Edgeworth and Walras equilibria of an arbitrage-free exchange economy

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Abstract

In this paper, we first give a direct proof of the existence of Edgeworth equilibria for exchange economies with (possibly) unbounded below consumption sets. The key assumption is that the individually rational utility set is compact. It is worth noticing that the statement of this result and its proof do not depend on the dimension or the particular structure of the commodity space. In a second part of the paper, we give conditions in order to decentralize Edgeworth allocations by continuous prices in a finite dimensional and in an infinite dimensional setting.

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1 Introduction

Since Hart (1974) [12], one knows that the existence of equilibrium in exchange economies with unbounded below consumption sets requires some nonarbitrage condition. For exchange economies consisting of a finite number of agents and defined on a finite dimensional commodity space, different variants of such a condition and different concepts of arbitrage have been formulated in [14], [21], [13], [15], [16]. The relations between these conditions are studied in [7], [16], [2]. All in turn imply the compactness of the individually rational utility set[1] when the preferences of agents are derived from utility functions. The few papers dealing with the equilibrium existence in an infinite dimensional setting ([5], [4], [6], [8]) assume the compactness of this set. Cheng [5],, Chichilnisky and Heal [6], Dana et al. [8] give also sufficient conditions on the primitives of the economy for this condition to be fulfilled.

This nonarbitrage condition is the central assumption of this paper. In order to model asset markets, we consider an exchange economy consisting of m agents, defined on a vector commodity space. Each agent is given with a (possibly unbounded below) consumption set, a utility function representing his preferences on his consumption set, an initial endowment. Our first concern is a direct proof of the existence for such models of Edgeworth equilibria as classically defined by Aliprantis et al. [1]. Since the set of attainable allocations needs not be bounded, this existence cannot be deduced from Debreu and Scarf's theorem or its extensions to an infinite dimensional setting ([1] and [10]). However, given the nonarbitrage condition, this existence is guaranteed under mild assumptions stated independently of the dimension of the commodity space or its particular structure.

The proof of this result is based on an extension to fuzzy coalitions of Scarf's theorem on the nonemptiness of the core of a nontransferable utility game game. The arguments of this preliminary result are inspired by a nice paper of Vohra [20]. The notion of balancedness for such a fuzzy game is borrowed from Florenzano [10]. The preliminary result is then applied to a proof (for any integer r) of the nonemptiness of the core of a fuzzy game appropriately associated to the r-replica of the exchange model. Finally, the existence of Edgeworth equilibria is proved using the compactness of the individually rational utility set.

A direct proof of the existence of Edgeworth equilibria open a room for using core-equilibrium equivalence theorems for proving the existence of Walras equilibria. The second part of the paper is devoted to some core-equilibrium equivalence theorems and to their consequences for the existence of Walras equilibria in asset market models.

[1] The individually rational utility set, sometimes simply called Utility set, is the set of utility vectors in which every agent receives no less than the utility of his initial endowment and no more than the utility of his consumption in a same attainable allocation.

Recall that the purpose of core-equilibrium equivalence theorems is to show that Edgeworth allocations can be supported as quasiequilibria by continuous prices. While the Edgeworth equilibrium existence theorem does not depend on the dimension and the structure of the commodity space, the techniques for obtaining the decentralizing continuous prices differ very much according to the dimension of the commodity space. In the finite dimensional case, the decentralizing vector price is obtained as a tangent linear functional supporting the set $\mathrm{co}(\bigcup_i \Gamma_i)$ where Γ_i is the set of preferred net trades of the ith consumer. The same argument is working in an infinite dimensional setting if the properties of preferred sets allow to use Hahn-Banach's theorem. In both cases, adding the assumptions of the core-equivalence theorem to the assumptions of the Edgeworth equilibrium existence result allow to extend most of known Walras equilibrium existence results.

At the end of the paper, to go further, we assume a vector lattice commodity space with a lattice ordered price space and propose to use a core-equilibrium equivalence result established by Tourky [19] with in mind the possibility of unbounded below consumption sets.

The paper is organized as follows: in Section 2, we prove the preliminary result. Section 3 contains the main result of the paper, the Edgeworth equilibrium existence result for an economy with (possibly) unbounded below consumption sets. Section 4 is devoted to decentralization results and to their consequences for the existence of Walras equilibria.

2 Abstract

In this paper, we first give a direct proof of the existence of Edgeworth equilibria for exchange economies with (possibly) unbounded below consumption sets. The key assumption is that the individually rational utility set is compact. It is worth noticing that the statement of this result and its proof do not depend on the dimension or the particular structure of the commodity space. In a second part of the paper, we give conditions for decentralizing Edgeworth allocations by continuous prices in a finite dimensional and in an infinite dimensional setting.

3 A preliminary nonemptiness theorem for the core of a fuzzy game

Let $M = \{1, \ldots, m\}$ be a finite set of players and $T = [0, 1]^m \setminus \{0\}$. An element $t \in T$ is interpreted as a *fuzzy coalition*, that is, a vector $t = (t_i)_{i=1}^m$ of rates of participation to the coalition t for the different players.

