Sunspot Fluctuations: a Way out of a Development Trap?

Sergey Slobodyan[1]
Department of Economics, Washington University in St. Louis

First version 06/20/99
This version 10/06/1999

[1]sergey@wueconc.wustl.edu. +1 (314) 935-7138 *For helpful comments we thank William Barnett, Gaetano Antinolfi, James Bullard, John Duffy, Heinz Schaettler, and participants of the Computing in Economics and Finance conference in Boston, MA, June 1999*

Abstract

It is well known that in dynamic general equilibrium economic models equilibrium may be indeterminate. Indeterminacy means that there is a continuum of equilibrium trajectories converging to the same steady state. Very often, the same mechanisms that are responsible for indeterminacy, like increasing returns to scale or market imperfections, lead to existence of multiple steady states, and development traps can arise. On the other hand, indeterminacy often allows construction of rational sunspot equilibrium as a randomization over different equilibrium trajectories or equilibria. The purpose of this paper is to study the possibility of "rescuing" an economy from a development trap through sunspot-driven self-fulfilling expectations.

1 Introduction

1.1 Indeterminacy and Poverty Traps

There are several types of models that produce "development traps" or "poverty traps". One group, best represented by (Azariadis and Drazen 1990), relies on "thresholds" to generate poverty traps. In this model, investing a non-zero amount of effort into accumulating human capital can lead to a balanced growth path with unlimited growth of all per capita quantities. Due to the presence of externalities, however, it is not optimal to invest into human capital accumulation until the average stock of it in the economy reaches some threshold value. Any economy that starts below threshold remains there forever. If, due to errors, some human capital is accumulated, it does not depreciate. The time of crossing the threshold is, therefore, a function of the magnitude of errors, but the crossing is inevitable if the magnitude is bounded above zero[1].

Other papers with similar dynamics include (Lee 1996) where financial intermediaries accumulate information about investment opportunities by making loans. In low information equilibrium, nobody lends. The paper proposes credit subsidies or inflow of relatively cheap foreign capital to overcome the trap. In (Ciccone and Matsuyama 1996), insufficient number of intermediate inputs hinders adoption of modern technologies. High start-up costs required to establish the production of necessary inputs mean that reallocating scarce resources from traditional production is inefficient, locking the economy in the poverty trap. It is sometimes possible for a large number of entrepreneurs expecting future growth to enter the specialized inputs markets, escaping from the poverty trap due to self-fulfilling prophecy, but for other parameter value the trap is inescapable. In another application of the same idea, (Burguet and Fernandez-Ruiz 1998) construct a development trap in an economy with publicly provided goods and public capital; sufficiently low world interest rate might be needed for escape.

General characteristics of the papers cited above is the existence of a certain threshold that separates poverty-trap-locked economies from developing ones. For an economy in the trap, there is no way out other than some change in parameters: consistent non-optimal accumulation of human capital, credit subsidy, or supply of external funds at low world interest rates.

The other strand of models with poverty traps has some kind of dynamic coordination failure or pessimistic expectations built in. Examples of such models include (Matsuyama 1991), (Gali and Zilibotti 1995), (Gans 1998), (Baland and Francois 1996). There, non-convexity in production function due to increasing returns, externalities, and/or market power lead to a possibility of multiple steady states. In these models, indeterminacy exists — for given values of stock variables like capital there are different choices of control variables like consumption, work effort, etc., such that a perfect foresight equilibrium trajectory

[1] Authors of (Arifovic, Bullard, and Duffy 1997) use revised version of the model. Instead of errors, it is random mutations forming part of the genetic algorithm learning mechanism that lead to the accumulation of human capital. Eventually, the threshold is passed.

converges to a steady state. Different choices of control variables might imply convergence to different steady states, and initial conditions do not necessarily determine to which steady state the economy converges. One can say that in the above models, the economy might be consigned to a poverty trap by the failure of economic agents to agree on the control variable value leading to the best equilibrium. The distinction between two groups is not strict, though, as majority of models in the second group allows parameter values leading to a threshold type poverty trap.

The major motivation of the current paper is to discuss an additional mechanism of overcoming coordination failures or pessimistic expectations in the models of the second type. As noted above, these models exhibit indeterminacy. There are different types of indeterminacy. One situation is when there are two (or more) saddle path stable steady states, and there are corresponding unique trajectories converging to them. This case is sometimes referred to as global indeterminacy. In this case, pessimistic or optimistic expectations simply select one trajectory out of two or other small number. This happens for some parameter values in (Gali and Zilibotti 1995), for example. On the other hand, it may happen that for one or more steady state the linearization of the law of motion has fewer unstable roots than "free" or control variables. In this case, the stable manifold of the steady state has less dimensions that the number of control variables, and there exists a continuum of values of control variables that put the system onto the stable manifold. Therefore, there exists a continuum of perfect foresight trajectories satisfying all the conditions for being an equilibrium trajectory, including transversality condition. This case is referred to as local indeterminacy, and it is the subject of this paper.

What happens if the system exhibits local indeterminacy? Suppose that we have a decentralized economy. Agents are free to choose initial values of the control variable(s) from some large set. Once the initial conditions are agreed upon and the dynamics of the system unfolds, none of the agents has an incentive to deviate from the optimal trajectory, which depends on the initial conditions[2]. However, the trajectory chosen can be a very bad one - it could include a very low level of, say, work effort, and a low growth rate as the result. Choosing a different initial condition with higher level of work effort could increase the growth rate and provide higher utility to every agent and thus be Pareto improving[3]. Different starting point can even imply convergence to a much better steady state with unbounded growth of all per capita variables,

[2] If agents are small compared to the size of the economy, their deviation will not significantly change variables that are arguments of their decision rules - interest rate and wage rate, for example. Thus, individual deviation from the optimal trajectory will reduce agent's payoff.

[3] Note that in the presence of increasing returns and/or externalities, the initial trajectory not necessary was Pareto optimal. In the process of solving such models, one usually assumes that every agents takes the current level of externality as exogenously given; every agent then faces a convex production function, and this decision problem is easily solved. To support increasing return, one usually assumes some degree of monopolistic competition. In any case, every agent makes decision under incomplete information and/or some market failures. Therefore, the solution is not required to be Pareto optimal to begin with.

as in endogenous growth models. A classical case of coordination failure can exist in situations with local indeterminacy of the steady state.

Imagine the situation where a low growth state is locally indeterminate. The decentralized economy develops along one of the trajectories leading to the low growth state, that is, the economy is in the development trap. Assume that there exists a high growth steady state which can also be locally indeterminate or saddle path stable. In any case, agents need some device to help them coordinate on a trajectory converging to the high growth steady state.

1.2 Sunspots as a Coordinating Mechanism

"Sunspot equilibria" are "rational expectations equilibria in which purely extrinsic uncertainty affects equilibrium prices and allocations" (Woodford 1990). "Purely extrinsic uncertainty" denotes some random variable which has no effect on preferences, endowments, or production possibilities. If this random variable and the resulting allocations and prices are stationary, one speaks about stationary sunspot equilibria, or SSE. In discrete time, one of the ways in which SSE are constructed is the randomization between different non-sunspot equlibria; alternatively, SSE can be a randomization over different trajectories converging to single non-sunspot equilibrium. This procedure can be performed when there exists indeterminate non-sunspot steady state. Indeed, in a simple OLG economy with constant supply of money (Azariadis 1981), a necessary condition for the existence of a particular kind of the SSE is exactly the condition for the indeterminacy of the non-sunspot rational expectations equilibrium (Woodford 1990). This connection between indeterminacy of a rational expectations equilibrium and the existence of some SSE (known as "Woodford's Conjecture") was established for a broad class of discrete time models, for example in (Woodford 1986), (Grandmont 1986), (Spear, Srivastava, and Woodford 1990).

Existence of sunspots is by no means limited to OLG or OLG-like discrete time models. In (Spear 1991), existence of sunspot equilibria in a pure capital accumulation model where production is subject to externality was shown. Switching to continuous time models allows complete understanding of the model's global dynamics, especially when the model reduces to a two-dimensional system of differential equations. In (Drugeon and Wigniolle 1996) a continuous-time endogenous growth model was studied. It was shown that when a balanced growth path is locally stable, sunspot equilibrium with a Poisson process as a sunspot variable exists. Finally, (Shigoka 1994) constructs a continuous time SSE in a variety of growth models (including the one used here), where a sunspot variable is a continuous time Markov process with finitely many states. Woodford's Conjecture holds in all three cases.

Stability under the equilibrium learning dynamics was proposed in (Lucas 1986) as a criterion in deciding which of the many equilibria in the OLG should be considered as more likely to occur . Lucas's conjecture was that only limited number of equilibria, and in particular locally determinate steady states, will survive such a test. If this conjecture were always true, sunspot equilibria could be considered esoteric theoretical constructs having no practical importance.

