An Order Theoretic Approach To Net Substitution Effects*

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Abstract

We revisit the analysis of discrete comparative statics effects in the classical consumer expenditure minimization framework, using techniques that exploit the order and lattice properties of the problem, without reference to topological properties. It is shown that these comparative statics effects give rise to classes of partial orders, which in turn induce lattice structures that define the critical points of comparability (for the behavior of the utility function), meets and joins, which are used to derive sufficient conditions, from the quasi-supermodular class of properties, for a good(s) to be a net substitute or complement of another. Examples demonstrate the analysis.

Key Words: Comparative Statics, Lattice Programming, Consumer Theory

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0. INTRODUCTION

The subject matter of this paper is a simple problem. Comparative statics in the classic expenditure minimization problem.[1] The proposed approach is novel and the ensuing comparative statics results both new and useful, it is hoped. The approach is perhaps best understood as an extension of revealed preference analysis.

Within a discrete comparative statics environment, the intuition for the proposed approach is very simple. We wish to give sufficient conditions for a good to be a net substitute of another good. Suppose we are presented with a pair of bundles, candidates for being expenditure minimizing bundles at a relevant pair of prices, which contradicted the net substitute condition for the good in question. Suppose also that in such a case we could pinpoint to another pair of bundles, which themselves were candidates for being expenditure minimizing at the same price-utility configurations, and which did indeed satisfy the net substitute condition for the same good. If the level of utility at the four bundles could be related in such a way as to exclude the possibility that the original pair were indeed expenditure minimizing, then we would have sufficient conditions for the comparative statics problem at hand.

This is indeed what the theorems of this paper state, albeit somewhat more formally. But this simple intuition is the guiding light. It suggests that in this discrete comparative statics framework, we are to seek informative binary comparisons between bundles. Such comparisons will be all the more meaningful if they are transitive, and indeed if the underlying binary relations are reflexive, antisymmetric and transitive (partial orders). Thus we exploit the order structure of the consumption set, as a partially ordered set with partial order(s) which give rise to the required comparative statics implications. Next, our intuition suggests that we are to search *systematically* for alternative pairs of bundles which can be used to exclude candidate pairs of bundles from being expenditure minimizing, if they do not satisfy the required comparative statics result. Joins and meets provide the natural candidates for such alternative pairs of bundles. Thus, in addition to the order structure of the consumption set, we also exploit the induced *lattice* structure of the set. The final step in our intuition is to relate the level of utility at an original pair of bundles, with the proposed alternative pair, their join and meet. We can do this with the *supermodular* class of properties on

[1]The producer cost-minimization problem is analogous. We refer only to the consumer problem in this paper for convenience.

functions (see Milgrom and Shannon (1994) and also Veinott (1992)). We have thus all the ingredients for the proposed *lattice programming* approach. It relies on the order/lattice structure of the problem without recourse to the topological properties of the problem (as in the standard implicit function theorem based comparative statics analysis).

Lattice programming methods have a relatively short, but remarkable, track record in economics. Milgrom and Shannon (1994) introduced ordinal lattice programming methods to economics.[2] Their main comparative statics theorem states that if a function which is being maximized is quasi-supermodular[3] and the constraint sets, within a lattice, are strong set comparable, then and only then, the corresponding optimizer sets are strong set comparable and the optimized objective is itself quasi-supermodular in the relevant parameters of the problem. As suggested above, quasi-supermodularity relates to the behavior of functions at the join (least upper bound) and meet (greatest lower bound) of a pair of points, vis-à-vis its values at the original pair of points. Strong set comparability is set comparability that again utilizes the lattice structure of the underlying set (unlike say set inclusion which does not) comparing the set inclusion of joins and meets of pairs of points in the sets to be compared.[4]

This is an impressive and general result. The issue is the applicability of this theorem (as well as adjacent theorems by Milgrom and Shannon (1994), Veinott, (1992), Antoniadou, (1996), and others) to a variety of economics problems, something which can be surprisingly difficult in problems with budgetary trade-offs between variables. The application presented here demonstrates how such difficulties may be overcome, by applying the methodological approach put forward in Antoniadou (1996), namely that a most critical element in the application of the lattice programming approach is the choice of appropriate underlying partial order(s), and corresponding lattice structure, appropriate to each specific problem.

The plan of the paper is as follows; section 1 gives order theoretic definitions of net substitutes and complements which are used to motivate the binary relations which

[2] Cardinal lattice theoretic methods, building primarily on Topkis (1978) were previously used in economics.

[3] Veinott (1992) also defined the same property under the term *lattice superextremal*. Here we use the more standard quasi-supermodular term.

[4] As its name indicates, it is a strong concept of comparability (we will give the formal definition later). In particular, if the optimizers are unique, strong set comparability of the sets of optimizers means that the optimizers are comparable with respect to the underlying order; for example, if this is the Euclidean order, then it simply says that one optimizer is smaller than another in every component.

it is argued must be satisfied by partial orders appropriate for comparative statics analysis in the consumer expenditure minimization problem. The approach is partly demonstrated in the special case of two goods. Section 2 uses the general principles laid out in the first section to state sufficient conditions for a good to be a net substitute or complement of another. Section 3 demonstrates in the case of three goods, which also gives the first description of *Net Substitute and Net Complement Partial Orders.* Section 4 extends the construction of such partial orders when there are many goods. Section 5 revisits the theorems of section 2 in light of the specific partial orders put forward. Section 6 concludes.

1. DISCRETE NET SUBSTITUTION EFFECTS AND SUITABLE PARTIAL ORDERS

In order to apply lattice programming techniques to the comparative statics analysis of the consumer expenditure minimization problem, the consumption set must be endowed with partial order(s), which can simultaneously accommodate the description of such effects, while enabling the application of these comparative statics techniques. For the latter the consumption set (and feasible sets therein) must be closed under relevant joins and meets. For, in this setting, joins and meets identify the crucial points of comparability. It is the nature of the behavior of the utility function at joins and meets, vis-à-vis its behavior at pairs of (incomparable) elements of the consumption set, which will enable the derivation of sufficient conditions for such comparative statics analysis. Accordingly, the underlying partial order(s) on the consumption set will not only be important in describing the comparative statics effect, but will also be very important in determining the strength of the sufficient conditions derived.[5] Thus, we begin by constructing partial orders describing net substitution effects from a minimal set of binary relations that must hold whenever net substitution effects can be analyzed. These binary relations are simply those that are implied by the weak axiom of revealed preference.

[5]Intuitively, the more parsimonious the description of a specific comparative statics effect enabled by a partial order, the stronger the sufficient conditions for it which can be derived from the application of lattice programming techniques with that partial order. For example, products of component-wise orders could be used to describe any number of comparative statics changes, but such 'descriptions' would not be parsimonious with respect to any particular comparative statics effect and the comparability enabled by such partial orders would not be particularly informative or useful.

Consumer preferences are represented by a utility function, $U : X \to \Re$, defined over the consumption set, X (element $x \in X$) subset of the commodity space $\Re^n$. In particular, the consumer has preferences over n goods, named $i = 1, \ldots, n$.[6] The discrete comparative statics question to be addressed is that of the net substitutability of good i for good j, in the expenditure minimization problem. More precisely, given prices $p = (p_1, \ldots, p_n)$ and $p' = (p_1, \ldots, p_{j-1}, p'_j, p_{j+1}, \ldots, p_n)$, with $p, p' \in \Re^n_{++}$, and such that without loss of generality (w.l.o.g. hereafter) $p_j < p'_j$, attainable utility level $\bar{u}$, and corresponding expenditure minimizing bundles, $\hat{x}$ and $\hat{x}'$ (i.e. $\hat{x} = argmin\,\{p \cdot x \mid x \in X, U(x) \geq \bar{u}\}$, and similarly for $\hat{x}'$), the comparability of $\hat{x}_i$ and $\hat{x}'_i$ ($\hat{x}_i < \hat{x}'_i$ or $\hat{x}'_i < \hat{x}_i$) is the issue.[7]

Unlike implicit function based comparative statics approaches, with the lattice programming approach optimizers need not be assumed to be unique. Therefore, we adjust the above statement accordingly. But the approach has also a drawback, namely that it does not allow for the investigation of strict inequality relations as suggested above. This is also taken into account in the formal definition below:

Definition 1.1.

(a) Good i is a *Strongly Net Substitute of good* j at price pair $p, p' \in \Re^n_{++}$, $p = (p_1, \ldots, p_n)$, $p' = (p_1, \ldots, p_{j-1}, p'_j, p_{j+1}, \ldots, p_n)$, with $p_j < p'_j$, and at attainable utility level $\bar{u}$, if:

$$\underset{x \in X}{argmin}\ \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_s \underset{x \in X}{argmin}\ \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$$

where $\leq_s$ is the *strongly-lower-than* set relation[8] compatible with a partial order, $\leq_{ns}$, on the consumption set X, which implies, whenever $x \leq_{ns} x'$:

$$\text{S1:} \quad x_i \leq x'_i$$

Such partial orders on the consumption set will be called *Net Substitutes Partial Orders* (denoted NSPOs).[9]

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[6]Below we will need to distinguish between the *index* $1, \ldots, n$ of a good and its *name* $1, \ldots, n$. Thus, an indexing of goods will be a one to one mapping from the names of goods onto their indices. However, where this is not confusing, we assume the natural indexing of goods, where the name and index of each good coincide.

[7]Statements relating to the symmetry of net substitution effects are avoided, since symmetry is a derived rather than an axiomatic property.

[8]A subset A of a poset $(S, \leq)$ is strongly-lower-than subset B, $A \leq_s B$, iff for each $x \in A$, and $y \in B$, $x \leq y$ in S. This definition is due to Veinott.

[9]Requiring the underlying binary relations to be partial orders is important for the proposed analysis. Nonetheless, reflexivity, antisymmetry and also transitivity are not unduly restrictive for

Good i is a *Pathwise Net Substitute of good* j at price pair $p, p' \in \Re^n_{++}$ and attainable utility level $\bar{u}$, if the set relation $\leq_s$ above is replaced with $\leq_P$, the *pathwise* compatible set relation.[10]

If $i = j$ in the definitions above, then good i is a Strongly (Pathwise) *Own* Net Substitute at price pair (p, p') and at attainable utility level $\bar{u}$.

Good i is a Strongly (Pathwise) Net Substitute of good j (*everywhere*), if it is a Strongly (Pathwise) Net Substitute of good j at every such price pair, $p, p' \in \Re^n_{++}$, and level of attainable utility $\bar{u}$.

(b) Good i is a *Strongly (Pathwise) Net Complement* of good j at price pair $p, p' \in \Re^n_{++}$ and attainable utility level $\bar{u}$ (everywhere), as in (a) above, if the underlying partial order $\leq_{ns}$ in (a) is replaced with a partial order, $\leq_{nc}$, on the consumption set X which implies, whenever $x \leq_{nc} x'$:

$$\text{C1:} \quad x'_i \leq x_i$$

Such partial orders on the consumption set will be called *Net Complements Partial Orders* (denoted NCPOs).

It will prove convenient below, in the construction of Net Substitute (Complement) Partial Orders, as in definition 1.1, to assume $i = 1$ and $j = n$, in the name, and every relevant indexing of goods. Since this involves no loss of generality we will adopt it from now on without further comment.

Definition 1.1 gives necessary restrictions on NSPOs and NCPOs in conditions S1 and C1, with respect to good 1 (good i more generally) comparability. However, using revealed preference, we can establish that definition 1.1 suggests two further restrictions on NSPOs and NCPOs, in order to perform their descriptive role within the context of the expenditure minimization problem. Whenever x, x' are expenditure minimizing bundles at prices p, p' respectively, as in definition 1.1:

$$\text{H1:} \quad p \cdot x \leq p \cdot x' \quad \text{and} \quad \text{H2:} \quad p' \cdot x' \leq p' \cdot x$$

the purposes of the definition alone, in as much as they relate to physical attributes of consumption bundles. The definition makes clear that such underlying partial orders need not be unique.

[10] A subset A of a poset $(S, \leq)$ is *pathwise-lower-than* subset B, $A \leq_P B$, iff for each $x \in A$ $(y \in B)$ there exists $y \in B$ $(x \in A)$ such that $x \leq y$ in S (Antoniadou, 1996).

For obvious reasons we will call *all* pairs satisfying H1 and H2, *Hicks Consistent* pairs (at prices p, p'), and argue that NSPO/NCPO comparability must incorporate Hicks Consistency on the consumption set.[11]

Thus, from now on it will be assumed that, whenever pair x, y are comparable with respect to a NSPO, with $x \leq_{ns} y$, then (S1) $x_1 \leq y_1$, (H1) $p \cdot x \leq p \cdot y$, and (H2) $p' \cdot y \leq p' \cdot x$. Similarly, whenever pair x, y are comparable with respect to a NCPO, with $x \leq_{nc} y$, then (C1) $y_1 \leq x_1$, (H1) $p \cdot x \leq p \cdot y$ and (H2) $p' \cdot y \leq p' \cdot x$ (where $p, p' \in \Re^n_{++}$, with $p = (p_1, \ldots, p_n)$, $p' = (p_1, \ldots, p_{n-1}, p'_n)$, and $p_n < p'_n$).[12]

In fact with two goods H1 and H2 themselves define a partial order, while with three goods H1 and H2 with either S1 or C1 define a partial order. However, these conditions are no longer sufficient to completely define partial orders with four or more goods and therefore, in these more general cases, we have the task of constructing partial orders, NSPOs and NCPOs, without further (unwarranted) descriptive content, but with useful normative content. Before we do so however, we will present the main comparatives statics theorems of the paper in the next section. These presume the existence of NSPOs and NCPOs, but do not use their specific properties. It is hoped that these will motivate the constructions of NSPOs and NCPOs that are suggested in subsequent sections.

Even though the two goods case is too special to avail itself to the theorems of the next section, it is useful for motivating and expositing the proposed lattice structures and is therefore presented here:

[11]Thus the property, Hicks Consistency, which must hold at expenditure minimizing bundles is extended to *all* comparable pairs of bundles. This is justified since comparability is determined a priori. Also, it must be noted that H1 and H2 depend on the price pair, and therefore the partial orders themselves will depend on the particular price pair. Thus, when we refer to a particular NSPO or NCPO this will be a class of partial orders and not a unique partial order.

[12]An elementary observation is that, given the restriction on the pair of prices, H1 and H2 jointly imply $y_n \leq x_n$ (strict inequality if at least one of H1, H2 is strict). This is a restatement of the compensated law of demand, when only the price of one good changes (which is equivalent to the weak axiom of revealed preference). In the context of this paper, its implication is that a partial order which enables the derivation of sufficient conditions for good 1 to be a strongly/pathwise net substitute/complement of good n, will also imply that the latter is a Strongly Own Net Substitute everywhere (under no further restrictions on the utility function).