We consider in this section m-person *fuzzy games* defined by $(\mathcal{T}, V)$ where $\mathcal{T}$ is a *finite* subset of T containing $\bar{1} = (1, \ldots, 1)$ and the canonical base (e^i) of R^m and $V : \mathcal{T} \to R^m$ is a nonempty-valued correspondence. For a fuzzy coalition $t \in \mathcal{T}$, let us denote

$$\operatorname{supp} t = \{i \in M \mid t_i > 0\}$$

the set of agents who participate in this coalition.

Definition 3.1 *The fuzzy core of the m-person fuzzy game $(\mathcal{T}, V)$ is the set*

$$\mathcal{C}(\mathcal{T}, V) = \{v \in V(\bar{1}) \mid \nexists t \in \mathcal{T} \text{ and } u \in V(t) \text{ s.t. } v_i < u_i, \forall i \in \operatorname{supp} t\}.$$

Consider the following set

$$\Delta^{\mathcal{T}} = \{\lambda = (\lambda_t)_{t \in \mathcal{T}} \mid \lambda_t \geq 0 \text{ and } \sum_{t \in \mathcal{T}} \lambda_t t = \bar{1}\}.$$

It is easily seen that $\Delta^{\mathcal{T}}$ is nonempty.

Definition 3.2 *A m-person fuzzy game $(\mathcal{T}, V)$ is said to be balanced whenever for every $\lambda \in \Delta^{\mathcal{T}}$,*

$$\bigcap_{\{t \in \mathcal{T} \mid \lambda_t > 0\}} V(t) \subset V(\bar{1}).$$

The following theorem extends Scarf's theorem [17] as stated by Aliprantis et al. [1]. The ideas of the proof are due to R. Vohra [20] (see also Shapley and Vohra [18]). This section is devoted to its proof.

Theorem 3.1 *If $\mathcal{T}$ is as above and if $(\mathcal{T}, V)$ is a balanced m-person fuzzy game such that*
a) each $V(t)$ is closed,
b) each $V(t)$ is comprehensive from below, i.e., $u \leq v$ and $v \in V(t)$ imply $u \in V(t)$,
c) $u \in R^m$, $v \in V(t)$ and $u_i = v_i$ $\forall i \in \operatorname{supp} t$ imply $u \in V(t)$,
d) for each $t \in \mathcal{T}$ there exists $c_t \in R$, such that $v \in V(t)$ implies $v_i \leq c_t$ for all $i \in \operatorname{supp} t$,
then

$$\mathcal{C}(\mathcal{T}, V) \neq \emptyset.$$

Proof. Each $V(t)$ is comprehensive from below. So for each $t \in \mathcal{T}$, there exists $a_t \in R^m$ such that $0 \in \operatorname{int}(a_t + V(t))$. If $a = \vee_{t \in \mathcal{T}} a_t$, it is obvious that $a + V$ satisfies the properties *a), b), c), d)* and that $\mathcal{C}(\mathcal{T}, a + V) = a + \mathcal{C}(\mathcal{T}, V)$. Hence, without loss of generality, we can (and we will) assume that $0 \in \operatorname{int} V(t)$ for each $t \in \mathcal{T}$.

Next, fix some constant $c > 0$ such that for each $t \in \mathcal{T}$ and each $v \in V(t)$ we have $v_i < c$ for all $i \in \operatorname{supp} t$, and then consider the set

$$W = (\bigcup_{t \in \mathcal{T}} V(t)) \bigcap]-\infty,\ c]^m.$$

Clearly, the set W is closed, comprehensive from below and contains 0 in its interior. Let ∂W denote the boundary of W.

Claim 3.1 *If $v \in \partial W \cap R^m_+$ and $v_i = 0$ for some i, then $v_j = c$ holds for some j.*

Proof of Claim 3.1. To see this, assume that $v_i = 0$ and $v_j < c$ holds for each j. Since $0 \in \operatorname{int} V(e^i)$, there exists some $u \in V(e^i)$ with $0 < u_i < c$. From Property c), we see that the vector x defined by $x_j = c$ for $j \neq i$ and $x_i = u_i$ belongs to $V(e^i)$ (and hence to $W \cap R^m_+$) and satisfies $v \ll x$. This implies $v \in \operatorname{int} W$, a contradiction. □

Let Δ be the unit-simplex of R^m.

