Spurious and hidden volatility

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Abstract

This paper analyzes the effects caused by the presence of outliers on the identification and estimation of GARCH models. We show that outliers can lead to detect spurious conditional heteroscedasticity and they can also hide genuine ARCH effects. First, we derive the asymptotic biases caused by outliers on the sample autocorrelations of squared observations and their effects on some popular homoscedasticity tests. Then, we obtain the asymptotic biases of the OLS estimates of the parameters of ARCH(p) models and analyze their

finite sample behavior by means of extensive Monte Carlo experiments. The finite sample results are also extended to GLS and ML estimates of ARCH(p) and GARCH(1,1) models. All the results are illustrated analyzing real series of financial returns.

1 Introduction

Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) models were introduced by Engle (1982) and Bollerslev (1986) to represent the dynamic evolution of conditional variances and they have been extensively used in the empirical analysis of high frequency financial returns. However, we often find that when these models are fitted to real time series, the residuals still have excess kurtosis, which could be explained, among other reasons, by the presence of outliers; see, for example, Bollerslev (1987), Teräsvirta (1996) and Franses and Ghijsels (1999).

Previous results on the effects of outliers on the identification and estimation of conditional heteroscedasticy are somehow confusing. Some authors argue that outliers generate spurious heteroscedasticity. For example, Balke and Fomby (1994) conclude that the outliers in several macroeconomic series of the US economy are able to explain most of the observed non-linearities. A similar conclusion is reached by Franses and Gijsels (1999) for macroeconomic series and Aggarwal *et al.* (1999) and Franses and Van Dijk (2000) for financial returns. On the other hand, other authors suggest that the presence of outliers may hide genuine heterocedasticity; see, for example, Mendes (2000) and Li and Kao (2002) for an empirical application with exchange rates returns.

The simultaneous presence of conditional heteroscedasticity and outliers in time series can be easily confused because both phenomena cause excess kurtosis and, if outliers appear in patches, autocorrelated squared observations. The objective of this paper is to show that the presence of additive

outliers in uncorrelated GARCH series may generate spurious heteroscedasticity when they appear in patches, and hide legitimate heterocedasticity when they are isolated. Consequently, both the size and power of tests for conditional homoscedasticity can be distorted in the presence of outliers. Also, they modify the estimated autocorrelations of squares and the estimated parameters of the conditional variance.

The paper is organized as follows. Section 2 defines different types of outliers in the context of the popular ARCH(p) and GARCH(1,1) models. In this paper, we focus on level outliers that do not affect the conditional variance. Section 3 analyses the effects of level outliers on the sample autocorrelations of squared observations and on several tests for conditional heteroscedasticity. In section 4, we derive the asymptotic bias of the Ordinary Least Squares (OLS) estimator of the parameters of ARCH(p) models contaminated by level outliers. Its finite sample properties are also analyzed by means of extensive Monte Carlo experiments. These results are also extended to the Maximum Likelihood (ML) estimator. Section 5 analyses the finite sample properties of the ML estimator of the parameters of GARCH(1,1) models. Section 6 illustrates the results by analyzing several real series of financial returns. Finally, Section 7 concludes the paper.

2 GARCH models and outliers

The most popular model to represent the dynamic evolution of conditional variances is the GARCH(1,1) model proposed independently by Bollerslev (1986) and Taylor (1986). This model specifies the series of interest, y_t, as

follows

$$y_t = \varepsilon_t \sigma_t \tag{1}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta \sigma_{t-1}^2$$

where ε_t is a Gaussian white noise with mean zero and variance one. The parameters α_0, α_1 and β are assumed to satisfy the usual restrictions to guarantee the positiveness, stationary and existence of the fourth order moment of y_t; see, for example, Bollerslev $et\ al.$ (1994). Alternatively, the conditional variance in (1) can be approximated assuming that $\beta = 0$ and adding $p - 1$ additional lags of squared observations. The resulting model was originally proposed by Engle (1982) and it is known as ARCH(p). In this case, the conditional variance is given by

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 \tag{2}$$

where the parameters α_i should also be restricted so that σ_t^2 is positive and y_t is stationary with finite fourth order moment.

It is well known that in both models the series y_t is an uncorrelated sequence. However, it is not independent because y_t^2 is correlated. To derive its autocorrelation function (acf), it is useful to write the ARCH(p) model as an AR(p) for squared observations as follows

$$y_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \nu_t \tag{3}$$

where the noise, $\nu_t = \sigma_t^2(\varepsilon_t^2 - 1)$, is a zero mean uncorrelated sequence. However, it is conditionally heteroscedastic and, consequently, it is non-independent and non-Gaussian. From equation (3), it is straightforward to

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see that the acf of y_t^2 has the same shape as the acf of an $AR(p)$ model with autoregressive parameters α_i, $i = 1, ..., p$. Similarly, the GARCH(1,1) model can be written as

$$y_t^2 = \alpha_0 + (\alpha_1 + \beta)y_{t-1}^2 + \nu_t - \beta\nu_{t-1} \tag{4}$$

and thus the acf of y_t^2 has the shape of the acf of an ARMA(1,1) model with autoregressive parameter $\alpha_1 + \beta$ and moving average parameter β. From (4) it is also clear that when the ARCH parameter $\alpha_1 = 0$, the parameter β is not identified. In this case, the series y_t is homoscedastic.

Outliers were originally defined for ARMA(p, q) models given by,

$$y_t = \frac{\theta_q(L)}{\phi_p(L)} a_t \tag{5}$$

where $\theta_q(L)$ and $\phi_p(L)$ are the moving average and autoregressive polynomials and a_t is an uncorrelated sequence. Fox (1982) proposed two types of outliers in model (5): additive (AO) and innovative (IO). An AO only affects one particular observation and the observed series, z_t, is given by

$$z_t = y_t + \omega I(t = \tau) \tag{6}$$

where ω is the size of the outlier and $I(t = \tau)$ is the indicator function that takes value one when $t = \tau$ and zero otherwise. On the other hand, the effect of an IO is transmitted through time with the same dynamics as the rest of the series and the observed series, z_t, is given by

$$z_t = \frac{\theta_q(L)}{\phi_p(L)}(a_t + \omega I(t = \tau)) \tag{7}$$

Given that ARCH(p) and GARCH(1,1) models are uncorrelated, $\theta_q(L) = \phi_p(L) = 1$ and, consequently, the distinction between AO and IO in this

context is not relevant. However, it is important to distinguish whether an outlier affects or not future conditional variances. In this sense, Hotta and Tsay (1998) introduce two types of outliers in GARCH models: level (LO) and volatility (VO). A LO affects only the level of the series and has no effect on the conditional variance. Therefore, if the series y_t is contaminated by a LO of size ω at time τ, the observed series is given by (6) but the conditional variance is like in (1) and depends on the underlying series y_t and not on the observed series z_t. Similarly, the conditional variance is given by (2) when dealing with an ARCH(p) model. On the other hand, a VO affects both the level and conditional variance of the series. In this case, the observed series is also given by expression (6) but if the model is a GARCH(1,1) model, the conditional variance is given by

$$\sigma_t^2 = \alpha_0 + \alpha_1 z_{t-1}^2 + \beta \sigma_{t-1}^2$$

while if it is an ARCH(p) model, it is given by

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i z_{t-i}^2.$$

In this paper we focus on LO. We expect that similarly to what happens to IO in the context of linear models, the effects of VO should be less important as they are transmitted by the same dynamics as the rest of the series; see, for example, Peña (2001).

3   Effects of outliers on the identification of conditional heteroscedasticity

ARCH models generate autocorrelated squared observations and conditional heteroscedasticity tests are mainly based on this property. In this section, we

derive the biases caused by level outliers on the sample autocorrelations of squared observations of ARCH(p) and GARCH($1, 1$) models and show how they affect some popular tests of conditional heteroscedasticity.

3.1 Effects on the correlogram of squares

Consider that the series of interest, y_t, is stationary and it has been contaminated from time τ ownwards by k consecutive outliers of the same size, ω. In this case, the observed series, z_t, is given by

$$
z_t = \begin{cases} y_t + \omega & \text{if} \quad t = \tau, \tau + 1, \ldots, \tau + k - 1 \\ y_t & \text{otherwise.} \end{cases} \tag{8}
$$

The autocorrelation of order h, $h = 1, 2, \ldots$, of squared observations is estimated by

$$
r(h) = \frac{\sum\limits_{t=h+1}^{T} z_t^2 z_{t-h}^2 - \frac{T-h}{T^2} \left(\sum\limits_{t=1}^{T} z_t^2 \right)^2}{\sum\limits_{t=1}^{T} z_t^4 - T^{-1} \left(\sum\limits_{t=1}^{T} z_t^2 \right)^2}. \tag{9}
$$

If the sample size, T, is large relative to the order of the estimated autocorrelation, h, the numerator of $r(h)$ can be written as follows

$$
\sum_{t \in \mathbb{T}(h)} y_t^2 y_{t-h}^2 + \sum_{i=0}^{h-1} (y_{\tau+i} + \omega)^2 y_{\tau+i-h}^2 + \sum_{i=h}^{k-1} (y_{\tau+i} + \omega)^2 (y_{\tau+i-h} + \omega)^2 +
$$

$$
+ \sum_{i=k}^{k+h-1} y_{\tau+i}^2 (y_{\tau+i-h} + \omega)^2 - T^{-1} \left[\sum_{t \in \mathbb{T}(0)} y_t^2 + \sum_{i=0}^{k-1} (y_{\tau+i} + \omega)^2 \right]^2 \tag{10}
$$

where $\mathbb{T}(s) = \{s + 1, \ldots, \tau - 1, \tau + k + s, \ldots, T\}$. Similarly, the denominator can be written as

$$
\sum_{t \in \mathbb{T}(0)} y_t^4 + \sum_{i=0}^{k-1} (y_{\tau+i} + \omega)^4 - T^{-1} \left[\sum_{t \in \mathbb{T}(0)} y_t^2 + \sum_{i=0}^{k-1} (y_{\tau+i} + \omega)^2 \right]^2 \tag{11}
$$

If the order of the autocorrelation is smaller than the number of consecutive outliers, i.e., $h < k$, then the third summation in (10) contains $k - h$ terms which depend on ω^4. Therefore, it is easy to see that expression (10) is equal to $(k - h - \frac{k^2}{T})\omega^4 + o(\omega^4)$. However, if $h \geq k$ then the third summation in (10) disappears and the numerator of $r(h)$ is equal to $-\frac{k^2}{T}\omega^4 + o(\omega^4)$. On the other hand, expression (11) is equal to $(k - \frac{k^2}{T})\omega^4 + o(\omega^4)$. Consequently, the limit of $r(h)$ when the size of the outliers tends to infinity and the sample size is large relative to k is $\frac{k-h}{k}$ if $h < k$ and zero if $h \geq k$. Therefore, one single large outlier biases towards zero all the autocorrelations of squares, making difficult the detection of genuine heteroscedasticity. However, several consecutive outliers can generate positive autocorrelations of squares even if they are truly zero. If, for example, y_t is a white noise series, two large consecutive outliers generate an autocorrelation of order one approximately equal to 0.5 and all the others close to zero. With three large consecutive outliers all the autocorrelations are approximately zero except $r(1)$ and $r(2)$ which are close to 0.6 and 0.3 respectively. In this case, outliers can be confused with conditional heteroscedasticity. Furthermore, it is important to notice that, in the presence of large outliers, the autocorrelations tend to the same values independently on the dynamics of the squared uncontaminated observations.