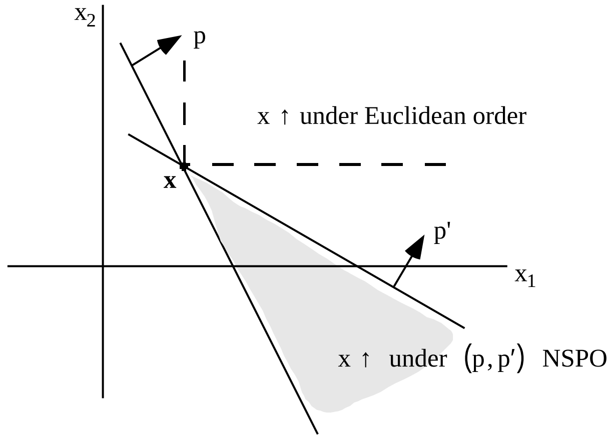

FIG. 1. The up-set of point x in the poset $\left(\Re^2, \leq_{ns}^2\right)$ with the (p,p') NSPO is depicted by the shaded area. Notice that the intersection of this set with the up-set of x under the Euclidean order is a singleton, x itself.

Special Case: Two Goods

It is easy to establish that with two goods H1 and H2 together imply S1 and that in fact H1 and H2 define a partial order on $\Re^2$. Therefore, a NCPO for two goods does not exist, while the NSPO (clearly unique) can be completely defined by H1, H2 alone. Thus, let us define:

DEFINITION 1.2. A pair x,y in $\Re^2$, is comparable with respect to *the* (p,p') *Net Substitutes Partial Order* (*NSPO* ^{2}) on $\Re^2$, w.l.o.g. $x \leq_{ns}^2 y$, if and only if:

$$\text{H1:} \;\; p_1 x_1 + p_2 x_2 \leq p_1 y_1 + p_2 y_2 \quad \text{and} \quad \text{H2}: \; p_1 y_1 + p_2' y_2 \leq p_1 x_1 + p_2' x_2$$

where $p = (p_1,p_2) \in \Re^2_{++}$ and $p' = (p_1,p_2')$ with $p_2 < p_2'$.

Verifying that the (p,p') Net Substitutes Partial Order on $\Re^2$ is indeed a partial order (reflexive, antisymmetric, and transitive) is immediate. It is also immediate to show that: $x \leq_{ns}^2 y$ implies $x_1 \leq y_1$ and $y_2 \leq x_2$, and, $x \leq_{ns}^2 y$ with $x \neq y$ imply $x_1 < y_1$ and $y_2 < x_2$. The up-set of any point in $\Re^2$ can be depicted graphically as in Figure 1.

In fact the poset $\left(\Re^2, \leq_{ns}^2\right)$ is a lattice with the (p,p') Net Substitutes Partial Order (but $\left(\Re^2_+, \leq_{ns}^2\right)$ is not a lattice). Given incomparable x,y in $\Re^2$ such that $p \cdot x < p \cdot y$ and $p' \cdot x < p' \cdot y$, their join is given by $x \vee y = \left(x_1 + \frac{p_2'(p \cdot y - p \cdot x)}{p_1\left(p_2' - p_2\right)},\; x_2 - \frac{p \cdot y - p \cdot x}{p_2' - p_2}\right)$,

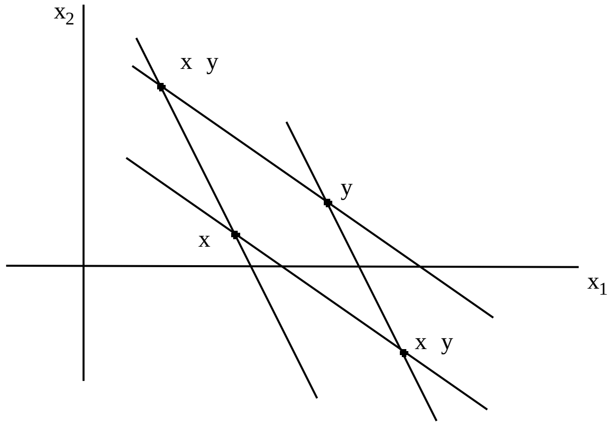

FIG. 2. The join and meet of incomparable x,y in $\left(\Re^2, \leq^2_{ns}\right)$ are such that $(x \wedge y)_1 < x_1$,$y_1 < (x \vee y)_1$ and $(x \vee y)_2 < x_2$,$y_2 < (x \wedge y)_2$. But notice that $\left(\Re^2_+, \leq^2_{ns}\right)$ is *not* a (sub)lattice.

and their meet is given by $x \wedge y = \left(x_1 - \frac{p_2\left(p' \cdot y - p' \cdot x\right)}{p_1\left(p'_2 - p_2\right)},\ x_2 + \frac{p' \cdot y - p' \cdot x}{p'_2 - p_2} \right)$. These are demonstrated in Figure 2.

Nonetheless, this lattice structure is not useful in the case of two goods. Because, no additional assumptions are needed in order to establish that whenever $\hat{x} \in \underset{x \in X}{argmin}\ \{p \cdot x \mid U(x) \geq \bar{u}\}$ and $\hat{x}' \in \underset{x \in X}{argmin}\ \{p' \cdot x \mid U(x) \geq \bar{u}\}$, then $\hat{x} \leq^2_{ns} \hat{x}'$, thus restating the elementary result that with two goods, good 1 is a Strongly Net Substitute of good 2 everywhere (and vice versa), and that each good is a Strongly Own Net Substitute, according to Definition 1.1 above, without further conditions.

The reader familiar with what has come to be known as the *Monotone Comparative Statics* literature may ponder what this may suggest about the quasi-supermodular class of properties of the utility function in $\left(X, \leq^2_{ns}\right)$. It should not be taken to suggest that such properties are not restrictive; rather the restictions that they impose are not relevant to the expenditure minimization problem since incomparable pairs in $\left(X, \leq^2_{ns}\right)$ are necessarily *not* Hicks Consistent.

This is a critical observation. The proposed NSPOs and NCPOs can induce a rich lattice structure. The expenditure minimization problem however does not need to make use of all this structure. Only the behavior of the utility function at Hicks Consistent pairs, their meets and joins, is relevant. The behavior of the utility function at incomparable pairs that are not Hicks Consistent will not add useful information for the expenditure minimization problem. Thus, we will use the following definitions:

Definition 1.3.

(a) $\Re^n$ endowed with a Net Substitutes Partial Order (satisfying S1, H1 and H2) is called a (p,p') *Net Substitutes Poset* and denoted $(\Re^n, \leq_{ns})$. Similarly, $\Re^n$ endowed with a Net Complements Partial Order (satisfying C1, H1 and H2) is called a (p,p') *Net Complements Poset* and denoted $(\Re^n, \leq_{nc})$.

(b) A subset X of $\Re^n$ is called a *Hicks Consistent Sublattice* of $(\Re^n, \leq_{ns})$ (alternatively of $(\Re^n, \leq_{nc})$), if the join and meet of every Hicks Consistent pair in $(\Re^n, \leq_{ns})$ (alternatively in $(\Re^n, \leq_{nc})$) taken in $\Re^n$ exists in X. It is a *Hicks Consistent Lattice* if the meet and join of every Hicks Consistent pair in X, taken in X, exists in X.

(c) A real-valued function $f : X \to \Re$ on a Hicks Consistent (sub)lattice is called *Hicks Consistent Quasi-Supermodular* if it is quasi-supermodular at Hicks Consistent pairs in X, i.e. for all Hicks Consistent pairs x,y such that (H1) $p \cdot x \leq p \cdot y$ and (H2) $p' \cdot y \leq p' \cdot x$:

$$f(x \wedge y) \underset{<}{\leq} f(x) \Longrightarrow f(y) \underset{<}{\leq} f(x \vee y) \;\; and \;\; f(x \wedge y) \underset{<}{\leq} f(y) \Longrightarrow f(x) \underset{<}{\leq} f(x \vee y)$$ [13]

Similarly, f is called *Hicks Consistent Strictly Quasi-Supermodular* if it is *strictly* quasi-supermodular at Hicks Consistent incomparable pairs of points in X, i.e.

$$f(x \wedge y) \leq f(x) \Longrightarrow f(y) < f(x \vee y) \;\; and \;\; f(x \wedge y) \leq f(y) \Longrightarrow f(x) < f(x \vee y)$$ [14]

(d) A real-valued function $f : X \to \Re$ on a Hicks Consistent (sub)lattice is called *Hicks Consistent Lower-Semi-Quasi-Supermodular* if for all Hicks Consistent pairs x,y such that (H1) $p \cdot x \leq p \cdot y$ and (H2) $p' \cdot y \leq p' \cdot x$:

$$f(x \wedge y) \underset{<}{\leq} f(x) \Longrightarrow f(y) \underset{<}{\leq} f(x \vee y)$$ [15]

Similarly f is called *Hicks Consistent Strictly Lower-Semi-Quasi-Supermodular* if instead:

$$f(x \wedge y) \leq f(x) \Longrightarrow f(y) < f(x \vee y)$$ [16]

(e) A real-valued function $f : X \to \Re$ on a Hicks Consistent (sub)lattice is called *Hicks Consistent Iso-Quasi-Supermodular* if it is quasi-supermodular, as in (c) above,

[13] Equivalently: $f(x \vee y) \underset{<}{\leq} f(y) \Rightarrow f(x) \underset{<}{\leq} f(x \wedge y)$ *and* $f(x \vee y) \underset{<}{\leq} f(x) \Rightarrow f(y) \underset{<}{\leq} f(x \wedge y)$.

[14] Equivalently: $f(x \vee y) \leq f(y) \Rightarrow f(x) < f(x \wedge y)$ *and* $f(x \vee y) \leq f(x) \Rightarrow f(y) < f(x \wedge y)$.

[15] Equivalently: $f(x \vee y) \underset{<}{\leq} f(y) \Rightarrow f(x) \underset{<}{\leq} f(x \wedge y)$.

[16] Equivalently: $f(x \vee y) \leq f(y) \Rightarrow f(x) < f(x \wedge y)$.

at Hicks Consistent pairs, x,y , which in addition satisfy $f(x) = f(y)$.

Similarly f is called *Hicks Consistent Strictly Iso-Quasi-Supermodular* if instead it is strictly quasi-supermodular at Hicks Consistent incomparable pairs, x,y , which in addition satisfy $f(x) = f(y)$. Equivalently, for all such x,y :

$$f\,(x) = f\,(y) < f\,(x \vee y) \;\; or \;\; f\,(y) = f\,(x) < f\,(x \wedge y)$$

The definition of a (strictly) quasi-supermodular function in Definition 1.3 is standard (Milgrom and Shannon 1994).[17] The definitions of a (strictly) lower-semi-quasi-supermodular and of (strictly) iso-quasi-supermodular function are new, weaker, variants of the standard definitions. A (strictly) iso-quasi-supermodular function is (strictly) semi-quasi-supermodular and a (strictly) semi-quasi-supermodular function is (strictly) quasi-supermodular.

We turn without further discussion to the main theorems of the paper, which, it is hoped, will make the usefulness of these definitions apparent.

2. THEOREMS FOR NET SUBSTITUTION EFFECTS

In this section we use the general properties of NSPOs and NCPOs, as discussed in the previous section, to derive sufficient conditions for net substitution effects as in definition 1.1. We have already seen how such partial order(s) can be constructed in the case of two goods. The NSPOs and NCPOs constructed in the following sections show how this can be done with three or more goods (for obvious reasons we assume in this section that there are at least three goods). It is on balance appropriate to give the theorems before discussing the construction of the relevant partial orders in order to indicate in advance their usefulness. The theorems will be revisited once the NSPOs and NCPOs have been discussed more fully in the following sections.

Theorem 2.1 (Net Substitutes).
Consider the consumer expenditure minimization problem, EM($p, \bar{u}$):

$$min\,\{p \cdot x \mid x \in X,\, U(x) \geq \bar{u}\,\}$$

(a) If:

[17] This is also the same as the definition of a (strictly) lattice superextremal function due to Veinott 1992.

(A) The consumption set X, $X \subset \Re^n$*, is a Hicks Consistent (Sub)lattice of a* (p,p') *Net Substitutes poset* $(\Re^n, \leq_{ns})$*, under some indexing of goods (with good 1 given index 1 and good n index n);*

(B) The consumer utility function $U : X \to \Re$ *is Hicks Consistent Iso-Quasi-Supermodular on* X*;*

Then, given feasible $\bar{u}$*, such that solutions to the problems EM*$(p, \bar{u})$ *and EM*$(p', \bar{u})$ *exist and are such that there is no excess utility:*

$$\underset{x \in X}{argmin} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_a \underset{x \in X}{argmin} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$$ [18]

and therefore

$$\underset{x \in X}{argmin} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_P \underset{x \in X}{argmin} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$$

i.e. good 1 is a Pathwise Net Substitute of good n at prices (p,p') *and at every such attainable utility level,* $\bar{u}$*.*

(b) If instead of (A) and (B) in part (a), (A), (B′) hold, where

(B′) The consumer utility function $U : X \to \Re$ *is Hicks Consistent* Strictly *Iso-Quasi-Supermodular on* X*, and*

Then, given feasible $\bar{u}$*, such that solutions to the problems EM*$(p, \bar{u})$ *and EM*$(p', \bar{u})$ *exist and are such that there is no excess utility:*

$$\underset{x \in X}{argmin} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_c \underset{x \in X}{argmin} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$$ [19]

If in addition:

(C) $I \equiv \underset{x \in X}{argmin} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \cap \underset{x \in X}{argmin} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$ *contains no more than one element,*

Then

$$\underset{x \in X}{\operatorname{argmin}} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_s \underset{x \in X}{\operatorname{argmin}} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$$

i.e. good 1 is a Strongly Net Substitute of good n at prices (p,p') *and at every such attainable utility level,* $\bar{u}$*.*

[18] Subset A is *strong set smaller* than subset B, $A \leq_a B$, if $A \wedge B \subseteq A$ and $A \vee B \subseteq B$.

[19] Subset A is *chain-lower-than* subset B, $A \leq_c B$, if $A \leq_a B$ and for each $a \in A$ and $b \in B$, a and b are comparable. Both this definition and the definition of strong set comparability are due Veinott.

Remark 2. 1. As stated the theorem does not give sufficient conditions for the minimization problems to have solutions, nor for these to imply no excess utility at all expenditure minimizing bundles. Standard assumptions can be employed for this. No excess utility is a standard assumption in the classical comparative statics analysis of this problem. However, it is important that the proposed approach, and in particular theorem 2.1, can accommodate discrete, finite consumption sets, where this assumption may not be justifiable. If so, the statment of the theorem can be adjusted by changing assumptions (B) and (B′) to: U is Hicks Consistent *Lower-Semi*-Quasi-Supermodular and Hicks Consistent *Strictly Lower-Semi*-Quasi-Supermodular, respectively. Statement $\underset{x\in X}{argmin}\,\{p\cdot x\,|\,U(x)\geq\bar{u}\}\leq_a \underset{x\in X}{argmin}\,\{p'\cdot x\,|\,U(x)\geq\bar{u}\}$ would be changed to $argmax\left\{U(\hat{x})\,\middle|\,\hat{x}\in\underset{x\in X}{argmin}\{p\cdot x|\,U(x)\geq\bar{u}\}\right\}\leq_a argmax\left\{U(\hat{x})\,\middle|\,\hat{x}\in\underset{x\in X}{argmin}\{p'\cdot x|\,U(x)\geq\bar{u}\}\right\}$ (with all subsequent statements adjusted correspondingly).

Proof (Theorem 2.1).