Claim 3.2 *For each $s \in \Delta$, there exists a unique $\alpha > 0$ (depending on s) such that $\alpha s \in \partial W \cap R^m_+$.*

Proof of Claim 3.2. Let $s \in \Delta$. We first prove that there exists at most one α such that $\alpha s \in \partial W \cap R^m_+$. Indeed, let αs and βs be elements of $\partial W \cap R^m_+$ such that $\alpha > \beta > 0$. If $s_i > 0$ holds for each i, then $\alpha s_i > \beta s_i$ holds for each i and so βs is an interior point of W, a contradiction. On the other hand, if $s_i = 0$ holds for some i, then by Claim 3.1 there exists some j such that $\beta s_j = c$ and so that $\alpha s_j > c$, a contradiction. Moreover, let $\alpha = \sup\{\beta \mid \beta s \in W \cap R^m_+\}$. From $0 \in \operatorname{int} W$, we deduce that $\alpha > 0$. From the definition of W, we deduce that α is finite. Since, W is closed it follows that $\alpha s \in \partial W \cap R^m_+$. □

Thus , a function $f : \Delta \longrightarrow \partial W \cap R^m_+$ can be defined by formula

$$f(s) = \alpha s \text{ where } \alpha = \sup\{\beta \in R_+ \mid \beta s \in W \cap R^m_+\}.$$

Claim 3.3 *f is continuous.*

Proof of Claim 3.3. It suffices to show that f has a closed graph. Let us consider a sequence $(s^n, f(s^n))$ in $\Delta \times (\partial W \cap R^m_+)$ that converges to (s, y). Write $f(s^n) = \alpha^n s^n \in \partial W \cap R^m_+$. Then $\alpha^n = \|\alpha^n s^n\|_1 = \|f(s^n)\|_1 \longrightarrow \|y\|_1$, and hence $f(s^n) = \alpha^n s^n \longrightarrow \|y\|_1 s$. By uniqueness of the limit $y = \|y\|_1 s$. Since $\partial W \cap R^m_+$ is a closed set, it follows from Claim 3.2 that $f(s) = y$. Consequently, f has a closed graph. □

Define a correspondence $\Psi : \Delta \to \Delta$ by

$$\Psi(s) = \left\{ \ \tfrac{t}{\|t\|_1} \mid t \in \mathcal{T} \text{ and } f(s) \in V(t) \ \right\}$$

Claim 3.4 *Ψ is nonempty-valued and has a closed graph.*

Proof of Claim 3.4. Since $f(s) \in W$, it follows immediately that $\Psi(s)$ is a nonempty subset of Δ. Furthermore, let us assume $s^n \to s$, $y^n \to y$ and $y^n \in \Psi(s^n)$. Since the range of Ψ is a finite set, there exists some n_0 such that for all $n \geq n_0$, $y^n = y$ and $f(s^n) \in \bigcup_{\{t \in \mathcal{T} \mid \frac{t}{\|t\|_1} = y\}} V(t)$. Since $\{t \in \mathcal{T} \mid \frac{t}{\|t\|_1} = y\}$ is a finite set, passing to a subsequence if necessary, we can assume that $f(s^n) \in V(t_0)$ for some $t_0 \in \{t \in \mathcal{T} \mid \frac{t}{\|t\|_1} = y\}$. Since f is continuous and $V(t_0)$ is a closed set, we deduce that $f(s) \in V(t_0)$ and consequently $y \in \Psi(s)$. □

Now we define the function $g : \Delta \times \Delta \to \Delta$ by

$$g_i(s, \mu) = \frac{s_i + (\mu_i - \frac{1}{m})^+}{1 + \sum_{j=1}^m (\mu_j - \frac{1}{m})^+}$$

where, as usual, $r^+ = \max\{r, 0\}$ for each real number r.
Clearly, g is a continuous function. Finally, we consider the correspondence $\Phi : \Delta \times \Delta \to \Delta \times \Delta$ defined by

$$\Phi(s, \mu) = \{g(s, \mu)\} \times \mathrm{co}\Psi(s).$$

Note that Φ is nonempty and convex-valued and has a closed graph. Thus by Kakutani's fixed point theorem, Φ has a fixed point $(\bar{s}, \bar{\mu})$. That is,

$$\bar{s} = g(\bar{s}, \bar{\mu}) \text{ and } \bar{\mu} \in \mathrm{co}\Psi(\bar{s}).$$

In other words,

$$\bar{s}_i = \frac{\bar{s}_i + (\bar{\mu}_i - \frac{1}{m})^+}{1 + \sum_{j=1}^m (\bar{\mu}_j - \frac{1}{m})^+} \ , \ i \in M \tag{1}$$

and there exist $\mathcal{T}' \subset \mathcal{T}$, $(a_t)_{t \in \mathcal{T}'} \in R^{\mathcal{T}'}$, with $a_t > 0$ $\forall t \in \mathcal{T}'$ and $\sum_{t \in \mathcal{T}'} a_t = 1$, such that

$$\bar{\mu} = \sum_{t \in \mathcal{T}'} a_t \frac{t}{\|t\|_1} \ , \ f(\bar{s}) \in V(t) \ \forall t \in \mathcal{T}' \tag{2}$$