As an illustration, we have simulated 1000 replicates of size $T = 1000$ of a Gaussian white noise process with zero mean and variance one. First, we have contaminated each series by one single LO of size 15 at time $t = 500$ and, second, by two consecutive outliers of the same size as before at times $t = 500$ and 501^1. The top panels of Figure 1 plot the mean correlogram

[1]Similar results have been obtained when outliers appear in other positions.

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of squared observations through all Monte Carlo replicates in both cases. It can be seen that although the series are uncorrelated, the mean of the first estimated autocorrelation is approximately 0.5 when they are contaminated by consecutive outliers.

We have also generated 1000 series by the GARCH(1,1) model in (1) with parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.1$ and $\beta = 0.8$[2]. In this case, the marginal variance is also one but the squares are autocorrelated. The simulated GARCH series have been contaminated by the same outliers as before. The bottom panels of Figure 1 plot the acf of squares together with the means of the sample autocorrelations estimated through the Monte Carlo replicates. When there is a single outlier, all the estimated autocorrelations are very close to zero and when outliers appear in patches, the autocorrelations are zero for all lags except the first one which is approximately 0.5. Notice that, as indicated by the theoretical analysis, in the presence of large outliers, the expected autocorrelations of squares are the same for GARCH and white noise series.

3.2   Testing for conditional heteroscedasticity

Many popular tests for conditional homoscedasticity are based on autocorrelations of squares. Therefore, if these autocorrelations are biased, the properties of the tests will be affected. In this subsection we analyze the behavior of four tests for conditional homoscedasticity, namely, the Lagrange Multiplier (LM) of Engle (1982), its robust version (RLM) proposed by Van Dijk *et al.* (1999), the McLeod and Li (1983) and the Peña and Rodriguez (2002) tests.

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[2]Similar results have been obtained generating series by alternative conditional heteroscedastic models like EGARCH and Stochastic Volatility.

The LM(p) test for ARCH effects proposed by Engle (1982) is given by TR^2, where R^2 is the determination coefficient of the regression of the squared observations, y_t^2, on a constant and p lags $y_{t-1}^2, ..., y_{t-p}^2$. Under the null hypothesis of conditional homoscedasticity, this statistic is asymptotically distributed as a χ^2 with p degrees of freedom. Lee (1991) shows that the LM(p) test for GARCH(p, q) is the same as for ARCH(p) since, as we have previously mentioned, the parameter β in (1) is not identified when $\alpha_1 = 0$. The finite sample properties of the LM test have been studied among many others by Lee and King (1993) who show that its empirical size is under the nominal while the power is reasonable.

Alternatively, McLeod and Li (1983) proposed to test for conditional heteroscedasticity using the Box-Ljung statistic for squared observations given by

$$Q(m) = T(T + 2) \sum_{j=1}^{m} \frac{r^2(j)}{(T - j)}. \tag{12}$$

If the eighth order moment of y_t exists and under the null hypothesis of conditional homoscedasticity, $Q(m)$ is approximately distributed as a χ^2 with m degrees of freedom.

Peña and Rodriguez (2002) propose to test for conditional homoscedasticity using the following statistic

$$D(m) = T(1 - |R_m|^{1/m}) \tag{13}$$

where

$$R_m = \begin{pmatrix} 1 & \tilde{r}(1) & \cdots & \tilde{r}(m) \\ \tilde{r}(1) & 1 & \cdots & \tilde{r}(m-1) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{r}(m) & \tilde{r}(m-1) & \cdots & 1 \end{pmatrix}$$

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and $\tilde{r}(j)$ is the standardized autocorrelation of order j, given by $\tilde{r}(j) = \sqrt{\frac{T+2}{T-j}} r(j)$. Under the null hypothesis of conditional homoscedasticity and if the eighth order moment of y_t exists, the asymptotic distribution of $D(m)$ can be approximated by a Gamma distribution, $\gamma(\alpha, \beta)$, with $\alpha = 3m(m+1)/4(2m+1)$ and $\beta = 3m/2(2m+1)$.

Finally, Van Dijk *et al.* (1999) show that, in the presence of consecutive AO, the LM test rejects the null hypothesis of conditional homoscedasticity too often. Furthermore, long isolated outliers lead to an asymptotic power loss of the LM test. They propose an alternative robust statistic (RLM) with better size and power properties; see also Franses *et al.* (2004) for an empirical illustration with series of financial returns. However, the performance of the robust test is adequate only when the proportion of outliers is less than 1% and it is seriously damaged when the proportion of outliers is greater than 5%.

Using the results in previous subsection on the effects of large outliers on the sample autocorrelations of squared observations, we analyze how these large outliers affect the size and power of the LM(m), $Q(m)$ and $D(m)$ tests. With respect to the LM(m) test, consider the equation of the AR(p) model for y_t^2 in (3) with $p = m$. It is well known that, for this model, the relationship among the parameters α_i and the first $p - 1$ autocorrelations of squared observations is given by

$$
\begin{pmatrix}
1 & \rho(1) & \cdots & \rho(p-1) \\
\rho(1) & 1 & \cdots & \rho(p-2) \\
\vdots & \vdots & \ddots & \vdots \\
\rho(p-1) & \rho(p-2) & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_p
\end{pmatrix}
=
\begin{pmatrix}
\rho(1) \\
\rho(2) \\
\vdots \\
\rho(p)
\end{pmatrix}
\tag{14}
$$

where $\rho(h)$ is the autocorrelation of order h of squared observations; see,

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for example, Brockwell and Davies (1996). The same relationship could be established between sample autocorrelations and parameters estimated by OLS. As we have seen before, one isolated outlier biases towards zero all estimated autocorrelations. Consequently, the $p \times p$ matrix on the left hand side of (14) becomes approximately the identity matrix while the vector on the right becomes approximately zero. Therefore, for large isolated outliers, all the estimated parameters α_i, $i = 1, ..., p$ are close to zero. It is straightforward to see that, in this case, the determination coefficient R^2 is zero. Consequently, if y_t is homoscedastic, the size of the LM test is zero. However, if the series follows an ARCH model, the power is also zero.

To consider the effects of consecutive outliers on the $LM(m)$ test, we are considering $m = 1$. Notice that if the null is rejected in this particular case, it is going to be rejected for any other value of m. The $LM(1)$ test is given by

$$LM(1) = T \frac{\widehat{\alpha}_1^2 \sum_{t=2}^{T} \left(y_{t-1}^2 - \frac{\sum y_{t-1}^2}{T-1} \right)^2}{\sum_{t=1}^{T} \left(y_t^2 - \frac{\sum y_t^2}{T} \right)^2}$$

We have seen before that k consecutive outliers bias the autocorrelation of order one and, consequently, the estimated ARCH(1) parameter, $\widehat{\alpha}_1$, towards $\frac{T(k-1)-k^2}{Tk-k^2}$. Consequently, the limit of the $LM(1)$ statistic is given by

$$\lim_{\omega \to \infty} LM(1) = T \left(\frac{T(k-1) - k^2}{Tk - k^2} \right)^2 \xrightarrow[T \to \infty]{} \infty$$

The null hypothesis is always rejected. In this case, if the true process is homoscedastic, the size tends to one while if it is heteroscedastic the power tends to one.

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Next, we consider the properties of the McLeod-Li test in (12) when the series y_t is affected by an isolated large outlier. In this case, the limit of the estimated autocorrelations of any order when the outlier size goes to infinity is $\frac{1}{1-T}$. Then,

$$\lim_{\omega \to \infty} Q(m) = T(T+2) \sum_{j=1}^{m} \frac{1}{(1-T)^2(T-j)} < \frac{T(T+2)m}{(1-T)^2(T-m)} \xrightarrow[T \to \infty]{} 0$$

Consequently, the null is never rejected. In this case, similarly to what happens to the LM test, if the series is homoscedastic the size is zero while if the series is heteroscedastic, the power is also zero.

On the other hand, we consider the effect of k consecutive outliers on the $Q(1) = \frac{T(T+2)r(1)^2}{T-1}$ test. As before, notice that if the null is rejected in this case, is going to be rejected for any other lag. The limit of the order one autocorrelation when the outliers size goes to infinity is in this case $\frac{T(k-1)-k^2}{Tk-k^2}$. Then

$$\lim_{\omega \to \infty} Q(1) = T(T+2) \frac{T(k-1)-k^2}{(Tk-k^2)(T-1)} \xrightarrow[T \to \infty]{} \infty$$

Therefore, even if the series is truly homoscedastic, patches of large outliers make the McLeod-Li test to reject always the null. The asymptotic size is, in this case, one. On the other hand, if the series is heteroscedastic, the power is also one.

The same kind of arguments can be use to show that the asymptotic size and power of the Peña-Rodriguez test in (14) are zero when series are contaminated by a large isolated outlier while they are one in the presence of large consecutive outliers.

The previous analytical results are asymptotic and for large outlier sizes. To analyze the effects of relatively moderate outliers on the sizes of the three

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tests considered above and of the robust test of Van Dijk *et al.* (1999), we have simulated 1000 Gaussian white noise series of sizes $T = 500$, 1000 and 5000 that have been contaminated first, by one single outlier and then, by two consecutive outliers of the same size. For each simulated series, we test the null hypothesis of conditional homoscedasticity using the $LM(m)$ and the corresponding robust test for $m = 1$ and the $Q(m)$ and $D(m)$ tests for $m = 20$. The performance of the LM and $D(m)$ tests are very similar to the $Q(m)$ test and, consequently, we only plot the size and power of the latter. The top panel on the left of Figure 2 plots the empirical sizes of the $Q(20)$ and $RLM(1)$ tests as a function of the outlier size, for the same sample sizes, when the nominal size is 5%. In this plot, it is possible to observe that, for small or moderate sample sizes, i.e. $T = 500$ or 1000, when the outlier size is greater than 7 standard deviations the size of the $Q(m)$ test is zero. However, if the sample size is large, i.e. $T = 5000$, the size only goes to zero if the outlier is as large as 10 standard deviations. On the other hand, the size of the robust test is around 9%, i.e. nearly double the nominal, independently on the outlier size. What is even worse is that its size strongly deteriorates when the sample size increases. For example, when $T = 5000$, the empirical size is around 25% independently on the outlier size. Therefore, although the robust test is clearly oversized, its size is not affected by the presence of outliers.