Using a simple adaptive learning rule, (Duffy 1994) has shown that indeterminate monetary steady state can be selected over the determinate one in an OLG economy with fiat money, thus rejecting the Lucas's conjecture. Furthermore, as was shown in (Woodford 1990), in a particular case when the sunspot variable can take on only two values and a monetary steady state is indeterminate, a particular learning scheme converges to the SSE with probability 1[4]. If the sunspot variable can take more than 2 values, it is unknown to which equilibrium the learning scheme will converge; but the probability that it will converge to the monetary steady state is still 0 when it is indeterminate[5]. Indeterminate and SSE equilibria are more than theoretical curiosity; one can observe them[6].

In this paper, we postulate the existence of SSE in a continuous-time model in which the sunspot variable is a sample-path continuous stochastic process[7]. Production technology in the model is subject to externality. It is also postulated that the learning mechanism like that described in (Woodford 1990) has taken place and has converged to a sunspot equilibrium. Agents simply add the sunspot variable to their optimal decision, and this is the SSE[8]. As a result, instead of simply moving along a particular trajectory, and, according to the assumptions about the agents being uninformed about the nature of the externality, choosing actions based on incomplete information about the state space outside that trajectory, agents coordinate on the sunspot and get to explore new regions of the state space.

Suppose the economy starts in the development trap. In the model used here, it means that consumption (and the work effort) are chosen to be too low because of the pessimistic expectations of the future wages and interest rates. It is possible to select a level of initial consumption which will push the system out of the trap and into the region of attraction of the positive steady state. However, no individual agent has an incentive to experiment, and everyone is coordinating on a trajectory leading to the origin. This coordination failure could be fixed if agents could form expectations corresponding to a trajectory converging to the positive steady state. Agents are unaware of existence of such a trajectory because the externality is assumed to be unknown. If a sunspot variable, modeled as a Wiener process, is included into the model, agents could take it into account when making their decisions. Coordinating on a sunspot white noise allows exploring new regions of the state space and can eventually

[4] The learning scheme studied in the paper is the one widely used in adaptive control - a "stochastic approximation" algorithm of (Robbins and Monro 1951).

[5] This passive adaptation approach was studied also in (Evans and Honkapohja 1994), (Evans and Honkapohja 1995), (Marcet and Sargent 1989) and others. There are different approaches to the problem, in particular active cognition (Evans and Ramey 1995). Convergence of the learning process to the rational expectations equilibrium was shown in a wide variety of models.

[6] Extracting information on the belief shocks from the financial markets data, (Salyer and Sheffrin 1998) show that model with self-fulfilling beliefs has incremental predictive power for key US economic time series.

[7] Taking into account (Shigoka 1994), this assumption does not seem to be too overstretched.

[8] More detailed description of the construction of a sunspot equilibrium is given in Section 4.

move the trajectory of the system out of the trap. As soon as the economy leaves the trap, agents become aware of the existence of a new non-stochastic steady state. It is assumed that in this case a regime change takes place and the agents stop taking sunspot variable into account. Therefore the further dynamics of the system reduces to convergence to the positive steady state. This will happen if zero steady state is stochastically unstable under sunspot fluctuations or an initial condition lies outside of the region of stochastic stability that might not coincide with the development trap of the deterministic system. In case when the economy eventually leaves the trap, it is possible to calculate expected first exit times from the region of attraction of a zero steady state depending on the initial conditions and magnitude of the sunspot process.

The rest of the paper is organized as follows. In Section 2, a brief summary of the model described in (Benhabib and Farmer 1994) is given. Construction of the phase portrait of the model is performed in Section 3 and existence of the poverty trap is proven. Properties of the model subject to sunspot fluctuations, in particular stochastic stability of the development trap, are studied in Section 4. Section 5 provides numerical estimates of escape probabilities and times, and Section 6 concludes.

2 The Model

As a basis for analysis, I use (Benhabib and Farmer 1994). This deterministic continuous-time model with infinitely lived agents is characterized by increasing social returns to scale due to externality in the production function of which the agents are assumed to be unaware. There are two steady states. One has zero capital and zero consumption (the origin), and positive levels of both capital and consumption characterize the other one. For some parameter values, both steady states are indeterminate, and the whole state space is separated into two regions of attraction of the steady states. The region of attraction of the origin is a development trap[9].

The economy consists of a large number of identical consumers seeking to maximize

$$\int_0^\infty \left(\frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1-\chi}}{1-\chi}\right)e^{-\rho t}dt$$

subject to

$$\dot{K} = (r-\delta)K + wN - C$$

[9] There are models with indeterminate interior poverty traps, (Gali 1994) being but one of them. For expositional clarity, the author has chosen the model reducing to the most simple mathematical form possible. Further work will focus on more realistic models.

where C is consumption, K capital, N work effort, r interest rate, and w the wage rate. There are a large number of identical firms with the production function

$$Y = K^a N^b \overline{K}^{\alpha - a} \overline{N}^{\beta - b} \tag{1}$$

where $a + b = 1$, $\alpha > a$, $\beta > b$, and $\overline{K}$ and $\overline{N}$ are economywide averages of K and N per firm, which are taken as given by every individual firm. From the profit maximization, the interest rate and the wage rate are given by

$$\begin{aligned} wN &= bY \\ rK &= aY \end{aligned} \tag{2}$$

Identical consumers take trajectories of wage and interest rates as given and solve their maximization problem. In equilibrium, all firms employ the same amount of labor and capital, and thus $K = \overline{K}$, $N = \overline{N}$. In the rational expectations equilibrium, consumers know the correct trajectories of r and w. Solving the problem and switching to logs, one gets the following system of equations:

$$\begin{aligned} \dot{c} &= [\frac{a}{\sigma} \exp(w - vk + uc) - \frac{\delta + \rho}{\sigma}] \\ \dot{k} &= [\exp(w - vk + uc) - \exp(c - k) - \delta] \end{aligned} \tag{3}$$

where w, v, and u are some functions of parameters that in particular depend on α and β, and are assumed to be unknown to every decision maker in the model. Under some parameter values, including the ones which are deemed "plausible" by the authors of (Benhabib and Farmer 1994), the steady state of this model is indeterminate - it has 2 stable roots. For the same parameter values, the (minus infinity, minus infinity) – origin in original (C, K) space - is also stable. The entire space is divided into two regions of attraction – one for the positive steady state, and another for the origin. The latter region will also be called "development trap" in the sequel. More detailed derivations and discussions are provided in Section 3.

3 Deterministic Dynamics of the Model.

3.1 Solving the Problem

Hamiltonian for the problem is

$$H = (\frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1-\chi}}{1-\chi})e^{-\rho t} + \lambda((r - \delta)K + wN - C) \tag{4}$$

First order conditions:

$$
\begin{aligned}
\frac{\partial H}{\partial C} &= 0 \Leftrightarrow C^{-\sigma}e^{-\rho t} = \lambda \\
\frac{\partial H}{\partial N} &= 0 \Leftrightarrow N^{-\chi}e^{-\rho t} = w\lambda \\
\frac{\partial H}{\partial K} &= -\dot{\lambda} \Leftrightarrow \dot{\lambda} = e^{-\rho t}(-\rho C^{-\sigma} - \sigma \dot{C} C^{-\sigma-1}) \qquad (5) \\
\lim_{t\to\infty} \lambda K &= 0 \\
\dot{K} &= (r-\delta)K + wN - C
\end{aligned}
$$

Plugging first FOC into the third and then using **(1)** and **(2)** to exclude N, and switching to logs (as usual, small letters denote logs), one arrives at

$$
\begin{aligned}
\dot{c} &= [\frac{a}{\sigma}\exp(w - vk + uc) - \frac{\delta+\rho}{\sigma}] \qquad (6) \\
\dot{k} &= [\exp(w - vk + uc) - \exp(c-k) - \delta]
\end{aligned}
$$

where

$$
\begin{aligned}
w &= -\frac{\beta\log(b)}{\beta+\chi-1} \\
v &= \frac{\beta-(1-\alpha)(1-\chi)}{\beta+\chi-1} \qquad (7) \\
u &= \frac{\sigma\beta}{\beta+\chi-1}
\end{aligned}
$$

The system does not look nice. Definitely, we do not have global Lipschitz conditions satisfied (for a system of equations $\dot{X} = b(t,X)$ we should have $|b(t,x)-b(t,y)| \le B|x-y|$ for any t and x,y). It is Lipschitz in every bounded domain in (c,k) space.

