Yield-Factor Volatility Models

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Abstract

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I Introduction

The term structure of interest rates is often summarized using a handful of yield factors that capture shifts in the shape of the yield curve, i.e., changes in the overall level, slope, and curvature of the yield curve (see Litterman and Scheinkman, 1991). This factor decomposition provides a parsimonious representation of the term structure and is extensively used in risk management (Pérignon and Villa, 2004), fixed-income derivative pricing (Driessen, Klaassen and Melenberg, 2003), and to model the linkages between interest rates and macroeconomic variables (see Ang and Piazzesi, 2003). Despite this wide application in financial economics, very little is known about the volatility of these factors. In this paper, we study the dynamics of yield-factor conditional volatility.

Yield factors are related to the latent factors implied by affine term structure models (See Duffie and Kan, 1996, and Dai and Singleton, 2000). Typically, the estimated loadings on the latent factors are very similar to the loadings on the yield factors and therefore estimated latent factors behave like yield factors (see De Jong, 2000, Bams and Schotman, 2003, and Dai and Singleton, 2003). More generally, any accurate dynamic term structure model, within or outside the affine class, must be consistent with movements in the yield curve. For instance, Andersen, Benzoni and Lund (2003) show that the observed shifts in bond yields can be adequately explained by a three-factor model of the short-term interest rate where the factors are the stochastic volatility, the mean drift, and jumps.

There are different ways to extract the yield factors. One approach uses fixed prespecified weights on yields of various maturities to capture economically meaningful characteristics of the yield curve, such as its overall level, slope, and curvature (see among others Ang, Piazzesi and Wei, 2003 and Brandt and Chapman, 2003). A second approach that is statistically motivated estimates the weights by decomposing the covariance matrix of bond yields through principal component analysis (see Litterman and Scheinkman, 1991) or factor analysis (see Knez, Litterman and Scheinkman, 1994).[1] The economic and statistic approaches produce factors which are very highly correlated.

Many studies have investigated the dynamics of short-term interest rates. The main

[1]Similar techniques can be applied to the implied covariance matrix which best fits a set of observed interest rate derivatives (see Longstaff, Santa-Clara and Schwartz, 2001).

conclusion from this literature is that a level effect, in which the volatility is a positive function of the level of interest rates, GARCH effects, and regime shifts are required to adequately model the short-rate volatility. The dependence of the interest rate volatility on the level of the short-rate was first systematically studied by Chan, Karolyi, Longstaff and Sanders (1992, hereafter CKLS). They found a very high elasticity parameter of around 1.5. This estimate was reconsidered, among others, by Bliss and Smith (1998) accounting for the possibility of a structural break in the data, by Smith (2002) allowing for regime shifts, and by Ronchetti, Dell'Aquila and Trojani (2003) using a robust version of GMM. The first model that combines both level and GARCH effects for the short-rate volatility was proposed by Longstaff and Schwartz (1992). Furthermore, Brenner, Harjes and Kroner (1996) show that models including both level and GARCH effects better predict volatility than models including only one of the effects (see also Bali, 2000). Gray (1996) extends the GARCH-level model to allow for multiple regimes in the short-rate volatility and finds that one needs all three effects to adequately model interest rate volatility.

Less research has been devoted to understanding the joint-dynamics of the yield factors. The role of conditional heteroscedasticity in the dynamics of the volatility of the yield factors has been highlighted by Christiansen (2004) for the short-rate and the slope of the U.S. term structure, and by Christiansen and Lund (2002) for the level, slope, and curvature factors. Unfortunately, these two papers do not include a level effect in the yield-factor volatility even though it has been shown to be extremely important in univariate models. The estimation of the latter effect turns to be particularly challenging in a multi-factor framework. Indeed, Boudoukh, Richardson, Stanton and Whitelaw (1998) find that the volatility of interest rates is increasing in the level of interest rates only for sharply, upward sloping term structures. On the other hand, using an empirical version of the Schaefer and Schwartz's (1984) model, Christiansen (2003) identifies a strong level effect for the volatility of the long rate but no level effect for the volatility of the slope of the yield curve. However, her specification of the level effect for the slope factor uses the slope of the yield curve itself. A difficulty with using the slope directly is that the standard-deviation becomes negative when the slope is negative.

Another strand of research examines the role of regime shifts in the dynamics of yield-factor volatilities. Using international data, Kugler (1996) and Ang and Bekaert (2002a,b) estimate a two-state regime-switching VAR for the level and the slope factors with a constant covariance matrix in each regime (i.e., without any level nor GARCH effects). Recently, Christiansen (2004) extended this latter approach by fitting a two-state regime-switching ARCH model to the level and the slope factors. A broad conclusion of this research is that regime shifts are a central feature of yield-factor volatilities.

An important contribution of our paper is the development of a comprehensive model for yield-factor volatilities that simultaneously includes level and GARCH effects along with regime shifts. Our approach is motivated by the observation that the volatility of all three yield factors tends to be higher when short-term interest rates are higher. We therefore include a level effect in which the volatility of the level, slope and curvature factors is positively related to the level of interest rates. Our model allows us to study the influence of the level of volatility on the conditional means of the factors. We employ a flexible specification that allows the mean of each factor to be a linear function of the conditional volatility of each factor and/or of the conditional volatility of the level factor. Our model explicitly includes regime shifts, a feature which has been demonstrated to be important in fitting short-term interest rates (see Gray, 1996). Each model we consider is nested within this encompassing model, so we are able to directly measure the marginal contribution of each component of the model.

We also contribute to the debate about the link between the level and the volatility of interest rates (see Chapman and Pearson, 2001). Using monthly bond yields over the 1970-2002 period, we show that all three yield factors display a significant level effect. In particular, we find that the level effect for the slope factor is better captured by the overall level of interest rates rather than by the level of the slope factor. A similar conclusion is reached for the curvature factor. Furthermore, both factors exhibit strong GARCH effects. Our empirical results identify some interesting characteristics of the dynamic behavior of the slope and curvature factors. Although there has been little attention devoted to analyzing these factors in the academic literature, they are important for the valuation of interest rate derivatives such as caps and swaptions (see Han, 2003), and they constitute an important component of bond portfolio risk and should be accounted for appropriately.

In addition to identifying the role of the level and GARCH effects in the dynamics of the yield-factor volatility, we examine the effect of volatility on the dynamics of the yield-factors. We find that the GARCH-based volatility of the overall level of interest rates is negatively related to the level and curvature and positively related to the slope of the yield curve, which is consistent with a "flight-to-cash". However, this volatility-in-mean effect becomes insignificant when a level effect is introduced. We also examine regime-switching models that recognize different regimes in the volatility of the yield factors. We find that allowing for regime shifts dramatically improves the model's fit and strengthens the level effect. The Bayesian information criterion suggests that the favored model is a regime-switching model with level but no GARCH effects.

Furthermore, a methodological contribution is to provide a novel specification for the conditional volatility of the factor residuals allowing simultaneously for GARCH and level effects. The main difference between our approach and current models (e.g. Brenner, Harjes and Kroner, 1996, and Gray, 1996) is conceptual. We endeavor to combine GARCH and level effects, while maintaining the traditional interpretation that a GARCH(1,1) model implies an ARMA(1,1) representation for the squared residual. The GARCH component of current interest-rate volatility models does not have this feature.

Finally, we discuss how the dynamics of yield factors we identify could potentially be used to discriminate between alternative term structure models. We propose a set of economic moments based on the empirical regularities identified in this paper. Following Brandt and Chapman (2003), we suggest how the simulated method of moments may be used in future research to contrast affine and quadratic term structure models. The actual implementation of the proposed test is beyond the scope of our paper though.

The remainder of the paper proceeds as follows. Section II details the model, section III describes the data with the central features of U.S. Treasury yields, and section IV presents the empirical results. Sections V and VI propose two extensions of our model accounting for GARCH-in-Mean effects and volatility regime shifts respectively. Section VII discusses how the key stylized facts of yield factors could potentially be used to compare term structure models. Section VIII offers some concluding comments and suggests some possible extensions.

II Model Development

We are interested in modeling the dynamics of the following three yield factors, the level of interest rates (L), the slope of the yield curve (S), and the curvature of the yield curve (C). Denote by Y_{it} the level of the i^{th} yield factor, with $i = L, S, C$, whose dynamics is modeled as:

$$dY_{it} = (a_i + b_i \cdot Y_{it})dt + \sigma_{it} Y_{jt}^{\gamma_i} dW_{it} \tag{1}$$

where W_{it} is a standard Brownian motion (W_{Lt}, W_{St}, and W_{Ct} may be correlated). This model is inspired by early models of the short-rate which include both mean reversion and allow the conditional volatility to be a function of the level of the short-rate (see among others Cox, Ingersoll and Ross, 1985). When allowing for a level effect in a multi-factor framework, one can either model the residual volatility of a given factor as a function of the value of this very factor ($j = i$) or, as we argue, of the level factor ($j = L$).[2] Given that the level effect is so important in modeling volatility of short-term interest rates, we test if the volatility of the slope and curvature are also functions of the level of interest rates. As will be seen below, this conjecture is born out by the data.

In our empirical work, we discretize the process in Equ. (1) as:

$$\Delta Y_{it} = \alpha_{0i} + \alpha_{1i} \cdot Y_{it-1} + e_{it} \tag{2}$$

for $i = L, S, C$ and $t = 1, ..., T$. We approximate dW_{it}, which is normally distributed with variance dt, by a normally distributed innovation e_{it}. We decompose the conditional volatility of e_{it} into the product of two terms, $E(e_{it}^2|\psi_{t-1}) = \delta_{it}^2 = \sigma_{it}^2 Y_{jt-1}^{2\gamma_i}$ with ψ_{t-1} denoting the information set at time $t-1$. This specification allows heteroscedasticity to enter through a time-varying coefficient σ_{it}^2, which depends on past shocks on the residuals factors, and through the level effect.

