}_{33}}\left[w_{23} + w_{13} - \frac{w_{33}\hat{w}_{13}}{\hat{w}_{33}}\right] + \\ + \frac{w_3}{\hat{w}_{33}}\left[\hat{w}_{312} - \frac{\hat{w}_{233}\hat{w}_{13}}{\hat{w}_{33}} - \frac{\hat{w}_{133}\hat{w}_{23}}{\hat{w}_{33}} + \frac{\hat{w}_{333}\hat{w}_{23}\hat{w}_{13}}{\hat{w}_{33}^2}\right] \end{aligned}$$

Taking the first bracketed term, the responsive complementarity assumption, (2) above, gives

$$\frac{w_{13}}{w_{33}} > \frac{\hat{w}_{13}}{\hat{w}_{33}} \tag{74}$$

so the first term is negative (recall that $w_{23} < 0$ and $w_{33} < 0$). Taking the second bracketed term, (2) again implies that

$$\hat{w}_{312} - \hat{w}_{233} \frac{\hat{w}_{13}}{\hat{w}_{33}} > 0 \tag{75}$$

and

$$\hat{w}_{133} - \hat{w}_{333} \frac{\hat{w}_{13}}{\hat{w}_{33}} < 0 \tag{76}$$

so that the term in brackets is positive and, as w is evaluated at a point of under-investment, we have $w_3 > 0$ which together with $\hat{w}_{33} < 0$ ensures that the second term in (74) is also negative. Thus d is negative: every term in the summation of (66) is positive and so $M > L$: the overall efficiency loss in the market is less than that which is possible by the under-investment of a single firm. ■

The intuition of Proposition 5 can be described as follows. As a result of the Bertrand competition game firms have incentive to invest in match specific investments with the purpose of improving their outside option: the maximum willingness to pay of the immediate competitor for the worker they match with. This implies that the under-investment of each firm is relatively small. The total inefficiency is then obtained by aggregating these relatively small under-investments. Given the decreasing returns to investment and the assumptions on how optimal firms' investments change across different matches, the sum of the loss in surplus generated by these almost optimal investments is clearly dominated by the loss in surplus generated by the unique under-investment of the best firm matched with the worst worker. Indeed, the firm's investment choice in the latter case is very far from the optimal level (returns from a marginal increase of investment are very high).

6. Concluding Remarks

When both sides to a market can undertake match specific investments Bertrand competition between these sides (workers and firms) for matches may help solve the hold-up problems generated by the absence of fully contingent contracts. In this paper, we have shown two results, quite different in their nature.

When workers choose investments that precede Bertrand competition then the workers' investment choices are constrained efficient. However, inefficiencies may arise that take the form of multiple equilibria. One of these equilibria is leads to efficient matches. However there may exist inefficient equilibria characterized by the difference between the order of workers' innate abilities and the order of their derived qualities.

If instead firms choose investments that precede the Bertrand competition game a different type of inefficiencies may arise. The equilibrium of the Bertrand competition game is unique and efficient: the order of firms' qualities coincide with the order of their innate abilities. However, firms choose an inefficient level of investment given the equilibrium match they are involved in. In this case, however, we are able to show that the aggregate inefficiency due to firms' under-investments is low in the sense that is bounded above by the inefficiency that would be induced by the sole under-investment of the best firm matched with the worst worker. In other words firms' investment choices are *near efficient*.

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