We have also contaminated the simulated Gaussian series by two consecutive outliers. The right panel on top of Figure 2 plots the empirical sizes of the two tests in this case. First of all, it is possible to observe that the behavior of the robust test is similar to the one observed when there was just

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one outlier. However, notice that for relatively small outliers sizes, like for example, 5 standard deviations, the size of the non-robust tests is almost 1 for any of the three sample sizes considered. Therefore, rather small outliers in homoscedastic series make the homoscedasticity tests to detect conditional heteroscedasticity even for relatively large samples.

To analyze the power of the considered homoscedasticity tests in the presence of outliers of finite size, the left bottom panel of Figure 2 plots the percentage of rejections of the null hypothesis as a function of the outlier size. These results are based on 1000 series of the same sample sizes as before generated by the same GARCH$(1, 1)$ model that have been contaminated by a single outlier. This figure shows that if the outlier size is smaller than 4 or 5 standard deviations, the power of the portmanteau test is larger than the power of the robust test when the sample size is $T = 500$ or 1000. For these sample sizes, the power of the $Q(m)$ test decreases very rapidly with the size of the outlier. If this size is larger than approximately 7 standard deviations, the power is negligible. However, if the sample size is rather large as, for example, $T = 5000$, then a very large outlier is needed for the RLM test to have more power than the $Q(m)$ test. In our experiments, the power of the $Q(m)$ test is affected only if the outlier is larger than 10 standard deviations. We have also contaminated the series with two consecutive outliers. The empirical powers have been plotted in the right bottom panel of Figure 2. As we can see, the power of the robust test is clearly lower than the power of the non-robust test considered for all sample sizes and outlier sizes chosen. However, given that their sizes are clearly distorted, these tests are worthless in the presence of consecutive outliers.

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Summarizing, relatively small consecutive outliers are able to generate spurious heteroscedasticity while larger sizes of outliers are required to hide genuine heteroscedasticity when standard tests are used for testing for conditional homoscedasticity. On the other hand, the available robust LM test seems to be of little help because it suffers from important size distortions that get worse with the sample size.

4 Effects of outliers on the estimation of ARCH models

The ARCH(p) model is hardly implemented for the analysis of real time series of financial returns because a very large number of lags, p, is often required to adequately represent the dynamic evolution of the conditional variances. However, this model is attractive because it is possible to obtain a closed-form expression for the OLS estimator of its parameters. In the following subsection, we quantify the effects of level outliers on the OLS estimates of ARCH(p) models. In subsection 4.2, we also analyze the effects of outliers on the GLS estimator. Finally, the results are extended in the next subsection to the ML estimators.

4.1 OLS estimator

The OLS estimator of the parameters of the ARCH(p) model defined in (3) is given by

$$\widehat{\alpha}^{OLS} = (X'X)^{-1}(X'Y) \tag{15}$$

where α is the vector of parameters given by $\alpha = (\alpha_0 \quad \alpha_1 \quad \ldots \quad \alpha_p)'$, $Y = (y_{p+1}^2 \quad y_{p+2}^2 \quad \ldots \quad y_T^2)'$ and X is the following matrix

$$X = \begin{pmatrix} 1 & y_p^2 & y_{p-1}^2 & \cdots & y_1^2 \\ 1 & y_{p+1}^2 & y_p^2 & \cdots & y_2^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_{T-1}^2 & y_{T-2}^2 & \cdots & y_{T-p}^2 \end{pmatrix}$$

Weiss (1986) shows that if the 4th order moment of y_t exists, $\widehat{\alpha}^{OLS}$ is consistent. Furthermore, if the 8th order moment is finite, then the asymptotic distribution of $\widehat{\alpha}^{OLS}$ is given by

$$\sqrt{T}(\widehat{\alpha}^{OLS} - \alpha) \xrightarrow{d} N(0, \Sigma_{XX}^{-1} \Sigma_{X\Omega X} \Sigma_{XX}^{-1})$$

where $\lim_{T \to \infty} \frac{X'X}{T} = \Sigma_{XX}$ and $\lim_{T \to \infty} \frac{X'VV'X}{T} = \Sigma_{X\Omega X}$; see Engle (1982) for sufficient conditions for the existence of higher moments of y_t when ε_t is Gaussian.

Consequently, a consistent estimator of the asymptotic covariance matrix of $\widehat{\alpha}^{OLS}$ is given by

$$(X'X)^{-1} S (X'X)^{-1} \tag{16}$$

where

$$S = \begin{pmatrix} \sum_{t=p+1}^{T} \widehat{\nu}_t^2 & \sum_{t=p+1}^{T} \widehat{\nu}_t^2 y_{t-1}^2 & \cdots & \sum_{t=p+1}^{T} \widehat{\nu}_t^2 y_{t-p}^2 \\ & \sum_{t=p+1}^{T} \widehat{\nu}_t^2 y_{t-1}^4 & \cdots & \sum_{t=p+1}^{T} \widehat{\nu}_t^2 y_{t-1}^2 y_{t-p}^2 \\ & & \ddots & \vdots \\ & & & \sum_{t=p+1}^{T} \widehat{\nu}_t^2 y_{t-p}^4 \end{pmatrix}$$

and $\widehat{\nu}_t$ are the residuals from the OLS regression in (3).

Alternatively, the covariance matrix of the OLS estimator can be estimated taking into account that the structure of the matrix $\Omega = E(V'V)$ is known. In particular, $\Omega = diag(\sigma_p^2 \quad \sigma_{p+1}^2 \quad \cdots \quad \sigma_T^2)$. Therefore, it is also possible to estimate the asymptotic covariance matrix of $\widehat{\alpha}^{OLS}$ by

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$$(X'X)^{-1}X'\widehat{\Omega}X(X'X)^{-1} \tag{17}$$

where $\widehat{\Omega}$ is given by the Ω matrix where the ARCH parameters have been substituted by the corresponding estimates.

Notice that although the OLS estimator is the best unbiased linear estimator, it is not efficient because, as we mentioned before, the noise ν_t is conditionally heteroscedastic.

Next, we analyze how a single outlier affects the asymptotic and finite sample properties of $\widehat{\alpha}^{OLS}$. We then consider the effects of patches of outliers.

4.1.1 Isolated outliers

Consider a series generated by an $\text{ARCH}(p)$ model which is contaminated at time τ by a single level outlier of size ω, as in (8) with $k = 1$. Then, $\widehat{\alpha}^{OLS}$ in (15) will be computed using the contaminated observations z_t^2 instead of y_t^2 and the matrix $X'X$ will become

$$\begin{pmatrix} T - p & (\omega^2 + o(\omega^2))\,\mathbf{1}' \\ (\omega^2 + o(\omega^2))\mathbf{1} & (\omega^2 + o(\omega^2))\mathbf{F} \end{pmatrix}$$

where $\mathbf{1}$ is a $p \times 1$ column vector equal to $(1 \quad 1 \quad \cdots \quad 1)'$, $\mathbf{F}$ is a $p \times p$ symmetric matrix with $f_{ij} = 1$ for $i = 1, \ldots, p$, $j = i + 1, \ldots, p$ and $f_{ii} = \omega^2$ for $i = 1, \ldots, p$. And all elements in $X'Y$ will be equal to $\omega^2 + o(\omega^2)$. Then, after some tedious algebra it can be seen that

$$\lim_{\omega \to \infty} \widehat{\alpha}_i^{OLS} = \begin{cases} \infty & \text{for } i = 0 \\ -\frac{1}{T - 2p} & \text{for } i = 1, \ldots, p. \end{cases} \tag{18}$$

The limit in (18) shows that, as expected, if the sample size is large enough, the estimated unconditional variance, given by $\widehat{\alpha}_0 / (1 - \sum_{i=1}^{p} \widehat{\alpha}_i)$,

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tends to infinity when the outlier size tends to infinity. Furthermore, as expected from the results on the autocorrelations obtained in the previous section, all the estimated ARCH parameters tend to zero and, consequently, the dynamic dependence in the conditional variance disappears. Finally, notice that the persistence of the volatility in an ARCH(p) model, measured by $\sum_{i=1}^{p} \alpha_i$, also decreases as the size of the outlier increases. It is also important to notice that if the sample size is not very large, it is possible to obtain estimates that do not satisfy the usual non-negativity restrictions.

To illustrate the behavior of the OLS estimator of the parameters of ARCH(p) models in the presence of a LO, we have generated 1000 series by an ARCH(1) model with parameters $\alpha_0 = 0.8$ and $\alpha_1 = 0.2$. The sample sizes are $T = 500$, 1000 and 5000 and all the series have been contaminated with a single LO of size ω. The values chosen for ω are $5, 10$ and 15 standard deviations. First and fourth rows of Figure 3 plot kernel estimates of the density of the $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$ OLS estimators respectively when the size of the outlier is $\omega = 5, 10$ and 15. Notice that, as expected, the bias caused by an outlier of fixed size is smaller the larger the sample size T. For example, the mean of the OLS estimates of α_0 and α_1 when there is an outlier of size 10, is 1.18 and 0.02 respectively when $T = 500$. However, if $T = 1000$, the corresponding means are 1.06 and 0.04 and they are 0.91 and 0.11 when $T = 5000$. Furthermore, Figure 3 also shows that the effect of the size of the outlier on the bias of the OLS estimator of α_1 is very quick when the sample sizes are $T = 500$ or 1000. Even for $T = 5000$, relatively small outliers generate strong negative biases on the OLS estimator of the ARCH parameter. In this figure it is also possible to see that the dispersion of $\widehat{\alpha}_1$

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decreases with the size of the outlier while it is approximately constant for $\widehat{\alpha}_0$.