Alternatively, the residual in Equ. (2) can be written as $e_{it} = Y_{jt-1}^{\gamma_i} \sigma_{it} z_{it}$ where z_{it} is i.i.d. $N(0,1)$. In this modeling, σ_{it}^2 is the volatility of the scaled residual $v_{it} = e_{it}/Y_{jt-1}^{\gamma_i} = \sigma_{it} z_{it}$ and is modeled as a GARCH process:

$$\sigma_{it}^2 = \beta_{0i} + \beta_{1i} \cdot v_{it-1}^2 + \beta_{2i} \cdot \sigma_{it-1}^2. \tag{3}$$

[2]While both specifications will be considered in the present study for exhaustivity, the own-level approach appears rather inconsistent since the residual standard-deviation becomes negative when the value of the factor (slope or curvature) is negative.

Our specification for σ_{it}^2 differs from previous processes proposed in the literature. For instance, in a univariate setting, Brenner, Harjes and Kroner (1996) use a standard GARCH model for the residual volatility of the short-rate:

$$\sigma_t^2 = \beta_0 + \beta_1 \cdot e_{t-1}^2 + \beta_2 \cdot \sigma_{t-1}^2. \tag{4}$$

Alternatively, Longstaff and Schwartz (1992), Brenner, Harjes and Kroner (1996), Gray (1996), and Hamilton and Kim (2002) add the level term directly to the GARCH model:

$$\sigma_t^2 = \beta_0 + \beta_1 \cdot e_{t-1}^2 + \beta_2 \cdot \sigma_{t-1}^2 + \beta_3 \cdot Y_{t-1}^{2\gamma} \tag{5}$$

where γ is either fixed or estimated. It is well known that "if u_t is described by a GARCH(r, m) process, then u_t^2 follows an ARMA(p, r) process, where p is the larger of r and m" (Hamilton, 1994, p. 666). We maintain this interpretation of the GARCH model when including a level effect. Indeed, we assume that v_{it}^2 evolves as an ARMA(1,1) process yielding a GARCH model for the scaled residual v_{it}. This model can be compared with the stochastic volatility literature (see Andersen and Lund, 1997, Ball and Torous, 1999, and Smith, 2002) where the conditional volatility of the short-rate is modeled as $\sigma_t^2 Y_{t-1}^{2\gamma}$ and the conditional volatility of the scaled residual σ_t^2 follows an autoregressive process. Here, we also model the conditional volatility of the scaled residual but using a GARCH model. Although there is nothing wrong with Equ. (4) and (5) as empirical models, they are somewhat ad hoc extensions of the GARCH model and are inconsistent with this traditional interpretation of the GARCH model. Model (4) is particularly difficult to motivate theoretically. It has the flavor of modeling the scaled residual using a GARCH model as in the stochastic volatility interest rate literature, but in this case the scaled residual clearly does not admit an ARMA representation since $E(e_{t-1}^2) = \delta_t^2 \neq \sigma_t^2$.

We assume that ΔY_t is a tri-dimensional vector of the changes in yield factors with conditional mean vector $\mu_t = E(\Delta Y_t | \psi_{t-1})$ and conditional covariance matrix $\Sigma_t = H_t^{1/2} \rho H_t^{1/2}$, where ρ is a (3×3) conditional correlation matrix and H_t is a (3×3) diagonal matrix with conditional volatility of the i^{th} factor on the i^{th} element of the principal diagonal (see Bollerslev, 1990). We estimate the parameter vector using quasi-maximum likelihood, where $\ln L = \sum_{t=1}^{T} \ln f(e_t | \psi_{t-1})$ is the quasi-loglikelihood function, and $f(e_t | \psi_{t-1})$ is the probability density function of the multivariate normal density with mean 0 and covariance matrix $\Sigma_{t|t-1}$. The initial observation is assumed to be drawn from the unconditional distribution of ΔY_t.

Many classical models for interest rates are nested in the model derived above. Firstly, a multivariate homoscedastic-AR(1) model, labeled as the NO GARCH-NO LEVEL model, can be derived by assuming that the residual volatility of each factor is constant through time ($\beta_{ki} = 0$, $i = L, S, C$ and $k = 1, 2$, and $\gamma_i = 0$, $i = L, S, C$). Secondly, a multivariate version of the CKLS model, which is called the LEVEL model, is obtained by assuming that σ_{it}^2 is constant. In the latter model, the volatility remains time-varying but depends solely on the level of the factor ($\beta_{ki} = 0$, $i = L, S, C$ and $k = 1, 2$). Thirdly, a multi-factor model, labeled as the GARCH model, allows σ_{it}^2 to follow a GARCH process but does not permit volatility to be a function of the level of the factors ($\gamma_i = 0$, $i = L, S, C$). Finally, the unrestricted version of the model, which is referred to as the GARCH-LEVEL model, permits both the coefficient σ_{it}^2 to vary through time as new information arrives and the residual volatility to depend on the level of the factors.

III Data

We use the Fama-Bliss (1987) monthly data on Treasury zero-coupon bond yields over the 1970:01 - 2002:12 period. We denote by $y_t^{(\tau)}$ the bond yield with a τ-month maturity observed at time t. Following a prevalent practice, we build the three yield-factor series from a short-term, medium-term, and long-term yields. Specifically, we associate the level factor with the 3-month yield ($L_t = y_t^{(3)}$), the slope factor with the difference between the 120-month yield and the 3-month yield ($S_t = y_t^{(120)} - y_t^{(3)}$), and the curvature factor with a linear transformation of the short, medium, and long-term yields ($C_t = y_t^{(3)} - 2y_t^{(24)} + y_t^{(120)}$).[3]

Table 1 presents some descriptive statistics for the yield factors and the yield-factor residuals extracted from a first-order autoregressive model. For each series, we provide the first four central moments, the Bera-Jarque normality test, the correlation with other factors (or factor residuals), the first-order (cross-)autocorrelation, and the Box-Pierce statistics to test for the k^{th}-order autocorrelation of the series.[4] We observe that the factor series

[3]We use the 3-month yield to proxy the level factor because the evolution of the 1-month yield is known to be idiosyncratic (Duffee, 1996). The 2-year maturity is an important intermediate maturity since the term structure of volatility peaks at this maturity (see Dai and Singleton, 2003). We proxy the long-term rate by the 10-year maturity since this yield is less subject to liquidity problems than longer-maturity yields. The transformation used to derive the curvature factor is a numerical measure of the second derivative of the yield curve, which captures its curvature.

[4]The Box-Pierce statistics are distributed as a chi-squared random variable with k degrees of freedom.

are strongly autocorrelated and depart from normality. The level factor turns out to be negatively correlated with the other two factors, while the slope and curvature factors exhibit a positive correlation. The factor residuals are negatively correlated among each other and far from being normal since their distributions are clearly leptokurtic. The Box-Pierce test suggests that the factor residuals are much less persistent than the factor levels.

The time series of each factor is plotted in Figure 1. While the level factor is always positive, both the slope and curvature factors take negative and positive values. There are several episodes when the yield curve is downward-sloping: the 1973 OPEC oil crisis, during a significant portion of the 1979-1982 monetary experiment, 1989, and towards the end of 2000. Note also that the monetary experiment had a great impact on the curvature of the term structure. Figure 2 displays the absolute value of the factor residuals and shows that all three series exhibit volatility clustering. This suggests that an appropriate model for the yield-factor volatility should include ARCH effects. Moreover, factor volatilities appear to depend on the level of interest rates as attested by the superimposed level series. Indeed, the volatility of both slope and curvature factors tends to be high when interest rates are high. This suggests that a diffusion model, in which volatility is a positive function of the level of interest rates, may be able to account for this effect.

< Insert Table 1 >

< Insert Figures 1 and 2 >

To initially assess the relative importance of the level and ARCH effects in yield-factor volatilities, we implement the robust, regression-based specification tests of Wooldridge (1990). The null hypothesis for these specification tests is homoscedasticity. If the data are homoscedastic, then $e_t^2 - \hat{\sigma}^2$ will be uncorrelated with any function of lagged information variables $\lambda(F_{t-1})$. Wooldridge's test is a conditional moment test that determines whether $E[(e_t^2 - \hat{\sigma}^2)\lambda(F_{t-1})] = 0_{K\times 1}$ for some K-dimensional vector $\lambda(F_{t-1})$. The alternative hypothesis is that the expectation is non-zero, which implies that at least one of the variables in $\lambda(F_{t-1})$ is useful in explaining conditional volatility. The size of each test statistic provides a crude metric of the relative ability of each component of $\lambda(F_{t-1})$ in explaining the time-varying volatility. Unlike Engle's (1982) test for ARCH, which proceeds by regressing e_t^2 on a number of lagged squared residuals, Wooldridge's test is robust to non-normality. A major advantage of this test is that it can be constructed using nothing more sophisticated than OLS. In this framework, a robust test for p^{th} order ARCH effects is obtained using $\lambda(F_{t-1})^{\intercal} = (e_{t-1}^2\ e_{t-2}^2\ ...\ e_{t-p}^2)$ and a robust test for level effects is derived with $\lambda(F_{t-1}) = |Y_{j,t-1}|^{\gamma}$ for some suitably defined γ, such as 0.5 or 1.