(a) Consider $\hat{x}\in\underset{x\in X}{\text{argmin}}\,\{p\cdot x\,|\,U(x)\geq\bar{u}\}$ and $\hat{x}'\in\underset{x\in X}{\text{argmin}}\,\{p'\cdot x\,|\,U(x)\geq\bar{u}\}$ which by assumption exist and are such that $U(\hat{x})=U(\hat{x}')=\bar{u}$. Clearly $\hat{x}$, $\hat{x}'$ are by definition Hicks Consistent, i.e. (H1) $p\cdot\hat{x}\leq p\cdot\hat{x}'$ and (H2) $p'\cdot\hat{x}'\leq p'\cdot\hat{x}$. If both of these are satisfied with equality, then either $\hat{x}\leq_{ns}\hat{x}'$ or $\hat{x}'\leq_{ns}\hat{x}$, but also in this case $\hat{x},\hat{x}'\in\underset{x\in X}{\text{argmin}}\,\{p\cdot x\,|\,U(x)\geq\bar{u}\}$ and $\hat{x},\hat{x}'\in\underset{x\in X}{\text{argmin}}\,\{p'\cdot x\,|\,U(x)\geq\bar{u}\}$. Thus, in either case $\hat{x}\wedge\hat{x}'\in\underset{x\in X}{\text{argmin}}\,\{p\cdot x\,|\,U(x)\geq\bar{u}\}$ and $\hat{x}\vee\hat{x}'\in\underset{x\in X}{\text{argmin}}\,\{p'\cdot x\,|\,U(x)\geq\bar{u}\}$ as required. Therefore, assume that at lease one of (H1), (H2) is a strict inequality. If $\hat{x}$, $\hat{x}'$ are comparable, $\hat{x}\leq_{ns}^{n}\hat{x}'$ and there is nothing further to prove. Hence assume that they are not comparable, i.e. $\hat{x}_1'<\hat{x}_1$. From assumption (A) their join and meet exist, and furthermore these satisfy (1) $p'\cdot(\hat{x}\vee\hat{x}')\leq p'\cdot\hat{x}'$, and (2) $p\cdot(\hat{x}\wedge\hat{x}')\leq p\cdot\hat{x}$, (also $\hat{x}_1\leq(\hat{x}\vee\hat{x}')_1$ and $(\hat{x}\wedge\hat{x}')_1\leq\hat{x}_1'$). Using the definition of $\hat{x}$, $\hat{x}'$ (under no excess utility) these imply (1′) $U(\hat{x})=U(\hat{x}')\geq U(\hat{x}\vee\hat{x}')$ and (2′) $U(\hat{x}')=U(\hat{x})\geq U(\hat{x}\wedge\hat{x}')$ respectively (strict inequalities if (a) or (b) are strict). But these and the assumption that U is Hicks Consistent iso-quasi-supermodular (assumption (B)) imply (1″) $U(\hat{x})=U(\hat{x}')\leq U(\hat{x}\wedge\hat{x}')$ and (2″) $U(\hat{x}')=U(\hat{x})\leq U(\hat{x}\vee\hat{x}')$ respectively (strict inequalities if (1′) or (2′) are strict). Therefore, (1) or (2) strict imply a contradiction and the hypothesis that $\hat{x}$, $\hat{x}'$ are incomparable is false. Otherwise, $U(\hat{x}) = U(\hat{x}') = U(\hat{x} \wedge \hat{x}') = U(\hat{x} \vee \hat{x}')$ and $\hat{x} \wedge \hat{x}' \in \underset{x \in X}{\operatorname{argmin}} \{p \cdot x \mid U(x) \geq \bar{u}\}$ and $\hat{x} \vee \hat{x}' \in \underset{x \in X}{\operatorname{argmin}} \{p' \cdot x \mid U(x) \geq \bar{u}\}$ as required to complete the proof.

(b) The proof of this part is very similar to that of part (a) and will therefore not be repeated here. We note that case assumption (C) implies that at least one of (H1) and (H2) is a strict inequality if $\hat{x} \neq \hat{x}'$. Hence if $\hat{x}, \hat{x}'$ are comparable there is nothing to prove. An analogous argument to that in (a) can be used to establish by contradiction that $\hat{x}, \hat{x}'$ cannot be incomparable, thus completing the proof.

∎

Theorem 2.2 below is the analog of theorem 2.1 in the case of net complements. But it bears a warning: as will be shown in the following sections, the assumption that the consumption set X is a (sub)lattice of the relevant poset is more difficult to establish here than in the case of net substitutes, especially if X is assumed to be a set bounded from below, a standard assumption in economics:

THEOREM 2.2 (Net Complements).
Consider the consumer expenditure minimization problem, EM($p, \bar{u}$), as in theorem 2.1 above.

(a) If:

(A) The consumption set X, $X \subset \Re^n$, is a Hicks Consistent (Sub)lattice of a (p,p') Net Complements poset $(\Re^n, \leq_{nc})$, under some indexing of goods giving good 1 index 1 and good n index n;

(B) The consumer utility function $U : X \to \Re$ is Hicks Consistent Iso-Quasi-Supermodular on X.

Then, given feasible $\bar{u}$, such that solutions to the problems EM($p, \bar{u}$) and EM($p', \bar{u}$) exist and are such that there is no excess utility:

$$\underset{x \in X}{argmin} \{p \cdot x \mid U(x) \geq \bar{u}\} \leq_a \underset{x \in X}{argmin} \{p' \cdot x \mid U(x) \geq \bar{u}\}$$

and therefore

$$\underset{x \in X}{\operatorname{argmin}} \{p \cdot x \mid U(x) \geq \bar{u}\} \leq_P \underset{x \in X}{\operatorname{argmin}} \{p' \cdot x \mid U(x) \geq \bar{u}\}$$

i.e. good 1 is a Pathwise Net Complement of good n at prices (p,p') and at every such attainable utility level, $\bar{u}$.

(b) If instead of (A) and (B) in part (a), (A), (B′) hold, where:

(B′) The consumer utility function $U : X \to \Re$ is Hicks Consistent Strictly *Iso-Quasi-Supermodular on X.*

Then given feasible $\bar{u}$, such that solutions to the problems EM($p, \bar{u}$) and EM($p', \bar{u}$) exist and are such that there is no excess utility:

$$\underset{x \in X}{\operatorname{argmin}} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_c \underset{x \in X}{\operatorname{argmin}} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$$

If in addition

(C) $\underset{x \in X}{\operatorname{argmin}} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \cap \underset{x \in X}{\operatorname{argmin}} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$ *contains no more than one element.*

Then

$$\underset{x \in X}{\operatorname{argmin}} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_s \underset{x \in X}{\operatorname{argmin}} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$$

i.e. Good 1 is a Strongly Net Complement of good n at prices (p,p') and every such attainable utility level, $\bar{u}$.

Proof. The proof is almost identical to the proof of theorem 2.1 and is omitted. ▮

It is straightforward to extend theorems 2.1 and 2.2 to give sufficient conditions for good 1 to be a Pathwise/Strongly Net Substitute or Complement of good n everywhere, by requiring conditions (A) and (B), or (B′), to hold in every such poset generated under all possible such price pairs (p,p'). It is also possible to weaken the sufficient conditions of theorems 2.1 and 2.2 so that good 1 is a Pathwise/Strongly Net Substitute or Complement of good n at a particular price pair (p,p') and particular attainable utility level (which may be more satisfactory especially in the case of net complements), by requiring that the consumption set be closed under joins and meets of Hicks Consistent pairs in the corresponding weakly preferred set (rather than all Hicks Consistent pairs). Also, these sufficient conditions can apply, and can be checked, with respect to more than one pair of goods simultaneously, thus establishing conditions for the nature of net substitutability of a group of goods with respect to any one good.

3. NSPOS AND NCPOS WITH THREE GOODS

The aim of this and the following section is to show how partial orders, that may be used to apply the comparative statics theorems of the previous section, can be constructed. We begin with the special case of three goods in this section.

Unlike the case of two goods, with three goods H1 and H2 alone do not define a partial order. Nonetheless, the three goods case is itself special because S1 (C1) suffices, along with H1 and H2, to define completely a partial order something which is not true with a larger number of goods. So let us begin by defining the NSPO and the NCPO (importantly these are unique) in three dimensions:

Definition 3.1.

(a) A pair x,y in $\Re^3$, is comparable with respect to *the* (p,p') *Net Substitutes Partial Order* (NSPO^3) on $\Re^3$, w.l.o.g. $x \leq^3_{ns} y$, if and only if:

$$\text{S1:} \quad x_1 \leq y_1 \qquad \text{H1:} \quad p \cdot x \leq p \cdot y \qquad \text{and} \qquad \text{H2 } : p' \cdot y \leq p' \cdot x$$

(b) A pair x,y in $\Re^3$, is comparable with respect to *the* (p,p') *Net Complements Partial Order* (NCPO^3) on $\Re^3$, w.l.o.g. $x \leq^3_{nc} y$, if and only if:

$$\text{C1:} \quad y_1 \leq x_1 \qquad \text{H1:} \quad p \cdot x \leq p \cdot y \qquad \text{and} \qquad \text{H2:} \quad p' \cdot y \leq p' \cdot x$$

where $p \in \Re^3_{++}$ and $p' = (p_1,p_2,p'_3)$ with $p_3 < p'_3$.

It is easy to verify that NSPO^3 and NCPO^3 are indeed partial orders,[20] and that $\left(\Re^3, \leq^3_{ns}\right)$ and $\left(\Re^3, \leq^3_{nc}\right)$ are Hicks Consistent lattices. In fact, given Hicks Consistent incomparable pair x,y in the (p,p') Net Substitutes Poset, $\left(\Re^3, \leq^3_{ns}\right)$, such that (w.l.o.g.): (NS1) $y_1 < x_1$, (H1) $p \cdot x \leq p \cdot y$, and (H2) $p' \cdot y \leq p' \cdot x$ (at least one of H1 and H2 strict inequality), then $x \vee y = \left(x_1,y_2 - \frac{p_1}{p_2}(x_1 - y_1),y_3\right)$ and $x \wedge y = \left(y_1,x_2 + \frac{p_1}{p_2}(x_1 - y_1),x_3\right)$.[21] Furthermore, the join and meet of x,y satisfy:

[20] One simple fact which distinguishes NCPO^3 from NSPO^3, is that $x \leq^3_{nc} y$ implies $x_2 \leq y_2$ (i.e. $y_1 \leq x_1$, $x_2 \leq y_2$ and $y_3 \leq x_3$).This again restates what we would have expected, namely that with three goods two goods cannot be both simultaneously net complements of the third one (with strict inequalities).

[21] Theproof thatthese arethe joinand meet of x,y is straightforward. Here wesketch theargument for the join. Let $z \equiv \left(x_1,y_2 - \frac{p_1}{p_2}(x_1 - y_1),y_3\right)$. Clearly $p \cdot z = p \cdot y$, $p' \cdot z = p' \cdot y$ and therefore $x,y \leq^3_{ns} z$, i.e. z is an upper bound. Consider any upper bound of x,y, say $r \equiv (r_1,r_2,r_3)$. By definition, $x_1 \leq r_1$, $p \cdot y \leq p \cdot r$ and $p' \cdot r \leq p' \cdot y$. Thus $z_1 \leq r_1$, $p \cdot z \leq p \cdot r$, and $p' \cdot r \leq p' \cdot z$, i.e. $z \leq^3_{ns} r$ as required, establishing $z = x \vee y$.

$x_2 < (x \vee y)_2 < y_2$ and $x_2 < (x \wedge y)_2 < y_2$, with $(x \vee y)_2 + (x \wedge y)_2 = x_2 + y_2$. Therefore, $x,y \in \Re^3_+$ implies $x \vee y, x \wedge y \in \Re^3_+$ and $\Re^3_+$ is closed under joins and meets of Hicks Consistent pairs taken in $\left(\Re^3, \leq^3_{ns}\right)$. The join and meet also satisfy: $p \cdot (x \vee y) = p \cdot y,\ \ p' \cdot (x \vee y) = p' \cdot y,\ \ p \cdot (x \wedge y) = p \cdot x,\ \ p' \cdot (x \wedge y) = p' \cdot x$.

The construction of joins and meets under NCPO^3 is analogous. Given Hicks Consistent incomparable pair x,y in the (p,p') Net Complements Poset, $\left(\Re^3, \leq^3_{nc}\right)$, such that (w.l.o.g.): (NC1) $x_1 < y_1$, (H1) $p \cdot x \leq p \cdot y$, and (H2) $p' \cdot y \leq p' \cdot x$ (at least one of H1, H2 strict inequality), then $x \vee y = \left(x_1, y_2 + \frac{p_1}{p_2}(y_1 - x_1), y_3\right)$ and $x \wedge y = \left(y_1, x_2 - \frac{p_1}{p_2}(y_1 - x_1), x_3\right)$.[22] Furthermore the join and meet of x,y satisfy $p \cdot (x \vee y) = p \cdot y,\ \ p' \cdot (x \vee y) = p' \cdot y,\ \ p \cdot (x \wedge y) = p \cdot x,\ \ p' \cdot (x \wedge y) = p' \cdot x$, $x_2, y_2 < (x \vee y)_2$ and $(x \wedge y)_2 < x_2, y_2$, with $(x \vee y)_2 + (x \wedge y)_2 = x_2 + y_2$. Thus $x,y \in \Re^3_+$ implies $x \vee y \in \Re^3_+$, but not necessarily so for the meet. Thus $\Re^3_+$ is closed under joins (but not meets) of Hicks Consistent pairs taken in $\left(\Re^3, \leq^3_{nc}\right)$.

The partial orders NSPO^3 and NCPO^3 are obviously related; whenever a Hicks Consistent pair x,y at prices p,p', is incomparable with respect to NSPO^3 it is comparable with respect to NCPO^3, and vice versa. Thus, even though joins and meets of Hicks Consistent pairs incomparable with respect to NCPO^3 take the same form as those of Hicks Consistent pairs incomparable with respect to NSPO^3, the points where these occur are mutually exclusive. Furthermore, it is important to note that despite their algebraic similarity, while $\Re^3_+$ is a Hicks Consistent sublattice of the (p,p') Net Substitutes Poset, $\left(\Re^3, \leq^3_{ns}\right)$, it is not so in the corresponding (p,p') Net Complements Poset, $\left(\Re^3, \leq^3_{nc}\right)$.

In fact this difference between NSPO^3 and NCPO^3 is even more pervasive. It is not difficult to verify that $\Re^3_+$ is not closed under meets of Hicks Consistent pairs under NCPO^3 even when these are taken in $\Re^3_+$ itself (poset $\left(\Re^3_+, \leq^3_{nc}\right)$).[23] This difference will extend to the n-dimensional extensions of NSPOs and NCPOs. What is critical about $\Re^3_+$, or $\Re^n_+$ more generally, is that it is bounded below, and indeed it can be established that this difference between the NSPOs and NCPOs extends to any such

[22] The argument establishing that these are the join and meet, respectively, of x,y is analogous to that under NSPO^3. We sketch the argument for the meet here. Let $w \equiv \left(y_1, x_2 - \frac{p_1}{p_2}(y_1 - x_1), x_3\right)$. Clearly $p \cdot w = p \cdot x$, $p' \cdot w = p' \cdot x$ and therefore $w \leq^3_{nc} x,y$, i.e. w is a lower bound. Consider any other lower bound of $x,yr \leq p \cdot x$ and $p' \cdot x \leq p' \cdot r$. Therefore, $w_1 \leq r_1$, $p \cdot r \leq p \cdot w$, and $p' \cdot w \leq p' \cdot r$, i.e. $r \leq^3_{nc} w$ as required, establishing $w = x \wedge y$.