An interesting problem that arises here is to study the effect that outliers have not only on the parameter estimates but also on the variance of the estimators. In order to have a first answer, we have computed, for each Monte Carlo replicate, the variance of $\widehat{\alpha}_0^{OLS}$ and $\widehat{\alpha}_1^{OLS}$ estimated with expressions (16) and (17). We denote the first one by White and the second one by ARCH. The first and fourth rows of Figure 4 plot, for $\widehat{\alpha}_0^{OLS}$ and $\widehat{\alpha}_1^{OLS}$ respectively, the ratio between the empirical variance, and the estimated asymptotic variance averaged through all Monte Carlo replicates. If the sample size is big enough and there are no outliers, we would expect this ratio to be close to one. With respect to the variance of $\widehat{\alpha}_0^{OLS}$, it is possible to observe that the White variances overestimate the empirical variances. The bias is larger, the larger the outlier size. The biases are less pronounced when the information on the structure of the volatility is used to estimate the variance of $\widehat{\alpha}_0^{OLS}$. In this case, very large outliers can even cause an underestimation of the empirical variance. In any case, for outliers smaller than 5, 8 and 13 standard deviations for $T = 500$, 1000 and 5000 respectively, the biases of the White variances are smaller. On top of this, when estimating the variance of $\widehat{\alpha}_1^{OLS}$, the results in Figure 4 suggest that biases are smaller when using the White estimator. Therefore, it seems more adequate to estimate the variance of the OLS estimator of the ARCH model using equation (16).

Summarizing the results of Figures 3 and 4 for the OLS estimator of an ARCH(1) model, we can say that a single outlier of a big size makes

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α_0 to be overestimated and α_1 to be underestimated. The presence of a big outlier in the sample also makes the variance of the OLS estimators to be overestimated. Similar results have been obtained when the series are generated by an ARCH(p) model, for different values of p.

4.1.2 Patches of outliers

When the original series, y_t, is contaminated by k consecutive outliers as given in (8), the effects on the OLS estimator depend on the relationship between the number of outliers and the order of the ARCH model. First, let us consider $k \geq p$, i.e., there are at least as many outliers as the number of lags in the ARCH model. In this case, it is necessary to consider separately the cases where $p = 1$ and $p > 1$. This is because in the first case, the parameter α_1 receives the whole effect of the outliers while in the latter, this effect is shared by all the parameters.

We consider first the effect of k consecutive outliers on the estimates of the parameters of an ARCH(1) model. In this case, taking into account that $\sum_{t=1}^{T-1} z_t^2 = k\omega^2 + o(\omega^2)$ and $\sum_{t=1}^{T-1} z_t^4 = k\omega^4 + o(\omega^4)$ it is possible to show that

$$\lim_{\omega \to \infty} \widehat{\alpha}_i^{OLS} = \begin{cases} \infty & \text{for } i = 0 \\ \frac{(T-1)(k-1)-k^2}{(T-1)k-k^2} & \text{for } i = 1 \end{cases} \tag{19}$$

Notice that if $k = 1$, we obtain the same result as in (16). The limit in (17) shows that isolated and consecutive outliers have the same effects in the limit on the estimated constant, $\widehat{\alpha}_0$. In both cases, it tends to infinity. Consequently, the estimated marginal variance increases with the size of the outliers. On the other hand, if the number of consecutive outliers is large, the estimated ARCH parameter, $\widehat{\alpha}_1$, tends to one when the outliers size tends

to infinity. Therefore, given that in ARCH(1) models, the persistence to shocks to volatility is measured by α_1, the presence of long patches of large outliers can lead to infer that the volatility is characterized by a unit root and, consequently, that y_t is not stationary. Finally, notice that patches of large outliers can overestimate or underestimate the ARCH parameter depending on its original value. If the sample size is moderate, $\widehat{\alpha}_1$ tends to 0.5 if there are two large consecutive outliers. Therefore, if $\alpha_1 < 0.5$, the OLS estimator will have a positive bias while if $\alpha_1 > 0.5$, the bias will be negative. However, notice that in cases of empirical interest in the context of financial time series, the ARCH parameter is usually rather small, never over 0.3. Consequently, in these cases, if there are patches of consecutive outliers, the OLS estimator will overestimate the ARCH parameter. It is also important to point out that the limit in (19) increases very quickly with the number of consecutive outliers. For example, if $k = 3$, $\widehat{\alpha}_1$ tends to 0.66 while if $k = 4$ the limit is 0.75.

To illustrate these results, we have generated 1000 series by the same ARCH(1) model as before with $\alpha_0 = 0.8$ and $\alpha_1 = 0.2$. Each series has been contaminated by 2 consecutive outliers. First and fourth rows of Figure 5 plot kernel estimation of the density of the $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$ OLS estimators respectively. Although in the limit, $\widehat{\alpha}_0$ increases with ω, notice in this figure that for small outliers, α_0 can be underestimated. For example, consider $T = 500$, then if the outlier size is 5 standard deviations, the mean of the estimates $\widehat{\alpha}_0$ is 0.75, below the true value of 0.8. If the size of the outlier is 10, the mean is also 0.75, however, if the size is 15, the mean is 0.98 and, therefore, bigger than the true value of 0.8. This effect is even stronger for

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larger sample sizes. Consequently, for the outlier sizes typically encountered in empirical applications, the constant can be underestimated in the presence of patches of outliers. Remember that in the presence of a single outlier, the OLS estimates of α_0 tend monotonically to infinity. Therefore, although the effect in the limit is the same, in practice, isolated outliers overestimate the constant while consecutive outliers underestimate the constant.

Looking at the results for $\widehat{\alpha}_1$, observe that in concordance with the limit in (19), they tend to 0.5 when $k = 2$. Furthermore, for all the sample sizes considered, the limit is reached for sizes of the outliers relatively small. For example, for $T = 500$, the mean of the estimates of α_1 is 0.31 when $\omega = 5$, 0.46 when $\omega = 10$ and 0.49 when $\omega = 15$.

Finally, the ratio of the empirical variance and the estimated asymptotic variance of the OLS estimators is plotted in rows 1 and 4 of Figure 6, where we can see that for both estimators, $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$, this ratio tends to zero with the size of the outlier. This meas that the asymptotic variance, estimated using (16) or (17), is overestimating the true variance, which tends to zero with the size of the outlier. Notice that in this case, the biases are larger than in the presence of a single outlier. And, as before, the biases are smaller when using equation (16) to estimate the asymptotic covariance matrix.

Next, we consider the effect of k consecutive outliers in an ARCH(p) model with $p > 1$ and $k > p$. Again, $\widehat{\alpha}^{OLS}$ in (15) will be computed using the contaminated observations z_t^2 instead of y_t^2 and the matrix $X'X$ will become

$$\begin{pmatrix} T - p & (k\omega^2 + o(\omega^2))\,\mathbf{1}' \\ (k\omega^2 + o(\omega^2))\mathbf{1} & (\omega^4 + o(\omega^4))\mathbf{M} \end{pmatrix}$$

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where $\mathbf{1}$ is a $p \times 1$ column vector equal to $(1 \quad 1 \quad \cdots \quad 1)'$, $\mathbf{M}$ is a $p \times p$ symmetric matrix with $m_{ij} = k + i - j$ for $i = 1, \ldots, p$, $j = i, \ldots, p$. And the vector $X'Y$ will become

$$\begin{pmatrix} k\omega^2 + o(\omega^2) \\ (\omega^4 + o(\omega^4))\mathbf{B} \end{pmatrix}$$

where $\mathbf{B}$ is a $(p + 1) \times 1$ column vector such that $b_i = k - i$ for $i = 1, \ldots, p$. After some tedious algebra it can be seen that the limit of the estimates when the size of the outliers tend to infinity is given by

$$\lim_{\omega \to \infty} \widehat{\alpha}_i^{OLS} = \begin{cases} \infty & \text{for } i = 0 \\ \frac{-2k^2 + (2k-p)(T-p)}{-2k^2 + (2k-p+1)(T-p)} & \text{for } i = 1 \\ 0 & \text{for } i = 2, \ldots, p-1 \\ \frac{-(T-p)}{-2k^2 + (2k-p+1)(T-p)} & \text{for } i = p \end{cases} \tag{20}$$

Like in previous cases, the estimated unconditional variance tends to infinity. The estimated parameters, $\widehat{\alpha}_i$, tend to zero, except $\widehat{\alpha}_1$ and $\widehat{\alpha}_p$. If the number of consecutive outliers is large relative to the order of the model, then $\hat{\alpha}_1$ tends to a quantity close to one and $\hat{\alpha}_p$ tends to zero. Consequently, the estimated persistence, given by $\sum_{i=1}^p \hat{\alpha}_i$, tends to $\frac{-2k^2 + (2k-p-1)(T-p)}{-2k^2 + (2k-p+1)(T-p)}$ which is close to one. Notice that if $p = 1$, the limit of the persistence coincides with the limit of $\hat{\alpha}_1$ given in (17). Consider, for example, an ARCH(2) series contaminated by 2 large consecutive outliers. In this case, if the sample size is moderately large, $\widehat{\alpha}_1$ tends approximately to 0.66 and $\widehat{\alpha}_2$ to -0.34 and, consequently, the persistence tends to 0.32. However, if the number of consecutive outliers is 5, $\widehat{\alpha}_1$ tends to 0.89 and $\widehat{\alpha}_2$ to -0.11 and the persistence tends to 0.78. On the other hand, if there are 5 consecutive outliers in an ARCH(4) series, $\widehat{\alpha}_1$ tends to 0.86 and $\widehat{\alpha}_4$ to -0.15 and the persistence to 0.71. It is also important to notice that in the presence of patches of outliers, the estimates may easily violate the non-negativity restrictions.

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Finally, it is not possible to derive a general result when the number of consecutive outliers is smaller than the order of the ARCH(p) process where obviously the order p should be at least 3 if there are two or more consecutive outliers. The biases are going to depend on the relationship between k and p.

4.2 Generalized Least Squares estimator

Taking into account the heterogeneity of the noise in equation (3), it is possible to obtain an alternative estimator of the parameters of the ARCH(p) model. Model (3) can be expressed in matrix form as follows,

$$Y = X\alpha + V$$

and premultiplying by P, where $P'P = \Omega$, the following expression is obtained

$$PY = PX\alpha + PV \tag{21}$$

The GLS estimator is obtained by estimating by OLS the parameters α in equation (21). In practice given that the matrix Ω is unknown, it can be substituted by

$$\widehat{\Omega} = \begin{pmatrix} \widehat{\sigma}_{p+1}^2 & 0 & \cdots & 0 \\ 0 & \widehat{\sigma}_{p+2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widehat{\sigma}_T^2 \end{pmatrix}$$

where $\widehat{\sigma}_t^2 = \widehat{\alpha}_0^{OLS} + \widehat{\alpha}_1^{OLS} y_{t-1}^2 + \ldots + \widehat{\alpha}_p^{OLS} y_{t-p}^2$. Therefore, the GLS estimator is given by

$$\widehat{\alpha}^{GLS} = (X'\widehat{\Omega}X)^{-1}(X'\widehat{\Omega}Y) \tag{22}$$

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The GLS estimator is very easy to obtain because it just involves two sets of linear equations. Another atractive is that its asymptotic efficiency is equivalent to the ML estimator. Bose and Mukherjee (2003) derive the asymptotic distribution of $\widehat{\alpha}^{GLS}$ and show that if the sixth order moment of y_t is finite, then

$$\sqrt{T}\left(\widehat{\alpha}^{GLS} - \alpha\right) \xrightarrow{d} N(0, \Sigma_\Omega^{-1}) \tag{23}$$

where $\lim\limits_{T\to\infty} \frac{X'\Omega^{-1}X}{T} = \Sigma_\Omega$.