We report the Wooldridge's test statistics with the associated p-values in Table 2. Since the largest p-value is 0.0228, there is some clear indication that factor changes are conditionally heteroscedastic. Further, the conditional volatility is strongly related to the overall level of interest rates, but not to the level of the slope or curvature factors. Indeed, when modeling the volatility of the slope (respectively curvature) as a function of the value of the slope (curvature) factor, no level effect can be detected. This result is consistent with the evidence reported by Christiansen (2003) for the slope factor using weekly data. It appears in Table 2 that setting the elasticity parameter γ equal to one is optimal for the three yield factors, though the significance remains even when $\gamma = 0.5$ as implied by the square root process of Cox, Ingersoll and Ross (1985). This preliminary analysis highlights the following features that a correctly specified volatility model should possess. First, the residuals exhibit volatility clustering, which suggests using a GARCH process to model the conditional residual volatility. Second, the volatility of the slope and curvature factors depends strongly on the level of interest rates, but much less on the levels of the slope and curvature factors. As a result, in the following empirical analysis, we primarily model the level effect using the overall level of interest rates.

< **Insert Table 2** >

IV Empirical Results

In this section, we report the results of fitting the competing models, i.e., the NO LEVEL-NO GARCH, LEVEL, GARCH, and GARCH-LEVEL models, to the U.S. term structure of interest rates over the 1970-2002 period.

We begin by estimating the univariate version of the four models. Table 3 reports for each yield factor the parameter estimates and Bollerslev-Wooldridge (1992) robust standard-errors. The first column of Table 3 reports the estimates of the homoscedastic model. There appears to be mean reversion in all three yield factors ($\hat{\alpha}_{1i} < 0$, $i = L, S, C$), though not statistically significant for the level. When a level effect is introduced, the loglikelihood function increases dramatically ($\Delta\,\mathrm{lnL}_L = 136.70$, $\Delta\,\mathrm{lnL}_S = 53.68$, and $\Delta\,\mathrm{lnL}_C = 11.40$). The elasticity parameter $\hat{\gamma}_i$ is significant for all three yield factors. Interestingly, if the level effect is modeled using the level of each factor, instead of the overall level of interest rates, the fit of the LEVEL model is significantly reduced ($\mathrm{lnL}_S = -297.58$ and $\mathrm{lnL}_C = -234.91$, not reported in the tables). Results of the GARCH model suggest that explicitly modeling the serial correlation in volatility leads to a superior fit. Further, the variance processes exhibit high persistence, though the persistence is lower for the slope and curvature factors ($\hat{\beta}_1 + \hat{\beta}_2 = 0.9897$ for L, 0.9316 for S, and 0.9070 for C). Finally, the GARCH-LEVEL model gives rise to lower estimates of the elasticity parameters than in the LEVEL model. The marginal contribution of the GARCH effect turns out to be stronger than the marginal contribution of the LEVEL effect for the three yield factors.

< Insert Table 3 >

Table 4 reports the parameter estimated and robust standard-errors for the multivariate models. The main difference between the specifications presented in Table 3 and the present specifications is that the residual factors can now be correlated. The point estimates for the correlation coefficients ($\rho_{L,S}$, $\rho_{L,C}$, and $\rho_{S,C}$) are negative in all models, while their magnitude varies across models. As pointed by Dai and Singleton (2000), negative correlation among risk factors is an important feature of the U.S. term structure of interest rates. Allowing the residuals to be correlated improves the fit of each model. For instance, the sum of the loglikelihoods for the three univariate GARCH models is equal to -602.24 and the loglikelihood for the trivariate GARCH model is as high as -477.18. However, correlation matters not only for fitting purposes but it also strongly impacts the point estimate for the elasticity parameters. Indeed, for the LEVEL and GARCH-LEVEL models, the point estimate of γ_i drops considerably for the level and slope factors after accounting for correlation. On the other hand, the point estimate of γ_i increases for the curvature factor. While the level of persistence is comparable to the univariate case, the response of volatility to lagged information shocks ($\hat{\beta}_1$) drops significantly for the level and slope factors.

Comparing the various multivariate volatility models yields some interesting conclusions. Consistent with the univariate results, the GARCH effect seems to dominate the level effect: When only a GARCH effect is introduced, the value of the loglikelihood function increases by 193, which exceeds the rise (126) observed when a level effect is introduced. However, the level effect only requires three extra parameters while the GARCH model requires six extra parameters. Another interesting observation is that the value of the elasticity parameter is weakened when a GARCH effect is introduced, though it remains significant for the level and slope factors. In conclusion, it seems that one needs both level and GARCH effects to adequately model yield-factor volatilities.

< Insert Table 4 >

V GARCH-in-Mean Effect

We extend the model presented in Section II by introducing a GARCH-in-Mean effect along the lines of Engle, Lilien and Robins (1987). This alternative specification allows us to analyze the impact of conditional volatility on the shape of the yield curve. This extension is motivated by several previous empirical findings. First, Engle, Ng and Rothschild (1990) find that excess returns on Treasury bills are strongly affected by the conditional volatility on an equally-weighted bill portfolio, which is taken as a unique common factor. Second, in a complementary study based on the Engle, Ng and Rothschild's model, Engle and Ng (1993) show that when volatility is high, the yield curve is likely to be upward sloped (see also Fong and Vasicek, 1991, and Longstaff and Schwartz, 1993). Third, Litterman, Scheinkman and Weiss (1991) find that the curvature of the yield curve and the implied volatility extracted from bond options are strongly related.

We model the dynamics of the conditional mean of each factor as:

$$\Delta Y_{it} = \alpha_{0i} + \alpha_{1i} \cdot Y_{it-1} + \alpha_{2i} \cdot \sigma_{Lt} + e_{it}. \tag{6}$$

Alternatively, we use the conditional volatility of the slope factor σ_{St} (in addition to or instead of σ_{Lt}) in the mean equation of the slope factor and the conditional volatility of the curvature factor σ_{Ct} (in addition to or instead of σ_{Lt}) in the mean equation of the curvature factor. Given the empirical results presented above, we model the level effect using the level of the short-term interest rate alone. Therefore, the residual volatility of each factor is given by $E(e_{it}^2|\psi_{t-1}) = \delta_{it}^2 = \sigma_{it}^2 Y_{Lt-1}^{2\gamma_i}$, where the scaled residual v_{it} is modeled as a GARCH process.

Table 5 reports the results for the GARCH model with volatility effects in the mean equation. We find that the conditional volatility of short-term interest rate is negatively related to the level of the short-term interest rate ($\hat{\alpha}_{2L} < 0$), positively related to the slope of the yield curve ($\hat{\alpha}_{2S} > 0$), and negatively related to the curvature of the yield curve ($\hat{\alpha}_{2C} < 0$), though the coefficient is not statistically significant for the curvature. These point estimates imply that when short-rate volatility is high, we expect short-term yields to decrease, intermediate yields to increase marginally and long-term yields to increase more strongly. We term this the "flight to cash" which has a neat economic interpretation because long-term bonds generally have higher exposure to interest-rate risk than short-term bonds.[5] An increase in the volatility of the level factor, i.e., the short-rate, will thus result in a larger increase in the risk of long-term bonds than short-term bonds. We would therefore expect investors to move funds out of these riskier long-term bonds and into safer short-term bonds. This, in turn, would cause the prices of intermediate and long-horizon bonds to decrease (and intermediate and long-horizon yields to correspondingly increase) in response to this selling pressure while the buying pressure on short-term bonds will increase their prices (and thus decrease their yields).

We find a similar pattern when the GARCH-M effect is modeled through the conditional volatility of each factor. When both conditional volatilities (σ_{Lt} and σ_{St} or σ_{Ct}) are included in the mean equation, the conditional volatility of the level seems to capture much of the volatility effect in the mean equation. Moreover, for the three alternative specifications, the point estimates for the correlation parameters are not substantially affected by the GARCH-M variables. Whatever the chosen specification for the mean equation, the increase in the loglikelihood value is very limited, though three and five new parameters are estimated respectively. We see in Table 6 that the volatility-in-mean effect is weaker when a level effect is introduced and that, consequently, it appears to be difficult to simultaneously estimate the GARCH-M and γ_i parameters. There appears to be little benefit to include a GARCH-M

[5] In particular, Campbell, Lo and MacKinlay (1997) show formally that both homoscedastic and heteroscedastic single-factor AR(1) lognormal affine models imply that the sensitivity of bond returns to the short-rate increases with maturity (see pages 431 and 437).

effect in the GARCH-LEVEL model presented in the previous section (see Table 4).

< Insert Tables 5 and 6 >

VI Regime-Switching Models

In this section, we extend our basic model to allow for different regimes in the volatility of the yield factors. This is motivated by the extensive empirical literature suggesting that regime-switching models describe historical interest rates better than single-regime models (see Hamilton, 1988, Gray, 1996, Bansal and Zhou, 2002, and Smith, 2002). Further, Ang and Bekaert (2002a,b) and Dai, Singleton and Yang (2003) show that regime shifts are also important in capturing the dynamics of interest rates using multi-factor term structure models.