Claim 3.5 . *For all $i \in M$, $\bar{\mu}_i = \frac{1}{m}$.*

Proof of Claim 3.5: Suppose that it is not true. Recalling that $\bar{\mu} \in \Delta$, it follows from (1) that $\sum_{j=1}^m (\bar{\mu}_j - \frac{1}{m})^+ > 0$ Then, the sets

$$I = \{i \in M \mid \bar{s}_i > 0\} = \{i \in M \mid \bar{\mu}_i > \frac{1}{m}\}$$

and

$$J = \{i \in M \mid \bar{s}_i = 0\} = \{i \in M \mid \bar{\mu}_i \leq \frac{1}{m}\}$$

are both nonempty. Indeed, from $\sum_{i=1}^m (\bar{\mu}_i - \frac{1}{m})^+ > 0$, it follows that $(\bar{\mu}_i - \frac{1}{m})^+ > 0$ for some i. On the other hand, if $\bar{\mu}_i > \frac{1}{m}$ for each i, then $\sum_{i=1}^m \bar{\mu}_i > 1$, a contradiction. Clearly, for all $j \in J$ we have $s_j = 0$ hence $f(s)_j = 0$. From (2), for all $i \in I$, there exists $t \in \mathcal{T}'$ such that $t_i > 0$, $f(\bar{s}) \in V(t)$, hence $f(\bar{s})_i < c$, which, together with $J \neq \emptyset$, contradicts Claim 3.1. □

Now, let us consider $\lambda \in R^T$ such that

$$\lambda_t = \begin{cases} \frac{m a_t}{\|t\|_1} & \text{if } t \in \mathcal{T}', \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\sum_{t \in \mathcal{T}} \lambda_t t = \bar{1}$, and therefore $\lambda \in \Delta^{\mathcal{T}}$.

Claim 3.6 $f(\bar{s}) \in \mathcal{C}(\mathcal{T}, V)$

Proof of Claim 3.6: Since $\lambda \in \Delta^{\mathcal{T}}$ we have

$$\bigcap_{t \in \mathcal{T}'} V(t) = \bigcap_{\{t \in \mathcal{T} | \lambda_t > 0\}} V(t) \ \subset \ V(\bar{1}).$$

Hence $f(\bar{s}) \in V(\bar{1})$. Suppose that there exists $t \in \mathcal{T}$ and $v \in V(t)$ such that $f(\bar{s})_i < v_i$ for all $i \in \operatorname{supp} t$. Let u be such that $u_i = v_i$ for all $i \in \operatorname{supp} t$ and $u_i = c$ otherwise. It follows from Property c) that $u \in V(t) \subset W$. But $f(\bar{s}) \ll u$ contradicts the fact that $f(\bar{s}) \in \partial W \cap R^m_+$. Therefore $f(\bar{s}) \in \mathcal{C}(\mathcal{T}, V)$, which ends the proof of theorem 3.1. □

4 Application to the existence of Edgeworth equilibria of an arbitrage-free exchange economy

4.1 Definitions

In order to apply the previous theorem, we consider an exchange economy defined on a commodity vector space L and recall some definitions. $M = \{1, \ldots, m\}$ is the set of consumers. Each consumer i is described by a consumption set $X_i \subset L$, an initial endowment $e_i \in X_i$, and a preference relation which is represented by a utility function $u_i : X_i \to R$. We normalize the utility functions by requiring $u_i(e_i) = 0$. To summarize, the economy $\mathcal{E}$ is a collection

$$\mathcal{E} = ((X_i, u_i, e_i)_{i \in M}).$$

Let $\mathcal{A}(\mathcal{E})$ be the set of all *attainable allocations* of the economy $\mathcal{E}$, that is:

$$\mathcal{A}(\mathcal{E}) = \{x = (x_i)_{i \in M} \in \prod_{i \in M} X_i \mid \sum_{i \in M} x_i = \sum_{i \in M} e_i\}.$$

Let also $\mathcal{M} = 2^M \setminus \{\emptyset\}$ be the family of all coalitions of consumers. The allocation $x \in \mathcal{A}(\mathcal{E})$ is improved upon by the coalition $S \in \mathcal{M}$ if there exists $(x'_i)_{i\in S} \in \prod_{i\in S} X_i$ satisfying $\sum_{i\in S} x'_i = \sum_{i\in S} e_i$ and such that $u_i(x_i) < u_i(x'_i)$ for every $i \in S$. The core of the economy $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$, is defined as the set of all allocations $x \in \mathcal{A}(\mathcal{E})$ which are improved upon by no coalition. Finally, following Aliprantis et al. [1], $x \in \mathcal{A}(\mathcal{E})$ is said to be an *Edgeworth equilibrium* if, for every integer $r \geq 1$, the r-repetition of x belongs to the core of the r-fold replica of $\mathcal{E}$. We will denote by $\mathcal{C}^E(\mathcal{E})$ the set of all Edgeworth equilibria of $\mathcal{E}$.