The effects of outliers on the GLS estimator of the ARCH parameters have been illustrated through Monte Carlo experiments. Second and fifth rows of Figure 3 show the results for the same ARCH(1) model considered before contaminated by one single outlier. Comparing these results with the ones shown in rows 1 and 4 of the same figure, it is possible to observe that the effects on the estimated constant are very similar to the ones observed for OLS estimates. However, the GLS estimator of α_1 is more robust against outliers than the OLS estimator. For example, when $T = 5000$, the GLS estimator is unbiased in the presence of outliers smaller than 15 standard deviations. However, this is not the case for the estimates of the variance. Rows 2 and 4 of Figure 4 plot the ratio of the empirical variance and the estimated asymptotic variance averaged through all Monte Carlo replicates for $\widehat{\alpha}_0^{GLS}$ and $\widehat{\alpha}_1^{GLS}$ respectively. As we can see, this ratio is bigger than one, meaning that the asymptotic variance underestimates the empirical variance.

The results for two consecutive outliers appear in rows 2 and 4 of Figures 5 and 6. In this cases, the biases on the estimated α_1 parameter are negligible for any of the sample sizes and outliers sizes considered. The biases on the

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estimated constant α_0 are slightly larger. However, it is important to point out that, as illustrated by Figure 6, the estimated asymptotic variances, strongly underestimates the empirical variances for consecutive outliers larger than 5 standard deviations.

4.3 Maximum likelihood estimator

Engle (1982) proposed to estimate the parameters of the ARCH(p) model by ML. The distribution of y_t conditional to $Y_{t-1} = \{y_{t-1}, y_{t-2}, \cdots, y_1\}$ is $N(0, \sigma_t^2)$ and consequently, ML estimation of their parameters is straightforward maximizing the log-likelihood function given by

$$L = -\frac{T-p}{2} \log(2\pi) - \frac{1}{2} \sum_{t=p+1}^{T} \left(\log \sigma_t^2 + \frac{y_t^2}{\sigma_t^2} \right). \qquad (24)$$

If the errors are not Gaussian, the estimates obtained by maximizing (24) are Quasi-Maximum Likelihood (QML) estimates. The consistency and asymptotic normality of the QML estimator was established by Weiss (1986) assuming that the fourth order moment of y_t is finite. Then,

$$\sqrt{T} \left(\widehat{\alpha}^{ML} - \alpha \right) \xrightarrow{d} N(0, [I(\alpha)]^{-1}) \qquad (25)$$

where $I(\alpha) = E[-\frac{\partial^2 L}{\partial \alpha \partial \alpha'}]$ is the Information matrix.

The QML is fully efficient when ε_t is Gaussian. However, there are not close form expressions of the QML estimators of the parameters α_0, α_1,...,α_p and the numerical maximization of the Gaussian log-likelihood function is difficult because it is rather flat unless the sample size is very large; see, for example, Shephard (1996).

There is reduced evidence on the effects of outliers on the ML estimator; see for example, Muller and Yohai (2002) who show that the Mean Squared

Error of the MLE of the parameters of ARCH(1) models are dramatically influenced by isolated outliers. However, this evidence is based on sample sizes which are too small ($T = 100, 200$) when compared with the sizes usually encountered in the empirical analysis of financial time series.

Rows 3 and 6 of Figure 3 illustrate the effect of a single outlier on the ML estimates of the parameters of the same ARCH(1) model considered before. For outliers of small size, such that 5 standard deviations, the effects on the estimated parameters α_0 and α_1 are similar to the ones observed for the OLS and GLS estimates. However, these effects are quite different for larger sizes of the outlier. Notice that, for sample sizes of $T = 500$ and 1000 and outliers of sizes 10 and 15 standard deviations, the kernel estimated density of both $\widehat{\alpha}_0^{ML}$ and $\widehat{\alpha}_1^{ML}$ are bimodal. Looking for example at the last row of Figure 3, we can see that in the presence of an outlier of size 15 standard deviations in a sample os size $T = 1000$, $\widehat{\alpha}_1^{ML}$ could take any value between 0 and 1, although values close to zero seem to be more probable, like what we had for $\widehat{\alpha}_1^{OLS}$ and $\widehat{\alpha}_1^{GLS}$.

Rows 3 and 6 of Figure 4 plot the ratio of the empirical variance and the estimated asymptotic variance averaged through all Monte Carlo replicates for $\widehat{\alpha}_0^{ML}$ and $\widehat{\alpha}_1^{ML}$ respectively. We can see, that similarly to what happened in the case of the GLS estimator, this ratio is bigger than one, meaning that the asymptotic variance underestimates dramatically the empirical variance. In the case of two consecutive outliers, the results appear in rows 3 and 6 of Figures 5 and 6. As we can see in the plots, the effects caused by two consecutive outliers on the ML estimators are very similar to the effects caused by a single outlier.

Another interesting fact observed in these plots is that the sample distribution of $\widehat{\alpha}_0^{ML}$ and $\widehat{\alpha}_1^{ML}$ are not symmetric but they have a positive skewness coefficient in the presence of both isolated and consecutive outliers. Hence, tests based on normality will be inadequate.

5   Maximum Likelihood estimator of GARCH(1,1) models

Using the same arguments as before, it is straightforward to derive the following expression of the Gaussian log-likelihood of a GARCH(p, q) model

$$L = -\frac{T-p}{2}\log(2\pi) - \frac{1}{2}\sum_{t=p+1}^{T}\left(\log\sigma_t^2 + \frac{y_t^2}{\sigma_t^2}\right). \qquad (26)$$

If the errors are not Gaussian, the estimates obtained by maximizing (22) are Quasi-Maximum Likelihood (QML) estimates. Bollerslev and Wooldridge (1992) show that under certain regularity conditions the QML estimator is consistent and asymptotically Normal if the first two conditional moments are correctly specified; see Lee and Hansen (1994) and Lumsdaine (1996) for weaker conditions for GARCH(1,1) models.

Previous studies on the effects caused by outliers on the QML estimators of GARCH models are based on Monte Carlo experiments. In any case, the evidence on these effects are scarce and contradictory. For example, Mendes (2000) carried out Monte Carlo experiments based on 100 replicates obtained by just one GARCH(1,1) model. She concludes that the QML estimates of α_1 are biased towards zero while the estimates of β are biased towards one. On the other hand, Gregory and Reeves (2001), based on Monte Carlo results with just one design and on the empirical analysis of a series of weakly

exchange rates returns, conclude that, in the presence of isolated LO, the QML estimators of the parameters α_0 and α_1 are positively biased while $\hat{\beta}$ underestimates β with a negative overall effect on the estimated persistence; see also Verhoeven and McAleer (2000) for an empirical application with the same conclusion.

In this section, we carry out detailed Monte Carlo experiments to analyze the biases caused by isolated and consecutive LO on the QML estimates of the parameters GARCH(1,1) models.

Figure 7 contains the kernel estimates of the density of $\widehat{\alpha}_0^{ML}$, $\widehat{\alpha}_1^{ML}$ and $\widehat{\beta}^{ML}$ based on 1000 replicates, for a GARCH(1,1) model with parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.1$ and $\beta = 0.8$, contaminated a single outlier of sizes $\omega = 5, 10$ and 15 standard deviations. As we can see in the plots, for large sample sizes, like $T = 5000$, ML estimators seem to be robust to the presence of outliers. Notice that they are unbiased even when the series is contaminated by an outlier of size 15 standard deviations. This is not true for smaller sample sizes, like $T = 500$ or 1000, where just one outlier seems to bias towards zero the estimated $\widehat{\alpha}_1$ and towards one the estimated value $\widehat{\beta}$, in the sense that the estimated density is bigger for those values. When we look at the results in Figure 8, containing kernel estimates of the density of $\widehat{\alpha}_0^{ML}$, $\widehat{\alpha}_1^{ML}$ and $\widehat{\beta}^{ML}$ based on 1000 replicates, for the same GARCH(1,1) model but now contaminated with two consecutive outliers of sizes $\omega = 5, 10$ and 15 standard deviations, it seems that $\widehat{\alpha}_0^{ML}$ and $\widehat{\alpha}_1^{ML}$ are overestimating the true parameters, and $\widehat{\beta}$ is underestimating the true β. Hence, the bias caused by outliers on the ML estimates of a GARCH(1,1) model depend also on the size of the outlier, and very important, on their position in the series.

The bias caused by isolated outliers are very different from the bias caused by consecutive ones. This could be one reason for the contradictory results found in the literature.

From the ARCH(∞) representation of a GARCH(1,1) model, we could find an explanation to the bias caused by a single outlier. Let us consider a series y_t generated by a GARCH(1,1) model where $\sigma_0^2 = \frac{\alpha_0}{1-\alpha_1-\beta}$, i.e., the initial value for σ_t^2 is the unconditional variance. For a sample size, T, fixed, the series can be approximated by the following ARCH($T-1$) model:

$$\begin{cases} y_t = \varepsilon_t \sigma_t \\ \sigma_t^2 = \tilde{\omega} + \sum_{i=1}^{T-1} \tilde{\alpha}_i y_{t-i}^2 \end{cases}$$

where $\tilde{\alpha}_0 = \alpha_0 \left[\frac{1+\beta^{T-1}}{1-\beta} + \frac{\beta^{T-1}}{1-\alpha_1-\beta} \right]$ y $\tilde{\alpha}_i = \alpha_1 \beta^{i-1}$ for $i = 1, 2, \ldots, T-1$.

If the observed series is contaminated with a single outlier of size ω, we have seen before that $\lim_{\omega \to \infty} \tilde{\alpha}_0 = \infty$ y $\lim_{\omega \to \infty} \tilde{\alpha}_i = \alpha_1 \beta^{i-1} = 1/(T-2)$, $\forall i$. In particular, for $i = 1$ we would have that $\widehat{\alpha}_1$ tends to $1/(T-2)$ and therefore, $\widehat{\beta}$ tends to one as the size of the outlier tends to infinity. In the limit, $\widehat{\alpha}_1$ is close to zero and $\widehat{\beta}$ is close to one, implying that the unconditional variance tends to infinity and α_0 would not be identified. This could explain why $\widehat{\omega}$ is not very affected by the outlier.

