Patience, Persistence and Properties of Two-Country Incomplete Market Models

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Abstract

Using a two-country endowment economy model with incomplete asset markets, we obtain closed-form solutions for consumption dynamics and welfare. We demonstrate that welfare of the incomplete market economy can be expressed as a convex combination of that under the complete market economy and autarky. The relative weights of the convex combination depend on the persistence and spillovers of the shocks and on the discount rate. As shocks become less persistent and as agents become more patient, the incomplete market economy approaches the complete market economy, vice versa. We also show that the convex combination property holds in a production economy model with labor by solving the model with the nonlinear certainty equivalence method.

1. Introduction

A number of open economy dynamic stochastic general equilibrium (DSGE) models with incomplete asset markets have been solved by numerical methods. The lack of an analytic solution due to the numerical approximation, while allowing a realistic depth of the model, makes it a challenge to fully understand the behavior of the incomplete market model. For example, the well-known property that the behavior of the incomplete market model with temporary shocks differs when shocks become persistent has not been analytically explained in the literature.[1] The effects of financial market structure on the macroeconomic variables, another important topic in international economics, also needs to be examined analytically.[2]

We use analytical methods to examine the behavior of a two-country incomplete market economy.[3] Using an endowment economy model, we obtain an analytically tractable closed-form expression of the exact solution for consumption and welfare. We show that the welfare of the incomplete market model can be expressed as a convex combination of those of the complete markets and autarky. We also extend the model by incorporating labor-leisure decision and use numerical methods to confirm the convex combination property. The closed-form solution enables us to algebraically analyze the role of each parameter of the model on the behavior of the economy, which has not been provided by previous papers using numerical approximations.

This paper is related to and improves on both international RBC literature and micro finance.[4] First, while most international RBC papers have difficulties in clearly explaining the dynamics of the model due to the use of numerical methods,

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[1] See Baxter and Crucini (1995) and Baxter (1997).

[2] Cole (1988), with a two-period model, and Heathcote and Perri (1999), with an infinite horizon loglinearized model, compare the economy under different financial markets using analytic solutions. However, no work has been done in the dimension of infinite horizon nonlinear model.

[3] Incomplete asset market assumption is more popular than the complete market assumption, as argued in Obstfeld and Rogoff (1996).

[4] Both literatures are based on a general-equilibrium framework. A number of other papers analyze the same issue in a partial-equilibrium framework. Consumption literature on permanent income hypothesis, beginning with Friedman (1957), points out that persistence of shocks is very important for understanding consumption dynamics. International finance literature on current account, such as Glick and Rogoff (1995), also emphasizes the importance of shock persistence. The same intuition for the importance of persistence in the calculation of present discounted value applies to the general-equilibrium model.

we characterize the economy analytically and demonstrate the effects of each parameter of the model on its behavior.[5] In the literature on international risk sharing gains, a number of papers have used numerical methods to calculate risk sharing gains, such as Tesar (1995) and Kim (1997). The existence of analytic solution for welfare allows us to calculate risk sharing gains analytically. Another contribution of this paper is that we do not use linearization method to solve the model. Linear approximation errors can be significant, especially in calculating welfare.[6] This paper approximates the dynamics of the variables up to the second order and, therefore, computes the level of welfare correctly up to the first order of shock variances.

Our paper also bears a relation to the risk sharing literature in theoretical microeconomics, such as Levine and Zame (1999), and empirical finance, such as Constantinides and Duffie (1996). They focus on the necessary and sufficient conditions of the model to make incomplete market complete. However, since these papers derive the implications of the model only qualitatively or numerically, we cannot use these models to derive analytical implications. On the other hand, we offer an example of an incomplete markets economy which admits an analytic expression for dynamics and welfare implications.[7] Another distinction of our paper from these micro papers is that we compare the incomplete markets not only with the complete markets but also with financial autarky, which gives insights on factors affecting welfare implications of the incomplete markets economy. We also analyze the welfare implications of growth in endowments and the endogenous labor supply, which are not treated much in microeconomics.

The main result of this paper is that welfare of the incomplete market economy is a convex combination of those of the complete market economy and autarky, with the weight depending on the persistence and spillover parameters of productivity shocks as well as the discount factor. As agents discount future consumption less, the incomplete market economy approaches the complete market economy.

[5] Most RBC papers use numerical simulations with linearized model. See, for example, Backus, Kehoe and Kydland (1992) and Baxter and Crucini (1995).

[6] As shown in Kim and Kim (1999), the linearization technique such as the one used in King, Plosser, and Rebelo (1988), very popular and conventional in macroeconomics and international finance, give rise to spurious results in terms of welfare. Specifically, the linearization method may generate the result that the complete market economy produces a lower welfare level than autarky.

[7] One exception is Willen (1999) that derives welfare implications analytically. However, its model specification, exponential utility function and normal endowment process, is not suitable for macroeconomics because it generates negative consumption.

When shocks are temporary, the behavior of the incomplete market economy is similar to that of the complete market economy. When shocks are permanent, the amount of international asset transactions becomes negligible and the economy behaves like an autarky economy.[8] This behavior is different from that of the complete market economy where people insure against risks before shocks occur, which makes the persistence of shocks irrelevant to the economy's behavior.[9]

We also prove that the relative weight of the convex combination property does not depend on intertemporal substitution by showing that the welfare of the incomplete market, complete market and autarky linearly depend on this parameter. Finally, we show that even with a growth in endowment, the convex combination property holds with a growth adjusted interest rate as the relative weight.