For each integer $r \geq 1$, using the notations of the previous section, if

$$\mathcal{T}^r = \{t \in T \mid rt_i \in \{0, \dots, r\}, \forall i \in M\},$$

let us define $\mathcal{C}^r(\mathcal{E})$ as the set of all attainable allocations $x \in \mathcal{A}(\mathcal{E})$ such that there exist no $t \in \mathcal{T}^r$ and no $x^t \in \prod_{i\in \operatorname{supp} t} X_i$ such that

$$\sum_{i\in \operatorname{supp} t} t_i x_i^t = \sum_{i\in \operatorname{supp} t} t_i e_i \text{ and } \forall i \in \operatorname{supp} t,\ u_i(x_i) < u_i(x_i^t).$$

As it is easily seen and proved in Florenzano [10], under convexity assumptions on preferences and consumption sets, $\mathcal{C}^E(\mathcal{E}) = \bigcap_{r\geq 1} \mathcal{C}^r(\mathcal{E})$. In other words, $\mathcal{C}^E(\mathcal{E})$ is the set of all $x \in \mathcal{A}(\mathcal{E})$ such that there exist no $t = (t_i)_{i\in M} \in T$, with rational rates of participation, and no $x^t \in \prod_{i\in \operatorname{supp} t} X_i$ such that

$$\sum_{i\in \operatorname{supp} t} t_i x_i^t = \sum_{i\in \operatorname{supp} t} t_i e_i \text{ and } \forall i \in \operatorname{supp} t,\ u_i(x_i) < u_i(x_i^t).$$

Following Aubin [3], the fuzzy core of the economy $\mathcal{E}$, $\mathcal{C}^F(\mathcal{E})$, is the set of all $x \in \mathcal{A}(\mathcal{E})$ such that there exist no $t = (t_i)_{i\in M} \in T$ and no $x^t \in \prod_{i\in \operatorname{supp} t} X_i$ such that

$$\sum_{i\in \operatorname{supp} t} t_i x_i^t = \sum_{i\in \operatorname{supp} t} t_i e_i \text{ and } \forall i \in \operatorname{supp} t,\ u_i(x_i) < u_i(x_i^t).$$

4.2 The existence result

Let us now denote by

$$\mathcal{U}(\mathcal{E}) = \{v = (v_i)_{i=1}^m \in R_+^m \mid \exists x \in \mathcal{A}(\mathcal{E}),\ s.t.\ 0 \leq v_i \leq u_i(x_i),\ \forall i\}$$

the *individually rational utility set*[2].

We make on $\mathcal{E}$ the following assumptions:

[A.1] For each i, X_i is convex;

[2]Recall that $u_i(e_i) = 0$, $i = 1, \dots, m$.

[A.2] For each i, $u_i : X_i \to R$ is quasi-concave;

[A.3] $\mathcal{U}(\mathcal{E})$ is compact.

For a fuzzy coalition $t \in \mathcal{T}^r$, let

$$\mathcal{A}^t(\mathcal{E}) = \{x^t \in \prod_{i \in \operatorname{supp} t} X_i \mid \sum_{i \in \operatorname{supp} t} t_i x_i^t = \sum_{i \in \operatorname{supp} t} t_i e_i\}$$

and

$$U^t = \{v^t \in R_+^{\operatorname{supp} t} \mid \exists x^t \in \mathcal{A}^t(\mathcal{E}),\ s.t.\ 0 \leq v_i \leq u_i(x_i^t),\ \forall i \in \operatorname{supp} t\}.$$

Finally, let

$$V(t) = (\overline{U^t - R_+^{\operatorname{supp} t}}) \times R^{M \setminus \operatorname{supp} t}.$$

Proposition 4.1 *Assume* [A.1]–[A.3]. *Then* $\mathcal{C}(\mathcal{T}^r, V)$ *is a nonempty subset of* $\mathcal{U}(\mathcal{E})$.

Proof. Since $\mathcal{U}(\mathcal{E})$ is compact, there exists $c > 0$ such that $\mathcal{U}(\mathcal{E}) \subset]-\infty, c[^m$. For each $t \in \mathcal{T}^r$, let us define

$$V^c(t) = ((\overline{\mathcal{U}(\mathcal{E})^t - R_+^{\operatorname{supp} t}}) \bigcap (]-\infty, c])^{\operatorname{supp} t}) \times R^{M \setminus \operatorname{supp} t}$$

We will keep in mind that $V^c(\bar{1}) = V(\bar{1}) = \mathcal{U}(\mathcal{E}) - R_+^m$ and that for every i, $V^c(e^i) = V(e^i) = -R_+ \times R^{M \setminus \{i\}}$. We first claim that the m-person fuzzy game $(\mathcal{T}^r, V^c)$ has a nonempty fuzzy core, that is, $\mathcal{C}(\mathcal{T}^r, V^c) \neq \emptyset$.

Clearly, $\mathcal{T}^r$ is a finite subset of T containing $\bar{1} = (1, \ldots, 1)$ and the canonical base (e^i) of R^m. The properties listed in Theorem 3.1 are also trivially satisfied. It suffices to verify that the m-person fuzzy game $(\mathcal{T}^r, V^c)$ is balanced in the sense of Definition 3.2.