6   Empirical application

This section illustrates the previous results by analyzing two world indexes. We consider the daily series of returns of the S&P 500 index of US and the Nikkei 225 index of Japan, observed from October 20, 1982 to May 17, 2004 and from January 4, 1984 to May 19, 2004 respectively[3]. Figures 9 and 10

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[3]Series have been obtained in the webpage http://finance.yahoo.com/.

plot the return series and the correlogram of the squared observations for the original series and for the series corrected for outliers. Series have been corrected by substituting the corresponding outliers by the unconditional mean. In the S&P 500 series there is one observation which is exactly 22 times the standard deviation. This observation corresponds to the 19th of October, 1987, also known as "October black monday", the biggest fall in all the history of Wall Street. The other two outliers correspond to the following days, October, 21 and 26. The size of these two observations is around 8.5 standard deviations. In the Nikkei 225, there are two consecutive outliers corresponding to the same "October black monday", 19th and 20th of October, 1987. In this case, the corresponding return was 11 times the standard deviation. The third outlier in this series corresponds to 2nd October 1990, and this observation is 8 times the standard deviation.

As we can observe in both figures, correcting the series by these extreme observations makes more clear the structure in the squared observations. In the case of the S&P 500, we can see how just one observation biases towards zero all correlation coefficients of squared observations, and in the Nikkei 225, two consecutive outliers overestimate the first order autocorrelation and underestimate all the others, as expected looking at the theoretical results. Table 1 contains descriptive statistics for both original and corrected series.

Tables 2 and 3 contain estimated parameters for and ARCH(9) and GARCH(1,1) model. As these examples illustrate, real time series can be affected by outliers which can bias the correlogram of squared observations and affecting in a decisive way the identification and estimation of conditional heteroscedasticity. Nevertheless, for these two particular series, outliers do

not change the conclusions of the conditional homoscedasticity tests analyzed in this paper, which reject the null hypothesis of conditional homoscedasticity for all the series, the original and the corrected ones. However, as Tables 2 and 3 show, estimated parameters are quite different in original and corrected series.

7 Conclusions

In the presence of isolated outliers, the sizes of the LM, $Q(k)$ and $D(k)$ tests for conditional homoscedasticty is always zero while their powers are also zero if the sample size is relatively small while it is recovered for large enough sample sizes. The effect of consecutive outliers is even worse because, in this case, these tests always reject the hosomoscedasticity hypothesis even if the series is truly homoscedastic and the sample size is large. On the other hand, the size and power of the test proposed by... are robust to the presence of both isolated and consecutive outliers. However, the size gets larger as the sample size increases and, consequently, this test can be misleading rejecting homoscedasticity too often.

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Tables and Figures

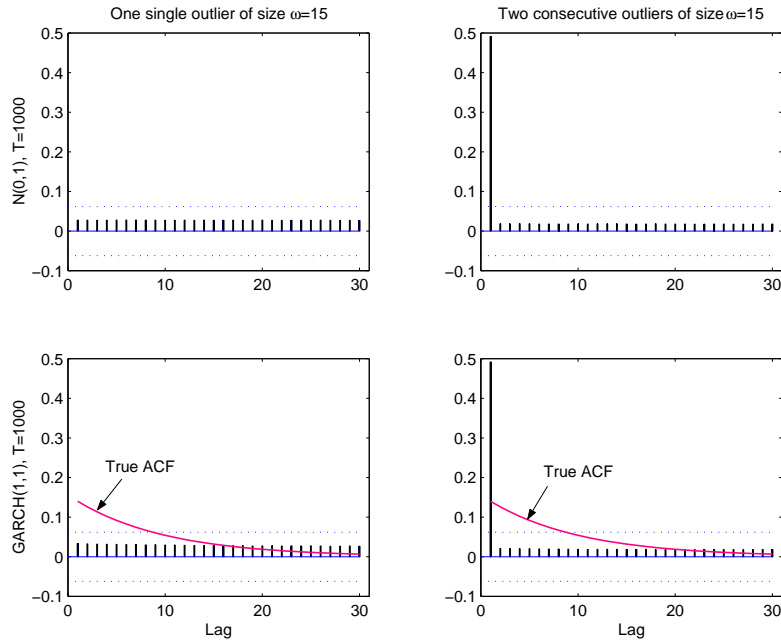Figure 1: Biases caused by outliers on the correlogram of squared observations

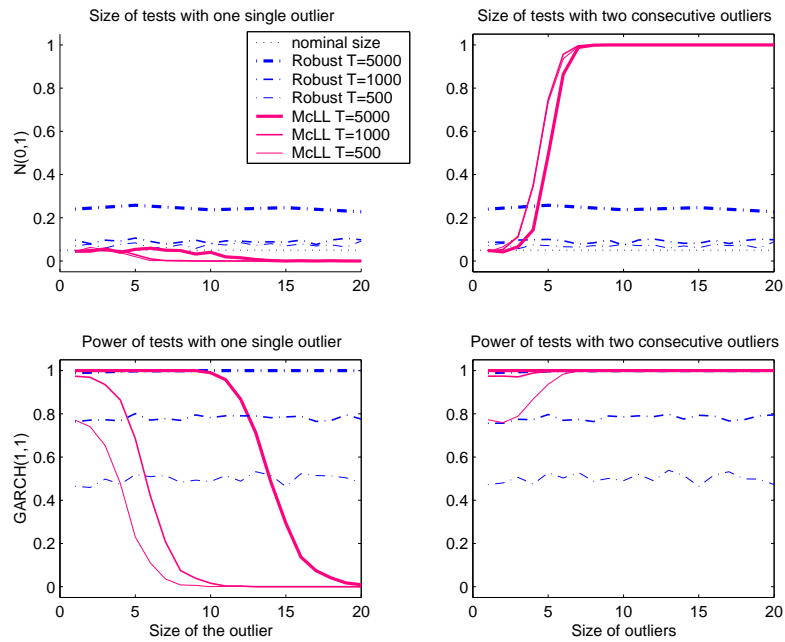Figure 2: Effects caused by outliers on the size and power of conditional homoscedasticity tests