The system **(6)** is extremely hard to analyze. It is therefore useful to change the coordinates to

$$
\begin{aligned}
x &= \exp(w - vk + uc) \qquad (8) \\
y &= \exp(c-k)
\end{aligned}
$$

After this change of variables, our system transforms into

$$
\begin{aligned}
\dot{x} &= x[(\frac{a}{\sigma}u - v)x + vy + v\delta - u\frac{\delta+\rho}{\sigma}] \qquad (9) \\
\dot{y} &= y[((\frac{a}{\sigma}-1)x + y + \delta - \frac{\delta+\rho}{\sigma})]
\end{aligned}
$$

By construction, x and y are nonnegative, therefore only the first quadrant of the (x,y) space should be considered.

3.2 Stability of Equilibria

The positive steady state of **(9)** is $\mathbf{A}=(x^*,y^*)=(\frac{\delta+\rho}{a},\frac{\delta+\rho}{a}-\delta)$. Linearization of (**9**) around this steady state produces

$$\mathbf{J}^* = \begin{bmatrix} x^*(\frac{a}{\sigma}u-v) & x^*v \\ y^*(\frac{a}{\sigma}-1) & y^* \end{bmatrix}$$

$$Det\ |\mathbf{J}^*| = x^*y^*(u-v), \quad Tr(\mathbf{J}^*) = x^*(\frac{a}{\sigma}u-v)+y^*$$

To get indeterminacy, we need 2 stable roots, which means $Det\ |\mathbf{J}^*|>0$, $Tr(\mathbf{J}^*)<0$.

Recalling definitions of u and v and simplifying, one gets:

$$u-v=\frac{(\sigma-1)\beta+(1-\alpha)(1-\chi)}{\beta+\chi-1} \tag{10}$$

Following the original paper, where $\alpha<1$, $\chi<0$, and assuming σ not too far away from 1 ($\sigma=1$ means utility logarithmic in consumption), necessary condition for indeterminacy is still $\beta+\chi-1>0$. The trace is

$$\frac{\delta+\rho}{a}(\frac{a}{\sigma}u-v+1)-\delta=\frac{\delta+\rho}{a}\frac{a\beta-\alpha(1-\chi)}{\beta+\chi-1}-\delta= \tag{11}$$
$$\frac{\delta+\rho}{a}\frac{a(\beta+\chi-1)-(\alpha-a)(1-\chi)}{\beta+\chi-1}-\delta=\rho-\frac{\delta+\rho}{a}\frac{(\alpha-a)(1-\chi)}{\beta+\chi-1}$$

If there is no capital externality ($\alpha=a$), trace equals ρ and is positive. The lowest α that makes trace negative is given by $\alpha=a(1+\frac{\rho}{\delta+\rho}\frac{\beta+\chi-1}{1-\chi})$. Combining all the conditions together, we see that if

$$\begin{array}{rcl} \beta+\chi-1 & > & 0; \\ a(1+\dfrac{\rho}{\delta+\rho}\dfrac{\beta+\chi-1}{1-\chi}) & < & \alpha<1; \\ (\sigma-1)\beta+(1-\alpha)(1-\chi) & > & 0, \end{array} \tag{12}$$

then the positive equilibrium is indeterminate. From now on, only parameter values satisfying conditions **(12)** will be considered.

There are other possible equilibria of **(9)**. Those are

$$\mathbf{B} = (0,0), \mathbf{C} = (0, \frac{\delta+\rho}{\sigma} - \delta), \text{ and } \mathbf{D} = (\frac{u\frac{\delta+\rho}{\sigma} - v\delta}{\frac{a}{\sigma}u - v}, 0)$$

For σ not too large, $\frac{\delta+\rho}{\sigma} - \delta$ is positive. In the expression for abscissa of **D**, denominator is given by

$$\frac{a}{\sigma}u - v = \frac{a\beta - \beta + (1-\alpha)(1-\chi)}{\beta+\chi-1} = \frac{a\beta - \alpha(1-\chi)}{\beta+\chi-1} - 1 = \alpha - 1 - \frac{\alpha - a}{\beta+\chi-1} \tag{13}$$

which is always negative if conditions **(12)** are true. For the numerator, one gets

$$u\frac{\delta+\rho}{\sigma} - v\delta = \frac{(\delta+\rho)\beta - \delta(\beta - (1-\alpha)(1-\chi))}{\beta+\chi-1} = \frac{\rho\beta + \delta(1-\alpha)(1-\chi)}{\beta+\chi-1} \tag{14}$$

which is always positive given **(12)**. Therefore, the third equilibrium lies in the second quadrant and does not interest us[10].

Linearizing **(9)** around the origin, one gets the following Jacobian:

$$\mathbf{J} = \begin{bmatrix} v\delta - u\frac{\delta+\rho}{\sigma} & 0 \\ 0 & \delta - \frac{\delta+\rho}{\sigma} \end{bmatrix} \tag{15}$$

The first non-zero element was estimated in **(14)** and is always negative, while the second is negative for σ not too large (and negative for $\sigma = 1$). Therefore, the origin is also stable in **(9)**. Finally, for the steady state **C**, one gets

$$\mathbf{J} = \begin{bmatrix} \frac{\delta+\rho}{\sigma}(v-u) & 0 \\ (\frac{a}{\sigma} - 1)(\frac{\delta+\rho}{\sigma} - \delta) & \frac{\delta+\rho}{\sigma} - \delta \end{bmatrix}$$

Here, the (2,2) element of **J** is positive, and taking into account **(10)** we conclude that the (1,1) element of **J** is negative. Therefore, **C** is a saddle.

[10] Steady states **B** and **C** both represent trajectories diverging to $(-\infty, -\infty)$ in the (c,k) space. Diverging trajectories have different asymptotic behavior. The change of variables collapses infinity points from the lower half of the (c,k) space onto the vertical half-axis in the (x,y) space. Trajectories with different asymptotic behavior at minus infinity are mapped into different points on the axis.

3.3 Dulac Criterion and Limit Cycles

To characterize the global dynamics of the system, we have to know whether limit cycles exist. The Dulac criterion states that if for the analytical two-dimensional system

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned}$$

in a simply connected region G there exists a continuously differentiable function $B(x,y)$, such that $\frac{\partial(PB)}{\partial x} + \frac{\partial(QB)}{\partial y}$ does not change sign in G, then there are no simple closed curves in G which are unions of paths of the system[11]. In particular, there are no limit cycles (Andronov, Leontovich, Gordon, and Maier 1973). For a system

$$\begin{aligned} \dot{x} &= x(a_1x + b_1y + c_1) \\ \dot{y} &= y(a_2x + b_2y + c_2) \end{aligned} \tag{16}$$

the Dulac function is $B(x,y) = x^{k-1}y^{h-1}$, where $k = \frac{b_2(a_2-a_1)}{\Delta}$, $h = \frac{a_1(b_1-b_2)}{\Delta}$, $\Delta = a_1b_2 - a_2b_1 \neq 0$. Then

$$\frac{\partial(PB)}{\partial x} + \frac{\partial(QB)}{\partial y} = (\frac{a_1c_2(b_1-b_2)}{\Delta} + \frac{b_2c_1(a_2-a_1)}{\Delta})x^{k-1}y^{h-1} \tag{17}$$

When $\xi = a_1c_2(b_1-b_2)+b_2c_1(a_2-a_1) \neq 0$, this function vanishes only along the integral curves $x = 0$ and $y = 0$. It does not change sign in the interior of any of four quadrants. Also, it can be shown that there can be no closed contours which are unions of paths in this case. After some algebraic transformations, it can be shown that the condition on ξ amounts to $a_1x^* + b_2y^* = Tr(\mathbf{J}^*) = 0$, where (x^*,y^*) denotes the non-trivial equilibrium. When $Tr(\mathbf{J}^*) = 0$, all trajectories of the system are closed orbits.

It is possible to have $Det\ |\mathbf{J}^*| > 0$ and $Tr(\mathbf{J}^*) = 0$ with two complex conjugate eigenvalues having zero real part. However, the system does not undergo Hopf bifurcation because there is no limit cycles when $Tr(\mathbf{J}^*) \neq 0$.

3.4 Global Behavior

The phase portrait of (**9**) is presented in Figure 1. The whole first quadrant is divided into 2 regions of attraction. The only trajectories that diverge to infinity are those that start on the vertical axis above **C**. The stable manifold of **C** serves as a separatrix between regions of attraction. In logged consumption and capital, the phase portrait is given by Figure 2. All trajectories that start

[11] Note that Bendixson's criterion is a special case of the Dulac's with $B(x,y) = 1$.

above the transformed stable manifold of **C** converge to the positive steady state corresponding to **A**. Trajectories with the initial conditions below it diverge to minus infinity. In the original (C,K) variables (Figure 3), the phase portrait looks very similar to that of (**9**), the only difference being that now the separatrix of the two regions of attraction starts at the origin rather than on the vertical axis. The stable manifold approaches the origin as a ray of constant positive tangent. Any other trajectory of the system which approaches the origin behaves as $C \sim K\exp(-\rho t)$. The distance between the stable manifold and any such trajectory expressed as a percentage of actual consumption level grows exponentially with time.