We denote by S_t the random state of the world at time t which can take two values, $s_t = \{1, 2\}$, where 1 denotes the "high-volatility regime" and 2 the "low-volatility regime". We assume that these regimes are common to the level, slope, and curvature of the yield curve. This assumption is primarily to keep the state space parsimonious[6], but it also seems more reasonable to assume that the state of the economy would affect all characteristics of the yield curve jointly rather than only affecting short-term interest rates without an effect on the slope and curvature. Furthermore, most term structure models assume that the entire yield curve be priced with the same underlying state variables, which would demand a common regime. Finally, a simple perusal of Figure 2 indicates that the interesting high-volatility episodes of one series appear also in the other two series. To also keep the model simple we allow only the unconditional mean and volatility of each series to be state-dependent. The conditional mean is given by:

$$\Delta Y_{it} = \alpha_{0is_t} + \alpha_{1i} \cdot Y_{it-1} + e_{it} \tag{7}$$

and the conditional volatility of the scaled residual is:

$$\hat{\sigma}^2_{it|s_t,s_{t-1}} = \beta_{0is_t} + \beta_{1i} \cdot \hat{v}^2_{it-1} + \beta_{2i} \cdot \hat{\sigma}^2_{it-1|s_{t-1}}. \tag{8}$$

[6]Without this assumption, the state space would enlarge to $2^3 = 8$ regimes. A similar assumption is made by Kugler (1996), Ang and Bekaert (2002a,b), and Christiansen (2004).

This specification of the GARCH model follows Dueker (1997) by defining $\hat{\sigma}^2_{it|s_t,s_{t-1}} = E(v_t^2|S_t = s_t, S_{t-1} = s_{t-1}, \psi_{t-1})$ and:

$$\hat{v}_{it-1} = \sum_{s_{t-1},s_{t-2}=1}^{2} P(S_{t-1} = s_{t-1}, S_{t-2} = s_{t-2}|\psi_{t-1}) v_{it|s_{t-1}}. \tag{9}$$

Note that for any time point t, the conditional volatility depends only on the regimes in the current period and in the previous period. The dependence of lagged volatility on states in previous periods is integrated out by substituting the entire path dependent $\sigma^2_{it-1|s_{t-1},s_{t-2},\ldots}$ with $\hat{\sigma}^2_{it-1|s_{t-1}} = E(v^2_{t-1}|S_{t-1} = s_{t-1}, \psi_{t-1})$:

$$\hat{\sigma}^2_{it|s_t} = \sum_{s_{t-1}=1}^{2} P(S_{t-1} = s_{t-1}|S_t = s_t, \psi_{t-1}) \sigma^2_{it|s_t,s_{t-1}} \tag{10}$$

$$P(S_{t-1} = s_{t-1}|S_t = s_t, \psi_{t-1}) = \frac{P(S_t = s_t, S_{t-1} = s_{t-1}|\psi_{t-1})}{\sum_{s_{t-1}=1}^{2} P(S_t = s_t, S_{t-1} = s_{t-1}|\psi_{t-1})}. \tag{11}$$

The transition between the two latent states is modeled as a first-order Markov process with constant transition probabilities.[7] We define $\xi_{t|t-1}$ as:

$$\xi_{t|\tau} = \begin{bmatrix} P(S_t = 1, S_{t-1} = 1|\psi_\tau) \\ P(S_t = 1, S_{t-1} = 2|\psi_\tau) \\ P(S_t = 2, S_{t-1} = 1|\psi_\tau) \\ P(S_t = 2, S_{t-1} = 2|\psi_\tau) \end{bmatrix} \tag{12}$$

with ψ_τ denoting three possible information sets. For $\tau = t - 1$, we get the forecast probabilities, which are used to construct the loglikelihood function; for $\tau = t$, we get the filtered probabilities, which are a product of the updating algorithm; for $\tau = T$, we get the full-sample smoothed probabilities, which use all information and are helpful when making inference regarding states. The transition matrix from one point to another is given by:

$$\xi_{t|t-1} = P\xi_{t-1|t-1}. \tag{13}$$

Imbedded in this formula is that the previous regime is integrated out at each point in time. The transition matrix P is given by:

$$P = \begin{bmatrix} p & p & 0 & 0 \\ 0 & 0 & 1-q & 1-q \\ 1-p & 1-p & 0 & 0 \\ 0 & 0 & q & q \end{bmatrix} \tag{14}$$

[7]Alternatively, the transition probabilities may depend on the level of interest rates (see Gray, 1996). However, as acknowledged by Ang and Bekaert (2002a, p. 172), multi-factor models with time-varying transition probabilities are likely to be overparameterized, which leads to many insignificant coefficients in the probability terms.

where p is $P(S_t = 1, S_{t-1} = 1)$ and q is $P(S_t = 2, S_{t-1} = 2)$. We follow Hamilton (1994) and set the initial probability vector $\xi_{1|0}$ to the ergodic steady state probabilities. The conditional density $f(\Delta Y_t | S_t = s_t, S_{t-1} = s_{t-1}, \psi_{t-1})$ is a multivariate normal density with conditional mean $\mu_{t|s_t,s_{t-1},t-1} = \left\{\Delta\hat{Y}_{it}\right\}_{i=L,S,C}$ and conditional covariance matrix $\hat{\Sigma}_{t|s_t,s_{t-1},t-1} = \hat{H}^{1/2}_{t|s_t,s_{t-1}} \rho \hat{H}^{1/2}_{t|s_t,s_{t-1}}$.

Although the states are latent, the forecast probabilities can be used to calculate the joint density of ΔY_t and the states as:

$$f(\Delta Y_t, S_t = s_t, S_{t-1} = s_{t-1} | \psi_{t-1}) =$$

$$f(\Delta Y_t | S_t = s_t, S_{t-1} = s_{t-1}, \psi_{t-1}) \times P(S_t = s_t, S_{t-1} = s_{t-1} | \psi_{t-1}). \tag{15}$$

The marginal density of ΔY_t is found by integrating the joint density of ΔY_t over all possible states and is given by:

$$f(\Delta Y_t | \psi_{t-1}) = \sum_{s_t, s_{t-1}=1}^{2} f(\Delta Y_t, S_t = s_t, S_{t-1} = s_{t-1} | \psi_{t-1}). \tag{16}$$

The loglikelihood function is calculated as lnL= $\sum_{t=1}^{T} \log f(\Delta Y_t | \psi_{t-1})$ and is maximized to estimate the parameters. Finally the updated filter probabilities of the latent states (the appropriate elements of $\xi_{t|t}$) can be obtained using the definition of the conditional probability:

$$P(S_t = s_t, S_{t-1} = s_{t-1} | \psi_t) = \frac{f(\Delta Y_t, S_t = s_t, S_{t-1} = s_{t-1} | \psi_{t-1})}{f(\Delta Y_t | \psi_{t-1})}. \tag{17}$$

The various models fitted in this section recognize diverse sources of conditional heteroscedasticity:

- In the RS-NO GARCH-NO LEVEL model, conditional heteroscedasticity can only be driven by switches between regimes.
- In the RS-LEVEL model, conditional heteroscedasticity comes from either time-variation in the level of interest rates or from switches between regimes.
- In the RS-GARCH model, conditional heteroscedasticity is driven by serial correlation in volatility or by switches between regimes.

- In the RS-GARCH-LEVEL model, conditional heteroscedasticity comes from the three different sources of time-variation.

Parameter estimates and robust-standard errors for the regime-switching models are reported in Table 7. The regime-switching models outperform the models estimated in section IV (see Table 4), which is not overly surprising given that these models have been estimated under the assumption that there is only one regime. Allowing for multiple regimes dramatically improves the fit of all four models. Interestingly, when the volatility is allowed to switch from low to high-volatility regimes, the level effect is strengthened and the volatility persistence drops significantly. Furthermore, the performance of the RS-LEVEL model is higher than the performance of the RS-GARCH model, whereas the single-regime GARCH model outperformed the single-regime LEVEL model. Because of its lack of parsimony (it requires the estimation of 26 parameters) and its loglikelihood value, the RS-GARCH level is dominated by the RS-LEVEL model according to the Bayesian information criterion ($\text{BIC}_{RS-LEVEL} = 972.48$ vs. $\text{BIC}_{RS-GARCH} = 1046.59$). In the same way, the RS-LEVEL model is also preferred to the general RS-GARCH-LEVEL model ($\text{BIC}_{RS-GARCH-LEVEL} = 991.07$).

The four panels of Figure 3 contain plots of the smoothed probabilities of high-volatility state for the four considered regime-switching models. The probabilities have been computed using the smoothing algorithm of Kim (1994). Because of their multi-factor nature, our models exploit complementary information on the slope and curvature of the term structure. Our models identify all the major well-known episodes of extreme volatility: the 1973 OPEC oil crisis and its aftermath, the 1979-1982 monetary experiment, the October 1987 stock market crash, and the Russian Ruble devaluation in August 1998. Furthermore, we also identify a period in 1985, which is also identified in Gray (1996), with no clear economic interpretation. Interestingly, the two models that include level effects identify a high-volatility episode following September 11, 2001. This illustrates the importance of the level effect. Indeed, during this period the volatility of all three yield factors was only trivially elevated above previous levels, yet the short-rate was at historically low levels. This coincidence of low interest rates and lightly elevated volatility is explained as a high-volatility episode. This demonstrates that we need to be cautious when interpreting these regimes. A more precise interpretation is that the scaled residuals $e_{it}/Y_{Lt-1}^{\gamma_L}$ have high volatility.

< Insert Table 7 >

< Insert Figure 3 >

VII Potential Application: Comparing Term Structure Models

In this section, we describe how the key stylized facts of yield factors identified in this paper could potentially be used to evaluate and compare existing term structure models. In the spirit of Brandt and Chapman (2003), we propose a set of economic moments based on the empirical regularities exhibited by yield factors. These moments can be used to contrast multi-factor term structure models using a simulated moments estimator of the type described in Duffie and Singleton (1993). The underpinning idea behind this approach is that the best model is the one that is doing the best job in capturing the central features of U.S. Treasury yields. An exhaustive empirical application is beyond the scope of our paper and we leave it for future research.