For the labor-production economy, we derive the closed form solution of consumption and labor by log-linearizing the model around its steady state. However, the loglinearization may not be accurate for the implications involving second-order dynamics such as welfare. Therefore, we use the Nonlinear Certainty Equivalence (NCE) solution method to derive welfare of the economy numerically. The results confirm that the convex combination property of welfare holds for the labor-production economy.

The remaining structure of the paper is as follows. In section two, we start with the basic model with log utility and derive the convex combination property of welfare, especially the role of patience of agents on welfare of the economy. Section three uses power utility function and analyzes the role of intertemporal elasticity of substitution on the properties of the incomplete market economy. Section four introduces growth in endowment and examine whether the convex combination property holds. Finally, in section five, we analyze the labor production economy using both log linearization and the NCE method and confirm the convex combination property. Appendix discusses the welfare gains and the size of approximation errors. Section six serves as a conclusion.

[8] Intuitively speaking, the international lending and borrowing do not take place when people know that the differences in wealth of the two countries increase indefinitely.

[9] It is an open question whether the convex-combination property holds with respect to other types of shocks, such as government spending shocks, monetary policy shocks, and labor supply shocks.

2. Patience and Properties of Incomplete Markets

This section introduces a two-country endowment economy model with incomplete asset markets where two countries trade one period risk-free bonds only. We make assumptions in a way that allows us to derive an analytically tractable closed form solution while maintaining infinite horizon, dynamic stochastic general equilibrium structure with rational expectation. In particular, we assume that shocks occur only at the initial period and analyze the dynamic responses of domestic and foreign economies over time. This is equivalent to using a perfect foresight assumption with shocks at every period.

In order to concentrate on the effects of patience of agents on the properties of incomplete market economy, we adopt a log utility function and assume that both countries face a pure temporary shock at the initial period 0. We derive the solutions for domestic and foreign consumptions and asset holdings. Then, we derive welfare using second order approximation of log consumption.

2.1. Model

Domestic country solves the following problem:

$$\max \mathtt{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t), \text{ where } U(C) = \log C, \tag{2.1}$$

subject to

$$C_t + B_t = R_{t-1} B_{t-1} + Y_t, \text{ for all } t. \tag{2.2}$$

where Y_t is endowment and B_t denotes the quantity of discount bonds purchased in period t maturing in $t+1$ with price R_t. Foreign country solves the same maximization problem and the two countries are assumed to be identical.

The solutions at time t consist of intertemporal Euler equations, budget constraints of the domestic and foreign countries and the bond market clearing condition:

$$\frac{1}{C_t} = \beta R_t \mathtt{E}_t \left(\frac{1}{C_{t+1}} \right), \tag{2.3}$$

$$C_t + B_t = R_{t-1} B_{t-1} + Y_t, \tag{2.4}$$

$$B_t + B_t^* = 0. \tag{2.5}$$

The foreign country has the same first order conditions (2.3), (2.4) where foreign variables are denoted with the asterisk. For the notational convenience, we adopt a different way of defining assets $A_t = R_{t-1}B_{t-1}$. Then, the budget constraint of the domestic country (2.4) can be written as

$$A_{t+1} = R_t \left(A_t + Y_t - C_t\right). \tag{2.6}$$

Using this definition, we can write the infinite horizon intertemporal budget constraint at time t as

$$C_t + \sum_{j=0}^{\infty} \left(\prod_{s=0}^{j} R_{t+s}\right)^{-1} C_{t+j+1} = Y_t + A_t + \sum_{j=0}^{\infty} \left(\prod_{s=0}^{j} R_{t+s}\right)^{-1} Y_{t+j+1}. \tag{2.7}$$

For simplicity, we assume that initial asset holding is zero, $A_0 = 0$.[10] Expectation operators are omitted for simplicity from this point.

2.2. Solution

Since we assume that shocks occur only at the initial period, Y_0 and Y_0^* are the only stochastic variables and $Y_t = Y_t^* = \bar{Y}$, for t=1,2,... where $\bar{Y}$ is steady state endowment. We assume that each shock follows a lognormal distribution with mean zero and variance σ_y^2 and shocks are not correlated across countries.[11] Since shocks are temporary, we can derive the solution using the fact that the economy reaches a new steady state at $t = 1$. The new steady state will be different from the initial state due to the well-know unit root property of asset holdings in the incomplete market models.[12] That is, $C_t = C_t^* = \bar{C}$, and $A_t = \bar{A}$, for t=1,2,... where $\bar{C}$ and $\bar{A}$ are the new steady state consumption and asset holdings, respectively, that are different from the initial values.

We solve the model backwards from $t = 2$ and the solutions, $C_0, \bar{C}, R_0, \bar{R}$, and $\bar{A}$, become:

$$C_0 = \beta \left(\frac{Y_0 + Y_0^*}{2}\right) + (1-\beta) Y_0, \tag{2.8}$$

$$\bar{C} = \beta \bar{Y} + (1-\beta)(\bar{Y} + \bar{A}), \tag{2.9}$$

$$R_0 = \frac{1}{\beta}\left(\frac{2\bar{Y}}{Y_0 + Y_0^*}\right), \tag{2.10}$$

[10] Appendix introduces the generalized model allowing non-zero initial asset holdings.