To obtain a point on the vertical axis $\{(x,y) : x = 0, y > 0\}$ of Figure 1, the following should be true:

$uc - vk = u(c-k) + (u-v)k \to -\infty, \;\; c - k = const$. This means that $k \to -\infty, c \to -\infty$, but $c-k$ is finite. This corresponds to going to the origin in the non-logged (C,K) space along a ray with finite tangent. In the (c,k) space any trajectory asymptotically linearly diverging to minus infinity satisfies the condition. A point on the horizontal axis $\{(x,y) : x > 0, y = 0\}$ is obtained when $uc - vk = u(c-k) + (u-v)k = const$, $c - k \to -\infty$. This is possible only when $k \to \infty$, c arbitrary, but c goes to infinity slower than k. In the (C,K) space, this corresponds to C going to infinity not faster than $log(K)$ or converging to a nonzero constant. What is the origin in the (x,y) space? Writing the change of coordinates (**8**) as $x = \left(\frac{C}{K}\right)^v C^{u-v}$, $y = \frac{C}{K}$, we see that the origin corresponds to $C < \infty$, $\frac{C}{K} = 0$. Any trajectory in the (C,K) space such that $C = o(K)$, $C \to 0$ corresponds to a trajectory converging to the origin in the (x,y) space. The trivial solution of Eq.(**9**) corresponds to a poverty trap, or imploding economy.

4 Stochastic Dynamics

4.1 Constructing SDE

From the previous Section, we know that the system (**9**) has two stable steady states, that there are no limit cycles, and no trajectory starting in the interior of the first quadrant escapes to infinity in the (x,y) space. A trajectory of (**9**) that starts on the vertical axis above **C** escapes to infinity; however, in the (C,K) space this corresponds to a trajectory going to the origin with ever increasing slope. Now we introduce a stochastic process into the system - the sunspot process. A key behavioral assumption is that agents observe a sunspot variable, which is a Wiener process. They simply add a "derivative" of the process to their decision rule. To justify such an approach, one has to remember that Itô stochastic differential equation can be obtained as a limit in probability of difference equations if the driving noise is Markov process with independent increments[12]. Existence of SSE of this form was shown in the current model by

[12] Construction of the SDE is very similar to that reported in (Shigoka 1994). Introducing a sunspot disturbance in this way has a simple justification. Adding $\widetilde{\sigma} dW_t$ to the equation for

(Shigoka 1994) for a continuous time Markov process with finitely many states. A Wiener process is a continuous time Markov process with infinitely many states.

We start with a deterministic differential equation (**6**) and formally add a "differential" of the Wiener process to the RHS of the equation for consumption. The result is

$$\begin{aligned} dc &= [\frac{a}{\sigma}\exp(w - vk + uc) - \frac{\delta+\rho}{\sigma}]dt + \widetilde{\sigma} dW_t \\ dk &= [\exp(w - vk + uc) + \exp(c-k) - \delta]dt \end{aligned} \tag{18}$$

This system of stochastic differential equations (SDE) does not satisfy linear growth condition or global Lipschitz conditions; however, for $\widetilde{\sigma}$ growing not very fast with x, y (and not exploding as they go to zero), those are satisfied in every bounded cylinder, and we can construct a sequence of processes that are solutions to the system **(18)** in cylinders and then construct a Markov process as a limit of those as n goes to infinity. The resulting process will be the solution of **(18)** everywhere**.** Doing the same change of variables as in the previous Section and applying the Itô theorem, one arrives at the following system of SDE:

$$\begin{aligned} dx &= [x((\frac{a}{\sigma}u - v)x + vy + v\delta - u\frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 u^2)]dt + ux\widetilde{\sigma} dW_t \\ dy &= [y((\frac{a}{\sigma} - 1)x + y + \delta - \frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2)]dt + y\widetilde{\sigma} dW_t \end{aligned} \tag{19}$$

4.2 Stochastic Stability of the Origin

The definition of stochastic stability used here comes from (Khasminskii 1980).

Definition 1 *The solution $x(t) \equiv 0$ is said to be asymptotically stable in probability if, for every $\varepsilon > 0$ and every $t > t_0$,*
$\lim\limits_{x_0 \to 0} \mathbf{P}\{\sup\limits_{t>t_0} |x(t, \omega, t_0, x_0)| > \varepsilon\} = 0$ *and moreover* $\lim\limits_{x_0 \to 0} \mathbf{P}\{\lim\limits_{t \to \infty} x(t, \omega, t_0, x_0) = 0\} = 1$.

In plain English, according to the definition, the origin is asymptotically stochastically stable if we can choose the $\delta-$neighborhood of the origin such that all trajectories starting in it will remain inside a given $\epsilon-$neighborhood of the origin with probability going to 1 as δ goes to 0. This definition is analogous to the definition of stability in deterministic case. Moreover, we want all such trajectories to converge to the origin as δ goes to 0, which has a close counterpart

$log(C)$ is approximately equivalent to adding $C\widetilde{\sigma} dW_t$ to the equation for C. C is the share of the net present wealth (future wages and interest income) agents choose to consume at time t. If agents consider increments of the sunspot variable as fluctuations in their present discounted wealth, $C\widetilde{\sigma} dW_t$ is simply an adjustment of this share due to the fact that perceived wealth has changed.

in the asymptotic stability in the deterministic case. To prove the asymptotic stability of the origin in **(19)** we will use the stability in first approximation.

A prominent role in the study of stochastic stability belongs to the operator L, a differential generator of the Markov process. Suppose that we are given a system

$$dX_t = b(t, X)X dt + \sigma(t, X)X dW_t \tag{20}$$

where $X \in R^N$. Suppose further that there exists a function $V(t, X)$ twice continuously differentiable with respect to X and continuously differentiable with respect to t. Then

$$LV(s,x) = \frac{\partial V(s,x)}{\partial s} + \sum_{i=1}^{N} b_i(s,x)\frac{\partial V(s,x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{N} \sigma_i(s,x)\sigma_j(s,x)\frac{\partial^2 V(s,x)}{\partial x_i \partial x_j}$$

LV plays the role of a full time derivative of the Lyapunov function, $\dot{V}(t, X)$, for stochastic differential equations.

Consider a linear system of SDEs

$$dX_t = BX dt + \sigma X dW_t \tag{21}$$

Suppose that a nonlinear system of SDEs has coefficients $B(X)$ and $\sigma(X)$ that are "close" to B and σ. Can we deduce the stability or instability of the origin for the nonlinear system from stability of the origin for **(21)**?

Theorem 1 *(Khasminskii 1980, Theorem 7.1.1) If the linear system with constant coefficients **(21)** is asymptotically stable in probability, and the coefficients of the system **(20)** satisfy an inequality*

$$|b(t, X) - B| + |\sigma(t, X) - \sigma| < \gamma|x| \tag{22}$$

in a sufficiently small neighborhood of the point x=0 and with sufficiently small constant γ, then the solution X=0 of the nonlinear system is asymptotically stable in probability.

Remark 1 *In the proof of the Theorem 7.1.1, Khasminskii actually shows that if the origin in **(21)** is exponentially p-stable for sufficiently small p and **(22)** holds, then the Theorem is true. For linear systems with constant coefficients, asymptotic stability in probability implies exponential p-stability for sufficiently small p (Theorem 6.4.1 in Khasminskii 1980).*

Definition 2 *Exponential p-stability (Khasminskii 1980). The solution X≡0 of the system **(20)** is said to be exponentially p-stable for $t \geq 0$, if for some positive constants A and α*

$$\mathbf{E}|\mathbf{x}(\mathbf{t}, \boldsymbol{\omega}, \mathbf{x}_0, t_0)|^p \leq A|x_0|^p \exp\{-\alpha(t - t_0)\}$$

Theorem 2 *(Khasminskii 1980, Theorem 6.3.1) The solution $X \equiv 0$ of the linear system with constant coefficients is exponentially p-stable if and only if there exists a function V(t,x), homogeneous of degree p in x, such that for some constants $k_i > 0$*

$$\begin{aligned} k_1|x|^p &\leq V(t,x) \leq k_2|x|^p;\ LV(t,x) \leq -k_3|x|^p; \\ |\frac{\partial V}{\partial x_i}| &\leq k_4|x|^{p-1};\ |\frac{\partial^2 V}{\partial x_i \partial x_j}| \leq k_4|x|^{p-2}; \end{aligned} \tag{23}$$