Figure 3: Kernel estimation of the density of estimators of an ARCH(1) model with a single outlier
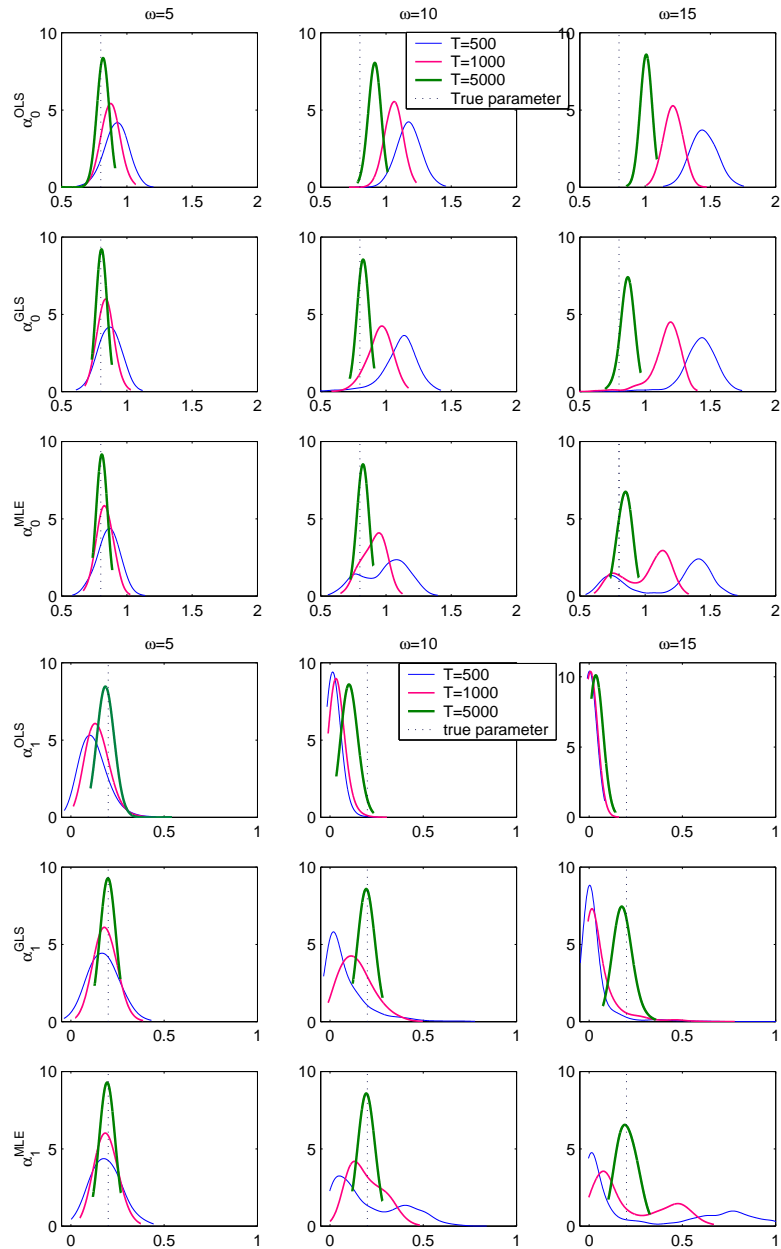
Figure 4: Ratio of variances of estimators of an ARCH(1) model with a single outlier
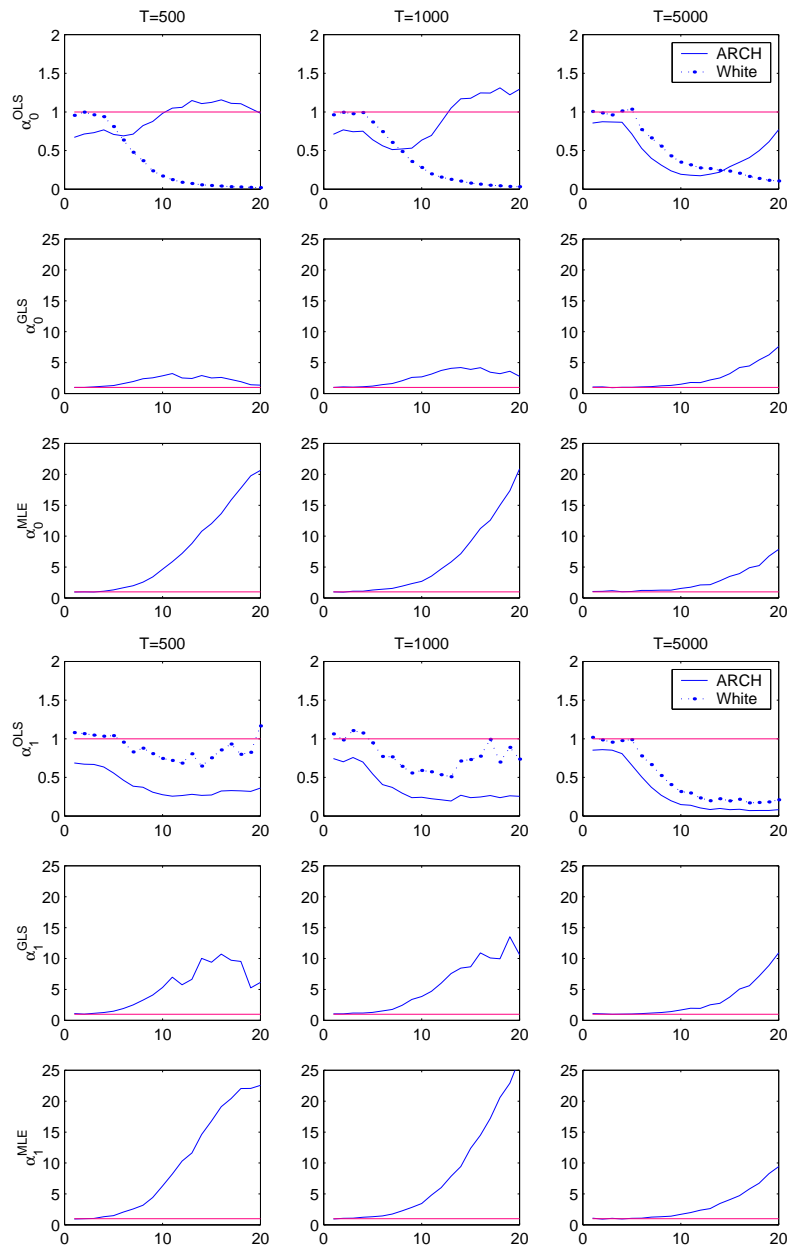
Figure 5: Kernel estimation of the density of estimators of an ARCH(1) model with two consecutive outliers
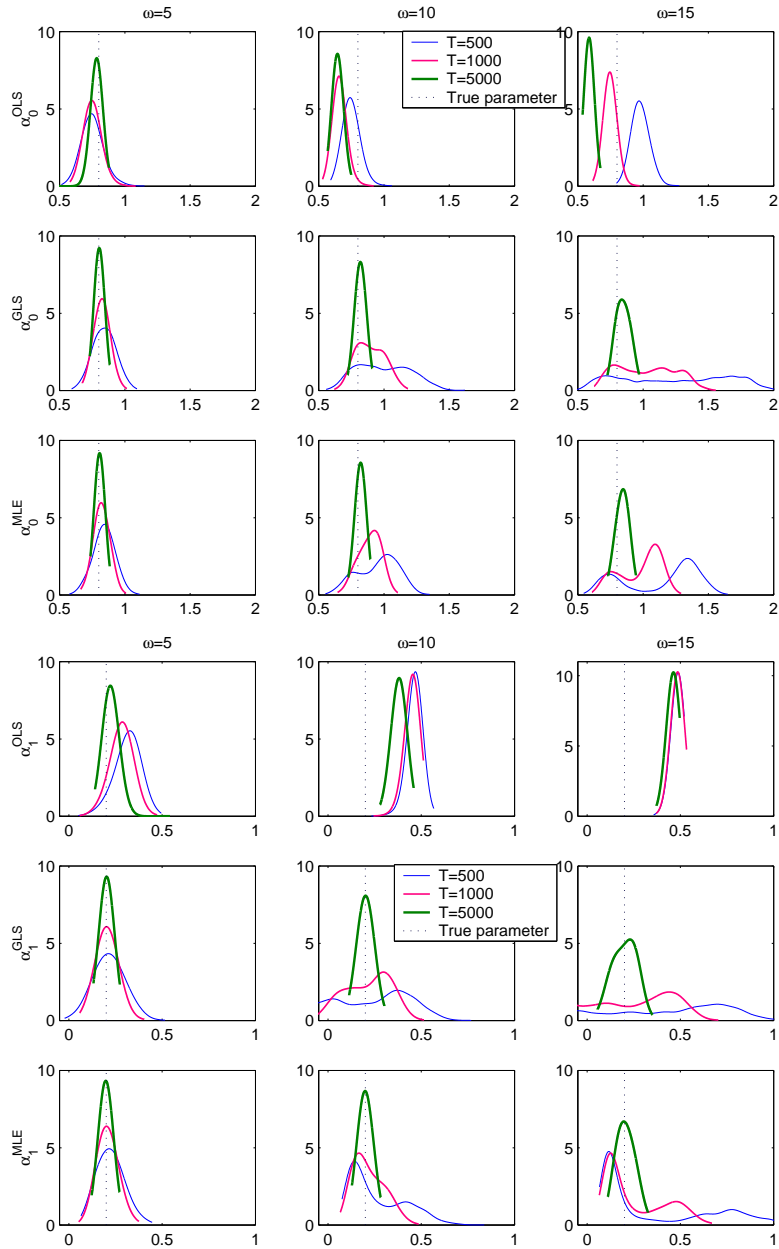
Figure 6: Ratio of variances of estimators of an ARCH(1) model with two consecutive outliers
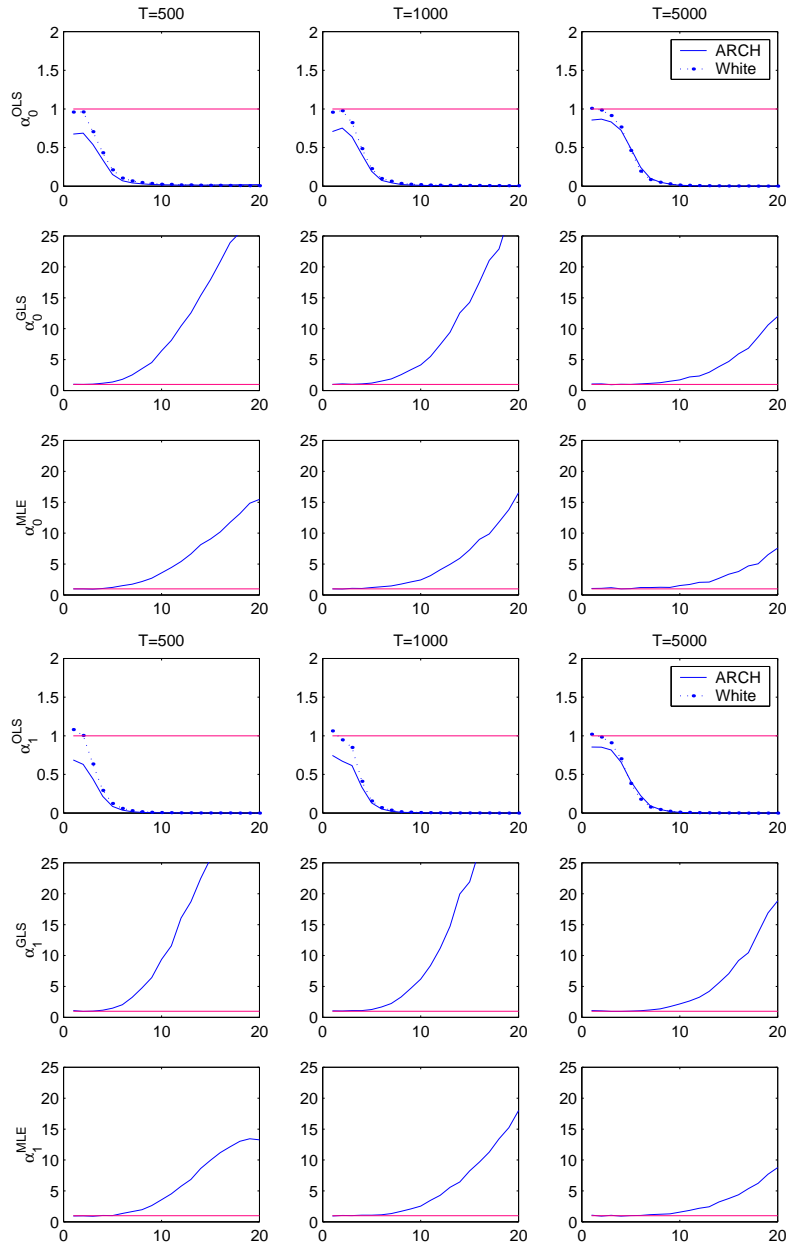
Figure 7: Kernel estimation of the density of ML estimators of GARCH models with a single outlier
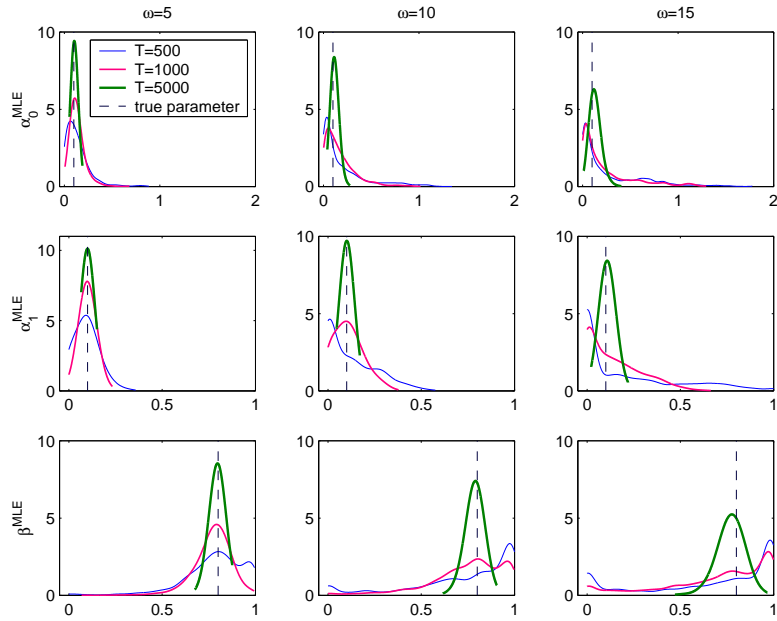