[11] In Appendix, we solve the model allowing a non-zero cross-country correlation of shocks.

[12] In open economy incomplete market models with a fixed discount factor, asset holdings follow a unit root process. Therefore, variables do not return to the initial steady state even with a temporary shock. See Kim and Kose (1999) for detailed analysis on this issue.

$$\bar{R} = \frac{1}{\beta}, \quad (2.11)$$

$$\bar{A} = \frac{R_0}{\bar{R}}\left(\frac{Y_0 - Y_0^*}{2}\right). \quad (2.12)$$

The foreign country has the symmetric consumption solutions for (2.8) and (2.9). Note that the consumption solution of the endowment economy model under complete markets is: $C_t^{complete} = \left(\frac{Y_t + Y_t^*}{2}\right)$. Under complete markets, people completely share risks across countries and consume an average world endowment at each period. Under autarky, agents consume the exact amount that they are endowed: $C_t^{autarky} = Y_t$.[13]

We can easily see that the initial consumption response, C_0, can be expressed as a convex combination of those of the complete markets economy and autarky. This property applies only to the initial period with zero asset holdings because the consumption onwards is affected by asset holdings, $\bar{A}$, that are not zero.

2.3. Welfare

It is not possible to derive an analytic solution for welfare if we plug the exact solution of consumption in (2.8) and (2.9) into utility function, because the statistical property of its distribution is not known. One way to calculate welfare is to use simulation method to numerically approximate the welfare. However, the analytic form of solution is not available, which is against the purpose of this paper. Therefore, we approximate consumption up to the second order and plug them into utility function to calculate welfare analytically. It is necessary to approximate consumption up to the second order since the effects of financial market structure on consumption appears at the second order.[14]

For simplicity, we assume that the deterministic steady state of endowment in both countries is unity ($\overline{Y} = \overline{Y}^* = 1$). Welfare is defined as a sum of discounted expected utility. Using the second order approximated solution for consumption

[13] See Appendix for the complete derivation of consumption solution and welfare for all three markets.

[14] See Kim and Kim (1999) for detailed example. In the next section when we relax the log utility assumption, we need to approximate utility up to the second order to have an analytic solution.

and lognormal property of shock, welfare of the incomplete markets becomes[15]

$$W^{inc} = \mathtt{E}\sum_{t=0}^{\infty}\beta^t \log C_t \approx \frac{\beta}{4}\sigma_y^2. \tag{2.13}$$

Welfare under complete markets (W^{comp}) and autarky (W^{aut}) is $\frac{1}{4}\sigma_y^2$ and zero, respectively. Therefore, welfare of the incomplete markets can be expressed as a convex combination of welfare under complete markets and autarky:

$$W^{inc} = \beta W^{comp} + (1-\beta)\,W^{aut}. \tag{2.14}$$

We can easily see that the relative weight on the complete markets economy is the discount factor β. Since β is a reciprocal of the steady state interest rate as in (2.11), we can express the convex combination property in terms of $\bar{R}$:

$$W^{inc} = \frac{1}{\bar{R}}W^{comp} + \left(1 - \frac{1}{\bar{R}}\right)W^{aut}. \tag{2.15}$$

The results imply that as we discount future consumption less (higher β or lower $\bar{R}$), the behavior of consumption under the incomplete markets approaches that of the complete markets. That is, as future consumption counts more in the utility function, agents concern more about intertemporal consumption smoothing and the consumption behavior generates similar results as in full risk sharing case under complete markets. On the other hand, as weights on future consumption in utility decrease (lower β or higher $\bar{R}$), agents become more concerned about the current period and the economy approaches autarky.

In order to further analyze the case when β is close to unity, we investigate the case with a finite time J. Plugging the interest rate solutions (2.10) and (2.11) into the J-period intertemporal budget constraint of (2.7) at time 0, then

$$\left(1-\beta^{J+2}\right)C_0 = (1-\beta)\left(\frac{Y_0 - Y_0^*}{2}\right) + \left(1-\beta^{J+2}\right)\left(\frac{Y_0 + Y_0^*}{2}\right) \tag{2.16}$$

If β approaches one, we have

$$C_0 = \left(\frac{Y_0 + Y_0^*}{2}\right) + \left(\frac{1}{J+2}\right)\left(\frac{Y_0 - Y_0^*}{2}\right). \tag{2.17}$$

As time horizon increases, the last term becomes insignificant. Then the behavior of the incomplete market is approaches that of the complete market.

[15] Expectation operator in calculating welfare is conditional on time -1 to capture the stochastic property of endowment shock at time 0.