Applying the Theorem 1, we can see that stability of the origin in **(19)** depends on the stability of the origin in the following linear system:

$$\begin{aligned} dx &= x(v\delta - u\frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 u^2)dt + ux\widetilde{\sigma}dW_t \\ dy &= y(\delta - \frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2)dt + y\widetilde{\sigma}dW_t \end{aligned} \tag{24}$$

To establish stability of **(24)**, set $V(t,x) = |x|^p + |y|^p$. Then

$$\begin{aligned} LV &= p|x|^p\left[v\delta - u\frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 u^2 + \frac{1}{2}\widetilde{\sigma}^2 u^2(p-1)\right] + \\ &\quad + p|y|^p\left[\delta - \frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 + \frac{1}{2}\widetilde{\sigma}^2(p-1)\right] \end{aligned}$$

or

$$LV = p|x|^p\left[v\delta - u\frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 u^2 p\right] + p|y|^p\left[\delta - \frac{\delta+\rho}{\sigma} + \frac{1}{2}\widetilde{\sigma}^2 p\right] \tag{25}$$

A quick look at **(15)** assures one that $LV(t,x) \leq -k_3|x|^p$ for p small enough. Therefore, by Theorem **2** the solution $X \equiv 0$ of the system **(24)** is exponentially $p - stable$, and by Remark **1**, the trivial solution of the system **(19)** is asymptotically stable in probability in a sufficiently small neighborhood of the origin.

The result means that for the economy that started very close to the origin, probability of escape from the trap is low and goes to zero as the initial point approaches the origin. There is no way out if expectations are very pessimistic. The sunspot variable cannot fix expectations if they are too low to begin with. The result should not come as a surprise because of the specification of the process that governs expectations. An addition to the derivative of consumption due to the sunspot variable is proportional to the current level of consumption itself. In the model, low expectations mean low consumption. Therefore, in a pessimistic state the sunspot variable exercises very small influence in absolute terms. As stated previously, for the economy converging to the origin the distance to the boundary of the poverty trap becomes very large as a percentage of the current level of consumption. The influence of the sunspot variable gets smaller as the level of consumption gets small. The only realistic chance of escape comes when the distance to the boundary is not exponentially large and the sunspot influence is not negligible. Both requirements are satisfied when the consumption level is not too low, which means expectations are not too pessimistic.

Now we have to make a distinction between the stability of the origin in the deterministic system **(9)** and the stochastic system **(19)**. The basin of attraction of the origin in the former system is a set in (x, y) space that for some values of y is unbounded in x. The solution of **(19)** is guaranteed to converge to the origin only as initial condition converges to zero. For any non-zero initial condition, there is a positive probability that the trajectory will not converge to the origin. A solution of **(19)** that started outside of the "sufficiently small neighborhood" of the origin is not guaranteed to converge to it or to remain near it at all. Therefore, following a sunspot variable leaves the possibility that the economy will escape poverty trap.

5 How good is a chance?

To understand how important sunspot-driven fluctuations could be for the economy's escape from the poverty trap, some numerical simulations of stochastic differential equation **(18)** were performed. First, to obtain a "realistic" noise magnitude, batches of 100 trajectories each with different noise intensities starting from the positive steady state **A** of the *deterministic* system were run for 300 time units (years). A noise intensity that resulted in approximately 14% standard deviation of the log consumption was chosen. This number is close to the average reported for several developing countries by (Mendoza 1995). The second step was to calculate the separatrix of the two regions of attraction. This separatrix is the stable manifold of the steady state **C** of the transformed system **(9).** A standard procedure was employed - calculate the eigenvector corresponding to the stable eigenvalue at **C** and run the system of differential equations **(9)** backwards in time from a point close to **C** in the direction of the eigenvector. Matlab5 procedure `ode45` was used to calculate the trajectory. Using the trasformation inverse to **(8),** this trajectory was transformed into (c, k) space in which further simulations were made. The separatrix is the thick solid line in Figures 1-3.

Numerical simulations of SDE are based on a stochastic Taylor expansion. The following brief exposition is taken from (Kloeden, Platen, and Schurz 1994). Suppose we are given a one-dimensional SDE **(20).** An equivalent integral representation is

$$X_t = X_{t_0} + \int_{t_0}^{t} b(X_s)ds + \int_{t_0}^{t} \sigma(X_s)dW_s$$

For any twice continuously differentiable function $f : R \to R$, It$\widehat{o}$'s formula gives

$$\begin{aligned} f(X_t) &= f(X_{t_0}) + \int_{t_0}^{t} (b(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s))ds + \\ &\quad + \int_{t_0}^{t} \sigma(X_s)f'(X_s)dW_s \qquad (26) \\ &= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)ds + \int_{t_0}^{t} L^1 f(X_s)dW_s \end{aligned}$$

where the two operators introduced are

$$\begin{aligned} L^0 f &= bf' + \frac{1}{2}\sigma^2 f'' \\ L^1 f &= \sigma f'. \end{aligned}$$

Now, if one applies It$\widehat{o}$'s formula to the functions $f = b$ and $f = \sigma$ under integral signs in **(26)**, one gets the following

$$\begin{aligned} X_t &= X_{t_0} + b(X_{t_0})\int_{t_0}^{t} ds + \sigma(X_{t_0})\int_{t_0}^{t} dW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z)dzds + \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(X_z)dW_z ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0\sigma(X_z)dzdW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^1\sigma(X_z)dW_z dW_s. \end{aligned}$$

The procedure can be repeated, for example by applying It$\widehat{o}$'s formula to $f = L^1\sigma$ in the above expression, and so on. On every step, the expansion will consist of multiple It$\widehat{o}$ integrals

$$\int_{t_0}^{t} ds, \quad \int_{t_0}^{t} dW_s, \quad \int_{t_0}^{t}\int_{t_0}^{s} dW_z dW_s$$

multiplied by some constants, and the remainder term involving higher-order multiple It$\widehat{o}$ integrals. Multiple integrals can be approximated numerically.

A usual problem in the numerical simulation of SDEs is to generate approximate values of the process X_t at the discretization times inside the interval $[0,T]$. For the uniform discretization $\tau_n = n\Delta$, $n = 1...N$ with the step size $\Delta = \frac{T}{N}$ the simplest approximation will look like

$$Y_{n+1} = Y_n + b(Y_n)\Delta_n + \sigma(Y_n)\Delta W_n,\ Y_0 = X_0. \tag{27}$$

The random variables ΔW_n are Wiener process increments, they are independently Gaussian distributed with zero mean and variance Δ.

If a particular approximation satisfies the condition

$$E(\left|X_T - Y_N^{\Delta}\right|) \leq K\Delta^{\gamma}$$

for all sufficiently small time steps Δ and some finite constant K, it is said that the approximation Y^{Δ} converges with strong order γ. For example, the stochastic Euler scheme **(27)** converges with strong order 0.5, while its deterministic counterpart has the order 1.0.

For purposes of the current paper, an explicit strong order 1.5 scheme was used. For a multi-dimensional process X with only one independent Wiener disturbance[13], the formula becomes

$$\begin{aligned} Y_{n+1}^k \;=\;& Y_n^k + b^k\Delta_n + \frac{1}{2}L^0 b^k \Delta_n^2 + \\ &+\sigma^k \Delta W_n + L^0\sigma^k(\Delta W_n \Delta_n - \Delta Z_n) + L^1 b^k \Delta Z_n + \\ &+L^1\sigma^k \frac{1}{2}\left((\Delta W_n)^2 - \Delta_n\right) + L^1 L^1 \sigma^k \frac{1}{2}\left(\frac{1}{3}(\Delta W_n)^2 - \Delta_n\right)\Delta_n. \end{aligned} \tag{28}$$

Here Y^k, $k = 1...K$ is the *k-th* component of the multidimensional vector Y and ΔZ_n is a random variable defined by $\Delta Z_n = \int\limits_{\tau_n}^{\tau_{n+1}} \int\limits_{\tau_n}^{s_2} dW_{s_1} ds_2$. This random variable is normally distributed with mean zero, variance $E\left((\Delta Z_n)^2\right) = \frac{1}{3}\Delta_n^3$, and covariance $E(\Delta Z_n \Delta W_n) = \frac{1}{2}\Delta_n^2$. Two random variables ΔW_n and ΔZ_n can be generated from two independent standard normal variables G_1 and G_2 as $\Delta W_n = \sqrt{\Delta_n}$, $\Delta Z_n = \frac{1}{2}\Delta_n^{3/2}(G_1 + \frac{1}{\sqrt{3}}G_2)$.