To this end, let $\lambda \in \Delta^{\mathcal{T}^r}$ and $v \in \bigcap_{\{t \in \mathcal{T}^r | \lambda_t > 0\}} V^c(t)$. For each integer n and for every $t \in \mathcal{T}^r$ such that $\lambda_t > 0$, there exists $x^{n,t} \in \mathcal{A}^t(\mathcal{E})$ such that $u_i(x_i^{n,t}) \geq 0\ \forall i \in \operatorname{supp} t$ and

$$v_i \leq u_i(x_i^{n,t}) + \frac{1}{n},\ \forall i \in \operatorname{supp} t. \tag{3}$$

For each $i \in M$, let

$$x_i^n = \sum_{t \in \mathcal{T}^r} \lambda_t t_i x_i^{n,t}.$$

Since $\sum_{t \in \mathcal{T}^r} \lambda_t t = \bar{1}$, we have for each $i \in M$, $x_i^n \in X_i$ (X_i is convex) and

$$\begin{aligned}
\sum_{i=1}^m x_i^n &= \sum_{i=1}^m \sum_{t \in \mathcal{T}^r} \lambda_t t_i x_i^{n,t} = \sum_{t \in \mathcal{T}^r} \lambda_t (\sum_{i \in \operatorname{supp} t} t_i x_i^{n,t}) \\
&= \sum_{t \in \mathcal{T}^r} \lambda_t (\sum_{i \in M} t_i e_i) = \sum_{i=1}^m (\sum_{t \in \mathcal{T}^r} \lambda_t t_i) e_i = \sum_{i=1}^m e_i
\end{aligned}$$

which shows that $x^n \in \mathcal{A}(\mathcal{E})$. Now, from relations (3) and in view of the definition of x^n and the quasi-concavity of utility functions u_i, we have

$$v_i \le u_i(x_i^n) + \frac{1}{n}, \ \forall i \in M.$$

Since for every $t \in \mathcal{T}^r$ such that $\lambda_t > 0$, $u_i(x_i^{n,t}) \ge 0 \ \forall i \in \operatorname{supp} t$, we have also $(u_i(x_i^n))_{i\in M} \in \mathcal{U}(\mathcal{E})$. Passing to a subsequence if necessary, it follows from the compactness of $\mathcal{U}(\mathcal{E})$ that there exists $x \in \mathcal{A}(\mathcal{E})$ such that

$$v_i \le \lim_{n\to+\infty} u_i(x_i^n) \le u_i(x_i), \ \forall i \in M.$$

Hence $v \in V(\bar{1}) = V^c(\bar{1})$, which shows that the game $(\mathcal{T}^r, V^c)$ is balanced. It then follows from Theorem 3.1 that $\mathcal{C}(\mathcal{T}^r, V^c) \neq \emptyset$.

To end the proof, let $v \in \mathcal{C}(\mathcal{T}^r, V^c)$. Note that $v \in V^c(\bar{1}) = V(\bar{1}) = \mathcal{U}(\mathcal{E}) - R_+^m$. Moreover, $v \in \mathcal{U}(\mathcal{E})$. Indeed if not, for some i, $\{0\} \times R^{M\setminus\{i\}} \in V(e^i)$ with $0 > v_i$. We now prove by contraposition that $v \in \mathcal{C}(\mathcal{T}^r, V)$. Let us assume on the contrary that there exist $t \in \mathcal{T}^r$ and $u \in V(t)$ such that $v_i < u_i \ \forall i \in \operatorname{supp} t$. Since $v_i < c \ \forall i \in M$, one can find $\lambda \in]0,1[$ such that

$$v_i < \lambda v_i + (1-\lambda)u_i < \min\{c, u_i\} \ \ \forall i \in \operatorname{supp} t.$$

Hence $(\lambda v_i + (1-\lambda)u_i)_{i\in \operatorname{supp} t} \in V^c(t)$. We have got a contradiction. □

Proposition 4.2 *Assume* [A.1]-[A.3] *on* $\mathcal{E}$. *Then* $\bigcap_{r\ge1} \mathcal{C}(\mathcal{T}^r, V) \neq \emptyset$.

Proof First, we show that $\mathcal{C}(\mathcal{T}^r, V)$ is closed. Let $v = \lim_{n\to+\infty} v^n$ with $v^n \in \mathcal{C}(\mathcal{T}^r, V)$. Suppose that $v \notin \mathcal{C}(\mathcal{T}^r, V)$. Then there exists $t \in \mathcal{T}^r$ and $u \in V(t)$ such that $v_i < u_i \ \forall i \in \operatorname{supp} t$. But, for n large enough, we have $v_i^n < u_i \ \forall i \in \operatorname{supp} t$, a contradiction. To end the proof, in view of the compactness of $\mathcal{U}(\mathcal{E})$, it suffices to show that for each integer $r \ge 1$ we have $\mathcal{C}(\mathcal{T}^{r+1}, V) \subset \mathcal{C}(\mathcal{T}^r, V)$. Let $v \in \mathcal{C}(\mathcal{T}^{r+1}, V)$ and suppose that $v \notin \mathcal{C}(\mathcal{T}^r, V)$. Then there exists $t \in \mathcal{T}^r$ and $u \in V(t)$ such that $v_i < u_i \ \forall i \in \operatorname{supp} t$. Let us consider $t' = \frac{r}{r+1}t$. Clearly $\mathcal{A}^{t'}(\mathcal{E}) = \mathcal{A}^t(\mathcal{E})$, $U^t = U^{t'}$, $V(t) = V(t')$. Since $t' \in \mathcal{T}^{r+1}$ and $u \in V(t')$, we have got a contradiction. □