[23] Themeetdoesnotexistfor Hicks Consistentincomparablepairs x,y such that $p_1x_1 + p_2x_2 < p_1y_1$ (whereby hypothesis $x_1 < y_1$). This could nothappen in the case of a Hicks Consistent incomparable pair under NSPO^3 since in that case the corresponding hypothesis would be $y_1 < x_1$.

set bounded from below. Since the non-negative quadrant (or other sets bounded from below) is often assigned to be the consumer consumption set, this contrast between the Net Substitutes and Net Complements Partial Orders is important and alludes to something intuitive, that it is more difficult to establish the net complementarity, that the net subsitutability, of a good for another at every attainable utility level.

Before going on to the more general case, with four or more goods, we conclude this section with two simple examples, Cobb-Douglas preferences and quasi-linear preferences. The comparative statics results in these examples are known, the method of arriving at these is clearly different. They show that having established the basic properties of NSPO^3, the relevant net substitute posets and Hicks Consistent (sub)lattices, the comparative statics analysis involves little more than elementary inequality manipulation:

Example 3.1. If the consumer utility function over three goods is Cobb-Douglas:

$$U : \Re^3_+ \to \Re \quad U(x) \equiv x_1^{\alpha} x_2 x_3^{\gamma} \quad \alpha,\gamma > 0$$

(a) The function U is Hicks Consistent weakly quasi-supermodular in the (p,p') Hicks Consistent (sub)lattice $\left(\Re^3_+, \leq^3_{ns}\right)$, for every pair of prices p,p' such that $p \in \Re^3_{++}$ and $p' = (p_1,p_2,p'_3)$ with $p_3 < p'_3$. It is strictly quasi-supermodular at Hicks Consistent pairs in the interior of the consumption set, $\Re^3_{++}$.

(b) Each good is a strongly net substitute of every other good everywhere (at positive utility levels).

Proof. (b) This follows from (a) and theorem 2.1, by observing that the indexing of goods is inconsequential in the proof of (a) (also the argmax is a singleton):
(a) Given Hicks consistent incomparable pair x,y, such that (NS1) $y_1 < x_1$, (H1) $p \cdot x \leq p \cdot y$, and (H2) $p' \cdot y \leq p' \cdot x$ (at least one of H1, H2 strict inequality), $x \vee y = \left(x_1,y_2 - \frac{p_1}{p_2}(x_1 - y_1),y_3\right)$ and $x \wedge y = \left(y_1,x_2 + \frac{p_1}{p_2}(x_1 - y_1),x_3\right)$. Also along with $y_1 < x_1$, we have: $y_3 < x_3$, $x_2 < y_2$, $x_2 < (x \wedge y)_2$, $(x \vee y)_2 < y_2$. Thus $x_1,x_3,y_2,(x \wedge y)_2,(x \vee y)_2$ are strictly positive. In order to prove the required result, assume first $U(x) \geq U(x \wedge y)$, i.e.

$$
\begin{aligned}
& x_1^\alpha x_2 x_3^\gamma \geq y_1^\alpha (x \wedge y)_2 \, x_3^\gamma \\
\Longleftrightarrow\ & x_1^\alpha x_2 \geq y_1^\alpha (x \wedge y)_2 \quad \text{since } x_3 > 0 \\
\Longleftrightarrow\ & x_1^\alpha (x \vee y)_2 \geq y_1^\alpha (x \wedge y)_2 + x_1^\alpha [y_2 - (x \wedge y)_2] \quad \text{since } (x \wedge y)_2 + (x \vee y)_2 = x_2 + y_2 \\
\Longrightarrow\ & x_1^\alpha (x \vee y)_2 > y_1^\alpha (x \wedge y)_2 + y_1^\alpha [y_2 - (x \wedge y)_2] \quad \text{since } x_1^\alpha > y_1^\alpha,\ y_2 - (x \wedge y)_2 > 0 \\
\Longrightarrow\ & x_1^\alpha (x \vee y)_2 \, y_3^\gamma \geq y_1^\alpha y_2 y_3^\gamma \quad \text{strict inequality if } y_3 > 0
\end{aligned}
$$

i.e. $U(x) \geq U(x \wedge y)$ implies $U(x \vee y) \geq U(y)$ for x,y in $\Re^3_+$, and it implies $U(x \vee y) > U(y)$ for x,y in $\Re^3_{++}$.

Thus U is Hicks Consistent Lower-Semi-Quasi-Supermodular on $\Re^3_+$ and stictly so on $\Re^3_{++}$. This suffices to establish (b). In order to prove that in addition U is Hicks Consistent Quasi-Supermodular on $\Re^3_+$, and strictly so on $\Re^3_{++}$, assume next that $U(x) \geq U(x \vee y)$. The remaining steps are as elementary as the steps above and are therefore omitted. ▮

Example 3.2. If the consumer utility function over three goods is given by:

$$U : \Re^3_+ \to \Re \quad U(x) = \min\{x_1, x_2\} + x_3$$

Then goods 1 and 2 are strongly net substitutes of good 3 everywhere.

Proof. Given theorem 2.1, the result follows if we prove that the utility function is strictly lower-semi-quasi-supermodular in every (p, p') Hicks Consistent (sub)lattice $\left(\Re^3_+, \leq^3_{ns}\right)$ (and observe that the indexing of goods 1 and 2 is inconsequential in the proof):

As in example 3.1, given Hicks consistent incomparable pair x,y, such that (NS1) $y_1 < x_1$, (H1) $p \cdot x \leq p \cdot y$, and (H2) $p' \cdot y \leq p' \cdot x$ (at least one of H1, H2 is a strict inequality), $x \vee y = \left(x_1,\ y_2 - \frac{p_1}{p_2}(x_1 - y_1),\ y_3\right)$ and $x \wedge y = \left(y_1,\ x_2 + \frac{p_1}{p_2}(x_1 - y_1),\ x_3\right)$, $y_1 < x_1$, we have: $y_3 < x_3$, $x_2 < y_2$, $x_2 < (x \wedge y)_2$, $(x \vee y)_2 < y_2$, and $x_1, x_3, y_2, (x \wedge y)_2, (x \vee y)_2$ are strictly positive. Suppose $U(x) \geq U(x \wedge y)$, i.e. $\min\{x_1, x_2\} + x_3 \geq \min\{y_1, (x \wedge y)_2\} + x_3$ which implies $\min\{x_1, x_2\} + y_3 \geq \min\{y_1, (x \wedge y)_2\} + y_3$. If $x_2 < x_1$, i.e. $\min\{x_1, x_2\} = x_2$, then $x_1 > x_2 \geq \min\{y_1, (x \wedge y)_2\}$ implies $\min\{y_1, (x \wedge y)_2\} = y_1$ since $(x \wedge y)_2 > x_2$. Hence $U(x \vee y) > U(y)$ as required since $\min\{x_1, (x \vee y)_2\} > \min\{x_1, x_2\} = x_2 \geq y_1 = \min\{y_1, (x \wedge y)_2\} = \min\{y_1, y_2\}$ or $\min\{x_1, (x \vee y)_2\} + y_3 > \min\{y_1, y_2\} + y_3$. If $x_1 \leq x_2$, i.e. $\min\{x_1, x_2\} = x_1$ then again it must be that $\min\{y_1, (x \wedge y)_2\} =$

y_1 since $(x \wedge y)_2 > x_2 \geq x_1 > y_1$. Hence in this case $U(x) > U(x \wedge y)$ and $\min\{x_1, (x \vee y)_2\} + y_3 > \min\{y_1, y_2\} + y_3$ or $U(x \vee y) > U(y)$ as required, since

$\min\{x_1, (x \vee y)_2\} = \min\{x_1, x_2\} = x_1$ and $\min\{y_1, (x \wedge y)_2\} = \min\{y_1, y_2\} = y_1$. ∎

4. NSPOS AND NCPOS WITH MORE THAN THREE GOODS

With more than three goods S1, H1 and H2 (alternatively C1, H1 and H2) do not define a partial order. There is now arbitrariness in the indexing of goods $2,\ldots,n-1$, and corresponding flexibility in their normative role. Therefore, as we have already alluded to above, with more than three goods we must differentiate between the *name*, $1,\ldots,n$, of a good, and its *index*, $1,\ldots,n$.[24]

We are faced with the challenge of constructing partial orders that are descriptively parsimonious and also normatively rich. Perhaps the most obvious and descriptively parsimonious way to construct partial orders from S1, H1 and H2 (respectively C1, H1 and H2) is by replacing S1 (C1) with a (generalized) lexicographic order. The lexicographic order does not add significantly, if at all, to the descriptive content of an NSPO or NCPO, as defined in the previous section. Its import (over and above S1/C1) in terms of the descriptive performance of the partial order is non-vacuous only at critical cases (when $y_1 = x_1$, where x,y is a Hicks Consistent pair). But, since the indexing of goods $(2,\ldots,n-1)$ is arbitrary and since we would expect good n to have at least one net substitute (strict inequality) this should not present a problem with the choice of an appropriate indexing. Furthermore, the (generalized) lexicographic order is particularly appealing in being a total binary relation (order), thus ensuring the existence of a partial order when combined with H1 and H2. We will call the partial orders so constructed (definition 4.1 below) the *Lexicographic* NSPO (denoted LNSPO) and the *Lexicographic* NCPO (denoted LNCPO), respectively.

LNSPOs and LNCPOs are important in their descriptive role and also as a useful benchmark. However, in the normative role of LNSPO and LNCPO, in the derivation of sufficient conditions on the utility function, the import of the lexicographic order is a lot more substantive, since it affects the nature of joins and meets of Hicks Consistent

[24] Recall that an indexing of the goods is a one-to-one mapping from the names of goods onto their indices. Obviously, the natural indexing corresponds to the identity map, and again unless otherwise warranted we will continue using the natural indexing of goods for simplicity. For obvious reasons, all relevant indexings of goods will be such that good 1 is mapped to index 1 and good n to index n.

incomparable pairs in the commodity space. It enables a particularly simple form for such joins and meets, such that only the quantities of the two goods in question and only one other good (the good with index $n-1$) are adjusted.[25] While the simplicity of the construction is attractive, its inflexibility is a drawback. Once the indexing of goods is determined, and in particular index $n-1$ is assigned, this good is given undue influence in the comparability of any pair of bundles without regard to any other information that the pair of bundles may contain.

The most obvious information that we can draw from a pair of bundles, is the list of goods that are in larger quantity in one bundle and those that are in larger quantity in the other. This gives some a-priori information about which goods are candidates for being net substitutes and which net complements of good n, and it would seem appropriate that it is used. We apply this intuition to provide extensions of LNSPO and LNCPO (definition 4.2 below) which utilize the arbitrariness of the indexing of goods $2,\ldots,n-1$, as suggested here, thus enhancing the normative performance of these partial orders, while maintaining the descriptive parsimony of LNSPOs and LNCPOs in binary comparisons. We call these partial orders *Augmented* LNSPO (denoted ALNSPO) and *Augmented* LNCPO (denoted ALNCPO).

Definition 4.1.

(a) A pair x,y in $\Re^n$, is comparable with respect to the *Lexicographic* (p,p') *Net Substitutes Partial Order* (LNSPO) on $\Re^n$, w.l.o.g. $x \leq^n_{ns} y$, if and only if:

$$\text{LS1:} \quad x \leq_L y \qquad \text{H1:} \quad p \cdot x \leq p \cdot y \quad \text{and} \quad \text{H2:} \quad p' \cdot y \leq p' \cdot x$$

where $\leq_L$ is the lexicographic order on $\Re^n$ under some indexing of goods (such that good 1 is given index 1 and good n index n), $p \in \Re^n_{++}$ and $p' = (p_1,p_2,\ldots,p_{n-1},p'_n)$ with $p_n < p'_n$.[26]

We call $(\Re^n, \leq^n_{ns})$ the *Lexicographic* (p,p') *Net Substitutes poset*.

(b) A pair x,y in $\Re^n$, is comparable with respect to the *Lexicographic* (p,p') *Net Complements Partial Order* (LNCPO) on $\Re^n$, w.l.o.g. $x \leq^n_{nc} y$, if and only if:

$$\text{LC1:} \quad x \leq_{LC} y \qquad \text{H1:} \quad p \cdot x \leq p \cdot y \quad \text{and} \quad \text{H2:} \quad p' \cdot y \leq p' \cdot x$$

[25] When the set is bounded below joins and meets can take more complicated form.

[26] There are $(n-2)!$ index sets which would give rise to distinct LNSPOs with a priori equivalent descriptive power in describing good 1 as a (strongly/pathwise) net substitute of good n. However, we suppress this fact in the notation, and we use the natural indexing whenever this involves no loss of generality, for notational simplicity.

where $\leq_{LC}$ is the generalized lexicographic order on $\Re^n$, under some indexing of goods (such that good 1 is given index 1 and good n index n), such that $x \leq_{LC} y$ iff $y_1 < x_1$, or $y_1 = x_1$ and $x_{-1} \leq_L y_{-1}$, where $x_{-1} \equiv (x_2,\ldots,x_n)$ and p,p' are as in (a) above. We call $(\Re^n, \leq^n_{nc})$ the *Lexicographic* (p,p') *Net Complements poset*.

Definition 4.2.

(a) A pair x,y in $\Re^n$, is comparable with respect to the *Augmented Lexicographic* (p,p') *Net Substitutes Partial Order* (ALNSPO) on $\Re^n$, w.l.o.g. $x \leq^n_{ans} y$, if and only if:

$$x \leq^n_{ns} y \quad \text{(i.e. LS1: } x \leq_L y; \quad \text{H1: } p \cdot x \leq p \cdot y; \quad \text{H2: } p' \cdot y \leq p' \cdot x)$$

(where $\leq_L$, p,p' are as in Definition 4.1(a) above)
and (whenever $n \geq 4$):

$$\text{S2: } S^k_{yx} < 0 \quad \Rightarrow \quad (y_{k+1}, \cdots, y_n) \leq_E (x_{k+1}, \cdots, x_n), \; k = 2,\ldots,n-2$$

where $S^k_{yx} \equiv \sum_{i=1}^{k} p_i\,(y_i - x_i)$, and $\leq_E$ is the usual (Euclidean) order.
We call $(\Re^n, \leq^n_{ans})$ the *Augmented Lexicographic* (p,p') *Net Substitutes poset*.

(b) A pair x,y in $\Re^n$, is comparable with respect to the *Augmented Lexicographic* (p,p') *Net Complements Partial Order* (ALNCPO) on $\Re^n$, w.l.o.g. $x \leq^n_{anc} y$, if and only if:

$$x \leq^n_{nc} y \quad \text{(i.e. LC1: } x \leq_{LC} y; \quad \text{H1: } p \cdot x \leq p \cdot y; \quad \text{H2: } p' \cdot y \leq p' \cdot x)$$

(where $\leq_{LC}$, p,p' are as in Definition 4.1(b) above)
and (whenever $n \geq 4$):

$$\text{C2: } S^{2,k}_{yx} < 0 \quad \Rightarrow \quad (y_{k+1}, \cdots, y_n) \leq_E (x_{k+1}, \cdots, x_n) \qquad k = 2,\ldots,n-2$$

where $S^{2,k}_{yx} \equiv \sum_{i=2}^{k} p_i\,(y_i - x_i)$, and $\leq_E$ is the usual (Euclidean) order.
We call $(\Re^n, \leq^n_{anc})$ the *Augmented Lexicographic* (p,p') *Net Complements poset*.