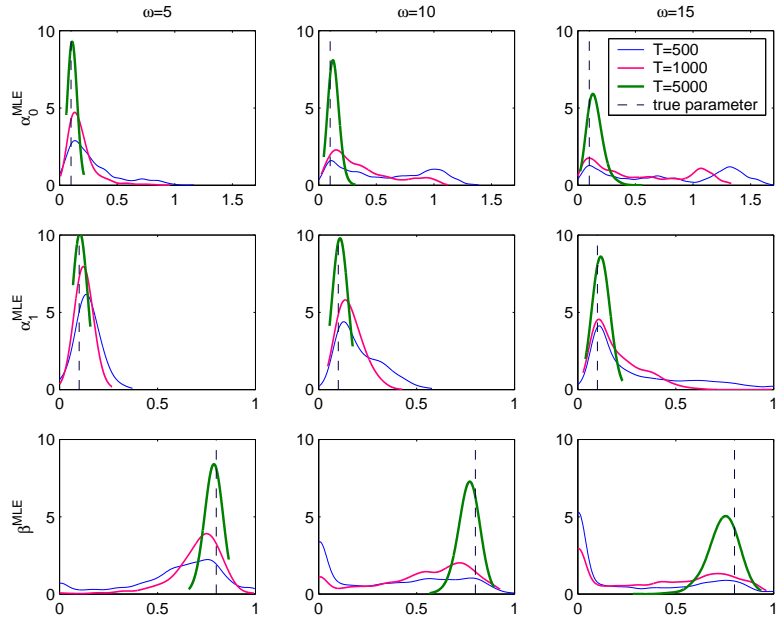Figure 8: Kernel estimation of the density of ML estimators of GARCH models with two consecutive outliers

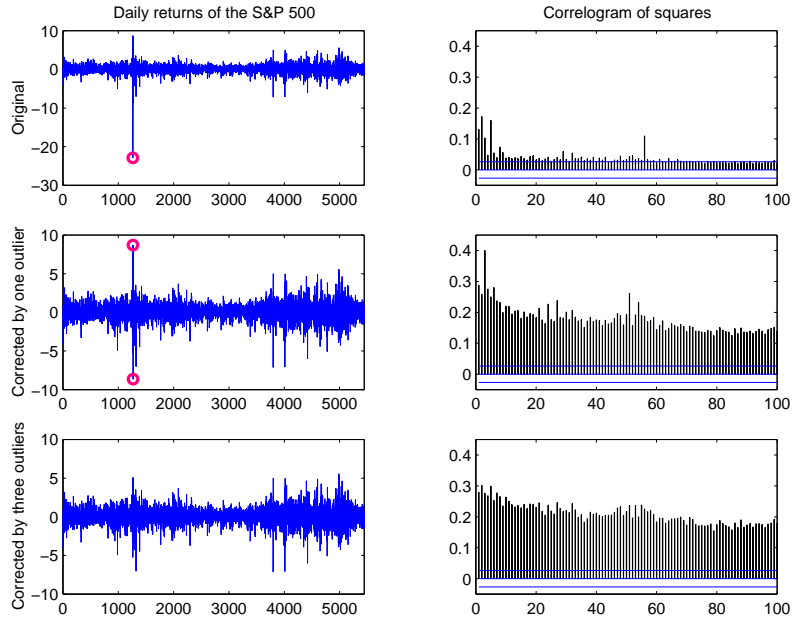Figure 9: Series and correlogram of squares of daily returns of S&P500 index

Figure 10: Series and correlogram of squares of daily returns of Nikkei 225 index
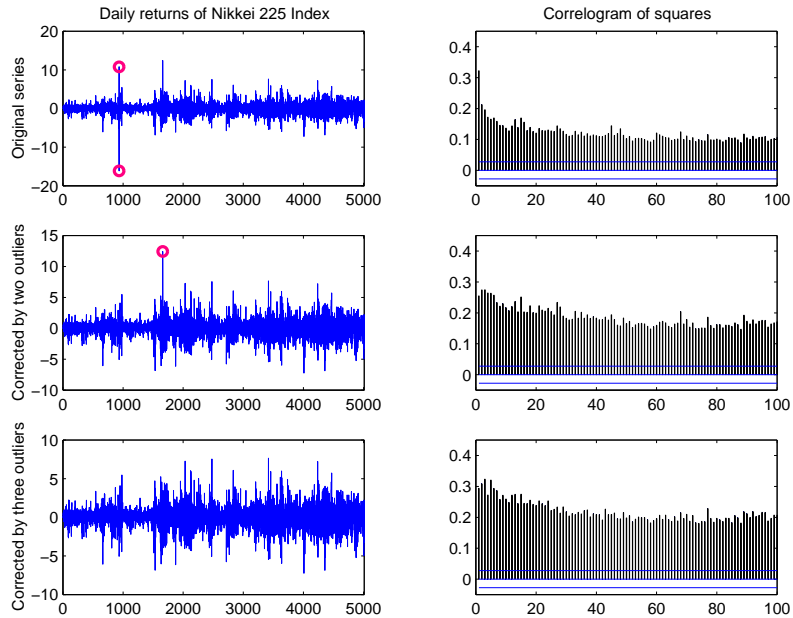
Table 1: Descriptive statistics of daily returns

Series		S&P 500	NIKKEI 225
T		5445	5012
Mean	original	0.0377*	0.0015
	corrected	0.0149	0.0001
S.D.	original	1.0761	1.4042
	corrected	1.0167	1.3660
Skewness	original	-1.9239*	-0.1109*
	corrected	-0.1872*	-0.0365
Kurtosis	original	44.8720*	10.6079*
	corrected	6.8724*	5.8164*
Autocorrelations of squared returns			
$r_2(1)$	original	0.1309*	0.3225*
	corrected	0.2791*	0.2938*
$r_2(2)$	original	0.1717*	0.2127*
	corrected	0.3022*	0.3085*
$r_2(5)$	original	0.1599*	0.1657*
	corrected	0.2996*	0.3205*
$r_2(10)$	original	0.0384*	0.1367*
	corrected	0.2641*	0.2594*
$Q_2(20)$	original	390*	797*
	corrected	1847*	1558*

T: Sample size.

$r_2(\tau)$: Order τ acf of y_t^2.

$Q_2(20)$: Box-Ljung statistic for y_t^2.

* Significative at 5% level.

Table 2: Estimates of the ARCH(9) Model

		S&P 500			NIKKEI 225		
		OLS	GLS	MLE	OLS	GLS	MLE
$\widehat{\alpha}_0$	original	0.71^* (0.11)	0.45^* (0.04)	0.27^* (0.01)	1.05^* (0.10)	0.52^* (0.07)	0.34^* (0.01)
	corrected	0.42^* (0.04)	0.26^* (0.04)	0.29^* (0.01)	0.78^* (0.07)	0.48^* (0.09)	0.35^* (0.02)
$\widehat{\alpha}_1$	original	0.09^* (0.01)	0.16^* (0.03)	0.10^* (0.01)	0.22^* (0.01)	0.19^* (0.04)	0.23^* (0.01)
	corrected	0.08^* (0.01)	0.13^* (0.03)	0.04^* (0.01)	0.07^* (0.01)	0.19^* (0.04)	0.11^* (0.01)
$\widehat{\alpha}_2$	original	0.13^* (0.01)	0.05 (0.03)	0.11^* (0.01)	0.05^* (0.01)	0.10^* (0.02)	0.12^* (0.02)
	corrected	0.11^* (0.01)	0.13^* (0.03)	0.11^* (0.01)	0.09^* (0.01)	0.15^* (0.04)	0.13^* (0.02)
$\widehat{\alpha}_3$	original	0.03^* (0.01)	0.06^* (0.02)	0.09^* (0.01)	0.05^* (0.01)	0.09^* (0.03)	0.11^* (0.01)
	corrected	0.07^* (0.01)	0.10^* (0.03)	0.07^* (0.01)	0.11^* (0.01)	0.11^* (0.04)	0.13^* (0.01)
$\widehat{\alpha}_4$	original	-0.02 (0.01)	0.00 (0.01)	0.10^* (0.01)	0.03 (0.01)	0.06^* (0.02)	0.08^* (0.01)
	corrected	0.06^* (0.01)	0.11^* (0.03)	0.10^* (0.01)	0.03^* (0.01)	0.08^* (0.03)	0.08^* (0.01)
$\widehat{\alpha}_5$	original	0.13^* (0.01)	0.11^* (0.03)	0.08^* (0.02)	0.03 (0.01)	0.11^* (0.02)	0.10^* (0.01)
	corrected	0.10^* (0.01)	0.09^* (0.03)	0.09^* (0.02)	0.11^* (0.01)	0.09^* (0.04)	0.11^* (0.01)
$\widehat{\alpha}_6$	original	-0.01 (0.01)	0.03 (0.02)	0.08^* (0.01)	0.04^* (0.01)	0.08^* (0.02)	0.06^* (0.01)
	corrected	0.03^* (0.01)	0.08^* (0.02)	0.08^* (0.01)	0.06^* (0.01)	0.05 (0.04)	0.07^* (0.01)
$\widehat{\alpha}_7$	original	-0.03^* (0.01)	0.02^* (0.01)	0.07^* (0.01)	0.02 (0.01)	0.09^* (0.02)	0.12^* (0.02)
	corrected	0.07^* (0.01)	0.07^* (0.03)	0.08^* (0.01)	0.06^* (0.01)	0.08^* (0.04)	0.13^* (0.02)
$\widehat{\alpha}_8$	original	0.03^* (0.01)	0.10^* (0.02)	0.10^* (0.01)	0.01 (0.01)	0.04 (0.02)	0.07^* (0.01)
	corrected	0.06^* (0.01)	0.09^* (0.03)	0.11^* (0.01)	0.01 (0.01)	0.05 (0.03)	0.06^* (0.01)
$\widehat{\alpha}_9$	original	0.03^* (0.01)	0.05^* (0.02)	0.07^* (0.01)	0.02 (0.01)	0.03 (0.02)	0.05^* (0.01)
	corrected	0.02 (0.01)	0.04^* (0.02)	0.06^* (0.01)	0.04^* (0.01)	0.04 (0.04)	0.06^* (0.01)

Table 3: Estimates of the GARCH(1,1) Model

	Series	S&P 500	NIKKEI 225
α_0	original	0.012^* (0.001)	0.076^* (0.003)
	corrected	0.006^* (0.001)	0.018^* (0.002)
α_1	original	0.078^* (0.002)	0.139^* (0.003)
	corrected	0.046^* (0.003)	0.101^* (0.005)
β	original	0.915^* (0.003)	0.863^* (0.004)
	corrected	0.948^* (0.004)	0.895^* (0.005)
Log-Likelihood	original	-7299.6	-8145.9
	corrected	-7217.2	-8040.9