We are now ready to prove the main result of this section.

Theorem 4.1 *Under Assumptions* [A.1]-[A.3], *the set of Edgeworth equilibria* $\mathcal{C}^E(\mathcal{E})$ *is nonempty.*

Proof Let $v \in \bigcap_{r\ge1} \mathcal{C}(\mathcal{T}^r, V)$ and $x \in \mathcal{A}(\mathcal{E})$ be such that $v_i \le u_i(x_i) \ \forall i \in M$. We claim that $x \in \mathcal{C}^E(\mathcal{E})$. Indeed, if for some r, $x \notin \mathcal{C}^r(\mathcal{E})$, then there exist $t \in \mathcal{T}^r$ and $x' \in \mathcal{A}^t(\mathcal{E})$ such that $v_i \le u_i(x_i) < u_i(x_i')$ for all $i \in \operatorname{supp} t$. Hence $v \notin \mathcal{C}(\mathcal{T}^r, V)$, a contradiction. □

Remark 4.1 Adding the assumption that the commodity space is finite dimensional, the consumption sets are closed and the utility functions are upper semicontinuous at every attainable consumption vector to the other assumptions of Theorem 4.1, it would be easy to deduce its conclusion from Proposition 3 in Florenzano [10]. The same proof[3], under analogous topological assumptions (relative to the weak*-topology on L), can be given if the commodity space is an infinite dimensional Banach space which has a predual. These two cases cover most of commodity spaces of economic interest. However, it should be noticed that the statement of Theorem 4.1 and its proof do not depend on the dimension of the commodity space or on its particular structure.

Remark 4.2 As easily seen and proved in Florenzano [10], if the commodity space L is a Hausdorff topological vector space and if the utility functions are lower semicontinuous at every attainable consumption vector, an Edgeworth equilibrium whose existence is proved in Theorem 4.1 is actually an element of the fuzzy core, $\mathcal{C}^F(\mathcal{E})$, of the economy $\mathcal{E}$.

5 Walras equilibria of an arbitrage-free exchange economy

Recall that a couple (x, p) is said to be a *quasiequilibrium* of $\mathcal{E}$ iff $x \in \mathcal{A}(\mathcal{E})$, p is a linear functional on L, with $p \neq 0$ and

$$\text{for every } i \in M,\ p \cdot x_i = p \cdot e_i \text{ and } u_i(x'_i) > u_i(x_i) \Rightarrow p \cdot x'_i \geq p \cdot x_i.$$

A quasiequilibrium such that $u_i(x'_i) > u_i(x_i)$ actually implies $p \cdot x'_i > p \cdot x_i$ is a *Walras equilibrium.* We will prove the existence of quasiequilibria by decentralizing Edgeworth equilibria obtained via Theorem 4.1.

5.1 Finite dimensional decentralization

Let us first assume that the commodity space L is R^ℓ, the l dimensional space. For each $x_i \in X_i$, we define the strictly preferred set to x_i by

$$P_i(x_i) = \{x'_i \in X_i \mid u_i(x_i) < u_i(x'_i)\}$$

and we set the two following assumptions:

[3]Truncating the economy by an increasing sequence of closed balls of L, centered at 0 and containing all initial endowments, one obtains a sequence $(x_i^\nu)_{i=1}^m$ of Edgeworth equilibria of the truncated economies. The sequence $(u_i(x_i^\nu))_{i=1}^m$ belongs to $\mathcal{U}(\mathcal{E})$ and has a converging subsequence. At the limit, from the definition of $\mathcal{U}(\mathcal{E})$, one gets an allocation $(x_i)_{i=1}^m$. Using the upper semicontinuity of functions u_i, it is easily proved that this allocation is an Edgeworth equilibrium of $\mathcal{E}$.

[A.4] For each $i \in M$, u_i is lower semicontinuous at every attainable consumption vector;

[A.5] If $x \in \mathcal{A}(\mathcal{E})$ then for each $i \in M$, $x_i \in \overline{P_i(x_i)}$ (the closure of $P_i(x_i)$).