Conditions S2 and C2 in definition 4.2 warrant comment. Their purpose is to force the choice of the indexing of goods to be such as to give potential complements of good n high indices. It may be suggested that a simpler way of achieving this is by using: (S2′) $(x_1, \cdots, x_k) \leq_E (y_1, \cdots, y_k)$, $(y_{k+1}, \cdots, y_n) \leq_E (x_{k+1}, \cdots, x_n)$, $k \in \{2, \cdots, n-1\}$ This condition implies (S1) and (S2), but it is not equivalent to them. The problem with it is that if k is not fixed for all pairs then it fails to be transitive, and fixing k is unduly limiting. An alternative condition that does give rise to a partial order is:

$$\text{(D)} \quad y_{k+1} - x_{k+1} \leq y_k - x_k \quad k = 2, \ldots, n-2$$

This, given (S1), implies (S2). Furthermore, it is not descriptively cumbersome if applied to each pair of bundles under suitable indexing, and together with (S1), (H1) and (H2) it defines a partial order. Indeed, it can be shown that $\Re^n$ is a Hicks Consistent lattice with the ensuing partial order. The difficulty is that $\Re^n_+$ is not a Hicks Consistent (sub)lattice. Therefore, we choose to proceed with (S2) (and (C2)). It is convenient to first give the properties of LNSPOs and LNCPOs and this is done in lemmas 4.1,4.2, and 4.3:

Lemma 4.1.

(a) *LNSPOs and LNCPOs are partial orders on* $\Re^n$.

(b) *(i)* $x \leq^n_{ns} y$ *implies* $x_1 \leq y_1$, $y_n \leq x_n$ *and* $S^{n-1}_{yx} \geq 0$;

(ii) $x \leq^n_{nc} y$ *implies* $y_1 \leq x_1$, $y_n \leq x_n$ *and* $S^{n-1}_{yx} \geq 0$.

(c) *When* $n = 3$ *LNSPO coincides with NSPO*^{3} *and LNCPO with NCPO*^{3} *(definition 3.1), and when* $n = 2$ *LNSPO coincides with NSPO*^{2} *(definition 1.2).*

Proof. See Appendix ▮

Lemma 4.2.

(a) *Given Hicks Consistent incomparable pair* x,y *in the Lexicographic* (p,p') *Net Substitutes poset* $(\Re^n, \leq^n_{ns})$, *such that w.l.o.g.: (NLS1)* $y <_L x$, *(H1)* $p \cdot x \leq p \cdot y$, *and (H2)* $p' \cdot y \leq p' \cdot x$ *(at least one of H1, H2 strict inequality), then*

$$x \vee y = \left(x_1, x_2, \ldots, x_{n-2},\; x_{n-1} + \frac{S^{n-1}_{yx}}{p_{n-1}},\; y_n \right) \tag{1}$$

$$x \wedge y = \left(y_1, y_2, \ldots, y_{n-2},\ y_{n-1} - \frac{S_{yx}^{n-1}}{p_{n-1}},\ x_n \right) \tag{2}$$

with $p \cdot (x \vee y) = p \cdot y$, $p' \cdot (x \vee y) = p' \cdot y$, $p \cdot (x \wedge y) = p \cdot x$, $p' \cdot (x \wedge y) = p' \cdot x$, *and* $(x \vee y)_{n-1} + (x \wedge y)_{n-1} = x_{n-1} + y_{n-1}$

(b) Given Hicks Consistent incomparable pair x,y *in the Lexicographic* (p,p') *Net Substitutes poset* $\left(\Re_+^n, \leq_{ns}^n\right)$, *as in (a) above, their join,* $x \vee y$, *is given by (1) above, and their meet,* $x \wedge y$, *is given by:*

$$x \wedge y = \begin{cases} \left(y_1, y_2, \ldots, y_{n-2}, y_{n-1} - \frac{S_{yx}^{n-1}}{p_{n-1}},\ x_n \right) & if\ S_{yx}^{n-1} \leq p_{n-1} y_{n-1} \\ \vdots & \\ \left(y_1, \ldots, y_{k-1}, y_k - \frac{S_{yx}^{n-1} - S_y^{k+1,n-1}}{p_k}, 0, \ldots, 0, x_n \right) & if\ S_y^{k+1,n-1} < S_{yx}^{n-1} \leq S_y^{k,n-1} \\ \vdots & \\ \left(y_1, y_2 - \frac{S_{yx}^{n-1} - S_y^{3,n-1}}{p_2}, 0, \ldots, 0,\ x_n \right) & if\ S_y^{3,n-1} < S_{yx}^{n-1} \end{cases} \tag{3}$$

where $S_y^{k,n-1} \equiv p_k y_k + \cdots + p_{n-1} y_{n-1}$ *and, as before,* $S_{yx}^k \equiv \sum_{i=1}^{k} p_i (y_i - x_i)$. *Also,* $p \cdot (x \vee y) = p \cdot y$, $p' \cdot (x \vee y) = p' \cdot y$, $p \cdot (x \wedge y) = p \cdot x$,$p' \cdot (x \wedge y) = p' \cdot x$.

Proof. See Appendix ▮

LEMMA 4.3.

(a) Given Hicks Consistent incomparable pair x,y *in the Lexicographic* (p,p') *Net Complements poset* $(\Re^n, \leq_{nc}^n)$, *such that w.l.o.g.: (NLC1)* $y <_{LC} x$, *(H1)* $p \cdot x \leq p \cdot y$, *and (H2)* $p' \cdot y \leq p' \cdot x$ *(least one of H1, H2 strict inequality), their join is given by 1, and their meet by 2 in lemma 4.2 above. Furthermore these satisfy* $p \cdot (x \vee y) = p \cdot y$, $p' \cdot (x \vee y) = p' \cdot y$, $p \cdot (x \wedge y) = p \cdot x$, $p' \cdot (x \wedge y) = p' \cdot x$, *and* $(x \vee y)_{n-1} + (x \wedge y)_{n-1} = x_{n-1} + y_{n-1}$.

(b) Given Hicks Consistent incomparable pair x,y *in the Lexicographic* (p,p') *Net Complements poset* $\left(\Re_+^n, \leq_{nc}^n\right)$, *as in (a) above, their join is given by 1 in lemma 4.2. Their meet is well defined and is given by 3 in lemma 4.2 if and only if* $p_1 y_1 \leq p_{-n} \cdot x_{-n}$.

Proof. See Appendix ▮

Lemmas 4.2 and 4.3 show that, as in the three goods case, the algebraic structure of the join and meet of Hicks Consistent pairs in $\Re^n$ under LNSPO and LNCPO is the same. Obviously the pairs where these occur are mutually exclusive. Consider a Hicks Consistent pair, x,y, in $\Re^n$ such that w.l.o.g. (H1) $p \cdot x \leq p \cdot y$, and (H2) $p' \cdot y \leq p' \cdot x$ (with at least one of H1, H2 a strict inequality): If $x_1 = y_1$ then x,y is comparable with respect to (p,p') LNSPO if and only if it is comparable with respect to (p,p') LNCPO. If $x_1 \neq y_1$ then x,y is comparable with respect to (p,p') LNSPO if and only if it is not comparable with respect to (p,p') LNCPO. Let $z \equiv \left(x_1,\ldots,x_{n-2},x_{n-1} + \frac{S_{yx}^{n-1}}{p_{n-1}},y_n\right)$, $r \equiv \left(y_1,\ldots,y_{n-2},y_{n-1} - \frac{S_{yx}^{n-1}}{p_{n-1}},x_n\right)$. If $y_1 < x_1$, so that x,y is incomparable with respect to LNSPO and comparable with respect to LNCPO, and $z \equiv x \vee_{ns} y$ and $r \equiv x \wedge_{ns} y$, (lemma 4.2). Then r,z are Hicks Consistent incomparable with respect to LNCPO, with $r \vee_{nc} z = y$ and $r \wedge_{nc} z = x$. Similarly, if $x_1 < y_1$, so that x,y is incomparable with respect to LNCPO and comparable with respect to LNSPO, and $z = x \vee_{nc} y$ and $r = x \wedge_{nc} y$ (lemma 4.3). Then r,z are Hicks Consistent incomparable with respect to LNSPO, with $r \vee_{ns} z = y$ and $r \wedge_{ns} z = x$.

We now turn to the discussion of ALNSPOs and ALNCPOs. Lemma 4.4 collects some basic properties of ALNSPOS and ALNCPOs and then lemma 4.5 shows that in comparing any pair of bundles they have no descriptive content over that of LNSPO and LNCPO respectively, if an appropriate indexing is chosen:

Lemma 4.4.

(a) *(i)* $x \leq_{ans}^n y$ *implies* $x_1 \leq y_1$, $y_n \leq x_n$ *and* $S_{yx}^k \geq 0$, $k = 2,\ldots,n-1$ *(and thus the conditions in S2 are not binding);*

(ii) $x \leq_{anc}^n y$ *implies* $y_1 \leq x_1$, $y_n \leq x_n$, $S_{yx}^{n-1} \geq 0$ *and* $S_{yx}^{2,k} \geq 0$, $k = 2,\ldots,n-1$ *(and thus the conditions of C2 are not binding).*

(b) *ALNSPOs and ALNCPOs are partial orders on* $\Re^n$.

(c) *When* $n = 3$ *ALNSPO coincides with NSPO*^{3} *and ALNCPO with NCPO*^{3} *(definition 3.1). When* $n = 2$ *ALNSPO coincides with NSPO*^{2} *(definition 1.2).*

Proof. See Appendix ▮

LEMMA 4.5.

(a) $x \leq^n_{ans} y$ *in* $(\Re^n, \leq^n_{ans})$ *implies* $x \leq^n_{ns} y$ *in* $(\Re^n, \leq^n_{ns})$ *under the same indexing, and* $x \leq^n_{anc} y$ *in* $(\Re^n, \leq^n_{anc})$ *implies* $x \leq^n_{nc} y$ *in* $(\Re^n, \leq^n_{nc})$ *also under the same indexing.*

(b) *(i)* *If* $x \leq^n_{ns} y$ *in* $(\Re^n, \leq^n_{ns})$ *under the natural indexing of goods (w.l.o.g.), then there exists an indexing of goods (where good 1 is given index 1 and good n index n), possibly different, such that* $x \leq^n_{ans} y$ *in* $(\Re^n, \leq^n_{ans})$ *under this indexing. In particular this will be so if goods are indexed according to:*

$$\text{(D)} \quad y_{k+1} - x_{k+1} \leq y_k - x_k \qquad k = 2,\ldots,n-2$$

(ii) *If* $x \leq^n_{nc} y$ *in* $(\Re^n, \leq^n_{nc})$ *under the natural indexing of goods (w.l.o.g.), then there exists an indexing of goods (where good 1 is given index 1 and good n index n), possibly different, such that* $x \leq^n_{anc} y$ *in* $(\Re^n, \leq^n_{anc})$ *under this indexing. In particular this will be so if goods are indexed according to (D) as in (i) above.*

(c) *(i)* *Given Hicks Consistent incomparable pair x,y in* $(\Re^n, \leq^n_{ns})$*, i.e. such that (NS1)* $y_1 < x_1$*, (H1)* $p \cdot x \leq p \cdot y$*, and (H2)* $p' \cdot y \leq p' \cdot x$ *(at least one of H1, H2 strict inequality), then x,y are Hicks Consistent incomparable in* $(\Re^n, \leq^n_{ans})$ *under the indexing according to (D) with: (NS1)* $y_1 < x_1$*, (H1)* $p \cdot x \leq p \cdot y$*, (H2)* $p' \cdot y \leq p' \cdot x$*, and (NS2)* $S^1_{yx} < 0,\ldots,S^{k-1}_{yx} < 0$ *and* $S^k_{yx} \geq 0,\ldots,S^{n-1}_{yx} \geq 0$*,* $k \in \{2,\ldots,n-1\}$*, under this indexing.*

(ii) *Given Hicks Consistent incomparable pair x,y in* $(\Re^n, \leq^n_{nc})$*, i.e. such that (NC1)* $x_1 < y_1$*, (H1)* $p \cdot x \leq p \cdot y$*, and (H2)* $p' \cdot y \leq p' \cdot x$ *(at least one of H1, H2 strict inequality), then x,y are Hicks Consistent incomparable in* $(\Re^n, \leq^n_{anc})$ *under the indexing according to (D) with: (NC1)* $x_1 < y_1$*, (H1)* $p \cdot x \leq p \cdot y$*, (H2)* $p' \cdot y \leq p' \cdot x$ *and (NC2)* $S^{2,2}_{yx} \geq 0$ *(if* $k > 2$*), ...,* $S^{2,k-1}_{yx} \geq 0$*, and* $S^{2,k}_{yx} < 0$*, ...,* $S^{2,n-1}_{yx} < 0$ *(if* $k < n$*), for* $k \in \{2,\ldots,n\}$*, under this indexing.*

Proof. See Appendix ▮

Lemma 4.5(b) shows that ALNSPOs and ALNCPOs do not add descriptive content over and above LNSPOs and LNCPOs, in the comparison of pairs of bundles. However, this does not extend to transitive comparisons, since the same indexing may not be suitable for two different binary comparisons under ALNSPO or ALNCPO. Thus, the basic Hicks Consistent comparability criterion remains under LNSPOs and LNCPOs. Based on this, lemma 4.5(c) suggests which Hicks Consistent incomparable pairs in the richer ALNSPOs and ALNCPOs must be considered, under suitable indexing. This is applied in the following two lemmas:

LEMMA 4.6. *Given Hicks Consistent incomparable pair* x,y *in the Augmented Lexicographic* (p,p') *Net Substitutes poset* $(\Re^n, \leq^n_{ans})$*, such that: (NS1)* $y_1 < x_1$*, (H1)* $p \cdot x \leq p \cdot y$*, and (H2)* $p' \cdot y \leq p' \cdot x$ *(at least one of H1, H2 strict inequality), and (NS2)* $S^1_{yx} < 0,\ldots,S^{k-1}_{yx} < 0$ *and* $S^k_{yx} \geq 0,\ldots,S^{n-1}_{yx} \geq 0$*,* $k \in \{2,\ldots,n-1\}$*. Then:*