The next step is to run batches of 100 trajectories with initial points inside the deterministic poverty trap. The percentage of trajectories crossing the trap boundary is interpreted as a probability that sunspot-driven fluctuations of a given magnitude will lead to the escape from the trap. For the purposes of the simulations, the time interval from 0 to 300 was chosen. All trajectories either

[13] Like the system **(18)**. We have chosen to simulate **(18)** instead of the equivalent system **(19)** because the former has noise intensity independent of state variables. For the approximation scheme chosen, this represented a major simplification.

crossed the boundary or moved very close to the origin in the (C, K) space during this time interval[14].

The basic result of the Section can be stated as follows: for the chosen level of the noise intensity, the probability of escaping the trap is not negligible only when the initial condition is very close to the trap boundary. The initial level of consumption, C, should not be less than 85% of the boundary level in order to see at least a couple escapes in a batch of 100 trajectories. The probability is not very sensitive to the initial level of capital. Figure 4 plots the probability of escape averaged over initial capital level versus the difference between the initial and borderline levels of consumption. As expected, it increases as expectations become more optimistic (the difference becomes smaller). Figure 5 presents similarly averaged mean and median escape times for trajectories that eventually leave the trap. For very optimistic expectations (initial consumption very close to the boundary) absolute majority of escapes happens within the first year. For the few trajectories that escape from pessimistic initial conditions (consumption far from the boundary) the time is much longer, 50 years or more. Given the structure of the SDE (**18**) the results are not surprising. In the (c,k) space, a typical trajectory in the development trap runs almost parallel to the separatrix for a sufficiently long time. Therefore, in the first approximation the escape happens if the stochastic process is able to cover the vertical distance between the initial point and the boundary. The variance of the increment of the Wiener process is linear in elapsed time. Larger distance to the boundary then implies larger expected time before the trajectory hits the boundary. When the expected time becomes too large, the fact that the separatrix and the trajectory are not exactly parallel comes into play. The distance that needs to be covered increases as time increases, and for large initial distances the stochastic process is unable to hit the boundary with large probability.

The answer to the question posted in the beginning of the Section then is: "Not very good". It is possible to miss the target level of consumption (and work effort) and still avoid falling into the poverty trap, but the error should not be large. Expected escape time and the escape probability are inversely related, and the prognosis for chronically trapped economies is not good. Highly probable escape happens very fast, and the imploding economies will probably continue the downward spiral. The outcome is brought about by the sunspot with magnitude proportional to the current level of consumption. A different specification of the sunspot variable might bring more optimistic results in this model. The same sunspot variable can be more effective in models where the economies trapped in poverty do not converge to the origin. Then the magnitude of the sunspot does not converge to zero as time goes to infinity, and the escape

[14] The origin is asymptotically stochastically stable in the system **(19).** Moreover, it can be shown that the trajectories that do not hit the separatrix in a finite time converge to the origin as time goes to infinity. A trajectory was considered as having converged to the origin in the (x,y) space and correspondingly in the (C, K) space if $\log(K)$ fell below 0. For the positive steady state **A**, $\log(K)$=7.32. Initial points for simulation purposes varied from $\log(K)$=4 to $\log(K)$=8. In practice, after approximately 100 years the non-escaped solutions would become numerically indistinguishable from the origin in the (C,K) space.

can be inevitable given enough time.

6 Conclusion

Poverty traps and indeterminacy in macroeconomic models may be caused by the same set of reasons, like externalities or increasing returns to scale. Wooford's conjecture, proven to hold in a broad set of discrete time and continuous time models, allows one to expect the presence of sunspot fluctuations whenever indeterminacy of the steady state is present. However, traditional approach to sunspot fluctuations is strictly local: the sunspot variable is assumed to behave in such a way that the economy subject to self-fulfilling beliefs shocks does not leave the region of the state space where the dynamics without sunspots takes place. This is usually achieved by chosing a random variable with bounded support as the sunspot variable. Considering a continous time model in only two dimensions allows one to describe fully both deterministic and stochastic dynamics of the system. Ability to discuss global properties of the stochastic process allowed us to raise a new question, that of connection between the sunspot driven fluctuations and escape from the poverty trap.

Taking a simple model that exhibits indeterminacy of both the positive steady state and zero steady state we were able to prove that the development trap is asymptotically stochastically stable under the chosen specification of the sunspot variable. The sunspot variable used here has a natural interpretation of a change in perceived present discounted wealth. Therefore the economy that starts with a very low initial capital and very pessimistic expectations of future interest rates and wages gets trapped. However, this analythical result is valid only asymptotitcally and economies starting with finite levels of capital and consumption have nonzero probability of escape. To estimate numerically this probability as a function of initial conditions, we assumed that the economies of several developing countries operated around the positive steady state with business cycle fluctuations caused by the sunspots described in our model. Allowing the sunspots of similar magnitude to act in the economy with initial conditions in the poverty trap, we were able to map the trap for initial conditions providing non-negligible probability of escape. The set of those initial conditions is not very large and is restricted to initial level of consumption within 85% of the level necessary to put the system right on the boundary between the poverty trap and the region of attraction of the positive steady state. At every finite level of the capital stock, there exists a level of consumption (and, accordingly, of the work effort) that withdraws the system from the poverty trap. However, for very low levels of capital the change from "pessimistic" optimal level of consumption to the "optimistic" one may constitute hundreds and thousands percent of the "pessimistic" level. This feature of the model is a direct consequence of the fact that the poverty trap is modeled as a point with zero capital and zero consumption. A model with interior development trap will produce more optimistic results under the same specification of the sunspot variable. This is the subject of future study.

References

Andronov, A. A., E. Leontovich, I. Gordon, and A. Maier (1973): *Qualitative Theory of Second-Order Dynamical Systems*. John Wiley & Sons, New York Toronto.

Arifovic, J., J. Bullard, and J. Duffy (1997): "The Transition from Stagnation to Growth: An Adaptive Learning Approach," *Journal of Economic Growth*, 2(2), 185–209.

Azariadis, C. (1981): "Self-Fulfilling Prophecies," *Journal of Economic Theory*, 25, 380–396.

Azariadis, C., and A. Drazen (1990): "Threshold Externalities in Economic Development," *The Quarterly Journal of Economics*, 105(2), 501–526.

Baland, J.-M., and P. Francois (1996): "Innovation, Monopolies, and the Poverty Trap," *Journal of Development Economics*, 49, 151–178.

Benhabib, J., and R. E. Farmer (1994): "Indeterminacy and Increasing Returns," *Journal of Economic Theory*, 63, 97–112.

Burguet, R., and J. Fernandez-Ruiz (1998): "Growth Through Taxes or Borrowing? A Model of Development Traps with Public Capital," *European Journal of Political Economy*, 14, 327–344.

Ciccone, A., and K. Matsuyama (1996): "Start-Up Costs and Pecuniary Externalities as Barriers to Economic Development," *Journal of Development Economics*, 49, 33–59.

Drugeon, J.-P., and B. Wigniolle (1996): "Continuous-Time Sunspot Equilibria and Dynamics in a Model of Growth," *Journal of Economic Theory*, 69, 24–52.

Duffy, J. (1994): "On Learning and the Nonuniqueness of Equilibrium in an Overlapping Generations Model with Fiat Money," *Journal of Economic Theory*, 64, 541–553.

Evans, G. W., and S. Honkapohja (1994): "Learning, Convergence and Stability with Multiple Rational Expectations Equilibria," *European Economic Review*, 38, 1071–98.

——— (1995): "Local Convergence of Recursive Learning to Steady States and Cycles in Stochastic Nonlinear Models," *Econometrica*, 63, 195–206.

Evans, G. W., and G. Ramey (1995): "Expectation Calculation, Hyperinflation and Currency Collapse," in *The New Macroeconomics: Imperfect Markets and Policy Effectiveness*, ed. by H. D. Dixon, and N. Rankin, pp. 307–336. Cambridge University Press, Cambridge, New York, and Melbourne.

Gali, J. (1994): "Monopolistic Competition, Endogenous Markups, and Growth.," *European Economic Review*, 38, 748–756.

Gali, J., and F. Zilibotti (1995): "Endogenous Growth and Poverty Traps in a Cournotian Model," *Annals D'economie et de Statistique*, 0(37/38), 197–213.

Gans, J. S. (1998): "Time Lags and Indicative Planning in a Dynamic Model of Industrialization," *Journal of the Japanese and International Economies*, 12, 103–130.

Grandmont, J.-M. (1986): "Stabilizing Competitive Business Cycles," *Journal of Economic Theory*, 40, 57–76.

Khasminskii, R. Z. (1980): *Stochastic Stability of Differential Equations*. Sijthoff & Noordhoof, The Netherlands.

Kloeden, P. E., E. Platen, and H. Schurz (1994): *Numerical Solution of SDE Through Computer Experiments*. Springer-Verlag, Berlin Heidelberg New York.