3. Intertemporal Substitution and Properties of Incomplete Markets

The consumption solution becomes more complicated if we deviate from the log utility case. In order to analyze the role of intertemporal elasticity of substitution on the properties of the incomplete markets, we adopt the following power utility function with the intertemporal elasticity $1/\gamma$:[16]

$$U(C) = \frac{C^{1-\gamma} - 1}{1-\gamma} \tag{3.1}$$

3.1. Solution

With pure temporary shocks in Y_0 and Y_0^*, consumption and asset holding solutions become:

$$\begin{aligned} C_0 &= \beta_\gamma \left(\frac{Y_0 + Y_0^*}{2} \right) + \left(1 - \beta_\gamma\right) Y_0, & (3.2) \\ \bar{C} &= \beta \bar{Y} + (1-\beta)\,(\bar{Y} + \bar{A}), & (3.3) \\ \bar{A} &= \frac{\beta_\gamma}{\beta} \left(\frac{2\bar{Y}}{Y_0 + Y_0^*} \right)^{\gamma} \left(\frac{Y_0 - Y_0^*}{2} \right), & (3.4) \end{aligned}$$

where

$$\beta_\gamma = \beta \left[\beta + (1-\beta) \left(\frac{2\bar{Y}}{Y_0 + Y_0^*} \right)^{\gamma - 1} \right]^{-1}.$$

The foreign country has the symmetric consumption solutions. The steady state interest rate is same as in the log utility case. Note that consumption depends on a nonlinear function of intertemporal elasticity and discount factor.

3.2. Welfare

Unlike the log utility case, in order to derive a tractable closed-form expression for welfare, we need to approximate the power utility function. We apply a modified version of the method used in Woodford () and approximate the utility function up to the second order with respect to the log endowment. First, using the

[16] A disadvantage of using power utility function is that risk aversion parameter and intertemporal elasticity are not distinguishable. That is, the intertemporal substitution is equal to the reciprocal of risk aversion parameter of the CRRA utility function.

consumption solution, (3.2) and (3.3), we derive the second order approximation of log consumption w.r.t. log endowment. Second, we derive approximated utility function by taking the second order Taylor expansion around the steady state endowment $\bar{Y}$. Note that we do not approximate the utility function directly to the initial consumption due to the unit root property of solution.[17] Finally, plug the approximated log consumption solution into the approximated utility function and derive welfare.

Welfare of the incomplete markets economy becomes:

$$W^{inc} \approx \frac{1}{2}\left[\beta\left(1-\frac{\gamma}{2}\right)+(1-\beta)(1-\gamma)\right]\sigma_y^2. \tag{3.5}$$

The level of welfare under complete markets and autarky become

$$W^{comp} \approx \frac{1}{2}\left(1-\frac{\gamma}{2}\right)\sigma_y^2, \tag{3.6}$$

$$W^{aut} \approx \frac{1}{2}(1-\gamma)\sigma_y^2. \tag{3.7}$$

We can easily see that welfare of the incomplete markets economy can be expressed in the same convex form with relative weight β:

$$W^{inc} = \beta W^{comp} + (1-\beta) W^{aut}. \tag{3.8}$$

This result suggests that even though γ affects the welfare levels under all three financial markets, the relative weight does not depend on γ and the same convex combination property holds irrespective of the shape of utility function whether it has a log or a power utility form. This is because levels of welfare of the three markets depend on γ linearly.

need intuitive explanation–what matters is the real interest rate and intertemporal substitution (risk aversion) does not affect it.

4. Growth and Properties of Incomplete Markets

While most micro studies on financial market completeness deal with stationary environment, many business cycle models incorporate growth as well as business

[17] While Woodford () adopts the second order approximation method with stationary processes, we need to modify his method due to the existence of unit root. See Appendix for detailed discussion on this approximation method.

cycles in the same model. In this section, we analyze the properties of the incomplete markets allowing growth in endowment. We still maintain the assumptions of CRRA utility and an initial-period temporary shock. We adopt the following description of growth that endowment grows at a constant rate μ:

$$Y_0,\ Y_t = \mu^t \overline{Y},\ \text{for t=1,2,...} \tag{4.1}$$

where only Y_0 and Y_0^* have shock component.

4.1. Solution

Steady state of this growing economy is defined as the state where all variables grow at the same rate as the endowment growth rate μ. That is, consumption as well as asset holdings grow at the same rate. We solve the model by using stationary variables derived from dividing each variable with its growth rate. Stationary variables are denoted with hat such as $\hat{C}_{t+s} = \frac{C_{t+s}}{\mu^s}$. We also define a new growth adjusted interest rate as $\hat{R}_t = \frac{R_t}{\mu}$. Then all the first order conditions go through with these transformed variables and the solution will be exactly the same as before with a new discount factor $\delta = \beta\mu^{1-\gamma}$ instead of β. This growth adjusted discount factor is widely used in the growth literature.[18] Note that in a growing economy, we need $\delta < 1$ to insure convergence of expected utility.

4.2. Welfare

We follow the same approximation procedure for log consumption and welfare, then the welfare levels of the three economies become

$$W^{inc} \approx \frac{1}{2}\left[\delta\left(1-\frac{\gamma}{2}\right) + (1-\delta)(1-\gamma)\right]\sigma_y^2 + \Delta, \tag{4.2}$$

$$W^{comp} \approx \frac{1}{2}\left(1-\frac{\gamma}{2}\right)\sigma_y^2 + \Delta, \tag{4.3}$$

$$W^{aut} \approx \frac{1}{2}(1-\gamma)\sigma_y^2 + \Delta, \tag{4.4}$$

where

$$\Delta = \frac{1}{1-\gamma}\left(\frac{\delta}{1-\delta} - \frac{\beta}{1-\beta}\right).$$

[18] See, for example, textbooks by Romer and Barro/Sala-i-Martin for the use of this notation as a typical form of growth adjusted discount factor.