The term structure models to be compared may consist of the following classes of models: (1) multi-factor affine term structure models of Duffie and Kan (1996) and Dai and Singleton (2000); (2) regime-switching Gaussian multi-factor term structure models of Dai, Singleton and Yang (2003); (3) and multi-factor quadratic term structure models of Ahn, Dittmar and Gallant (2002) and Leippold and Wu (2002). To find the term structure model that is the most consistent with yield-factor dynamics, the simulated methods of moments can be used. The idea in this approach is to simulate yields from a candidate term structure model and see how closely the moments from these artificial yields compare with the important economic moments we observe in the real data. As Brandt and Chapman (2003) point out, the key advantages of the simulated-moment approach are that it can be used to contrast models that are not nested and does not require that the likelihood function be known in closed-form. Furthermore, it can explicitly identify those moments which each term structure model has difficulty matching. The approach can therefore suggest directions in which term structure models may be extended.

We suggest using the following set of moments which capture the stylized facts identified in the empirical term structure literature and the key features of the yield-factor volatility that we identify. In particular, we recommend including for each of the three yield factors: the unconditional mean, residual standard deviation, and (residual) autocorrelation coefficient, along with the contemporaneous correlations and (residual) cross-autocorrelations. The next moments are the slope coefficients from a regression of yield changes on the slope of the yield curve, which is termed the LPY regression. In disagreement with the expectations hypothesis, Dai and Singleton (2003), among others, find that this slope coefficient is positive and increases for longer maturities. If regressions are based on two and ten year maturities we obtain two extra moments. Moments related to conditional volatility can also be used. Brandt and Chapman (2003) suggest using the slope coefficients from a regression of the squared holding period return on the three yield factors, which is termed the LPV regression. If the regression is run using two bond maturities then six moments are identified. However, in light of the empirical results reported above, we suggest the LPV slope coefficients be replaced with some combination of the following moments:

- the GARCH parameters β_{1i} and β_{2i} of the three yield factors,
- the elasticity parameters γ_i of the three yield factors,
- the correlation parameters ρ_{ij} between the yield-factor residuals,
- and the unconditional probability of being in the high-volatility regime.

The level-based moment (γ_i) extends the CKLS testing procedure to the multivariate framework of yield factors. An empirically successful term structure models should imply persistence in the yield-factor volatility, level effect, correlations, and switches between volatility regimes that are comparable to the ones observed in historical data.

VIII Conclusion

In this paper, we develop a comprehensive model for volatility dynamics in the level, slope, and curvature factors that simultaneously includes level and GARCH effects along with regime shifts. The analysis in this paper leads to the following conclusions. First, we show that the level of the short-rate is useful in modeling the volatility of the level, slope, and curvature factors. Second, there is significant GARCH effects present even after including a level effect. Third, we find that the GARCH-based volatility of the overall level of interest rates is negatively related to the level and curvature and positively related to the slope of the yield curve, which is consistent with a "flight-to-cash". Fourth, when the volatility is allowed to switch from low to high-volatility regimes, the model's fit improves dramatically, the level effect is strengthened, and the volatility persistence drops significantly. Finally, we discuss how the central features of U.S. Treasury yields identified in this paper can be used to compare existing term structure models.

The encouraging results obtained with our regime-switching models strengthen the need for including regime-shifts in theoretical term structure models, from both the affine and quadratic classes. From this respect, the recent contribution of Dai, Singleton and Yang (2003), which develops a regime-switching, Gaussian dynamic term structure model, is a promising endeavor on this challenging avenue of research.

The present econometric model can be expanded to include additional features, such as non-linear drifts, asymmetric effects both in the drift and in the diffusion, or jumps in the yield-factor dynamics. Perhaps more importantly, the present version of the model offers enough flexibility to allow some state variables, such as macroeconomic variables, to enter into the dynamics of the yield factors. These extensions are left for future research.

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IX Tables and Figures

Table 1: Descriptive Statistics

	L	S	C	e_L	e_S	e_C
Mean	6.4947	1.3590	-0.0428	-	-	-
Variance	7.7562	2.0399	0.5712	0.3681	0.2633	0.1938
Skewness	1.0471	-0.5966	0.0435	-1.1751	0.7211	-0.6965
Kurtosis	4.4361	3.1934	4.0297	14.6185	9.3072	7.8707
BJ	106.39	24.11	17.62	2318.47	690.70	423.45
Corr(L,i)	-	-0.6180	-0.4070	-	-0.8134	-0.0533
Corr(S,C)	-	-	0.1404	-	-	-0.1448
CACorr(L_{t-1},i)	0.9711	-0.5928	-0.4043	0.1219	-0.0336	-0.2118
CACorr(S_{t-1},i)	-0.5721	0.9299	0.1794	-0.1074	0.0463	0.1779
CACorr(C_{t-1},i)	-0.3930	0.1339	0.8044	-0.0385	-0.0624	-0.0920
BP_1	376.24	345.01	258.20	5.93	0.86	3.38
BP_{12}	3325.19	2063.31	1051.63	42.29	22.12	35.48

Note: This table presents the mean, variance, skewness, and kurtosis of the three yield factors (level L, slope S, curvature C) and yield-factor residuals (e_L, e_S, e_C). BJ stands for the Bera-Jarque normality test, Corr for correlation, CACorr for first-order cross-autocorrelation, and BP_1 and BP_{12} for the Box-Pierce test with one and twelve lags respectively. The latter two statistics are distributed as chi-squared with 1 and 12 degrees of freedom and then the 5 percent critical values are 3.84 and 21.03 respectively.

Table 2: Robust Tests for Level Effect and Heteroscedasticity

$\lambda(F_{t-1})$	L	S	C
$e^2_{t-1}\ e^2_{t-2}\ ...\ e^2_{t-6}$	14.6891 [0.0228]	15.9148 [0.0142]	21.8449 [0.0013]
${Y_{L,t-1}}^{0.5}$	7.2969 [0.0069]	6.8220 [0.0090]	6.5559 [0.0105]
$Y_{L,t-1}$	7.8533 [0.0051]	8.8714 [0.0029]	7.1608 [0.0075]
$\lvert Y_{j,t-1}\rvert^{0.5}$	-	0.6067 [0.4360]	2.8213 [0.0930]
$\lvert Y_{j,t-1}\rvert$	-	0.2095 [0.6472]	3.2472 [0.0715]

Note: This table reports the Wooldridge (1990) robust specification tests for level and ARCH effects in yield-factor volatilities. This conditional moment test determines whether $E[(e_t^2 - \hat{\sigma}^2)\lambda(F_{t-1})] = 0_{K\times 1}$ for some K-dimensional vector $\lambda(F_{t-1})$. The robust test for p^{th} order ARCH effects is obtained using $\lambda(F_{t-1})^\intercal = (e^2_{t-1}\ e^2_{t-2}\ ...\ e^2_{t-p})$ and the robust test for level effects is derived with $\lambda(F_{t-1}) = |Y_{j,t-1}|^\gamma$ with $\gamma = 0.5, 1$. We report the Wooldridge's statistic with the associated p-values into brackets.

Table 3: Parameter Estimates for Univariate Models

	NO LEVEL-NO GARCH			LEVEL			GARCH			GARCH-LEVEL		
	L	S	C	L	S	C	L	S	C	L	S	C
α_0	0.1349 (0.1094)	0.0889 (0.0556)	-0.0045 (0.0214)	-0.0001 (0.0118)	0.0744 (0.0387)	0.0126 (0.0203)	0.1135 (0.0549)	0.0424 (0.0426)	-0.0150 (0.0177)	0.0326 (0.0161)	0.0330 (0.0386)	-0.0157 (0.0182)
α_1	-0.0220 (0.0196)	-0.0661 (0.0281)	-0.1820 (0.0417)	-0.0020 (0.0012)	-0.0448 (0.0168)	-0.1487 (0.0384)	-0.0230 (0.0128)	-0.0420 (0.0208)	-0.1121 (0.0254)	-0.0046 (0.0023)	-0.0356 (0.0197)	-0.1146 (0.0260)
β_0	0.3681 (0.0699)	0.2633 (0.0379)	0.1938 (0.0253)	2.5210 (1.0076)	20.404 (6.4583)	63.638 (23.870)	5.8937 (3.4347)	15.947 (9.8622)	23.613 (7.4734)	0.1443 (0.1915)	1.7201 (1.8911)	14.066 (15.773)
β_1	-	-	-	-	-	-	0.2935 (0.0790)	0.2318 (0.1024)	0.3516 (0.0995)	0.2263 (0.1087)	0.1604 (0.0728)	0.3353 (0.0905)
β_2	-	-	-	-	-	-	0.6962 (0.0795)	0.6998 (0.1184)	0.5554 (0.0779)	0.7516 (0.1147)	0.7737 (0.1023)	0.5617 (0.0799)
γ	-	-	-	1.2022 (0.1069)	0.6410 (0.0872)	0.2961 (0.1046)	-	-	-	1.1062 (0.3342)	0.5754 (0.2296)	0.1563 (0.3401)
$\ln L_i$	-365.59	-298.68	-237.60	-228.70	-245.00	-226.20	-199.81	-216.15	-186.28	-185.27	-210.32	-185.94

Notes: The general model used is the GARCH-LEVEL model:

$$\Delta Y_{it} = \alpha_{0i} + \alpha_{1i} \cdot Y_{it-1} + e_{it}$$

for $i = L, S, C$ and $t = 1, ..., T$. The conditional volatility of e_{it} is $E(e_{it}^2|\psi_{t-1}) = \sigma_{it}^2 Y_{Lt-1}^{2\gamma_i}$. We model σ_{it}^2, the volatility of the scaled residual $v_{it} = e_{it}/Y_{Lt-1}^{\gamma_i}$, as:

$$\sigma_{it}^2 = \beta_{0i} + \beta_{1i} \cdot v_{it-1}^2 + \beta_{2i} \cdot \sigma_{it-1}^2.$$

The restricted versions of the model impose the following constraints: NO LEVEL-NO GARCH model: $\beta_{ki} = 0$, $i = L, S, C$ and $k = 1, 2$, and $\gamma_i = 0$, $i = L, S, C$; LEVEL model: $\beta_{ki} = 0$, $i = L, S, C$ and $k = 1, 2$; GARCH model: $\gamma_i = 0$, $i = L, S, C$. For each univariate model, we report the maximum likelihood parameter estimates, the value of the loglikelihood, and robust standard errors into parentheses. β_0 parameters are multiplied by 1000 and so are they in Tables 4-7.