(a)

$$x \vee y = \begin{cases} \left(x_1,x_2,\ldots,x_{n-2},y_{n-1} + \frac{S^{n-2}_{yx}}{p_{n-1}},y_n\right) & if\ S^{n-2}_{yx},\ldots,\ S^2_{yx} < 0 \\ \vdots & \\ \left(x_1,\ldots,x_{k-1},y_k + \frac{S^{k-1}_{yx}}{p_k},y_{k+1},\ldots,y_n\right) & if\ S^{n-2}_{yx},\ldots,S^k_{yx} \geq 0 \,\&\, S^{k-1}_{yx},\ldots,S^2_{yx} < 0 \\ \vdots & \\ \left(x_1,x_2 + \frac{S^2_{yx}}{p_2},y_3,\ldots,y_n\right) & if\ S^{n-2}_{yx},\ \ldots\ ,S^2_{yx} \geq 0 \end{cases} \tag{4}$$

$$x \wedge y = \begin{cases} \left(y_1,y_2,\ldots,y_{n-2},x_{n-1} - \frac{S^{n-2}_{yx}}{p_{n-1}},x_n\right) & if\ S^{n-2}_{yx},\ldots,S^2_{yx} < 0 \\ \vdots & \\ \left(y_1,\ldots,y_{k-1},x_k - \frac{S^{k-1}_{yx}}{p_k},x_{k+1},\ldots,x_n\right) & if\ S^{n-2}_{yx},\ldots,S^k_{yx} \geq 0 \,\&\, S^{k-1}_{yx},\ldots,S^2_{yx} < 0 \\ \vdots & \\ \left(y_1,y_2 - \frac{S^2_{yx}}{p_2},x_3,\ldots,x_n\right) & if\ S^{n-2}_{yx},\ldots,S^2_{yx} \geq 0 \end{cases} \tag{5}$$

with $p \cdot (x \vee y) = p \cdot y$*,* $p' \cdot (x \vee y) = p' \cdot y$*,* $p \cdot (x \wedge y) = p \cdot x$*,* $p' \cdot (x \wedge y) = p' \cdot x$*.*

(b) *If* $S^{n-2}_{yx},\ldots,S^2_{yx} < 0$ *then* $x_{n-1} < (x \wedge y)_{n-1}$*,* $(x \vee y)_{n-1} < y_{n-1}$ *and* $(x \vee y)_{n-1} + (x \wedge y)_{n-1} = x_{n-1} + y_{n-1}$*. And in general, if* $S^{n-2}_{yx},\ldots,S^k_{yx} \geq 0$ *and* $S^{k-1}_{yx},\ldots,S^2_{yx} < 0$*,* $k \in \{2,\ldots,n-2\}$*, then* $x_k \leq (x \vee y)_k < y_k$*,* $x_k < (x \wedge y)_k \leq y_k$ *and* $(x \wedge y)_k + (x \vee y)_k = x_k + y_k$*.*

(c) $\left(\Re^n_+, \leq^n_{ans}\right)$ *is closed under joins and meets of Hicks Consistent incomparable pairs, satisfying (NS1), (H1), (H2), and (NS2), taken in the Augmented Lexicographic* (p,p') *Net Substitutes poset* $(\Re^n, \leq^n_{ans})$*.*

Proof. See Appendix ▌

LEMMA 4.7. *Given Hicks Consistent incomparable pair* x,y *in the Augmented Lexicographic* (p,p') *Net Complements poset* $(\Re^n, \leq^n_{anc})$*, such that: that (NC1)* $x_1 < y_1$*, (H1)* $p \cdot x \leq p \cdot y$*, (H2)* $p' \cdot y \leq p' \cdot x$ *(at least one strict inequality) and (NC2) (NC2)* $S^{2,2}_{yx} \geq 0$ *(if* $k > 2$*), …,* $S^{2,k-1}_{yx} \geq 0$*, and* $S^{2,k}_{yx} < 0$*, …,* $S^{2,n-1}_{yx} < 0$ *(if* $k < n$*), for* $k \in \{2,\ldots,n\}$*. Then:*

(a)

$$x \vee y = \begin{cases} \left(x_1, x_2, \ldots, x_{n-2}, x_{n-1} + \frac{S^{n-1}_{yx}}{p_{n-1}}, y_n\right) & if\ S^{2,n-1}_{yx}, \ldots, S^{2,2}_{yx} < 0 \\ \vdots & \\ \Big(x_1, y_2, \ldots, y_{k-1}, x_k - \frac{S^{2,k-1}_{yx}}{p_k}, & if\ S^{2,n-1}_{yx}, \ldots, S^{2,k}_{yx} < 0\ \& \\ \quad x_{k+1}, \ldots, x_{n-2}, x_{n-1} + \frac{S^{n-1}_{yx}}{p_{n-1}}, y_n\Big) & \quad S^{2,k-1}_{yx}, \ldots, S^{2,2}_{yx} \geq 0 \\ \vdots & \\ \left(x_1, y_2, \ldots, y_{n-2}, y_{n-1} + \frac{p_1(y_1 - x_1)}{p_2}, y_n\right) & if\ S^{2,n-1}_{yx}, \ldots, S^{2,2}_{yx} \geq 0 \end{cases} \tag{6}$$

$$x \wedge y = \begin{cases} \left(y_1, y_2, \ldots, y_{n-2}, y_{n-1} - \frac{S^{n-1}_{yx}}{p_{n-1}}, x_n\right) & if\ S^{2,n-1}_{yx}, \ldots, S^{2,2}_{yx} < 0 \\ \vdots & \\ \Big(y_1, x_2, \ldots, x_{k-1}, y_k + \frac{S^{2,k-1}_{yx}}{p_k}, & if\ S^{2,n-1}_{yx}, \ldots, S^{2,k}_{yx} < 0\ \& \\ \quad y_{k+1}, \ldots, y_{n-2}, y_{n-1} - \frac{S^{n-1}_{yx}}{p_{n-1}}, x_n\Big) & \quad S^{2,k-1}_{yx}, \ldots, S^{2,2}_{yx} \geq 0 \\ \vdots & \\ \left(y_1, x_2, \ldots, x_{n-2}, x_{n-1} - \frac{p_1(y_1 - x_1)}{p_2}, x_n\right) & if\ S^{2,n-1}_{yx}, \ldots, S^{2,2}_{yx} \geq 0 \end{cases} \tag{7}$$

with $p \cdot (x \vee y) = p \cdot y$*,* $p' \cdot (x \vee y) = p' \cdot y$*,* $p \cdot (x \wedge y) = p \cdot x$*,* $p' \cdot (x \wedge y) = p' \cdot x$*.*

(b) $\left(\Re^n_+, \leq^n_{anc}\right)$ *is closed under joins but not under meets of Hicks Consistent incomparable pairs, satisfying (NC1), (H1), (H2) and (NC2), in the Augmented Lexicographic* (p,p') *Net Complements poset* $(\Re^n, \leq^n_{anc})$*.*

Proof. The proof is similar to the proof of lemma 4.6 and is therefore omitted. ▌

Some of the results of this section can be summarized as follows: $(\Re^n, \leq^n_{ns})$, $(\Re^n, \leq^n_{nc})$, $(\Re^n, \leq^n_{ans})$, $(\Re^n, \leq^n_{anc})$, $\left(\Re^n_+, \leq^n_{ns}\right)$ and $\left(\Re^n_+, \leq^n_{ans}\right)$ are Hicks Consistent lattices.[27] In fact, lemma 4.6 shows that $\left(\Re^n_+, \leq^n_{ans}\right)$ is a Hicks Consistent sublattice of $(\Re^n, \leq^n_{ans})$, while neither is $\left(\Re^n_+, \leq^n_{ns}\right)$ a Hicks Consistent sublattice of $(\Re^n, \leq^n_{ns})$, nor is $\left(\Re^n_+, \leq^n_{anc}\right)$ a Hicks Consistent Sublattice of $(\Re^n, \leq^n_{anc})$.

5. SUFFICIENT CONDITIONS FOR NET SUBSTITUTION EFFECTS REVISITED

In section 2 we gave sufficient conditions on preferences for a good to be a pathwise/strongly net substitute of another, based on the general properties of NSPOs and NCPOs developed in section 1. The results of the previous section show that the two main theorems can be applied using LNSPOs and LNCPOs. However, we also argued in the previous section that these class of partial orders is not normatively entirely satisfactory. In Corollary 5.1 below we show how the ALNSPOs and can be used instead (the analogous result with respect to ALNCPO is left to the interested reader):

Corollary 5.1 (Net Substitutes).
Consider the consumer expenditure minimization problem, EM$(p, \bar{u})$ as in theorem 2.1:

(a) If:

(A) The consumption set X, $X \subset \Re^n$, is closed under joins and meets of Hicks Consistent incomparable pairs satisfying (NS2) in the Augmented Lexicographic (p,p') Net Substitutes poset $(\Re^n, \leq^n_{ans})$ for every *indexing of goods* $2,\ldots,n \quad -1$*;*

(B) The consumer utility function $U : X \rightarrow \Re$ is Hicks Consistent Iso-Quasi-supermodular on X at all such pairs of points in (A), in every Augmented Lexicographic (p,p') Net Substitutes poset $(\Re^n, \leq^n_{ans})$.

Then, given feasible $\bar{u}$, such that solutions to the problems EM$(p, \bar{u})$ and EM$(p', \bar{u})$ exist and are such that there is no excess utility:

$$\underset{x\in X}{\operatorname{argmin}} \left\{ p \cdot x \,|\, U(x) \geq \bar{u} \right\} \leq_P \underset{x\in X}{\operatorname{argmin}} \left\{ p' \cdot x \,|\, U(x) \geq \bar{u} \right\}$$

[27] Of course in the case of $(\Re^n, \leq^n_{ans})$ and $(\Re^n, \leq^n_{anc})$, the relevant Hicks Consistent pairs satisfy (NS2) and (NC2) respectively.

(where the underlying order on the consumption set is $\leq_{ns}^{n}$ *for* some *indexing of goods* $2,\ldots,n-1$*), and good 1 is a Pathwise Net Substitute of good* n *at prices* (p,p') *and at every such attainable utility level* $\bar{u}$*.*

(b) *If instead of (A) and (B) in part (a), (A), (B′) and (C) hold, where*

(B′) *The consumer utility function* $U : X \to \Re$ *is Hicks Consistent* Strictly *Iso-Quasi-supermodular on* X *at all such pairs of points in (A), in every Augmented Lexicographic* (p,p') *Net Substitutes poset*$(\Re^n, \leq_{ans}^{n})$*;*

(C) $I \equiv \underset{x\in X}{\text{argmin}} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \cap \underset{x\in X}{\text{argmin}} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$ *contains no more than one element, where* $\bar{u}$ *is an attainable utility level,*

Then

$$\underset{x\in X}{\text{argmin}} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_s \underset{x\in X}{\text{argmin}} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$$

(where the underlying order is $\leq_{ns}^{n}$ *for* some *indexing of goods* $2,\ldots,n-1$*), and good 1 is a Strongly Net Substitute of good* n *at prices* (p,p') *and at every such attainable utility level* $\bar{u}$*.*

Remark 5. 1. Notice that unlike in theorem 2.1, it is no longer possible to show in (a) that $\underset{x\in X}{argmin} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_a \underset{x\in X}{argmin} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$, and in (b) that $\underset{x\in X}{argmin} \{p \cdot x \,|\, U(x) \geq \bar{u}\} \leq_c \underset{x\in X}{argmin} \{p' \cdot x \,|\, U(x) \geq \bar{u}\}$, with underlying order $\leq_{ns}^{n}$, under *any* -re-indexing of goods $2,\ldots,n-1$.

Proof. Follows easily from theorem 2.1 and the results of the previous section. ∎

It is tempting to consider the conditions in corollary 5.1 above, which restrict the behavior of the utility function within not just one Hicks Consistent poset, but in all possible ones under some re-indexing of goods $2,\ldots,n-1$, more restrictive than the conditions of theorem 2.1. However, the Hicks Consistent incomparable pairs at which the behavior of the utility function is restricted in this class of posets is also restricted. And, as will be shown with the example of additive preferences, it is possible that quasi-supermodularity in this class of posets is satisfied whereas it is not in any single poset under LNSPO:

Corollary 5.2 (Additive Preferences). *If consumer preferences are additive such that* $U : \Re^n_+ \to \Re \quad U(x) \equiv U_1(x_1) + \ldots + U_n(x_n)$ *where each* U_i *is a monotone, concave function, then*

(a) The function U *is Hicks Consistent quasi-supermodular at all incomparable pairs satisfying (NS2) in the* (p,p') *Hicks Consistent poset* $(\Re^n_+, \leq^n_{ans})$, *for every re-indexing of goods* $2,\ldots,n-1$, *and for every* (p,p') *such that* $p \in \Re^n_{++}$ *and* $p' = (p_1,\ldots,p_{n-1},p'_n)$ *with* $p_n < p'_n$.[28]

(b) Each good is a pathwise net substitute of every other good everywhere.

Proof. Part (b) follows from (a) and theorem 2.1 above, by observing that the choice of goods 1 and n is inconsequential in the proof of (a) (note however, that the quasi-supermodularity established (a) is more than is needed for (b)):
Consider Hicks Consistent incomparable pair x,y, such that (refer to lemma 4.5): (NS1) $y_1 < x_1$, (H1) $p \cdot x \leq p \cdot y$, and (H2) $p' \cdot y \leq p' \cdot x$ (where at least one is a strict inequality), and (NS2) $S^1_{yx} < 0,\ldots,S^{k-1}_{yx} < 0$ *and* $S^k_{yx} \geq 0,\ldots,S^{n-1}_{yx} \geq 0$, $k \in \{2,\ldots,n-1\}$. In order to prove the required result, assume that $U(x) \geq U(x \wedge y)$ i.e. $U_1(x_1) + \ldots + U_n(x_n) \geq U_1(y_1) + \ldots + U_{k-1}(y_{k-1}) + U_k\left(y_k - \frac{S^k_{yx}}{p_k}\right) + U_{k+1}(x_{k+1}) + \ldots + U_n(x_n)$. Hence $U_1(x_1) + \ldots + U_k(x_k) \geq U_1(y_1) + \ldots + U_{k-1}(y_{k-1}) + U_k\left(y_k - \frac{S^k_{yx}}{p_k}\right)$ or $U_1(x_1) + \ldots + U_k(x_k) + U_k\left(x_k + \frac{S^k_{yx}}{p_k}\right) + U_{k+1}(y_{k+1}) + \ldots + U_n(y_n) \geq U(y) - U_k(y_k) + U_k\left(y_k - \frac{S^k_{yx}}{p_k}\right) + U_k\left(x_k + \frac{S^k_{yx}}{p_k}\right)$ or $U(x \vee y) \geq U(y) + U_k\left(y_k - \frac{S^k_{yx}}{p_k}\right) + U_k\left(x_k + \frac{S^k_{yx}}{p_k}\right) - U_k(x_k) - U_k(y_k)$. Similarly $U(x) \geq U(x \vee y)$ implies $U(x \wedge y) \geq U(y) + U_k\left(y_k - \frac{S^k_{yx}}{p_k}\right) + U_k\left(x_k + \frac{S^k_{yx}}{p_k}\right) - U_k(x_k) - U_k(y_k)$. Therefore, if $U_k\left(y_k - \frac{S^k_{yx}}{p_k}\right) + U_k\left(x_k + \frac{S^k_{yx}}{p_k}\right) - U_k(x_k) - U_k(y_k) \geq 0$ then $U(x) \geq U(x \wedge y)$ implies $U(x \vee y) \geq U(y)$ (strict inequality if $U(x) > U(x \wedge y)$) and $U(x) \geq U(x \vee y)$ implies $U(x \wedge y) \geq U(y)$ (again strict inequality if $U(x) > U(x \vee y)$) i.e. U is quasi-supermodular as required. From lemma 4.6 of the previous section $x_k \leq \left(x_k + \frac{S^k_{yx}}{p_k}\right) < y_k$, $x_k < \left(y_k - \frac{S^k_{yx}}{p_k}\right) \leq y_k$ and $(x \wedge y)_k + (x \vee y)_k = x_k + y_k$. Therefore $\left(x_k + \frac{S^k_{yx}}{p_k}\right) = \alpha x_k + (1-\alpha) y_k$ and $\left(y_k - \frac{S^k_{yx}}{p_k}\right) = (1-\alpha) x_k + \alpha y_k$, for some $\alpha \in$

[28] It is possible to construct examples of Hicks Consistent incomparable pairs with LNSPO, with just four goods, where the quasi-supermodular property may fail for both of the indexings of goods relevant for the description of good 1 as a net substitute of good 4.