These results are the same as the ones without growth except that we have the new discount factor δ and a constant term Δ. The level of welfare eventually depends on whether the economy is growing or shrinking ($\mu \lesseqgtr 1$),discount factor (β) and the intertemporal elasticity of substitution ($1/\gamma$).

With growth, the convex combination property of welfare of the incomplete markets economy holds with a growth adjusted discount factor δ:

$$W^{inc} = \delta W^{comp} + (1-\delta) W^{aut}. \tag{4.5}$$

We can intuitively interpret the growth adjusted discount factor δ by expressing δ in terms of the steady state interest rate, $\bar{R} = \beta^{-1}\mu^{\gamma}$. Then, $\delta = \left(\bar{R}/\mu\right)^{-1}$ which is simply a reciprocal of the growth adjusted steady state interest rate. Therefore, the growth case is subject to the same explanation as used previously in terms of steady state interest rate: Agents discount future consumption considering the additional endowment due to growth. The important parameter affecting the behavior of incomplete market is the interest rate, not discount factor.

Need more explanation in terms of annuity value, etc.

Effects of γ on δ in terms of intertemporal elasticity of substitution.

5. Persistence and Properties of Incomplete Markets

In this section, we generalize the model including endowment shocks with an AR(1) form and analyze the role of persistence of shocks on properties of the incomplete markets economy. We maintain the current structure of model: CRRA utility, growth in endowment, and initial-period shocks.

We define the detrended stationary endowment $\hat{Y}_t = Y_t/\mu^t$ and assume the log $\hat{Y}_t$ follows an AR(1) process:

$$\log \hat{Y}_t = \rho \log \hat{Y}_{t-1},\ \ t = 1, 2, ... \tag{5.1}$$

where ρ is persistence of shocks.

5.1. Solution and Welfare

The consumption path at time t can be derived from the first order conditions and the intertemporal budget constraint (2.7):

$$C_0 = \frac{Y_0 + \left(\frac{Y_0+Y_0^*}{2}\right)^{\gamma} \sum_{t=1}^{\infty} \delta^t \left(\frac{Y_t+Y_t^*}{2}\right)^{-\gamma} \hat{Y}_t}{1 + \left(\frac{Y_0+Y_0^*}{2}\right)^{\gamma-1} \sum_{t=1}^{\infty} \delta^t \left(\frac{Y_t+Y_t^*}{2}\right)^{1-\gamma}}. \tag{5.2}$$

The second order log approximation of consumption can be derived using the same method. Following the same approximation procedure of utility, we can derive the welfare of the three economies:

$$W^{inc} \approx \frac{1}{2\left(1-\delta\rho^2\right)}\left[(1-\gamma)+\frac{\gamma}{2}\left(\frac{1-\rho}{1-\delta\rho}\right)^2\delta\right]\sigma_y^2+\Delta, \tag{5.3}$$

$$W^{aut} \approx \frac{1}{2\left(1-\delta\rho^2\right)}(1-\gamma)\sigma_y^2+\Delta, \tag{5.4}$$

$$W^{com} \approx \frac{1}{2\left(1-\delta\rho^2\right)}(1-\frac{\gamma}{2})\sigma_y^2+\Delta. \tag{5.5}$$

The convex combination property holds in the following form:

$$W^{inc} = \left[\left(\frac{1-\rho}{1-\delta\rho}\right)^2\delta\right]W^{comp}+\left[1-\left(\frac{1-\rho}{1-\delta\rho}\right)^2\delta\right]W^{aut}. \tag{5.6}$$

Even with CRRA utility and AR(1) shock, welfare of incomplete markets follows an exact convex combination with a constant weight as far as we approximate welfare up to the first order of the variances. The relative weight is a complex nonlinear function of growth adjusted discount factor δ and persistence of shock ρ. The result also shows that the risk aversion parameter does not affect the welfare weight as in the case of temporary shock. This property suggests that the same convex combination property holds with log utility.

In a case without growth, all the equations hold with β replacing δ. The convex combination property becomes:

$$W^{inc} = \left[\left(\frac{1-\rho}{1-\beta\rho}\right)^2\beta\right]W^{comp}+\left[1-\left(\frac{1-\rho}{1-\beta\rho}\right)^2\beta\right]W^{aut}. \tag{5.7}$$

We use this case to interpret the welfare weight intuitively.

Figure 1 plots the weight on the complete markets economy with respect to the persistence parameter ρ under different values of β. Figure 2 draws the same graph in the three dimensional space. The graphs show that as ρ approaches one—permanent shock—the weight on the complete markets economy approaches zero. That is, the incomplete markets economy approaches autarky where no asset transactions occur across countries. Intuitively speaking, the domestic country does not have any incentive to smooth consumption by saving since the positive shocks are permanent—permanent wealth effects. Permanent shocks cannot be smoothed by bond transactions. However, under the complete markets, any gain in a country is shared by both countries and the consumption behavior does not change as the persistence of shock changes.