Table 4: Parameter Estimates for Multivariate Models

	NO GARCH-NO LEVEL			LEVEL			GARCH			GARCH-LEVEL		
	L	S	C	L	S	C	L	S	C	L	S	C
α_0	0.1256 (0.0575)	0.0744 (0.0347)	-0.0023 (0.0212)	0.0266 (0.0274)	0.0547 (0.0275)	0.0153 (0.0204)	0.1200 (0.0559)	0.0427 (0.0385)	-0.0109 (0.0174)	0.0461 (0.0462)	0.0245 (0.0208)	-0.0130 (0.0172)
α_1	-0.0212 (0.0106)	-0.0541 (0.0147)	-0.1586 (0.0416)	-0.0069 (0.0051)	-0.0359 (0.0114)	-0.1435 (0.0373)	-0.0215 (0.0127)	-0.0391 (0.0164)	-0.1169 (0.0260)	-0.0073 (0.0076)	-0.0305 (0.0130)	-0.1189 (0.0271)
β_0	0.3682 (0.0697)	0.2633 (0.0383)	0.1941 (80.0254)	6.0018 (1.9586)	38.621 (10.609)	54.799 (20.112)	7.9016 (7.0820)	9.1544 (11.1589)	21.153 (6.8654)	0.5619 (0.4937)	4.7968 (5.2924)	9.3270 (10.768)
β_1	-	-	-	-	-	-	0.1406 (0.1138)	0.0960 (0.0659)	0.3550 (0.0977)	0.0905 (0.0546)	0.0901 (0.0673)	0.3375 (0.0918)
β_2	-	-	-	-	-	-	0.8117 (0.1350)	0.8522 (0.1206)	0.5700 (0.0783)	0.8463 (0.0897)	0.8257 (0.1516)	0.5718 (0.0862)
γ	-	-	-	0.9688 (0.0800)	0.4684 (0.0729)	0.3384 (0.1034)	-	-	-	0.7997 (0.1865)	0.3144 (0.1666)	0.2536 (0.3422)
$\rho_{L,S}$	-0.8076 (0.0324)	-	-	-0.7123 (0.0346)	-	-	-0.6611 (0.0375)	-	-	-0.6508 (0.0364)	-	-
$\rho_{i,C}$	-0.0465 (0.0999)	-0.1550 (0.0959)	-	-0.1068 (0.0825)	-0.1612 (0.0788)	-	-0.1616 (0.0645)	-0.1323 (0.0729)	-	-0.1754 (0.0623)	-0.1330 (0.0697)	-
lnL	-670.637			-544.633			-477.183			-458.124		

Notes: The general model used is the GARCH-LEVEL model:

$$\Delta Y_{it} = \alpha_{0i} + \alpha_{1i} \cdot Y_{it-1} + e_{it}$$

for $i = L, S, C$ and $t = 1, ..., T$. The conditional volatility of e_{it} is $E(e_{it}^2|\psi_{t-1}) = \sigma_{it}^2 Y_{Lt-1}^{2\gamma_i}$. We model σ_{it}^2, the volatility of the scaled residual $v_{it} = e_{it}/Y_{Lt-1}^{\gamma_i}$, as:

$$\sigma_{it}^2 = \beta_{0i} + \beta_{1i} \cdot v_{it-1}^2 + \beta_{2i} \cdot \sigma_{it-1}^2.$$

The restricted versions of the model impose the following constraints: NO LEVEL-NO GARCH model: $\beta_{ki} = 0$, $i = L, S, C$ and $k = 1, 2$, and $\gamma_i = 0$, $i = L, S, C$; LEVEL model: $\beta_{ki} = 0$, $i = L, S, C$ and $k = 1, 2$; GARCH model: $\gamma_i = 0$, $i = L, S, C$. For each multivariate model, we report the maximum likelihood parameter estimates, the value of the loglikelihood, and robust standard errors into parentheses.

Table 5: Parameter Estimates for GARCH Models with Volatility Effect in the Mean Equation

	Level Volatility in the Mean			Own Volatility in the Mean			Both Volatilities in the Mean		
	L	S	C	L	S	C	L	S	C
α_0	0.1448 (0.0609)	-0.0043 (0.0189)	0.0120 (0.0456)	0.1364 (0.0562)	-0.0075 (0.0248)	-0.0501 (0.0454)	0.1428 (0.0620)	-0.0017 (0.0167)	-0.0442 (0.0502)
α_1	-0.0214 (0.0090)	-0.0371 (0.0115)	-0.1233 (0.0305)	-0.0200 (0.0083)	-0.0413 (0.0138)	-0.1149 (0.0257)	-0.0210 (0.0091)	-0.0358 (0.0129)	-0.1274 (0.0323)
α_2	-0.1585 (0.0710)	0.2003 (0.0877)	-0.0618 (0.1381)	-0.1627 (0.0691)	-	-	-0.1625 (0.0706)	0.2146 (0.1936)	-0.1439 (0.1551)
α_3	-	-	-	-	0.2192 (0.1124)	0.1237 (0.1294)	-	-0.0269 (0.2130)	0.2610 (0.1782)
β_0	7.3735 (5.3312)	8.3114 (8.3508)	21.113 (6.9928)	7.4291 (5.0282)	7.5847 (6.3685)	20.9037 (6.9970)	7.5215 (5.5351)	8.5162 (9.2012)	19.916 (7.7929)
β_1	0.1411 (0.0825)	0.0912 (0.0475)	0.3543 (0.0977)	0.1373 (0.0728)	0.0900 (0.0415)	0.3542 (0.1020)	0.1438 (0.0849)	0.0925 (0.0514)	0.3362 (0.1072)
β_2	0.8137 (0.0984)	0.8617 (0.0898)	0.5697 (0.0807)	0.8169 (0.0878)	0.8677 (0.0712)	0.5721 (0.0837)	0.8102 (0.1014)	0.8591 (0.0989)	0.5895 (0.1014)
$\rho_{L,S}$	-0.6580 (0.0354)	-	-	-0.6565 (0.0351)	-	-	-0.6573 (0.0352	-	-
$\rho_{i,C}$	-0.1683 (0.0627)	-0.1260 (0.0724)	-	-0.1667 (0.0630)	-0.1292 (0.0729	-	-0.1623 (0.0618)	-0.1287 (0.0717)	-
lnL		-471.340			-471.936			-470.200	

Notes: The general model used is the GARCH-in-Mean model:

$$\begin{aligned} \Delta Y_{Lt} &= \alpha_{0L} + \alpha_{1L} \cdot Y_{Lt-1} + \alpha_{2L} \cdot \sigma_{Lt} + e_{Lt} \\ \Delta Y_{jt} &= \alpha_{0j} + \alpha_{1j} \cdot Y_{jt-1} + \alpha_{2j} \cdot \sigma_{Lt} + \alpha_{3j} \cdot \sigma_{jt} + e_{jt} \end{aligned}$$

for $j = S, C$, and $t = 1, ..., T$. The conditional volatility of e_{it} is $E(e_{it}^2|\psi_{t-1}) = \sigma_{it}^2$ and is modeled as:

$$\sigma_{it}^2 = \beta_{0i} + \beta_{1i} \cdot e_{it-1}^2 + \beta_{2i} \cdot \sigma_{it-1}^2$$

for $i = L, S, C$. The restricted versions of the model impose $\alpha_{3j} = 0$ and $\alpha_{2j} = 0$, respectively. For each model, we report the maximum likelihood parameter estimates, the value of the loglikelihood, and robust standard errors into parentheses.