$(0,1]$. Hence by concavity $U_k\left(x_k+\frac{S^k_{yx}}{p_k}\right)\geq \alpha U_k\left(x_k\right)+\left(1-\alpha\right)U_k\left(y_k\right)$ and similarly $U_k\left(y_k-\frac{S^k_{yx}}{p_k}\right)\geq\left(1-\alpha\right)U_k\left(x_k\right)+\alpha U_k\left(y_k\right)$. Hence adding these two inequalities $U_k\left(y_k-\frac{S^k_{yx}}{p_k}\right)+U_k\left(x_k+\frac{S^k_{yx}}{p_k}\right)\geq U_k\left(x_k\right)+U_k\left(y_k\right)$ as required, establishing that indeed U is quasi-supermodular. ▮

6. CONCLUDING REMARKS

It is perhaps surprising that almost half a century of consumer theory has produced very little in the way of conditions on consumer preferences that suffice to sign net substitution effects. We hope that the sufficient conditions offered in this paper contribute in this direction, and that in the process the versatility of the proposed order theoretic framework is established. The methods exploit the order/lattice structure of a problem, and it is important therefore to identify the appropriate order structures inherent to the problem in hand before applying the general lattice programming theorems and results. The strength of the results relies on the stregth of these order/lattice structures.

It is hoped that the results of this paper can be helpful in both theoretical and applied work. On the applied side, it can be envisaged that the methods developed can be used to develop market research methods based on the questionnaire approach. It is not easy to structure questionnaires that may confirm or otherwise that consumers have, for example, additive preferences. But it would seem possible to construct a finite set of questions, which can suggest with some degree of accuracy whether consumer preferences satisfy quasi-supermodularity conditions as developed here. On the theoretical side, at least part of the attraction of the proposed approach is that it can be applied in cases with non-convexities and indivisibilities, to name but a couple of its differences with standard approaches. In this paper we have worked within a conventional setting and hinted only at possible extensions. It would be useful to carry out such extensions explicitly.

Milgrom and Shannon (1994) provide an order theoretic (and a differential) generalization of the (Spence-Mirlees) Single Crossing Property, paving the way for more versatile extensions of asymmetric information models beyond the standard two variable, one dimensional information characteristic, set-up. One such problem is that of the incentive compatible profit maximizing, or optimal, non-linear price schedule for a good, when consumers preferences over many goods are explicitly modeled, as opposed to the standard practice of using reduced demand functions. This example is suggestive because it raises the question of the applicability of lattice programming techniques to problems where there are budgetary trade-offs between the variables. This is a difficult problem. The difficulty manifests itself in the fact that constraint sets involving budgetary trade-offs between more than two variables do not avail themselves to strong set comparability (or even weaker forms of set comparability) under commonly used partial orders, most notably the Euclidean order, as lattice programming theorems require. It is the aim of the research agenda, of which this paper is a part, to provide the required machinery for such applications.

APPENDIX

Proof (Lemma 4.1).

(a) Obvious.

(b) (i) and (ii): $x_1 \leq y_1$ ($y_1 \leq x_1$) follows from LS1 (LC1). $y_n \leq x_n$ and $S_{yx}^{n-1} \geq 0$ follow from H1 and H2.

(c) It is obvious that when $n = 3$ comparability with respect to LNSPO (LNCPO) implies comparability with respect to NSPO^3 (NCPO^3). For the converse, assume first $x \leq_{ns}^3 y$ and therefore $x_1 \leq y_1$ and $y_3 \leq x_3$. If $x_1 < y_1$ then clearly $x \leq_{ns}^n y$. If $x_1 = y_1$ then $S_{yx}^2 \geq 0$ implies $x_2 \leq y_2$. If the inequality is strict there is again nothing further to prove. If $x_2 = y_2$ then from H1 and H2 $y_3 = x_3$. Hence $x \leq_{ns}^3 y$ implies $x \leq_L y$ and therefore $x \leq_{ns}^n y$ as required. The argument establishing that $x \leq_{nc}^3 y$ implies $x \leq_{nc}^n y$ is very similar. The case $n = 2$ is trivial.

∎

Proof (Lemma 4.2).

(a) Considering first the join of x,y: Let $z \equiv (x_1, x_2, \ldots, x_{n-2}, z_{n-1}, y_n)$ with $z_{n-1} \equiv x_{n-1} + \frac{S_{yx}^{n-1}}{p_{n-1}}$. Clearly $x_{n-1} < z_{n-1}$, and $y <_L x <_L z$. Also $p \cdot z = p \cdot y$ and $p' \cdot z = p' \cdot y$. Hence $x \leq_{ns}^n z$ and $y \leq_{ns}^n z$, i.e. z is an upper bound of x,y. Consider next any upper bound, $w \equiv (w_1, \ldots, w_n)$. By definition $y <_L x \leq_L w$, $p \cdot x \leq p \cdot y \leq p \cdot w$ and $p' \cdot w \leq p' \cdot y \leq p' \cdot x$ and in particular $w_n \leq y_n = z_n$. If $x_1 < w_1$, or if $x_{i-1} = w_{i-1}$

and $x_i < w_i$, $i = 2,\ldots,n-2$ then $z <_L w$ and $z \leq^n_{ns} w$ as required. Therefore, suppose $x_i = w_i$, $i = 1,\ldots,n-2$. If $w_{n-1} < z_{n-1}$ then $p \cdot z = p \cdot y \leq p \cdot w$ implies $y_n = z_n < w_n$ contradicting $w_n \leq y_n = z_n$. Therefore, $z_{n-1} \leq w_{n-1}$. If $z_{n-1} < w_{n-1}$ then $z <_L w$ as required and if $z_{n-1} = w_{n-1}$, then $w_n \leq y_n = z_n$ and $p \cdot z = p \cdot y \leq p \cdot w$ imply $w_n = y_n = z_n$, i.e. $z = w$, thus completing the proof that $z \leq^n_{ns} w$, i.e. z is the join of x,y.

Considering next the meet of x,y:Let $r \equiv (y_1,y_2,\ldots,y_{n-2},r_{n-1},x_n)$ with $r_{n-1} \equiv y_{n-1} - \frac{S^{n-1}_{yx}}{p_{n-1}}$. Clearly $r_{n-1} < y_{n-1}$, and $r <_L y <_L x$. Also $p \cdot r = p \cdot x$ and $p' \cdot r = p' \cdot x$. Hence $r \leq^n_{ns} x$ and $r \leq^n_{ns} y$, i.e. r is a lower bound of x,y. Consider any lower bound $s \equiv (s_1,\ldots,s_n)$. By definition $s \leq_L y <_L x$, $p \cdot s \leq p \cdot x \leq p \cdot y$, and $p' \cdot y \leq p' \cdot x \leq p' \cdot s$ and in particular $r_n = x_n \leq s_n$. If $s_1 < y_1$, or if $y_{i-1} = s_{i-1}$, and $s_i < y_i$, $i = 2,\ldots,n-2$, then $s <_L r$ and $s \leq^n_{ns} r$ as required. Therefore suppose $y_i = s_i$, $i = 1,\ldots,n-2$. If $r_{n-1} < s_{n-1}$ then $p \cdot s \leq p \cdot x = p \cdot r$ implies $s_n < r_n = x_n$, contradicting $r_n = x_n \leq s_n$. Therefore, $s_{n-1} \leq r_{n-1}$. If $s_{n-1} < r_{n-1}$, then $s <_L r$ and $s \leq^n_{ns} r$ as required. If $s_{n-1} = r_{n-1}$, then $r_n = x_n \leq s_n$ and $p \cdot s \leq p \cdot x = p \cdot r$, imply $s_n = r_n = x_n$, i.e. $s = r$, thus completing the proof that $s \leq^n_{ns} r$, i.e. r is the meet of x,y.

(b) Since $x \vee y = \left(x_1,x_2,\ldots,x_{n-2},x_{n-1} + \frac{S^{n-1}_{yx}}{p_{n-1}},y_n\right) \in \Re^n_+$ whenever $x,y \in \Re^n_+$, the proof for the join is the same as in part (a). Therefore we only need to amend the proof of part (a) in the case of the meet:

Case 1; $S^{n-1}_{yx} \leq p_{n-1}y_{n-1}$. The proof in part (a) applies.

Case 2; $S^{k+1,n-1}_{y} < S^{n-1}_{yx} \leq S^{k,n-1}_{y}$, $k = 3,\ldots,n-2$:

Let $r^k \equiv \left(y_1,\ldots,y_{k-1},y_k - \frac{S^{n-1}_{yx} - S^{k+1,n-1}_{y}}{p_k},0,\ldots,0,x_n\right)$. The hypotheses of this case implies $r^k \in \Re^n_+$ and furthermore $r^k_k < y_k$, $p \cdot r^k = p \cdot x$ and $p' \cdot r^k = p' \cdot x$. Hence $r^k <_L y <_L x$ and $r^k \leq^n_{ns} x,y$, i.e. r^k is a lower bound of x,y. Consider any lower bound $s \equiv (s_1,\ldots,s_n)$. By definition $s \leq_L y <_L x$, $p \cdot s \leq p \cdot x \leq p \cdot y$, and $p' \cdot y \leq p' \cdot x \leq p' \cdot s$, and in particular $x_n = r^k_n \leq s_n$. If $s_1 < y_1$, or if $y_{i-1} = s_{i-1}$, and $s_i < y_i$, $i = 2,\ldots,k-1$, then $s <_L r^k$ and $s \leq^n_{ns} r^k$ as required. Therefore, suppose $y_i = s_i$, $i = 1,\ldots,k-1$. If $r^k_k < s_k$ then this along with $x_n = r^k_n \leq s_n$, in $p \cdot s \leq p \cdot r^k = p \cdot x$ or $p_k s_k + \cdots + p_n s_n \leq p_k r^2_k + p_n r^k_n$, imply $s_i < 0$ some $k < i < n$, contradicting $s \in \Re^n_+$. Therefore $s_k \leq r^k_k$. If $s_k < r^k_k$ then $s <_L r^k$ and $s \leq^n_{ns} r^k$ as required. Therefore suppose $s_k = r^k_k$. But then $p_{k+1}s_{k+1} + \cdots + p_n s_n \leq p_n r^k_n$, given $x_n = r^k_n \leq s_n$, implies $s_i = 0$, $i = k+1,\ldots,n-1$, and $x_n = r^k_n = s_n$, i.e. $s = r^k$, thus completing the proof that $s \leq_{ns}^{n} r^k$, i.e. r^k is the meet of x,y.

Case 3; $S_y^{3,n-1} < S_{yx}^{n-1}$: Let $r^2 \equiv \left(y_1, y_2 - \frac{S_{yx}^{n-1} - S_y^{3,n-1}}{p_2}, 0, \ldots, 0, x_n\right)$. $r_2^2 \geq 0$ is equivalent to $S_{yx}^{n-1} \leq S_y^{2,n-1}$ or $p_1 y_1 \leq p_{-n} \cdot x_{-n}$ which is clearly true since by hypothesis $y <_L x$. Therefore, $r^2 \in \Re_+^n$. The rest of the proof is identical to the general proof in case 2 above.

▮

Proof (Lemma 4.3).

The proof is very similar to the proof of lemma 4.2 and is therefore not repeated here. We only remark that the meet of x,y does not exist when $p_{-n} \cdot x_{-n} < p_1 y_1$ since the meet, if it exists, must be such that $(x \wedge y)_1 = y_1$ and $(x \wedge y)_n = x_n$. But this is incompatible with $p \cdot (x \wedge y) \leq p \cdot x$ in this case. ▮

Proof (Lemma 4.4).

(a) (i) $x_1 \leq y_1$, $y_n \leq x_n$ and $S_{yx}^{n-1} \geq 0$ are consequences of $x \leq_{ns}^{n} y$ (see lemma 4.1). Suppose $S_{yx}^{k} < 0$, some $k \in \{2,\ldots,n-2\}$. But then by (S2), $S_{yx}^{k+1} < 0$, ..., $S_{yx}^{n-1} < 0$, contradicting, $S_{yx}^{n-1} \geq 0$. Thus, $S_{yx}^{k} \geq 0$, $k = 2,\ldots,n-2$ whenever $x \leq_{ans}^{n} y$.

(ii) $y_1 \leq x_1$, $y_n \leq x_n$, $S_{yx}^{n-1} \geq 0$ and $S_{yx}^{2,n-1} \geq 0$ are consequences of $x \leq_{nc}^{n} y$. Suppose $S_{yx}^{2,k} < 0$, some $k \in \{2,\ldots,n-2\}$. But then by (C2), $S_{yx}^{2,k+1} < 0$, ..., $S_{yx}^{2,n-1} < 0$, contradicting, $S_{yx}^{2,n-1} \geq 0$. Thus, $S_{yx}^{2,k} \geq 0$, $k = 2,\ldots,n-2$, whenever $x \leq_{anc}^{n} y$.

(b) ALNSPO: Reflexivity and antisymmetry are obvious. In order to establish transitivity assume $x \leq_{ans}^{n} y$ and $y \leq_{ans}^{n} z$. Hence $x \leq_{ns}^{n} z$. From (a)(i) above $S_{yx}^{k} \geq 0$, and $S_{zy}^{k} \geq 0$, $k = 2,\ldots,n-2$, implying $S_{zx}^{k} \geq 0$, $k = 2,\ldots,n-2$, rendering the conditions of (S2) non-binding. Hence $x \leq_{ans}^{n} z$ as required. The argument for ALNCPO is analogous.

(c) Follows from lemma 4.1(c) since by construction (S2) and (C2) are inapplicable when $n = 2, 3$.

▮

Proof (Lemma 4.5).

(a) This is obvious from definitions 4.1 and 4.2.

(b) (i) We show that the indices of goods $2,\ldots,n-2$ can be chosen so that $S_{yx}^{k} \geq 0$, $k = 2,\ldots,n-1$. Suppose this is not true under the natural indexing. An indexing that will work (not necessarily unique) is one which indexes goods according to (D) $y_{k+1} - x_{k+1} \leq y_k - x_k$, $k = 2,\ldots,n-2$. Under this indexing, if $S_{yx}^{k} < 0$ for some k, then $S_{yx}^{k+1} < 0,\ldots,$ $S_{yx}^{n-1} < 0$, contradicting, $S_{yx}^{n-1} \geq 0$. Furthermore, under this indexing clearly $x \leq_L y$, and (H1) and (H2) are unaffected by the re-indexing.

(ii) The proof is almost identical to (i) and is therefore omitted.

(c) Obvious, using the same argument as in (b) above.

▮

Proof (Lemma 4.6).