As shocks become less persistent—a decrease in ρ—the incomplete markets economy approaches the complete markets economy. With a pure temporary shock, consumption smoothing through bond transactions can produce almost identical welfare as the one from perfect risk sharing observed in the complete markets. However, as long as β is strictly less than unity, the incomplete markets cannot be identical to the complete markets. The behavior of the incomplete markets economy is very sensitive to the parameter choice of β and ρ since both of them are very close to unity in most applications.[19]

6. Production Economy with Labor

This section extends the endowment economy model to the production economy with labor. However, in order to derive the closed form solution, we need to linearize the model.[20] First, using the closed-form solution from the linearized model, we show that consumption follows a similar form as in the endowment economy case. Next, we solve the model nonlinearly by adopting the NCE solution method. We confirm the results from the linearized model by comparing the impulse responses from both solution methods. Finally, the convex combination property of welfare of the incomplete markets economy is shown by numerical simulations using the NCE method.

6.1. Model

Each country faces the following maximization problem with a separable utility function:

$$\max \mathtt{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t), \text{ where } U(C,L) = \frac{C^{1-\gamma}-1}{1-\gamma} + \nu_0 \nu \left(1 - L^{\frac{1}{\nu}}\right), \tag{6.1}$$

subject to

$$C_t + B_t = R_{t-1} B_{t-1} + Y_t, \tag{6.2}$$

[19] Mathematically speaking, the weight function is not continuous where both β and ρ are at unity.

[20] Since deriving a closed-form solution is crucial to the purpose of this paper, we do not analyze the production economy with both labor and capital.

$$Y_t = X_t L_t^{1-\alpha}, \text{ for all } t. \tag{6.3}$$

where X_t is productivity, L_t is the amount of labor. The labor production technology can be nonlinear with a nonzero α. The parameter ν, taking a value between 0 and 1, represents how elastic the labor supply is. The elasticity of labor supply is $\nu / (1-\nu)$, which is increasing in ν and takes a value between 0 and ∞.

Since we use linearization, we can generalize the solution to a time t problem instead of concentrating on time 0 *solution with an initial period shock as before. That is, we solve the model so that the model is readily available for multiple period shocks.* We also use a shock process including spillovers of shocks with a period lag, because many international business cycle models have incorporated spillovers of shocks. Assume the vector Markov process for productivity:

$$\begin{bmatrix} \log X_t \\ \log X_t^* \end{bmatrix} = \begin{bmatrix} \rho & \theta \\ \theta & \rho \end{bmatrix} \begin{bmatrix} \log X_{t-1} \\ \log X_{t-1}^* \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \varepsilon_t^* \end{bmatrix}, \tag{6.4}$$

where $E(\varepsilon_t) = E(\varepsilon_t^*) = 0$ and $E(\varepsilon_t^2) = E(\varepsilon_t^{*2}) = \sigma_\varepsilon^2$, and $\rho(\varepsilon_t, \varepsilon_t^*) = \psi$ for all t.[21] ρ is the persistence parameter of the shock and θ represents the spillover effects—the degree of transmission of productivity with one period lag. The stability condition is that $|\rho + \theta| < 1$ and $|\rho - \theta| < 1$. A non-zero ψ means that the innovations are contemporaneously correlated across countries.

6.2. Solution of the Linearized Model

We solve the model using Lagrangian method assuming $\nu_0 = 1 - \alpha$ for notational simplicity. Since the closed form solution such as (5.2) is not available, we solve the model by linearizing the first order conditions and budget constraint around the deterministic steady states, $\bar{X} = \bar{C} = \bar{Y} = 1$, $\bar{B} = 0$, and $\bar{R} = \beta^{-1}$.[22] After solving for each country, we use the bond market clearing condition and endogenize the interest rate process. Use the shock process in (6.4) and derive the following equations for consumption and asset holdings at time t:

$$c_t \approx \left(\frac{1-\beta}{\beta}\right) \frac{\Upsilon}{\Gamma} B_{t-1} + \left[\omega \left(\frac{x_t + x_t^*}{2\Gamma}\right) + (1-\omega)\left(\frac{x_t}{\Gamma}\right)\right], \tag{6.5}$$

$$B_t \approx B_{t-1} + \frac{\omega}{2\Upsilon}(x_t - x_t^*), \tag{6.6}$$

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[21] An alternative expression of the error terms, more convenient for simulation, is $\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} \begin{bmatrix} \eta_t \\ \eta_t^* \end{bmatrix}$ where $\rho(\eta_t, \eta_t^*) = 0$ and a satisfies $\frac{2a}{1+a^2} = \psi$.

[22] Linearization around the deterministic steady state generates additional inaccuracy due to unit root property of asset holdings. Later, we use the NCE method to overcome this problem.

where

$$\Gamma = 1-(1-\alpha)(1-\gamma)\nu,$$
$$\Upsilon = 1-(1-\alpha)\nu,$$
$$\omega = \frac{\beta(1-(\rho-\theta))}{1-\beta(\rho-\theta)},$$

and lower case variables denote log variables. The foreign country has the symmetric consumption solution. Note that the equation (6.6) shows a unit root process of asset holdings.

The equation for the interest rate is

$$r_t \approx \frac{\gamma}{2\Gamma}\left[\mathrm{E}_t\left(x_{t+1}+x^*_{t+1}\right)-(x_t+x^*_t)\right]. \tag{6.7}$$

which is equivalent to that of the complete markets economy and so independent of the discount factor.