Table 6: Parameter Estimates for GARCH-LEVEL Models with Volatility Effect in the Mean Equation

	Level Volatility in the Mean			Own Volatility in the Mean			Both Volatilities in the Mean		
	L	S	C	L	S	C	L	S	C
α_0	0.0607 (0.0579)	-0.0020 (0.0169)	0.0234 (0.0552)	0.0570 (0.0500)	0.0010 (0.0152)	-0.0394 (0.0467)	0.0583 (0.0591)	-0.0002 (0.0123)	-0.0291 (0.0560)
α_1	-0.0092 (0.0088)	-0.0308 (0.0112)	-0.1309 (0.0351)	-0.0086 (0.0075)	-0.0333 (0.0125)	-0.1170 (0.0265)	-0.0088 (0.0089)	-0.0295 (0.0134)	-0.1351 (0.0367)
α_2	-0.0443 (0.0658)	0.1208 (0.0831)	-0.1060 (0.1662)	-	-	-	-0.0474 (0.0642)	0.1444 (0.2297)	-0.2119 (0.2065)
α_3	-	-	-	-0.0479 (0.0622)	0.1156 (0.0848)	0.0791 (0.1394)	-	-0.0344 (0.2372)	0.2732 (0.2003)
β_0	0.6309 (0.5931)	5.4258 (6.6441)	9.7330 (10.4790)	0.6313 (0.5664)	4.9752 (5.6952)	9.5943 (11.1897)	0.6498 (0.6286)	5.5824 (6.9829)	9.4357 (10.0757)
β_1	0.0973 (0.0633)	0.0913 (0.0722)	0.3318 (0.0938)	0.0919 (0.0537)	0.0883 (0.0662)	0.3380 (0.0941)	0.0993 (0.0680)	0.0932 (0.0758)	0.3052 (0.1165)
β_2	0.8384 (0.1033)	0.8255 (0.1680)	0.5792 (0.0918)	0.8457 (0.0888)	0.8345 (0.1536)	0.5735 (0.0891)	0.8343 (0.1116)	0.8216 (0.1748)	0.6125 (0.1314)
γ	0.7749 (0.2012)	0.2778 (0.1615)	0.2328 (0.3143)	0.7649 (0.1935)	0.2812 (0.1725)	0.2416 (0.3511)	0.7760 (0.2005)	0.2764 (0.1603)	0.2098 (0.3135)
$\rho_{L,S}$	-0.6502 (0.0361)	-	-	-0.6501 (0.0361)	-	-	-0.6502 (0.0358)	-	-
$\rho_{i,C}$	-0.1780 (0.0619)	-0.1284 (0.0702)	-	-0.1748 (0.0623)	-0.1318 (0.0703)	-	-0.1700 (0.0610)	-0.1314 (0.0701)	-
lnL		-456.312			-456.911			-455.218	

Notes: The general model used is the GARCH-LEVEL-in-Mean model:

$$\begin{aligned}\Delta Y_{Lt} &= \alpha_{0L} + \alpha_{1L} \cdot Y_{Lt-1} + \alpha_{2L} \cdot \sigma_{Lt} + e_{Lt} \\ \Delta Y_{jt} &= \alpha_{0j} + \alpha_{1j} \cdot Y_{jt-1} + \alpha_{2j} \cdot \sigma_{Lt} + \alpha_{3j} \cdot \sigma_{jt} + e_{jt}\end{aligned}$$

for $j = S, C$, and $t = 1, ..., T$. The conditional volatility of e_{it} is $E(e_{it}^2|\psi_{t-1}) = \sigma_{it}^2 Y_{Lt-1}^{2\gamma_i}$. We model σ_{it}^2, the volatility of the scaled residual $v_{it} = e_{it}/Y_{Lt-1}^{\gamma_i}$, as:

$$\sigma_{it}^2 = \beta_{0i} + \beta_{1i} \cdot v_{it-1}^2 + \beta_{2i} \cdot \sigma_{it-1}^2$$

for $i = L, S, C$. The restricted versions of the model impose $\alpha_{3j} = 0$ and $\alpha_{2j} = 0$, respectively. For each model, we report the maximum likelihood parameter estimates, the value of the loglikelihood, and robust standard errors into parentheses.

Table 7: Parameter Estimates for Regime-Switching Models

	RS-NO GARCH-NO LEVEL			RS-LEVEL			RS-GARCH			RS-GARCH-LEVEL		
	L	S	C	L	S	C	L	S	C	L	S	C
$\alpha_{0,S1}$	-0.0041 (0.1577)	0.0866 (0.1199)	-0.0970 (0.0905)	-0.0133 (0.0312)	0.1077 (0.0473)	0.0517 (0.0599)	-0.0172 (0.0966)	0.1247 (0.0785)	-0.0650 (0.0525)	-0.0186 (0.0713)	0.1423 (0.0793)	0.0140 (0.0643)
$\alpha_{0,S2}$	0.0589 (0.0403)	0.0238 (0.0221)	0.0197 (0.0160)	0.0988 (0.0669)	0.0058 (0.0199)	-0.0021 (0.0237)	0.0707 (0.0066	0.0109 (0.0113)	0.0013 (0.0146)	0.1180 (0.2338)	0.0143 (0.0148)	-0.0083 (0.0207)
α_1	-0.0091 (0.0065)	-0.0255 (0.0107)	-0.1317 (0.0287)	-0.0123 (0.0086)	-0.0302 (0.0120)	-0.1435 (0.0401)	-0.0101 (0.0010)	-0.0240 (0.0094)	-0.1165 (0.0244)	-0.0166 (0.0322)	-0.0328 (0.0204)	-0.1281 (0.0386)
$\beta_{0,S1}$	1171.5597 (139.3382)	690.2253 (103.8130)	665.8439 (122.0522)	5.7746 (2.2116)	45.3573 (14.2848)	87.7479 (37.1780))	413.0212 (81.3628)	313.8057 (61.2716)	268.0241 (97.5286)	5.1311 (2.2832)	44.4600 (18.5021)	62.0588 (34.3799)
$\beta_{0,S2}$	100.8820 (15.9410)	104.8961 (9.7203)	93.3185 (11.1286)	1.1366 (0.5002)	10.8774 (3.7007)	12.4080 (6.1394)	5.2555 (3.4850)	18.7982 (14.0511)	11.2662 (5.8319)	0.1904 (0.2545)	5.7374 (5.6103)	2.9259 (2.4358)
β_1	-	-	-	-	-	-	0.2403 (0.0978)	0.0944 (0.0466)	0.3148 (0.0952)	0.1372 (0.0770)	0.0457 (0.0459)	0.2506 (0.0820)
β_2	-	-	-	-	-	-	0.1161 (0.0218)	0.1863 (0.0477)	0.1381 (0.0462)	0.1940 (0.0533)	0.1440 (0.1087)	0.1282 (0.0440)
γ	-	-	-	1.1248 (0.1034)	0.5604 (0.0882)	0.4141 (0.1177)	-	-	-	1.0380 (0.1151)	0.5033 (0.0969)	0.3837 (0.1391)
$\rho_{L,S}$	-0.6356 (0.0358)	-	-	-0.6069 (0.0379)	-	-	-0.6648 (0.0403)	-	-	-0.6102 (0.0386)	-	-
$\rho_{i,C}$	-0.2162 (0.0558)	-0.0959 (0.0591)	-	-0.1630 (0.0616)	-0.1794 (0.0628)	-	-0.2053 (0.0602)	-0.1247 (0.0648)	-	-0.1981 (0.0604)	-0.1493 (0.0593)	-
p_{S1}		2.1466 (0.5189)			2.0022 (0.3964)			2.4406 (0.6495)			2.2527 (0.5224)	
p_{S2}		3.7348 (0.6152)			2.4183 (0.3518)			3.3285 (0.5144)			2.7390 (0.5095)	
lnL		-463.894			-417.453			-445.539			-408.8048	

Notes: The general model used is the Regime-Switching GARCH-LEVEL model:

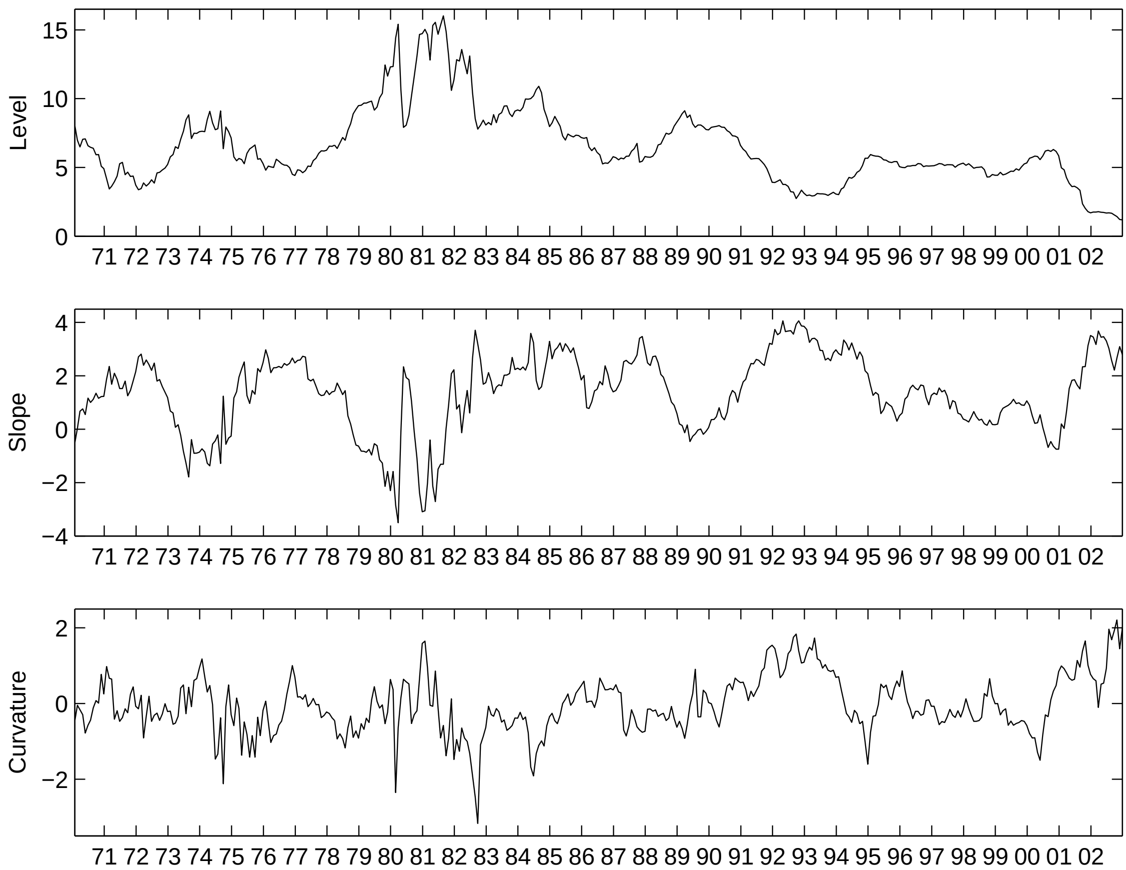
$$\Delta Y_{it} = \alpha_{0is_t} + \alpha_{1i} \cdot Y_{it-1} + e_{it}$$

for $i = L, S, C$ and $t = 1, ..., T$, where $s_t = 1$ denotes the "high-volatility regime" and $s_t = 2$ the "low-volatility regime". The conditional volatility of e_{it} is $E(e_{it}^2|\psi_{t-1}) = \hat{\sigma}^2_{it|s_t,s_{t-1}} Y_{Lt-1}^{2\gamma_i}$ where $\hat{\sigma}^2_{it|s_t,s_{t-1}} = E(v_t^2|S_t = s_t, S_{t-1} = s_{t-1}, \psi_{t-1})$ is modeled as:

$$\hat{\sigma}^2_{it|s_t,s_{t-1}} = \beta_{0is_t} + \beta_{1i} \cdot v_{it-1}^2 + \beta_{2i} \cdot \hat{\sigma}^2_{it-1|s_{t-1}}.$$