Proposition 5.1 *Under* [A.1]-[A.5], *the economy* $\mathcal{E} = ((X_i, u_i, e_i)_{i \in M})$ *has a quasiequilibrium.*

Proof Let $\bar{x} \in \mathcal{C}^E(\mathcal{E})$. In view of Assumption [A.5] , we have already remarked that $\bar{x} \in \mathcal{C}^F(\mathcal{E})$. Let $G = \mathrm{co}(\bigcup_{i \in M}(P_i(\bar{x}_i) - e_i))$. G is nonempty since $x \in \mathcal{A}(\mathcal{E})$ and the assumption [A.5] imply that $P_i(\bar{x}_i) \neq \emptyset$. We first prove that $0 \notin G$. Indeed if not, there exists $\lambda = (\lambda_i)_{i \in M}$ such that $\lambda_i \geq 0$, for all i and $\sum_{i \in M} \lambda_i = 1$ and $(x_i) \in \prod_{i \in M} X_i$ such that

$$\sum_{i \in M} \lambda_i x_i = \sum_{i \in M} \lambda_i e_i$$

$$x_i \in P_i(\bar{x}_i), \ \forall i \text{ such that } \lambda_i > 0.$$

Thus the fuzzy coalition λimprove upon $\bar{x}$, which contradicts $\bar{x} \in \mathcal{C}^F(\mathcal{E})$.

Now, by the separation theorem for finite dimensional vector spaces, there exists $p \in R^\ell \setminus 0$ such that $p \cdot g \geq 0$, for all $g \in G$. From [A.5], one deduces that $p \cdot \bar{x}_i \geq p \cdot e_i$ for all $i \in M$. Since $\bar{x} \in \mathcal{C}^E(\mathcal{E})$, $\sum_{i \in M} = \sum_{i \in M} e_i$. Thus $p \cdot \bar{x}_i = p \cdot e_i$ for all $i \in M$ and $(\bar{x}, p)$ is a quasi-equilibrium of $\mathcal{E}$. □

Remark 5.1 In view of [A.5], assuming either that each $e_i \in \mathrm{int}\, X_i$ or that $e = \sum_{i \in M} e_i$ is an interior point of $\sum_{i \in M} X_i$ and that $\mathcal{E}$ satisfies some irreducibility assumption, then $(\bar{x}, p)$ is a Walras equilibrium.

5.2 Infinite dimensional decentralization

If the commodity space, L, is infinite dimensional, let us first assume that L is a Hausdorff topological vector space and that $\mathrm{int}(\bigcup_{i \in M}(P_i(\bar{x}_i) - e_i)) \neq \emptyset$. Using Hahn-Banach's theorem, we can mimic the proof of Proposition 5.1 in order to obtain:

Proposition 5.2 *Under* [A.1]-[A.5], *the economy* $\mathcal{E} = ((X_i, u_i, e_i)_{i \in M})$ *has a quasiequilibrium. Under the same additional assumptions as in remark 5.1, this quasiequilibrium is an equilibrium.*

The previous result extends the results of Cheng [5], Brown and Werner [4], Theorem 1 of Dana et al.[8]. Actually, Brown and Werner, Dana et al. do not assume [A.4] but prove only the existence of a quasiequilibrium.

To go further, we now assume that L is a vector lattice with a topological dual which is a sublattice of its order dual. We restrict ourselves in this last part of the paper to the case of a noncompact attainable set, $\mathcal{A}(\mathcal{E})$. The reader can find a study of the case $\mathcal{A}(\mathcal{E})$ compact in Deghdak and Florenzano [9] for consumption sets equal to the positive cone and in Florenzano and Marakulin [11] for more general consumption sets (see also the references quoted in both papers). Recall that $e = \sum_{i \in M} e_i$ and that $P_i(x_i) = \{x'_i \in X_i \mid u_i(x_i) < u_i(x'_i)\}$ defines a preference correspondence (preference relation) $P_i : X_i \to X_i$. The following definition is borrowed from Tourky [19]:

Definition 5.1 *A preference relation P_i is M-proper at x_i if there are a convex lattice Z_{x_i} and a convex set $\widehat{P_i(x_i)}$ such that:*

1. $\widehat{P_i(x_i)} \bigcap Z_{x_i} = P_i(x_i)$
2. *$x_i + e$ is an interior point of $\widehat{P_i(x_i)}$ and $P_i(x_i)$ is open in Z_{x_i}*
3. *0, $e_i \in Z_{x_i}$ and $Z_{x_i} + L_+ = Z_{x_i}$*
4. *$(1+\alpha)x_i \in Z_{x_i}$ for some $\alpha > 0$.*

Using Theorem 2.1 of Tourky [19], we obtain the following existence result:

Proposition 5.3 *Assume* [A.1]-[A.5] *and that, in addition, $e > 0$, each u_i is strictly increasing, each P_i is M-proper at every component of an attainable allocation. Then $\mathcal{E}$ has a quasiequilibrium. Under the same additional assumptions as in remark 5.1, this quasiequilibrium is a Walras equilibrium.*

This Walras equilibrium existence theorem has no antecedent in the literature on trade in financial assets The economic meaning of M-properness is a purpose for future work.

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