Use the assumption of zero initial asset holdings, then consumption and asset holdings at time 0 become

$$c_0 \approx \omega\left(\frac{x_0+x^*_0}{2\Gamma}\right)+(1-\omega)\left(\frac{x_0}{\Gamma}\right), \tag{6.8}$$

$$B_0 \approx \frac{\omega}{2\Upsilon}\left(x_0-x^*_0\right), \tag{6.9}$$

Note that the consumption solution of the complete markets is $\frac{x_0+x_0^*}{2\Gamma}$ and under autarky, consumption solution is $\frac{x_0}{\Gamma}$.[23]

The consumption solution (6.8) has the similar form to (2.8) in the endowment economy case and follows the same convex combination form with a weight ω on the complete market. Without spillovers θ, the weight ω is the same in the first order as the weight of the consumption process in the endowment economy model with persistent shocks. Therefore, the same propositions regarding β and ρ apply to this case. In addition, we can see that as the spillover parameter rises (higher θ), the behavior of the incomplete market becomes similar to that of the complete market economy.

By including spillover parameters as well as persistence of shocks, we can confirm the results from numerical simulations in previous studies. Baxter and

[23] See Appendix for derivation of solutions of the three markets.

Crucini (1995) compared the consumption responses to permanent productivity shocks ($\rho = 1, \theta = 0$) and temporary shocks with spillovers ($\rho = 0.906, \theta = 0.088$) using impulse responses from numerical approximation. They found that these two shock processes generate different consumption responses under the incomplete markets. They argued that the spillover effects, which influence the fluctuations in productivity to become nearly identical across countries, are crucial in explaining the different behaviors.

Our work reveals that the results by Baxter and Crucini (1995) is only a special case of the general results shown above. Their work is equivalent to the one-dimensional analysis of consumption by fixing ρ and changing θ only. The above equation shows that the weight ω depends on the difference between ρ and θ. What is important in determining the behavior of the incomplete markets economy is the net transfers of shocks over time. Spillovers from foreign shocks decrease the difference between domestic and foreign productivity, which allows the two economies to borrow and lend more easily.

6.3. Nonlinear Certainty Equivalent Solution

Since the above solution is derived from the linearized model, we need to examine whether the convex combination property holds for the exact solution of the nonlinear equation system. We use the Nonlinear Certainty Equivalence method for the exact solution. The NCE method solves this system using a numerically efficient algorithm for solving non-linear equations.[24] As with the log-linearization method, we assume certainty equivalence; that is, the technology process in future periods is treated as deterministic rather than stochastic. However, rather than log-linearizing around the initial steady state, the NCE method works directly with the non-linear first-order conditions and resource constraints.[25] In particular, we consider the set of equations that describe the entire path of all model variables from the current period through some finite horizon in the future, at which point the system is assumed to have reached its new steady state. Thus,

[24] Compared with log-linearization, the nonlinear certainty equivalence (NCE) method is computationally more intensive, but may provide a substantially more accurate solution if the nonlinear system is relatively far from being log-linear, or if the steady state exhibits large shifts in response to exogenous shocks of reasonable magnitude.

[25] The solution to this system of non-linear equations can be obtained using standard numerical methods. We use the stacked-Newton algorithm in Troll 1.03 because of its computational efficiency, but one could also use an alternative algorithm such as Gauss-Seidel algorithm used in Fair and Taylor (1983).

these terminal conditions replace the corresponding Euler equations in the horizon period. Finally, we confirm that the solution is invariant to using a longer horizon. See Appendix for detailed explanation of solution method.

In order to confirm the consumption response at the initial period, c_0, we compare the impulse responses from exact solutions and approximate solutions. Figure 3 draw the impulse responses of consumption to the initial productivity shocks with various specifications for ρ and ν. The figures from the linearized model and the NCE solution exhibit similar initial responses of consumption. We also confirm that the initial consumption response of the incomplete markets model lies between those of the complete markets and autarky. Under permanent shocks, the incomplete market economy generates the same impulse responses as those under autarky, which shows that the fact that the weight on autarky is one when $\rho = 1$ holds both in the exact and loglinear approximate solutions. Since the loglinear approximate solution of autarky is same as the exact solution, there is no approximation errors under the incomplete markets when shocks are permanent.

6.4. Welfare

It is well-known that the welfare calculation using linearized model is inaccurate as shown in Kim and Kim (1999).[26] Therefore, we use only the NCE method and test the convex combination property of welfare of the incomplete markets using numerical approximations.

Tables 1 and 2 present the welfare levels from the simulation using the NCE method. Shocks occur only at the initial period as before and are assumed to have two states only, a good shock when $X_t = 1.21$ and a bad shock when $X_t = 1/1.21$, so that their logged values are symmetric around 0. We experiment with various parameter values—$\gamma = 1$ and 2, $\nu = 0$, 0.5 and 0.9, and $\rho = 0$, 0.9, 0.95 and 1. Welfare is measured as an average certainty equivalent world consumption of the good and bad cases. Certainty equivalent consumption is defined as the permanent consumption which would give the same amount of the sum of discounted utilities assuming labor stays at the steady state level.