Lee, J. (1996): "Financial Development by Learning," *Journal of Development Economics*, 50, 147–164.

Lucas, Jr, R. E. (1986): "Adaptive Behavior and Economic Theory," *Journal of Business*, 59, S401–S426.

Marcet, A., and T. J. Sargent (1989): "Convergence of Least Squares Learning Mechanisms in Self Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–68.

Matsuyama, K. (1991): "Increasing Returns, Industrialization, and Indeterminacy of Equilibrium," *The Quarterly Journal of Economics*, 106(2), 617–650.

Mendoza, E. G. (1995): "The Terms of Trade, the Real Exchange Rate, and Economic Fluctuations," *International Economic Review*, 36, 101–137.

Robbins, H., and S. Monro (1951): "A Stochastic Approximation Method," *Annals of Mathematical Statistics*, 22, 400–407.

Salyer, K. D., and S. M. Sheffrin (1998): "Spotting Sunspots: Some Evidence in Support of Models with Self-Fulfilling Prophecies," *Journal of Monetary Economics*, 42, 511–523.

Shigoka, T. (1994): "A Note on Woodford's Conjecture: Constructing Stationary Sunspot Equilibria in a Continuous Time Model," *Journal of Economic Theory*, 64, 531–540.

Spear, S. E. (1991): "Growth, Externalities, and Sunspots," *Journal of Economic Theory*, 54, 215–223.

Spear, S. E., S. Srivastava, and M. Woodford (1990): "Indeterminacy of Stationary Equilibrium in Stochastic Overlapping Generations Models," *Journal of Economic Growth*, 50, 265–284.

Woodford, M. (1986): "Stationary Sunspot Equilibria in a Finance Constrained Economy," *Journal of Economic Theory*, 40, 128–137.

——— (1990): "Learning to Believe in Sunspots," *Econometrica*, 58, 277–307.

Figure 1. Phase portrait of the transformed system in (x,y) variables.