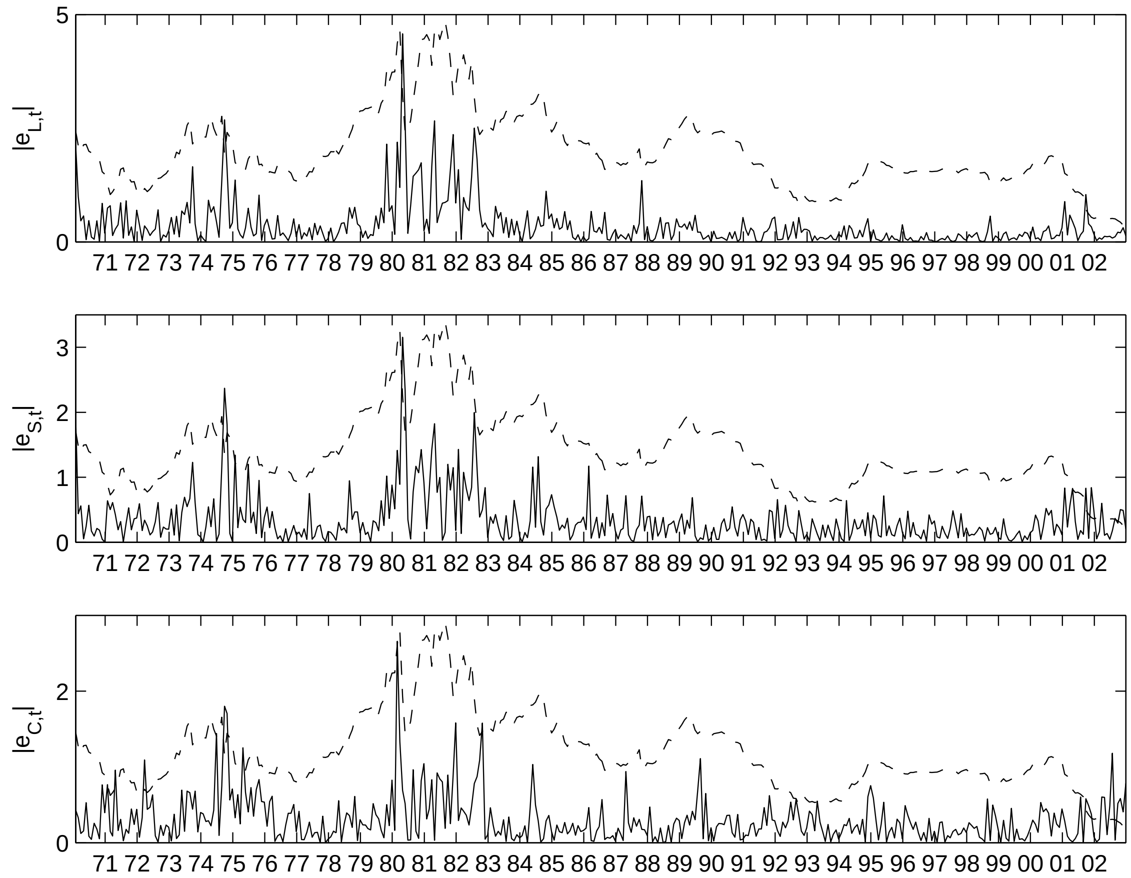
The probability of staying in state j is modeled as $P(S_t = j|P_{t-1} = j) = \exp(p_{Sj})/(1+\exp(p_{Sj}))$. The restricted versions of the model impose the following constraints: RS-NO LEVEL-NO GARCH model: $\beta_{ki} = 0$, $i = L, S, C$ and $k = 1, 2$, and $\gamma_i = 0$, $i = L, S, C$; RS-LEVEL model: $\beta_{ki} = 0$, $i = L, S, C$ and $k = 1, 2$; RS-GARCH model: $\gamma_i = 0$, $i = L, S, C$. For each model, we report the maximum likelihood parameter estimates, the value of the loglikelihood, and robust standard errors into parenthesis.

Figure 1: Value of the Level, Slope, and Curvature Factors



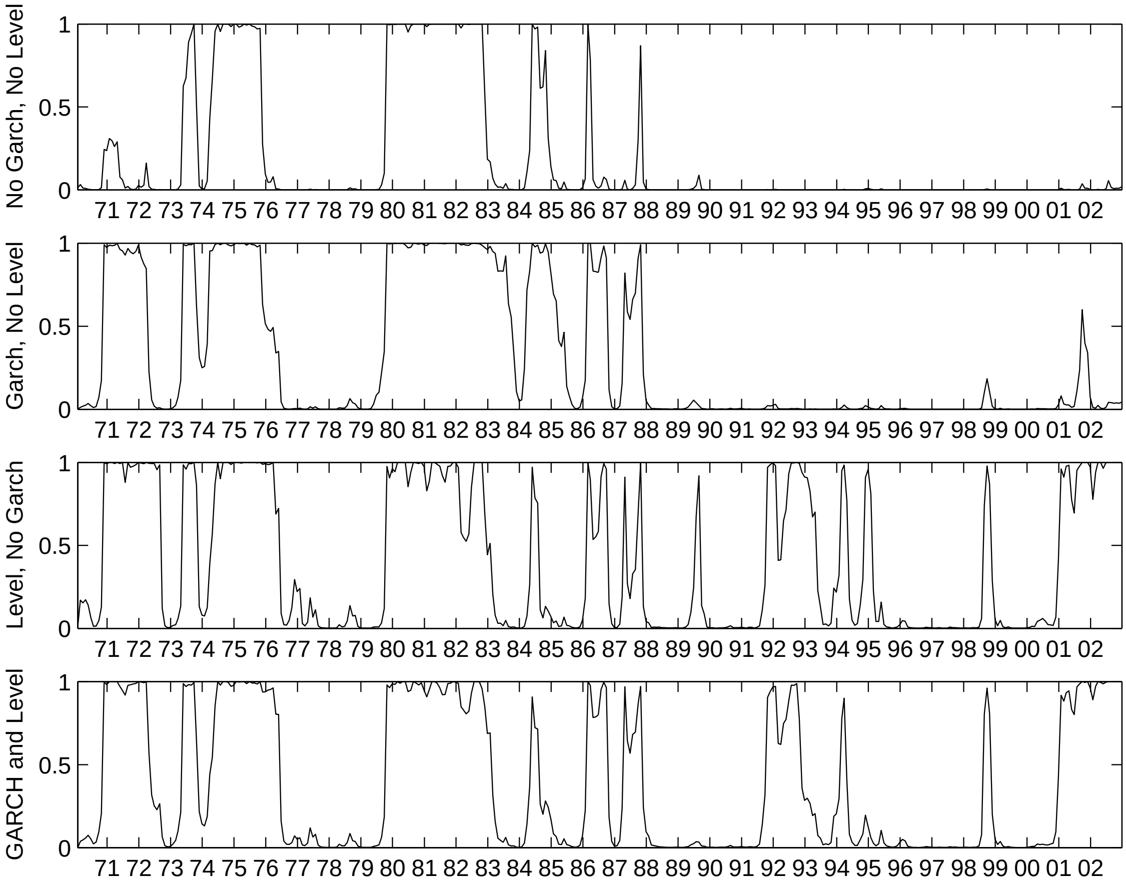
Notes: This figure displays the series for the three yield factors. The level factor is associated with the 3-month yield, the slope factor with the difference between the 120-month yield and the 3-month yield, and the curvature factor with a linear transformation of the short (1-month), medium (24-month), and long-term (120-month) yields.

Figure 2: Factor Volatilities and the Level Factor



Notes: This figure displays the absolute value of the factor residuals (left Y-axis, solid line) with the level factor (right Y-axis, dashed line). $|e_{L,t}|$ denotes the absolute level residual, $|e_{S,t}|$ the absolute slope residual, and $|e_{C,t}|$ the absolute curvature residual.

Figure 3: Smoothed Probability of High-Volatility State $P(S_t|\psi_T)$



Notes: The four panels contain the time series of the smoothed probabilities that the level factor is in the high-volatility regime at time t according to the Regime Switching (RS) NO GARCH - NO LEVEL model, the RS LEVEL model, the RS GARCH model, and the RS GARCH - LEVEL model. The smoothed probability is based on the entire sample: $P(S_t = 1|\psi_T)$.