In all cases, the exact solution presents a convex combination property of the incomplete markets; a welfare ordering follows (complete $>$ incomplete $>$ autarky) except when shocks are permanent. When $\rho = 1$, the incomplete market and autarky economies generate the same level of welfare as shown in the closed-form

[26] Kim and Kim (1999) well documents the welfare reversal between complete market and autarky economies in the loglinearized solution.

solutions. The differences of welfare between complete and incomplete market economies increases as shocks become more persistent, which confirms the convex combination property that the incomplete market economy approaches autarky as ρ increases.

However, using the linearized model, the welfare ordering can be reversed. Appendix shows the complete set of welfare analysis and the possibility of welfare reversal using both exact and linearized solution methods. Two sources of approximation error account for the magnitude of these inaccuracies: (1) log-linearizing around the initial steady state ignores the non-stationarity of consumption, labor and net foreign assets; and (2) the true global resource constraint is severely distorted by log-linearization as confirmed in Kim and Kim (1999). However, these approximation errors do not affect the first-order dynamics of the model significantly, as shown in the impulse responses.

7. Conclusion

will be added later

References

[1] Backus, D., P. Kehoe, and F. Kydland, 1992, International Real Business Cycles, Journal of Political Economy 100, 745-775.

[2] Baxter, M. and M. Crucini, 1995, Business Cycles and Asset Structure of Foreign Trade, International Economic Review 36, 821-54.

[3] Baxter, M., 1997, International Trade and Business Cycles, in: G.H. Grossman and K. Rogoff, eds., Handbook of international economics, Vol. 3 (North-Holland, Amsterdam).

[4] Cole, H., 1988, Financial structure and international trade, International Economic Review 29, 237-259.

[5] Constantinides and Duffie (1996)

[6] Fair R. and J. Taylor, 1983, Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models, Econometrica 51, 1169-85.

[7] Friedman (1957)

[8] Glick, R. and K. Rogoff, 1995, Global versus Country-specific Productivity Shocks and the Current Account, Journal of Monetary Economics 35, 159-192.

[9] Heathcote, J. and F. Perri, 1999, Financial Autarky and International Business Cycles, mimeo.

[10] Kim, S. H., 1997, International Business Cycles in a Partial-Risk-Sharing Market with Capital Income Taxation, chapter 3 in Ph.D. dissertation, Yale University.

[11] Kim, J. and S.H. Kim, 1999, Spurious Welfare Reversals in International Business Cycle Models, mimeo.

[12] Kim, S.H. and A. Kose, 1999, Dynamics of Open Economy Business Cycle Models: Understanding the Role of the Discount Factor, mimeo.

[13] King, R., C. Plosser, and S. Rebelo, 1988, Production, growth and business cycles: I. the basic neoclassical mode, Journal of Monetary Economics 21, 195–232.

[14] Levine, D. and W. Zame, 1999, Does Market Incompleteness Matter?, mimeo.

[15] Obstfeld, M. and K. Rogoff, 1996, Foundations of International Macroeconomics, MIT Press.

[16] Tesar, L., 1995, Evaluating the Gains from International Risksharing, Carnegie-Rochester Conference Series on Public Policy 42, 95–143.

[17] van Wincoop, E., 1999, How Big are Potential Gains from International Risksharing?, Journal of International Economics 47, 109–135.

[18] Willen, P., 1999, Welfare, Financial Innovation and Self Insurance in Dynamic Incomplete Market Models, mimeo.

Table 1. Welfare with the initial period's shock
<Production economy with labor–exact solution>
<$\alpha = 0$, $\beta = 0.95$, $\gamma = 2$, and $\theta = 0$>

< $\gamma = 1$, case of log utility>

$\nu = 0$ (endowment economy)				
ρ	0	0.9	0.95	1
complete	1.000098	1.000428	1.000691	1.001972
incomplete	1.000093	1.000194	1.000173	1
autarky	1	1	1	1

$\nu = 0.5$				
ρ	0	0.9	0.95	1
complete	1.000196	1.000853	1.001379	1.003929
incomplete	1.000186	1.000387	1.000346	1
autarky	1	1	1	1

$\nu = 0.9$				
ρ	0	0.9	0.95	1
complete	1.000866	1.003950	1.006374	1.0174589
incomplete	1.000835	1.001879	1.001694	1
autarky	1	1	1	1

Numbers are the average certainty equivalent world consumption of good and bad cases calculated from 400 period simulation with symmetric initial period's shocks at the domestic country. Certainty equivalent consumption is defined as the permanent consumption which would give the same amount of the sum of discounted utilities.

Table A1. continued

$< \gamma = 2 >$

$\nu = 0$ (endowment economy)				
ρ	0	0.9	0.95	1
complete	1	1	1	1
incomplete	0.999977	0.998922	0.997618	0.990971
autarky	0.999545	0.998029	0.996819	0.990971

$\nu = 0.5$				
ρ	0	0.9	0.95	1
complete	1.000300	1.001307	1.002113	1.006028
incomplete	1.000270	0.999874	0.998940	0.993972
autarky	0.999697	0.998687	0.997879	0.993972

$\nu = 0.9$				
ρ	0	0.9	0.95	1
complete	1.00186	1.008580	1.013895	1.038526
incomplete	1.001787	1.003532	1.002429	0.995238
autarky	0.999761	0.998963	0.998326	0.995238