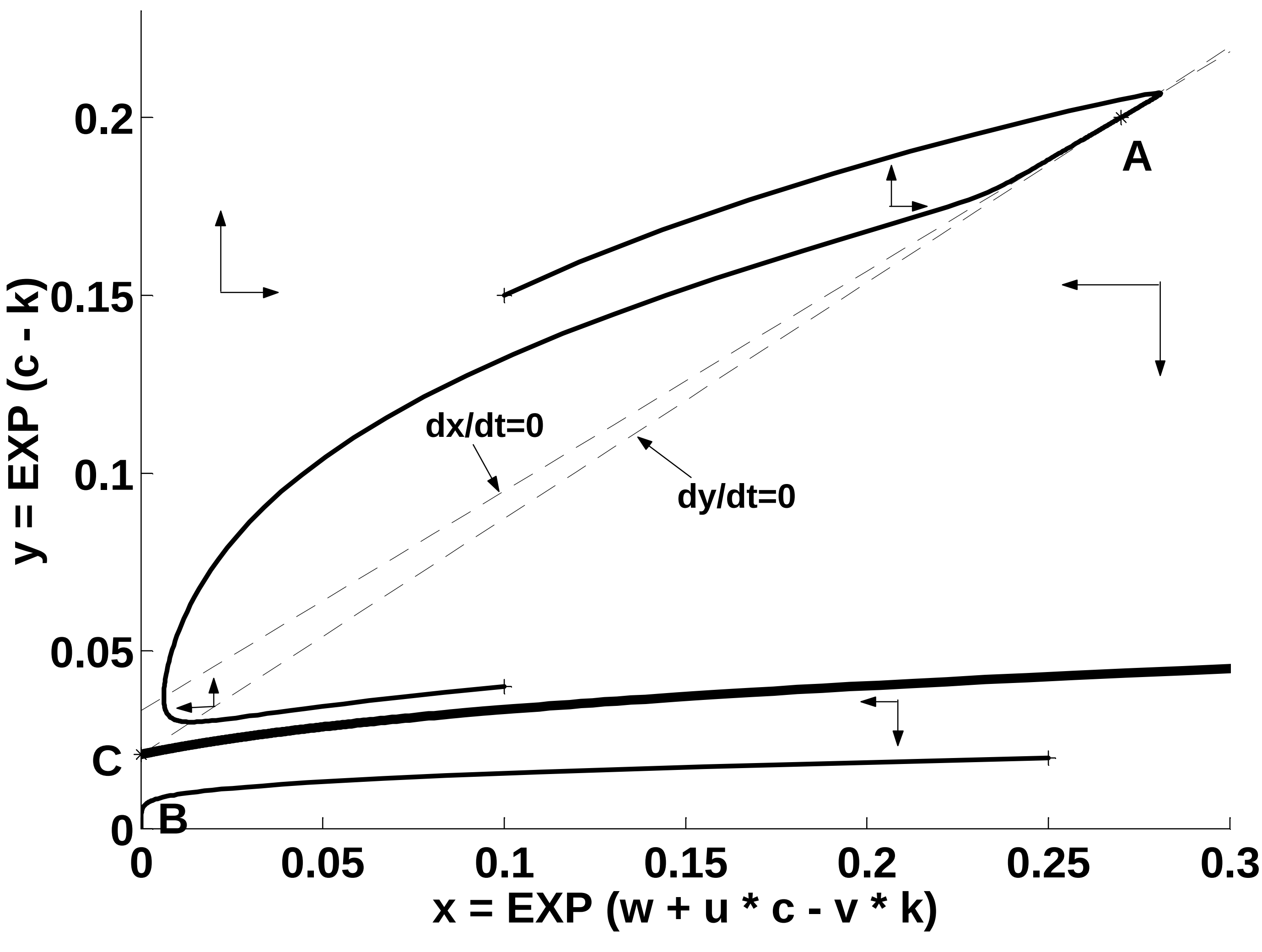

Fig. 2 Phase portrait in log capital and log consumption

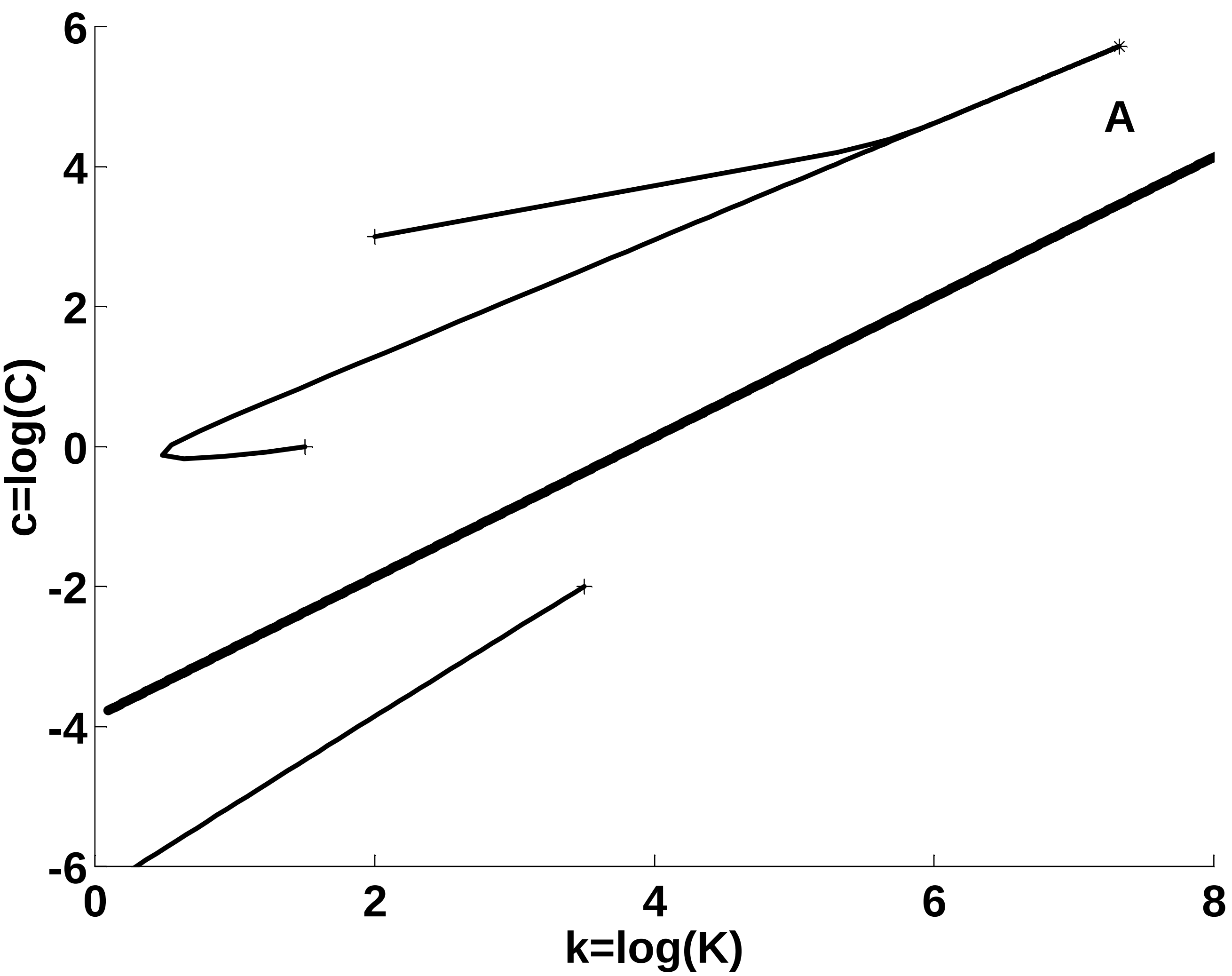

Figure 3. Dynamics of the system in original, nonlogged variables

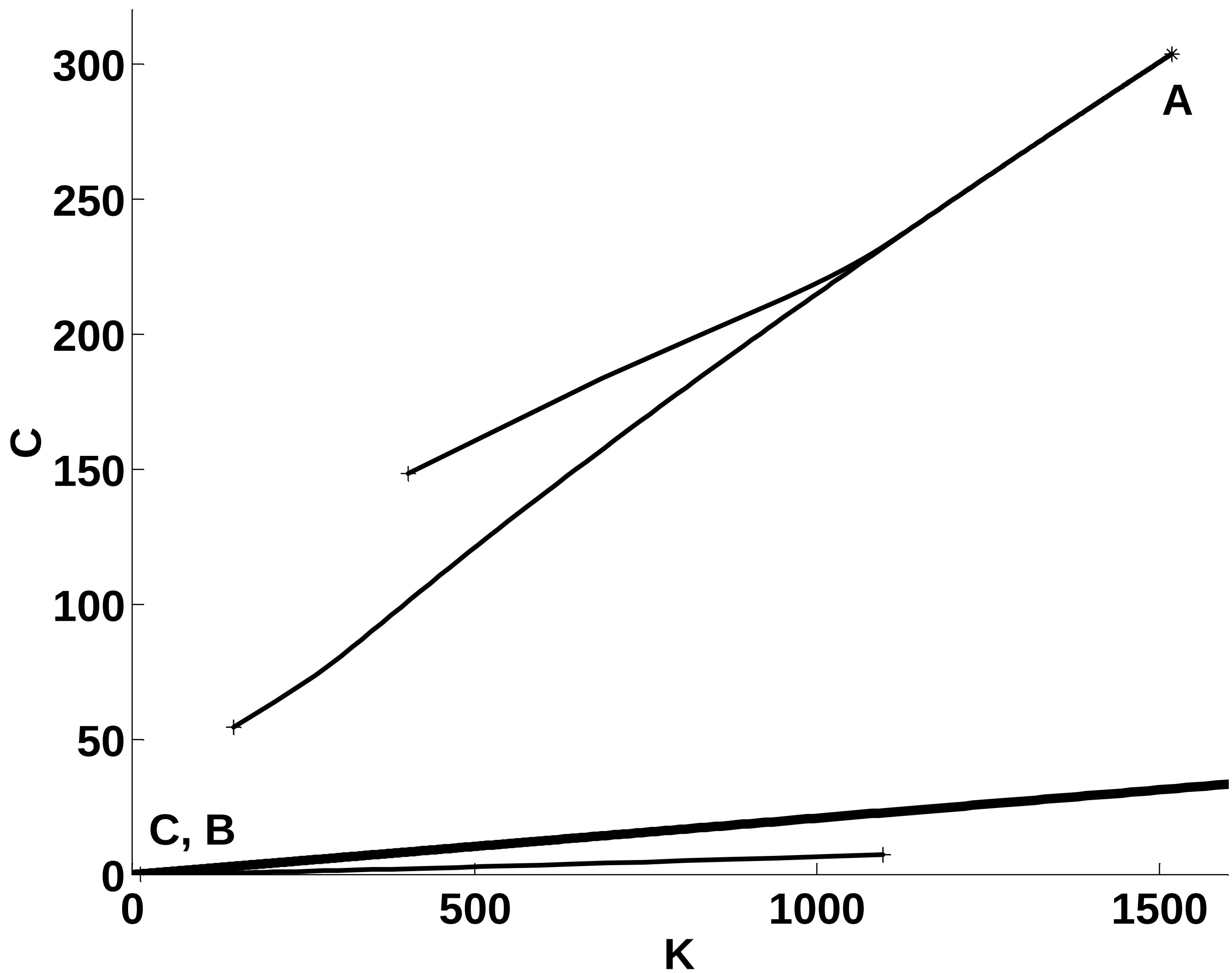

Figure 4. Probability of escape as function of the distance to the poverty trap boundary

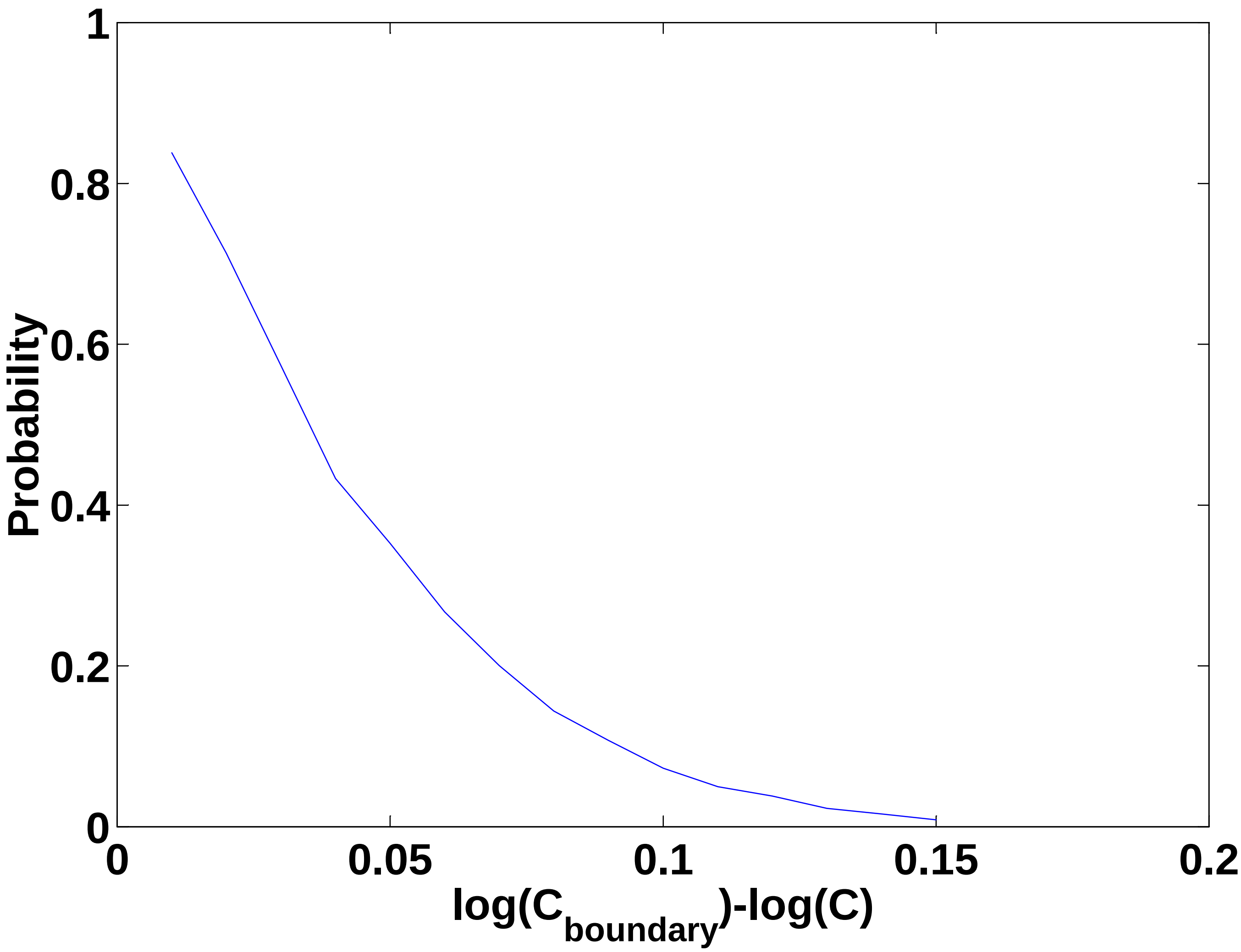

Figure 5. Mean and median escape time as function of the distance to the poverty trap boundary