(a) Considering first the join of x,y: Let

$$z = \begin{cases} \left(x_1,x_2,\ldots,x_{n-2},y_{n-1} + \frac{S_{yx}^{n-2}}{p_{n-1}},y_n\right) & if\ S_{yx}^{n-2},\ldots,S_{yx}^{2} < 0 \\ \vdots & \\ \left(x_1,\ldots,x_{k-1},y_k + \frac{S_{yx}^{k-1}}{p_k},y_{k+1},\ldots,y_n\right) & if\ S_{yx}^{n-2},\ldots,S_{yx}^{k} \geq 0 \&\ S_{yx}^{k-1},\ldots,S_{yx}^{2} < 0 \\ \vdots & \\ \left(x_1,x_2 + \frac{S_{yx}^{2}}{p_2},y_3,\ldots,y_n\right) & if\ S_{yx}^{n-2},\ldots,S_{yx}^{2} \geq 0 \end{cases}$$

By construction $p \cdot z = p \cdot y$ and $p' \cdot z = p' \cdot y$. Next we show: $y <_L x <_L z$: Firstly, $y_k + \frac{S_{yx}^{k-1}}{p_k} = x_k + \frac{S_{yx}^{k}}{p_k} \geq x_k$, $k = 2,\ldots,n-1$, whenever $S_{yx}^{k} \geq 0$. If this is strict inequality the result follows immediately. If $S_{yx}^{k} = 0$, $S_{yx}^{k+1} \geq 0$ is equivalent to $z_{k+1} = y_{k+1} \geq x_{k+1}$. If this is strict inequality the result again follows. Similarly, if $S_{yx}^{k+1} = 0$,..., $S_{yx}^{n-2} = 0$ (i.e. $y_{k+1} = x_{k+1},\ldots,y_{n-2} = x_{n-2}$) then $S_{yx}^{n-1} > 0$ (implied by H1, H2 where at least one is strict inequality) implies $x_{n-1} < y_{n-1}$. Hence $x <_L z$ as required. Therefore $x \leq_{ns}^{n} z$ and $y \leq_{ns}^{n} z$. Furthermore,

$$S_{zx}^{n-2} = \begin{cases} 0 \\ S_{yx}^{n-2} \geq 0 \\ S_{yx}^{n-2} \geq 0 \end{cases} \quad S_{zy}^{n-2} = \begin{cases} -S_{yx}^{n-2} > 0 & if\ S_{yx}^{n-2},\ldots,S_{yx}^{2} < 0 \\ 0 & if\ S_{yx}^{n-2} \geq 0\ \&\ S_{yx}^{n-3},\ldots,S_{yx}^{2} < 0 \\ 0 & if\ S_{yx}^{n-2},\ldots,S_{yx}^{k} \geq 0 \&\ S_{yx}^{k-1},\ldots,S_{yx}^{2} < 0 \end{cases}$$

and in general, for $k = 2,\ldots,n-3$

$$S_{zx}^k = \begin{cases} 0 \\ 0 \\ \\ S_{yx}^k \geq 0 \\ \end{cases} \quad S_{zy}^k = \begin{cases} -S_{yx}^k > 0 & if\ S_{yx}^{n-2},\ldots,S_{yx}^2 < 0 \\ -S_{yx}^k > 0 & if\ S_{yx}^{n-2} \geq 0,\ \&\ S_{yx}^{n-3},\ldots,S_{yx}^2 < 0\ \ldots \\ & S_{yx}^{n-2},\ldots,S_{yx}^{k+1} \geq 0,\ \&\ S_{yx}^k,\ldots,S_{yx}^2 < 0 \\ 0 & if\ S_{yx}^{n-2},\ldots,S_{yx}^k \geq 0,\ \&\ S_{yx}^{k-1},\ldots,S_{yx}^2 < 0 \\ & \ldots\ S_{yx}^{n-2},\ldots,S_{yx}^2 \geq 0 \end{cases}$$

Therefore the conditions of S2 do not apply and $x \leq_{ans}^n z$, and $y \leq_{ans}^n z$, i.e. z is an upper bound of x,y. Consider any other upper bound of x,y, say r;

Case 1; Suppose $S_{yx}^{n-2},\ldots,S_{yx}^2 < 0$; clearly $S_{rz}^k = S_{rx}^k - S_{zx}^k = S_{rx}^k \geq 0$, $k = 2,\ldots,n-2$, since $S_{zx}^k = 0$ from above and $S_{rx}^k \geq 0$ since $x \leq_{ans}^n r$ (refer to lemma 4.4). This means that if $z \leq_{ns}^n r$ then $z \leq_{ans}^n r$ since the conditions of S2 are inapplicable. But $z \leq_{ns}^n r$ follows from the proof of lemma 4.2 and therefore $z \leq_{ans}^n r$ as required.

Case 2; Suppose $S_{yx}^{n-2},\ldots,S_{yx}^k \geq 0$, $S_{yx}^{k-1},\ldots,S_{yx}^2 < 0$, $k = 2,\ldots,n-2$. As in case 1 above $S_{rz}^m = S_{rx}^m - S_{zx}^m = S_{rx}^m \geq 0$ for $m = 2,\ldots,k-1$ since $S_{zx}^m = 0$ from above and $S_{rx}^m \geq 0$ since $x \leq_{ans}^n r$. Similarly $S_{rz}^m = S_{ry}^m - S_{zy}^m = S_{ry}^m \geq 0$ for $m = k,\ldots,n-2$ since again $S_{zy}^m = 0$ from above and $S_{ry}^m \geq 0$. Therefore the conditions of S2 do not apply and $z \leq_{ns}^n r$ implies $z \leq_{ans}^n r$. If $r_1 < x_1$ or if $r_{i-1} = x_{i-1}$ and $r_i < x_i$, $i = 2,\ldots,k-1$ then $r <_L x$, contradicting $x \leq_{ans}^n r$. If $x_1 < r_1$ or if $r_{i-1} = x_{i-1}$ and $x_i < r_i$, $i = 2,\ldots,k-1$ then $z \leq_{ns}^n r$ and therefore $z \leq_{ans}^n r$. Hence assume $r_i = x_i$, $i = 1,\ldots,k-1$ and suppose that $x_k \leq r_k < z_k$. Therefore $S_{ry}^k < S_{zy}^k = 0$ contradicting $S_{ry}^k \geq 0$. Hence $z_k \leq r_k$; if the inequality is strict then $z \leq_{ns}^n r$ and $z \leq_{ans}^n r$. Therefore assume $z_k = r_k$. If $r_{k+1} < z_{k+1} = y_{k+1}$ or if $r_{i-1} = z_{i-1} = y_{i-1}$ and $r_i < z_i = y_i$, $i = k+2,\ldots,n-2$ then again $S_{ry}^i < S_{zy}^i = 0$ contradicting $S_{ry}^i \geq 0$. If$z_{k+1} = y_{k+1} < r_{k+1}$ or if $r_{i-1} = z_{i-1} = y_{i-1}$ and $z_i = y_i < r_i$, $i = k+2,\ldots,n-2$ then $z \leq_{ns}^n r$ and therefore $z \leq_{ans}^n r$. Therefore assume further $r_i = z_i = y_i$, $i = k+1,\ldots,n-2$. If next $r_{n-1} < z_{n-1} = y_{n-1}$ then using $p \cdot z = p \cdot y \leq p \cdot r$, $z_n = y_n < r_n$ which contradicts $r_n \leq y_n$ (implied by $p \cdot y \leq p \cdot r$ and $p' \cdot r \leq p' \cdot y$). Hence $z_{n-1} = y_{n-1} \leq r_{n-1}$. Again if the inequality is strict then $z \leq_{ns}^n r$ and $z \leq_{ans}^n r$. If $y_{n-1} = r_{n-1}$ then $r_n = y_n$ and $r = z$, thus completing the proof that $z \leq_{ans}^n r$ for each upper bound of x,y and thus $z = x \vee y$.

Considering next the meet of x,y (the proof is analogous to the argument establishing the join and may be omitted). Let

$$w=\begin{cases}\left(y_1,y_2,\ldots,y_{n-2},x_{n-1}-\frac{S_{yx}^{n-2}}{p_{n-1}},x_n\right) & if\ S_{yx}^{n-2},\ldots,S_{yx}^{2}<0\\ \vdots & \\ \left(y_1,\ldots,y_{k-1},x_k-\frac{S_{yx}^{k-1}}{p_k},x_{k+1},\ldots,x_n\right) & if\ S_{yx}^{n-2},\ldots,S_{yx}^{k}\geq 0\&\ S_{yx}^{k-1},\ldots,S_{yx}^{2}<0\\ \vdots & \\ \left(y_1,y_2-\frac{S_{yx}^{2}}{p_2},x_3,\ldots,x_n\right) & if\ S_{yx}^{n-2},\ldots,S_{yx}^{2}\geq 0\end{cases}$$

By construction $p\cdot w=p\cdot x$ and $p'\cdot w=p'\cdot x$. Next we show $w<_L y<_L x$: Firstly, $x_k-\frac{S_{yx}^{k-1}}{p_k}=y_k-\frac{S_{yx}^{k}}{p_k}\leq y_k$, $k=2,\ldots,n-1$, whenever $S_{yx}^{k}\geq 0$. If this is strict inequality the result follows. If $S_{yx}^{k}=0$, $S_{yx}^{k+1}\geq 0$ is equivalent to $w_{k+1}=x_{k+1}\leq y_{k+1}$. If this is strict inequality then again the result follows. Similarly, if $S_{yx}^{k+1}=0,\ldots,\ S_{yx}^{n-2}=0$ (i.e. $y_{k+1}=x_{k+1},\ldots,y_{n-2}=x_{n-2}$) then $S_{yx}^{n-1}>0$ (implied by H1, H2 where at least one is strict inequality) implies $x_{n-1}<y_{n-1}$ and $w<_L y$ as required. Therefore $w\leq_{ns}^{n} x$ and $w\leq_{ns}^{n} y$. Furthermore,

$$S_{yw}^{n-2}=\begin{cases}0\\ S_{yx}^{n-2}\geq 0\\ S_{yx}^{n-2}\geq 0\end{cases}\quad S_{xw}^{n-2}=\begin{cases}-S_{yx}^{n-2}>0 & if\ S_{yx}^{n-2},\ldots,S_{yx}^{2}<0\\ 0 & if\ S_{yx}^{n-2}\geq 0,\ \&\ S_{yx}^{n-3},\ldots,S_{yx}^{2}<0\\ 0 & if\ S_{yx}^{n-2},\ldots,S_{yx}^{k}\geq 0,\ \&\ S_{yx}^{k-1},\ldots,S_{yx}^{2}<0\end{cases}$$

and in general, for $k=2,\ldots,n-3$

$$S_{yw}^{k}=\begin{cases}0\\ 0\\ \\ S_{yx}^{k}\geq 0\\ \end{cases}\quad S_{xw}^{k}=\begin{cases}-S_{yx}^{k}>0 & if\ S_{yx}^{n-2},\ldots,S_{yx}^{2}<0\\ -S_{yx}^{k}>0 & if\ S_{yx}^{n-2}\geq 0,\ \&\ S_{yx}^{n-3},\ldots,S_{yx}^{2}<0\ \ldots\\ & S_{yx}^{n-2},\ldots,S_{yx}^{k+1}\geq 0,\ \&\ S_{yx}^{k},\ldots,S_{yx}^{2}<0\\ 0 & if\ S_{yx}^{n-2},\ldots,S_{yx}^{k}\geq 0,\ \&\ S_{yx}^{k-1},\ldots,S_{yx}^{2}<0\ \ldots\\ & S_{yx}^{n-2},\ldots,S_{yx}^{2}\geq 0\end{cases}$$

Therefore the conditions of S2 do not apply; $w\leq_{ans}^{n} x$, and $w\leq_{ans}^{n} y$, i.e. w is a lower bound of x,y. Consider any other lower bound of x,y, say r;

Case 1; Suppose $S_{yx}^{n-2},\ldots,S_{yx}^{2}<0$; clearly $S_{wr}^{k}=-S_{yw}^{k}+S_{yr}^{k}=S_{yr}^{k}\geq 0$, $k=2,\ldots,n-2$, since $S_{yw}^{k}=0$ from above and $S_{yr}^{k}\geq 0$ since $r\leq_{ans}^{n} y$ (refer to lemma 4.4). This means that if $r\leq_{ns}^{n} w$ then $r\leq_{ans}^{n} w$ since the conditions of S2 are inapplicable. But $r\leq_{ns}^{n} w$ follows from lemma 4.2 and therefore $r\leq_{ans}^{n} w$ as required.

Case 2; Suppose $S_{yx}^{n-2},\ldots,S_{yx}^{k}\geq 0$, $S_{yx}^{k-1},\ldots,S_{yx}^{2}<0$, $k=2,\ldots,n-2$. As in case 1 above $S_{wr}^{m}=-S_{yw}^{m}+S_{yr}^{m}=S_{yr}^{m}\geq 0$ for $m=2,\ldots,k-1$ since from above and

$S^m_{yr} \geq 0$ since $r \leq^n_{ans} y$. Similarly $S^m_{wr} = -S^m_{xw} + S^m_{xr} = S^m_{xr} \geq 0$ for $m = k,\ldots,n-2$ since again $S^m_{xw} = 0$ from above and $S^m_{xr} \geq 0$. Therefore the conditions of S2 do not apply and $r \leq^n_{ns} w$ implies $r \leq^n_{ans} w$. If $y_1 < r_1$ or if $y_{i-1} = r_{i-1}$ and $y_i < r_i$, $i = 2,\ldots,k-1$ then $y <_L r$, contradicting $r \leq^n_{ans} y$. If $r_1 < y_1$ or if $y_{i-1} = r_{i-1}$ and $r_i < y_i$, $i = 2,\ldots,k-1$ then $r \leq^n_{ns} w$ and therefore $r \leq^n_{ans} w$. Hence assume $r_i = y_i$, $i = 1,\ldots,k-1$ and suppose that $w_k < r_k \leq y_k$. Therefore $S^k_{xr} < S^k_{xw} = 0$ contradicting $S^k_{xr} \geq 0$. Hence $r_k \leq w_k$; if the inequality is strict then $r \leq^n_{ns} w$ and $r \leq^n_{ans} w$. Therefore assume $r_k = w_k$. If $x_{k+1} = w_{k+1} < r_{k+1}$ or if $x_{i-1} = w_{i-1} = r_{i-1}$ and $x_i = w_i < r_i$, $i = k+2,\ldots,n-2$ then again $S^i_{xr} < S^i_{xw} = 0$ contradicting $S^i_{xr} \geq 0$. If $r_{k+1} < w_{k+1} = x_{k+1}$ or if $r_{i-1} = w_{i-1} = x_{i-1}$ and $r_i < w_i = x_i$, $i = k+2,\ldots,n-2$ then $r \leq^n_{ns} w$ and therefore $r \leq^n_{ans} w$. Therefore assume further $r_i = w_i = x_i$, $i = k+1,\ldots,n-2$. If next $w_{n-1} = x_{n-1} < r_{n-1}$ then using $p \cdot r \leq p \cdot w = p \cdot x$, $r_n < w_n = x_n$ which contradicts $x_n \leq r_n$ (implied by $p \cdot r \leq p \cdot x$ and $p' \cdot x \leq p' \cdot r$). Hence $r_{n-1} \leq w_{n-1} = x_{n-1}$. Again if the inequality is strict then $r \leq^n_{ns} w$ and $r \leq^n_{ans} w$. If $r_{n-1} = w_{n-1}$ then $r_n = x_n$ and $r = w$, thus completing the proof that $r \leq^n_{ans} w$ for each lower bound of x,y. Thus $w = x \wedge y$.

(b) and (c) : Obvious from the proof of (a)

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